Scalars and Vectors.

1. Scalar → Magnitude.

Example $T = 300\ k$

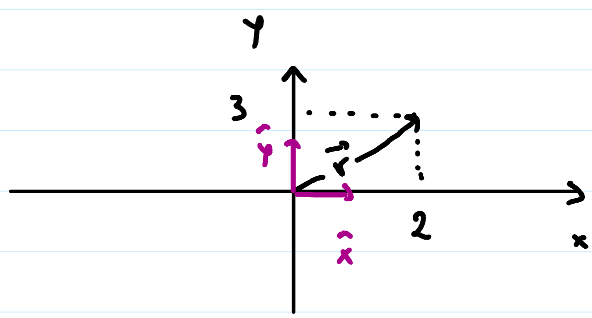
$m = 80\ kg$

2. Vector → Magnitude and direction.

Example $\vec{v} = 60$ miles / hour nourth

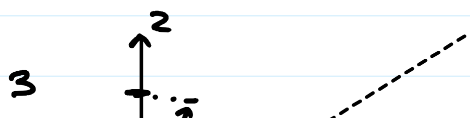
Vectors.

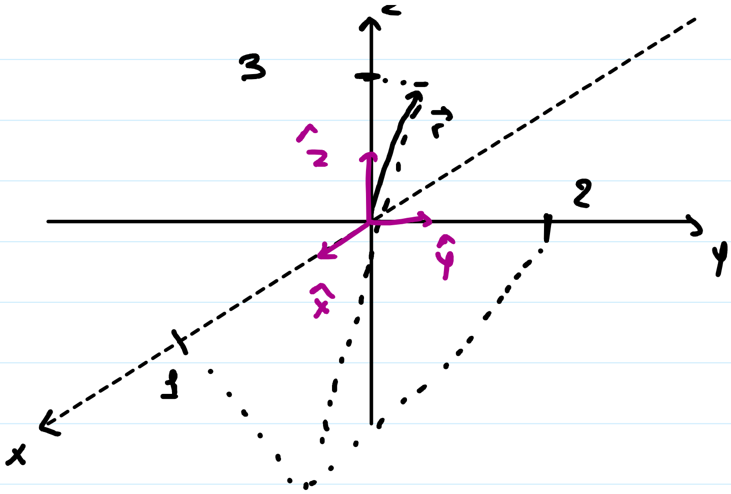
2D Universe:



$$\vec{r} = 2\hat{x} + 3\hat{y} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

3D Universe





$$\vec{r} = 1\hat{x} + 2\hat{y} + 3\hat{z} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

4D Universe : We can't draw it !

$$\vec{r} = \begin{pmatrix} 0.7 \\ 0 \\ 0.7 \\ 0 \end{pmatrix}$$

We need matrices to understand Quantum mechanics

Heissenberg
Born
Jordan

$\Big\}$ 1925 → Matrix Formulation of QM

1. Definition of a matrix

$$A = \begin{pmatrix} a_{00} & a_{01} & \cdots & a_{0m} \\ a_{10} & a_{11} & \cdots & a_{1m} \\ \vdots & \vdots & & \vdots \\ a_{n0} & a_{n1} & \cdots & a_{nm} \end{pmatrix}$$

$A \in \mathbb{R}^{n \times m}$ or $A \in \mathbb{C}^{n \times m}$

$n :=$ rows

$m :=$ columns

Example 1. $A = \begin{pmatrix} 1 & 4 \\ 6i & 5 \\ 5 & 7 \end{pmatrix}$ $A \in \mathbb{C}^{3 \times 2}$

$a_{01} = 4$
$a_{20} = 5$

Example 2. $B = \begin{pmatrix} 1 & 6 & 5 \\ 4 & 5 & 7 \end{pmatrix}$ $B \in \mathbb{R}^{2 \times 3}$

Example 3. $\varphi = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\varphi \in \mathbb{R}^{2\times 1}$

$\varphi_{00} = 1$
$\varphi_{10} = 0$

2. Basic Operations

1. Addition / Substraction

Restriction : Must have the same size

Rule : Add / Substract elements that are in the same position

Example 1. $A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ $B = \begin{pmatrix} 0 & -j \\ j & 0 \end{pmatrix}$

$\varphi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\varphi_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$A + \varphi_1$: Not defined!

$A + \varphi_2$: — " —

$$A + B = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} + \begin{pmatrix} 0 & -j \\ j & 0 \end{pmatrix} =$$

$$= \begin{pmatrix} 1 & -j \\ j & 2 \end{pmatrix}$$

$$\varphi_1 + \varphi_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Example 2.

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$I - Z = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$$

2. Scalar Multiplication

Rule: Each element is multiplied by the scalar

$$A = \begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix}$$

$$cA = c \begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix} = \begin{pmatrix} c\,a_{00} & c\,a_{01} \\ c\,a_{10} & c\,a_{11} \end{pmatrix}$$

Example 1.

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$c = j$$

$$cH = j \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} \frac{j}{\sqrt{2}} & \frac{j}{\sqrt{2}} \\ \frac{j}{\sqrt{2}} & \frac{j}{\sqrt{2}} \end{pmatrix}$$

Example 2.

$$\varphi = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad c\,\varphi = \begin{pmatrix} -j \\ 0 \end{pmatrix}$$

$c = -j$

4. Transposition

Denoted : A^T

Rule : Turn rows into columns

If $A \in \mathbb{R}^{n \times m}$ $\qquad A^T \in \mathbb{R}^{m \times n}$

Example 1. $\varphi = \begin{pmatrix} 0.7 \\ 0 \\ 0 \\ 6.7 \end{pmatrix}$

$$\varphi^T = \begin{pmatrix} 0.7 & 0 & 0 & 0.7 \end{pmatrix}$$

Example 2.

$$P = \begin{pmatrix} 1 & 0 \\ 0 & e^{j\theta} \end{pmatrix} \qquad P^T = \begin{pmatrix} 1 & 0 \\ 0 & e^{j\theta} \end{pmatrix}$$

$(P^T = P \,!)$

5. Dagger : Complex Conjugate and Transpose

$$A^\dagger = (\bar{A})^T$$

Example 1. $\psi = \begin{pmatrix} j/\sqrt{2} \\ 0 \\ 0 \\ j/\sqrt{2} \end{pmatrix}$

$$\psi^\dagger = \begin{pmatrix} -j/\sqrt{2} & 0 & 0 & -j/\sqrt{2} \end{pmatrix}$$

Example 2.

$$A = \begin{pmatrix} j & 0 \\ -3j & 4 \end{pmatrix} \rightarrow A^\dagger = \begin{pmatrix} -j & 3j \\ 0 & 4 \end{pmatrix}$$

6. Matrix Multiplication

Quantum Gates $\rightarrow$ Operates $\longrightarrow$ Quantum States

$\downarrow$ $\qquad\qquad$ $\downarrow$ $\qquad\qquad$ $\downarrow$

$n \times m$ matrices $\qquad$ matrix Multiplication $\qquad$ $n \times 1$ matrices

$$A \quad B \quad = \quad C$$

$$\begin{array}{ccc} A & B & C \\ \downarrow & \downarrow & \uparrow \\ n \times m & p \times q & n \times q \end{array}$$

Restriction: The matrix multiplication is defined only if $m = p$

The result is an $n \times q$ matrix

Rule: In order to find the c_{ij}, multiply the i-th row of A by the j-th column of B element by element, and then add the results

Example 1. $\varphi = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $\quad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$\underset{(2\times 1)(2\times 2)}{\varphi \cdot X}$: Is not defined

$$\underset{(2\times 2)(2\times 1)}{X \cdot \varphi} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0\cdot 0 + 1\cdot 1 \\ 1\cdot 0 + 0\cdot 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Example 2.

$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\quad Y = \begin{pmatrix} 0 & -j \\ j & 0 \end{pmatrix}$

$$IY = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -j \\ j & 0 \end{pmatrix} =$$

$$= \begin{pmatrix} 1\cdot 0 + 0\cdot j & 1\cdot(-j) + 0\cdot 0 \\ 0\cdot 0 + 1\cdot j & 0\cdot(-j) + 1\cdot 0 \end{pmatrix} =$$

$$= \begin{pmatrix} 0 & -j \\ j & 0 \end{pmatrix}$$

Example 3.

$$SW = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \varphi = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$\varphi\, SW$: Is not defined

$$SW\,\varphi = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

Example 4

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \qquad \varphi = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$H\varphi = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1\cdot 1 + 1\cdot 0 \\ 1\cdot 1 - 1\cdot 0 \end{pmatrix} =$$

7. Tensor Product (Kronecker product)

$$A \otimes B = C$$

$$\begin{array}{ccc} A & B & C \\ \downarrow & \downarrow & \downarrow \\ (n \times m) & (p \times q) & (np \times mq) \end{array}$$

Rule: Multiply each entry a_{ij} of A with the entire matrix B

Example 1.

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$$

$$A \otimes B = \begin{pmatrix} 1 \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} & 2 \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \\ 3 \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} & 4 \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \end{pmatrix} =$$

$$= \begin{pmatrix} 1 & 1 & 2 & 2 \\ 1 & 2 & 2 & 4 \\ 3 & 3 & 4 & 4 \\ 3 & 6 & 4 & 8 \end{pmatrix}$$

Example 2.

$$\varphi_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (2 \times 1)$$

$$\varphi_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (1 \times 2)$$

$$\varphi_1 \otimes \varphi_2 = \begin{pmatrix} 0 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \\ 1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad (4 \times 1)$$

Example 3.

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$I \otimes X = \begin{pmatrix} 1 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & 0 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ \\ 0 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & 1 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \end{pmatrix} =$$

$$= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Eigenvalues and Eigenvectors

Let A be a $n \times n$ matrix and let X be a nonzero vector for which

$$AX = \lambda X$$

for some scalar λ. The λ is called an eigenvalue and X eigenvector of A.

The set of all eigenvalues is denoted by $\lambda(A)$ and is referred as the spectrum of A.

Hermitian and Unitary Matrices

1. Hermitian Matrices

$$A = A^{\dagger} = (\bar{A})^{T}$$

Observables physical quantities are represented by Hermitian operators. Measument outcomes are the real eigenvalues of these operators

2. Unitary Matrices

A matrix is Unitary if

$$U^{\dagger}U = UU^{\dagger} = I \Rightarrow U^{\dagger} = U^{-1}$$

Quantum gates are represented by unitary matrices So unitary matrices describe how quantum states are change.