Q4: Vehicle routing problem

The goal of vehicle routing problem (VRP) is planning routes of each vehicle to finish certain tasks (pick up passengers here) with some constraints (capacity constraints here) so that minimum total cost is reached. However, due to the complexity of VRP, it's really hard to solve it directly especially for larger scale problem. So, we treat this VRP here as two phases like Fisher *(1)*, namely generalized assignment problem (GAP) and travelling salesman problem (TSP), where depots are assigned to specific vehicle in GAP and the route of each vehicle is solved in TSP. In this way, the origin problem is decomposed into two classic problems and the complexity can be significantly reduced.

First, we build the GAP formulation as follow.

$$min\ z = \sum_{k=1}^{K}\sum_{i=1}^{n} c_{ik}y_{ik} \tag{1}$$

$$\sum_{i}^{n} a_i y_{ik} \le b_k \qquad k = 1,\ldots,K \tag{2}$$

$$\sum_{k}^{K} y_{ik} = \begin{cases} K & i = 0 \\ 1 & i = 1,\ldots,n \end{cases} \tag{3}$$

$$y_{ik} = 0\ or\ 1 \qquad i = 0,\ldots,n\ k = 1,\ldots,K \tag{4}$$

where c_{ik} is the estimated cost if depot i is assigned to vehicle k (please refer to *(1)* for detailed calculation process, depot 2, 5,9 and 10 are selected as seed). One thing has to be mentioned here is that since the cost c_{ik} is an estimated value, the final results of this method may have tiny gaps compared with optimal solution of origin VRP. y_{ik} is a binary variable ($y_{ik} = 1$ if depot i is assigned to vehicle k, otherwise 0). a_i denotes the demand of depot i and b_k denotes the capacity of vehicle k. Constraints (3) guarantees that all depots are assigned to vehicles. To obtain the optimal solution of the model above, we solve it using gurobi (version 8.0.1) in python environment. The results are shown as below.

Table 1 Assignment results of GAP

Vehicle	Depot	Total demand	Capacity
1	1,2,3,4	100	100
2	5,6,7	98	100
3	8,9,10	84	100
4	11,12,13,14,15	100	100

As can be seen from table 1, all depots have been assigned to vehicles with no capacity constraint violation. Now, what we should do is planning the route of each vehicle so that each vehicle can pass through all depots assigned to it and go back to origin depot, which can be viewed as a TSP for each vehicle. The formulation of TSP for each vehicle is given as below.

$$min\ z = \sum_{ij} d_{ij}x_{ij} \tag{5}$$

$$\sum_{j} x_{ij} = 1 \qquad i = 0,\ldots,n' \tag{6}$$

$$\sum_{j} x_{ji} = 1 \qquad i = 0,\ldots,n' \tag{7}$$

$$t_i + 1 - M * (1 - x_{ij}) \le t_j \qquad i = 1,\ldots,n'\ j = 1,\ldots,n' \tag{8}$$

$$x_{ij} = 0\ or\ 1 \qquad i = 0,\ldots,n'\ j = 0,\ldots,n' \tag{9}$$

where d_{ij} denotes the distance between depot i and depot j. x_{ij} is a binary variable ($x_{ij} = 1$ if vehicle chooses arc ij, otherwise 0). Constraint (8) guarantees no sub tours in the final solution, and M is a larger enough value. Using gurobi solver in python again, vehicle routes are given in table 2.

Table 2 vehicle routes of TSP

Vehicle	Route	Cost	Total cost
1	0-4-3-2-1-0	55.75	231.4
2	0-7-6-5-0	45.56	
3	0-10-9-8-0	64.95	
4	0-12-11-13-14-15-0	65.14	

In order to calculate the gap between the result of the model above, we formulate the entire VRP here.

$$min\ z = \sum_{ijk} d_{ij}x_{ijk} \tag{10}$$

$$\sum_i a_i y_{ik} \leq b_k \qquad k = 1, \dots, K \tag{11}$$

$$\sum_k y_{ik} = \begin{cases} K & i = 0 \\ 1 & i = 1, \dots, n \end{cases} \tag{12}$$

$$y_{ik} = 0\ or\ 1 \qquad i = 0, \dots, n\ k = 1, \dots, K \tag{13}$$

$$\sum_j x_{ijk} = y_{ik} \qquad i = 0, \dots, n\ k = 1, \dots, K \tag{14}$$

$$\sum_j x_{jik} = y_{ik} \qquad i = 0, \dots, n\ k = 1, \dots, K \tag{15}$$

$$t_{ik} + 1 - M * (1 - x_{ijk}) \leq t_{jk} \qquad i = 1, \dots, n\ j = 1, \dots, n\ k = 1, \dots, K \tag{16}$$

$$x_{ijk} = 0\ or\ 1 \qquad i = 0, \dots, n\ j = 0, \dots, n\ k = 1, \dots, K \tag{17}$$

where x_{ijk} is a binary variable ($x_{ijk} = 1$ if vehicle k chooses arc ij, otherwise 0). Constraint (16) guarantees no sub tours in each vehicle's route. By using gurobi in python, we can get the final results as below.

Table 3 vehicle routes of VRP

Vehicle	Route	Cost	Total cost
1	0-4-3-2-1-0	55.75	231.4
2	0-8-9-10-0	64.95	
3	0-5-6-7-0	45.56	
4	0-12-11-13-14-15-0	65.14	

Obviously, the results are same with two-phase model, so the gap for this problem is 0%. Except solutions, we also record the CPU running time for these two models. Two-phase (GAP+TSP) model took 1.2 seconds while VRP model took 4.0 second. Although the difference here was tiny, there would be significant difference if the problem become more complex.

[1] Fisher M L, Jaikumar R. A generalized assignment heuristic for vehicle routing[J]. Network, 2006, 11(2):109-124.