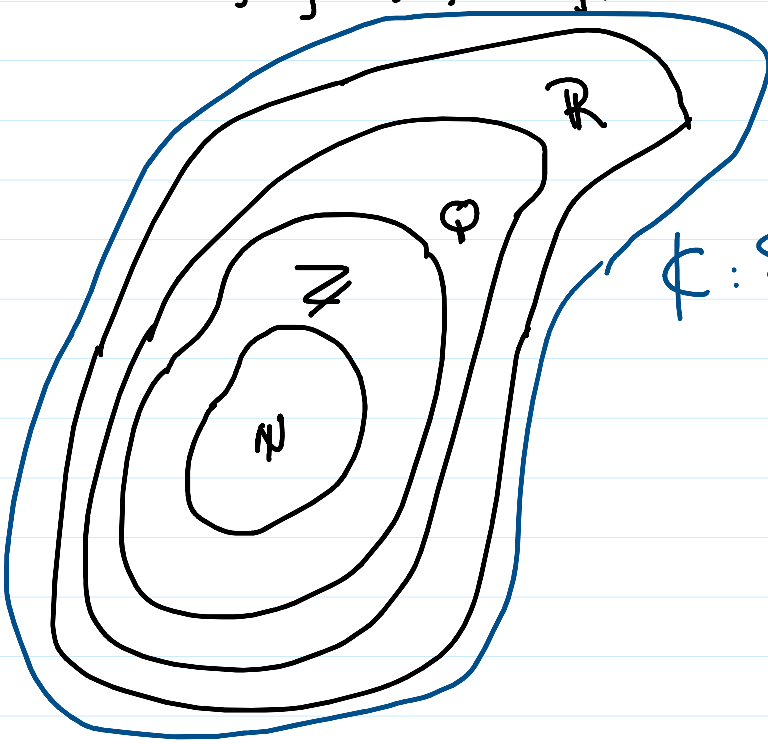
Introduction to Complex Numbers

$\mathbb{N}$ Natural Numbers : 1, 2, 3, 4, ...

$\mathbb{Z}$ Integers : Naturals + Zero + Negative Intergers

$\mathbb{Q}$ Rationals : -3/2, ..., -2/2, ..., -1/2, ... 5/3, ...

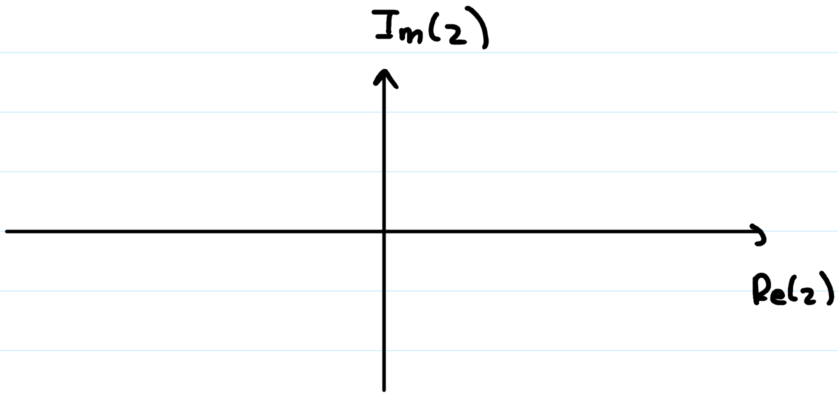
$\mathbb{R}$ Reals : 2.72, ..., 3.14, ..., 6.66, ...



$\mathbb{C}$: $\{$ 2D and $i^2 = \underline{1}$ $\}$

Example: Now you can define $\sqrt{-25} = \sqrt{i^2 5^2} = i5$

Cartesion Form of Complexs



$$z = (Re(z), Im(z)) = Re(z) + j\,Im(z)$$

Example : $z = \frac{\sqrt{2}}{2} + j$

$Re(z) = \frac{\sqrt{2}}{2}$ $\qquad$ $Im(z) = 1$

COMPLEX ARITHMETICS

1. Addition and Substraction :

$z_1 = a + jb$ $\qquad$ $z_1 + z_2 = (a+c) + j(b+c)$

$z_2 = c + jd$ $\qquad z_1 - z_2 = (a-c) + j(b-c)$

Example: $\quad z_1 = 0.707 + j\,0.707$

$z_2 = j = 0 + j1$

$z_1 + z_2 = 0.707 + 1.707j$

$z_1 - z_2 = 0.707 - 0.293j$

2. Multiplication

$$\left.\begin{aligned} z_1 &= a + jb \\ z_2 &= c + jd \end{aligned}\right\} \quad z_1 \cdot z_2 = (a+jb)(c+jd) =$$

$$= ac + jad + jbc + j^2 bd =$$

$$= (ac - bd) + j(ad + bc)$$

Example: $\quad z_1 = 1 + 2j$

$z_2 = 5 - j$

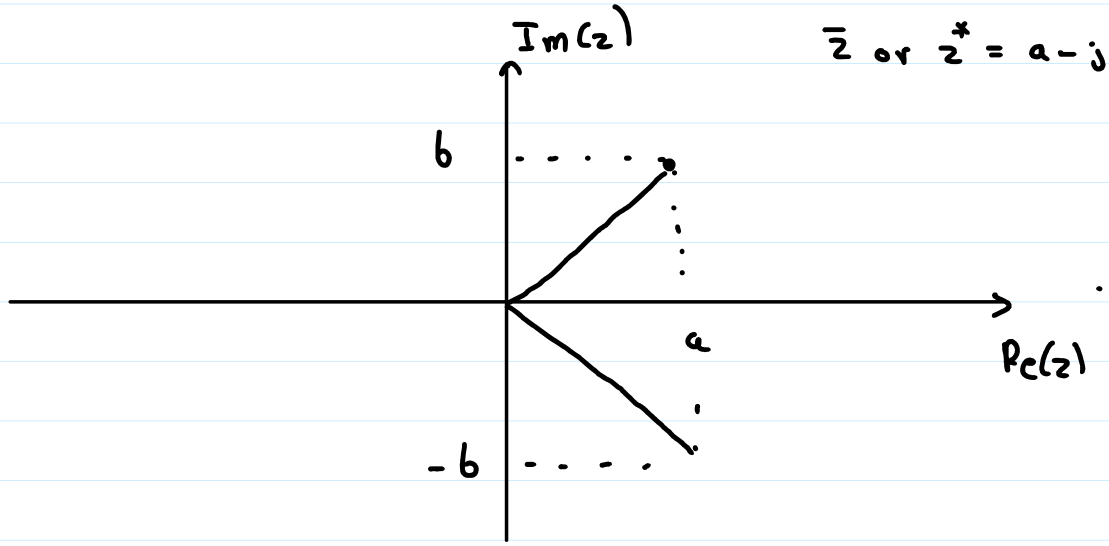
$z_1 z_2 = (1 + 2j)(5 - j) = 5 - j + 2 \cdot 5j - 2j^2 = 5 - j + 10j + 2 =$

$= 7 + 9j$

3. Complex Conjugate

$z = a + jb$

$\bar{z}$ or $z^* = a - jb$



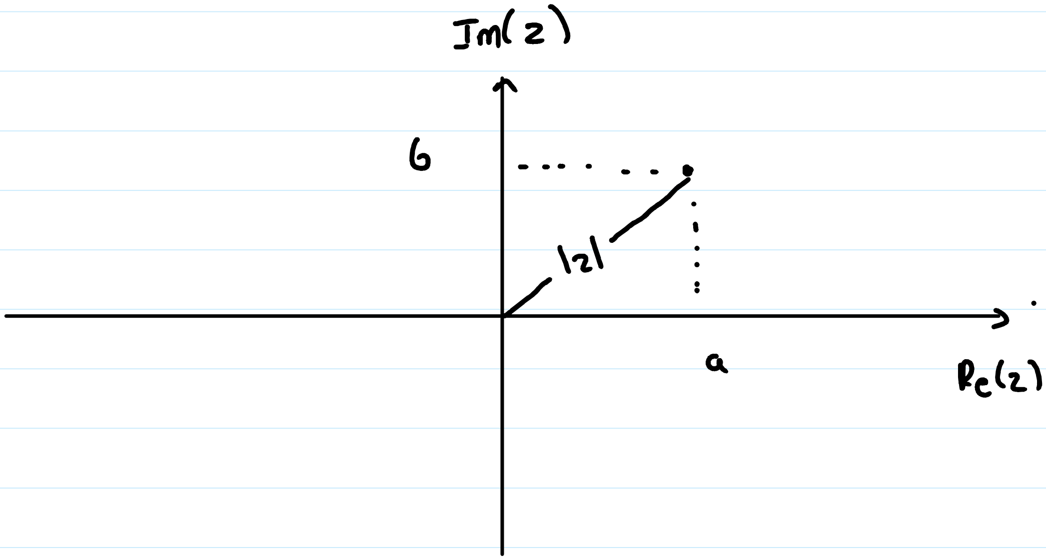
Properties :

1. $z + \bar{z} = (a + jb) + (a - jb) = 2a = 2\,Re(z) \Rightarrow Re(z) = \dfrac{z + \bar{z}}{2}$

2. $z - \bar{z} = (a + jb) - (a - jb) = 2jb \Rightarrow 2j\,Im(z) \Rightarrow Im(z) = \dfrac{z - \bar{z}}{2j}$

3. $z \cdot \bar{z} = (a + jb)(a - jb) = a^2 - (jb)^2 = a^2 + b^2 = |z|^2$

Πυθαγόρειο



Example: $z_1 = 5 + 3j$

$\overline{z_1} = 5 - 3j$

4. Division

$z_1 = a + jb$

$z_2 = c + jd$

$$\frac{z_1}{z_2} = \frac{a+jb}{c+jd}\,\frac{c-jd}{c-jd} = \frac{ac - jad + jbc - j^2 bd}{c^2 - (jd)^2} =$$

$$= \frac{ac + bd + j(bc - ad)}{c^2 + d^2} = \frac{ac + bd}{c^2 + d^2} + j\,\frac{(bc - ad)}{c^2 + d^2}$$

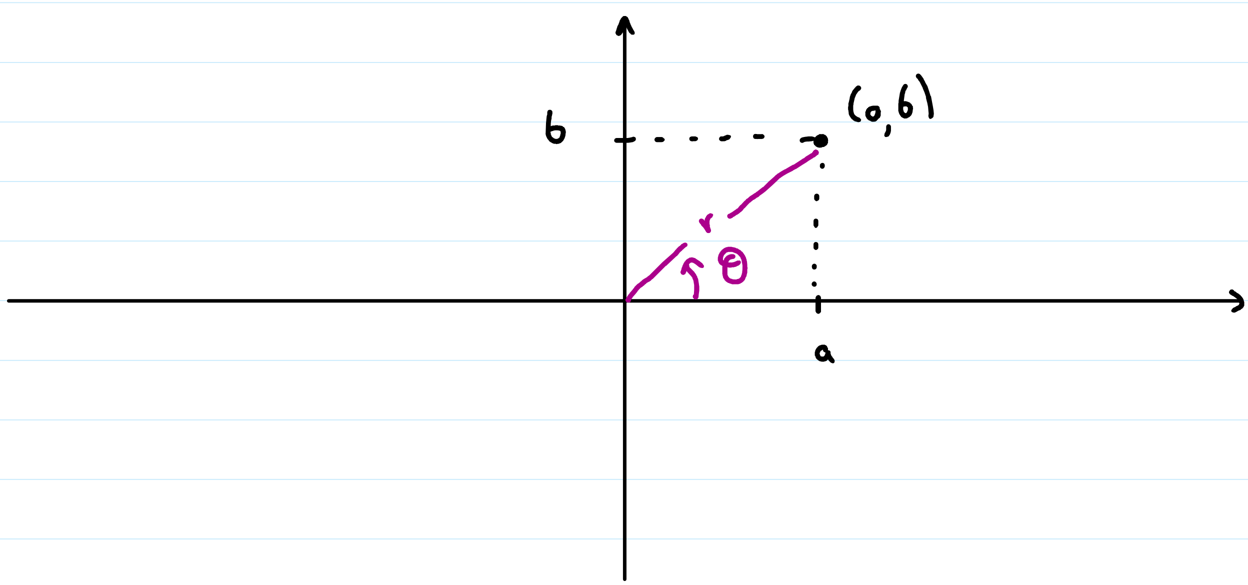
Example. $z_1 = 3 + 4j$

$z_2 = 1 + 3j$

$$\frac{z_1}{z_2} = \frac{3+4j}{1+3j} = \frac{3+4j}{1+3j} \cdot \frac{1-3j}{1-3j} = \frac{3\cdot 1 - 3\cdot 3j + 4\cdot 1j - 4\cdot 3\cdot j\cdot j}{1^2 - (3j)^2} =$$

$$= \frac{3 - 9j + 4j + 12}{1+9} = \frac{15 - 5j}{10} = \frac{15}{10} - \frac{5}{10}j = \frac{3}{2} - \frac{1}{2}j$$

Polar Form of Complex Numbers



$$\left.\begin{aligned} \sin\theta &= \frac{b}{r} \Rightarrow b = r\sin\theta \\ \cos\theta &= \frac{a}{r} \Rightarrow a = r\cos\theta \end{aligned}\right\} \quad z = r\cos\theta + j\, r\sin\theta = r\,(\cos\theta + j\sin\theta)$$

Euler's Formula:

$$e^{j\theta} = \cos\theta + j\sin\theta$$

$$e^{-j\theta} = e^{j(-\theta)} = \cos(-\theta) + j\sin(-\theta) = \cos(\theta) - j\sin\theta$$

So $\boxed{z = re^{j\theta}}$

$$\left.\begin{aligned} r &= \sqrt{a^2+b^2} = |z| \\ \tan\theta &= \frac{b}{a} \Rightarrow \theta = \tan^{-1}\left(\frac{b}{a}\right) \end{aligned}\right\} \quad z = r\,e^{j\theta}$$

Example. Let's convert the

$$z = \frac{\sqrt{2}}{2} + j\frac{\sqrt{2}}{2}$$

in Polar form

$$r = \sqrt{\left(\frac{\sqrt{2}}{2}\right)^2 + \left(\frac{\sqrt{2}}{2}\right)^2} = \sqrt{\frac{2}{4} + \frac{2}{4}} = \sqrt{\frac{1}{2} + \frac{1}{2}} = \sqrt{1} = 1$$

$$\tan\theta = \frac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} = 1 \Rightarrow \theta = \tan^{-1}(1) \Rightarrow \theta = 45^\circ \text{ or } \frac{\pi}{4}$$

$$z = 1\,e^{j\frac{\pi}{4}}$$