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Exercise Sheet 6

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Computational Fluid Dynamics (Summer Term 2021)

Exercise 1 (Programming). Pressure Stabilization (15 Points)
Consider the Stokes equations:

$$\begin{aligned} -\nu\Delta\mathbf{v} + \nabla p &= \mathbf{f} \;\text{ in } \Omega := [0,1]\times[0,1] \\ \nabla\cdot\mathbf{v} &= g \;\text{ in } \Omega \\ \mathbf{v} &= \mathbf{v}^* \;\text{ on } \partial\Omega. \end{aligned}$$

Here, the right-hand sides $\mathbf{f}$, g are chosen such that

$$\begin{aligned} \mathbf{v}_x^*(x,y) &= \sin(\pi x)^2\sin(2\pi y) \\ \mathbf{v}_y^*(x,y) &= -\sin(2\pi x)\sin(\pi y)^2 \\ p^*(x,y) &= \cos(2\pi y) \end{aligned}$$

is a classical solution, i.e. $f(x,y) := -\nu\Delta\mathbf{v}^*(x,y) + \nabla p^*(x,y)$ and $g(x,y) := \nabla\cdot\mathbf{v}^*(x,y)$.

The goal of this exercise is to investigate different ways to stabilize the discrete system: find $\mathbf{v}_h \in V_h$, $p_h \in Q_h$ such that

$$\begin{aligned} \nu(\nabla\mathbf{v}_h,\nabla\mathbf{w}_h) - (p_h,\nabla\cdot\mathbf{w}_h) &= (f,\mathbf{w}_h) \qquad\quad \forall\mathbf{w}_h\in V_h \\ (\nabla\cdot\mathbf{v}_h,q_h) + s(\mathbf{v}_h,p_h,q_h) &= (g,q_h) + l(q_h)\;\forall q_h\in Q_h. \end{aligned}$$

Here, the finite element spaces are given by $V_h = Q_h^2$, with Q_h denoting the space of continuous, piecewise linear polynomials. For this choice, the pair $V_h\backslash Q_h$ does not satisfy the inf-sup condition. For this reason, we introduce the stabilization forms $s(\mathbf{v}_h,p_h,q_h)$, $l(q_h)$.

We consider the following choices:

a. no stabilization:

$$\begin{aligned} s(\mathbf{v}_h,p_h,q_h) &= 0 \\ l(q_h) &= 0 \end{aligned}$$

b. mass stabilization:

$$\begin{aligned} s(\mathbf{v}_h,p_h,q_h) &= \gamma\sum_K h_K^2(p_h,q_h)_K \\ l(q_h) &= 0 \end{aligned}$$

c. diffusion stabilization:

$$s(\mathbf{v}_h, p_h, q_h) = \gamma \sum_K h_K^2 (\nabla p_h, \nabla q_h)_K$$
$$l(q_h) = 0$$

d. consistent diffusion stabilization:

$$s(\mathbf{v}_h, p_h, q_h) = \gamma \sum_K h_K^2 (-\nu \Delta \mathbf{v}_h + \nabla p_h, \nabla q_h)_K$$
$$l(q_h) = \gamma \sum_K h_K^2 (f, \nabla q_h)_K$$

Here, $(p, q)_K := \int_K pq \mathrm{d}V$ for each mesh cell K. The last option is called consistent, since the exact solution $\mathbf{v}^*, p^*$ is also a solution of the stabilized discrete system.

a. Modify the assembly routines of the class LocalStokesAssembler at the locations marked by TODO exercise A in ex5_StabilizedStokes.h to compute the matrix and vector corresponding to the stabilized variational formulation. Include all presented stabilization option 1. - 4. The selection of the applied stabilization routine is managed by the variable pressure_stab_type_. This type can be changed in the xml file.

b. Modify the evaluation routines at the locations marked by TODO exercise B in ex5_StabilizedStokes.h to evaluate the gradient and Laplace operator of the exact solution $\mathbf{v}^*, p^*$ and the right-hand side terms f, g.

c. Modify the assembly routines of the class WkpErrorIntegrator at the locations marked by TODO exercise C in ex5_StabilizedStokes.h to compute the errors

$$e_{L2,\mathbf{v}} := \int_\Omega |(\mathbf{v}^* - \mathbf{v}_h) \cdot (\mathbf{v}^* - \mathbf{v}_h)|$$
$$e_{H1,\mathbf{v}} := \int_\Omega \sum_{i=1}^d |\nabla(\mathbf{v}_i^* - \mathbf{v}_{h,i}) \cdot \nabla(\mathbf{v}_i^* - \mathbf{v}_{h,i})|$$
$$e_{L2,p} := \int_\Omega |(p^* - p_h)|^2$$
$$e_{H1,p} := \int_\Omega |\nabla(p^* - p_h) \cdot \nabla(p^* - p_h)|.$$

d. Run the code on up to 4 MPI processes for all presented stabilization methods. In each case, set $\nu = 1$, $\gamma = 0.1$ and visualize the results ex5_solution4.pvtu - ex5_solution7.pvtu with *paraview*. To do so, visualize the pressure field and combine the filters Calculator and Stream Tracer to visualize the streamlines of the velocity field $\mathbf{v}$ and submit the corresponding screenshots.

Further, for each stabilization method, visualize the calculated errors in a common plot (x-axis: maximal cell diameter h, y-axis: $e_{L2,\mathbf{v}}$, $e_{H1,\mathbf{v}}$, $e_{L2,p}$, $e_{H1,p}$.

Note that all parameters are defined in the parameter file ex5_CavityStokes.xml and can be changed without recompilation. The corresponding errors are printed out in the routine compute_error in ex5_CavityStokes.cc.

e. Redo **d.** for the consistent diffusion stabilization and vary the strength of the stabilization by adjusting the parameter $\gamma \in \{1, 0.1, 0.01, 0.001\}$.

The corresponding code framework is ex5_StabilizedStokes. Don't forget to the modify the file exercises/CMakeLists.txt accordingly, see the first exercise sheet for details.

Submission: until 04.06.2021, 11 am, moodle upload