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Exercise Sheet 8

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Computational Fluid Dynamics (Summer Term 2021)

Exercise 1 (Programming). Lid-driven Cavity (15 Points)
In this exercise you will solve the nonlinear Navier-Stokes equations and compare the result with the corresponding Stokes system:

$$\begin{aligned} -\nu\Delta\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} + \nabla p &= \mathbf{f} \;\text{ in } \Omega \\ \nabla\cdot\mathbf{v} &= 0 \;\text{ in } \Omega \\ \mathbf{v} &= \mathbf{v}_D \;\text{ on } \partial\Omega. \end{aligned}$$

with $\Omega = [0,1]\times[0,1]$, $\partial\Omega = \Gamma_D + \Gamma_0$, with $\Gamma_D = [0,1]\times\{1\}$ and $\Gamma_0 = \partial\Omega \setminus \Gamma_D$. Further,

$$\mathbf{f}(t,x,y) = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

$$\mathbf{v}_D(t,x,y) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot \sin(10t) \cdot \begin{cases} 0, & (x,y)\in\Gamma_0 \\ v_X, & (x,y)\in\Gamma_D, x\in[\epsilon, 1-\epsilon] \\ \frac{x}{\epsilon}v_X, & (x,y)\in\Gamma_D, x\in[0,\epsilon] \\ \frac{1-\epsilon-x}{\epsilon}v_X + v_X, & (x,y)\in\Gamma_D, x\in[1-\epsilon,1] \end{cases}$$

for some constants $\epsilon, v_X \geq 0$, $\nu > 0$.

The corresponding nonlinear variational formulation is given by: find $(\mathbf{v},p)$ such that for all test functions $(\mathbf{w},q)$:

$$\begin{aligned} 0 = F_\nu((\mathbf{v},p);(\mathbf{w},q)) &= \int_\Omega \nu\nabla\mathbf{v}\cdot\nabla\mathbf{w} + \mathbf{v}(\nabla\mathbf{v})^T\mathbf{w} - p(\nabla\cdot\mathbf{w}) + q(\nabla\cdot\mathbf{v}) - \mathbf{f}\cdot\mathbf{w} && (1) \\ &= (\nabla\mathbf{v},\nabla\mathbf{w}) + ((\mathbf{v}\cdot\nabla)\mathbf{v},\mathbf{w}) - (p,\nabla\cdot\mathbf{w}) + (q,\nabla\cdot\mathbf{v}) - (\mathbf{f},\mathbf{w}) && (2) \end{aligned}$$

Nonlinear equations of this type can be solved by using a combination of continuation and Newton's method:

a. Choose initial value $(\mathbf{v}^0, p^0) = (0,0)$ and let ν_1, ν_N be given.

b. for $k = 1,\ldots,N$:

(a) set $\nu_k = \nu_1 \left(\frac{\nu_N}{\nu_1}\right)^{\frac{k-1}{N-1}}$

(b) solve $F_{\nu_k}((\mathbf{v},p),\cdot) = 0$ by applying Newton's method with starting value $(\mathbf{v}^{k-1}, p^{k-1})$. The solution is denoted by $(\mathbf{v}^k, p^k)$.

Newton's method requires the evaluation of the linear form $F_\nu((\mathbf{v}_l, p_l), \cdot)$ and the bilinear form $D_{(\mathbf{v},p)}F_\nu((\mathbf{v}_l, p_l); \cdot, \cdot)$ for the current Newton iterate $(\mathbf{v}_l, p_l)$. Here, the bilinear form $D_{(\mathbf{v},p)}F_\nu((\mathbf{v}_l, p_l); \cdot, \cdot)$ denotes the Jacobian of F_ν w.r.t. argument $(\mathbf{v}, p)$ at point $(\mathbf{v}_l, p_l)$:

$$D_{(\mathbf{v},p)}F_\nu((\mathbf{v}_l, p_l); (\mathbf{v}, p), (\mathbf{w}, q)) = (\nabla \mathbf{v}, \nabla \mathbf{w}) + ((\mathbf{v}_l \cdot \nabla)\mathbf{v}, \mathbf{w}) + ((\mathbf{v} \cdot \nabla)\mathbf{v}_l, \mathbf{w}) - (p, \nabla \cdot \mathbf{w}) + (q, \nabla \cdot \mathbf{v}).$$

HiFlow provides an implementation of Newton's method for solving nonlinear problems of type (1). In order to use this method, the user has to provide assembly routines for $D_{(\mathbf{v},p)}F_\nu((\mathbf{v}_l, p_l); \cdot, \cdot)$ (matrix assembly) and $F_\nu((\mathbf{v}_l, p_l), \cdot)$ (vector assembly).

a. Modify the matrix assembly routine of the class LocalFlowAssembler at the location marked by TODO exercise A in ex7_CavityNavierStokes.h to compute $D_{(\mathbf{v},p)}F_\nu((\mathbf{v}_l, p_l); \cdot, \cdot)$. Use the member variable equation_type to distinguish between the Stokes equations (value 0) and the Navier-Stokes equations (value 1).

Hint: In this routine, the data fields sol_ns_[q], grad_sol_ns_[q] store the values of the previous Newton iterate at quadrature point q, i.e. $(\mathbf{v}_l, p_l)(x_q)$ and $(\nabla \mathbf{v}_l, \nabla p_l)(x_q)$, respectively.

b. Modify the vector assembly routine of the class LocalFlowAssembler at the location marked by TODO exercise B in ex7_CavityNavierStokes.h to compute $F_\nu((\mathbf{v}_l, p_l), \cdot)$.

c. Implement the presented continuation / Newton method at the location marked by TODO exercise C in ex7_CavityNavierStokes.cc. Within each iteration, call the visualize(k) function after the Newton solver to visualize the solution.

d. Run the code and load the provided paraview state file. This will automatically perform the setup of visualization filters. See the pdf paraview_notes.pdf for further details. Save an animation for the loaded solution files.

e. What is the smallest viscosity ν for which you are able to compute a solution? Hint: modify the parameters InitialNu ($= \nu_1$), FinalNu($= \nu_N$) and StepsNu($= N$) in ex7_CavityNavierStokes.xml Feel free to modify the way ν_k is computed.

The corresponding code framework is ex7_CavityNavierStokes. Don't forget to the modify the file exercises/CMakeLists.txt accordingly, see the first exercise sheet for details.

Submission: until 25.06.2021, 11 am, moodle upload