Laboratory 1

Coordinate systems. Geometrical transformations

One of the fundamental necessities in robotics, is the localisation of objects in three dimensional space. These objects can be elements that compose the physical structure of a robot, parts and components that the robots manipulate, tools or in general any body that exists in the space of operation of the robot. (See figure 1.1).

These objects can be described fundamentally with roughly two attributes: their position and orientation. An immediate aspect of interest would be how to represent these and, furthermore, how can we manipulate mathematically these properties.

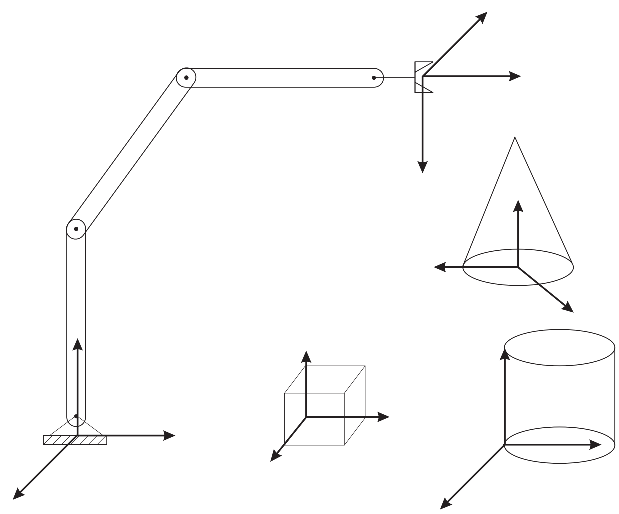


Figure 1.1: Coordinate systems in the space of operation of a robot

For the description of position and orientation of a solid in space, we will attach on it a rigid Cartesian coordinate system. Since any coordinate system can server as a reference system on which we can express the position and orientation of a body, there is the question of how to change or transform these attributes of a body from a Cartesian system of reference to another one. This exercise presents conventions and methodology for the description of positions and orientations, as well as the mathematical formalisation that is used for the manipulation of these quantities in several coordinate systems.

The manipulation using a robot assumes the fact that objects and tools will be displaced in space with the use of a mechanism. This fact determines the need to represent the position and orientation of such objects that we want to manipulate as well as the position of the mechanism of manipulation. To define and manipulate mathematically the quantities that represent the position and orientation it is necessary to define a coordinate system and the corresponding conventions used for their representation.

1.1 Cartesian coordinate systems

In robotics, the *standard* Cartesian system is obtained through the application of the rule of the right hand, that is presented in figure 1.2. The thumb of the right hand indicates the positive direction of the Z axis, and the extended fingers indicate the positive direction for the X axis. By flexing the fingers by 90°, we are obtaining the positive direction for axis Y.

For the definition of positive directions of rotations, we use again the rule of the right hand. When the thumb is pointing the positive direction of an axis, the rest of the fingers indicate the positive rotation around that axis.

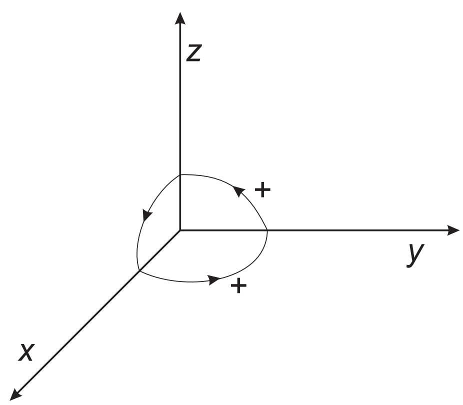
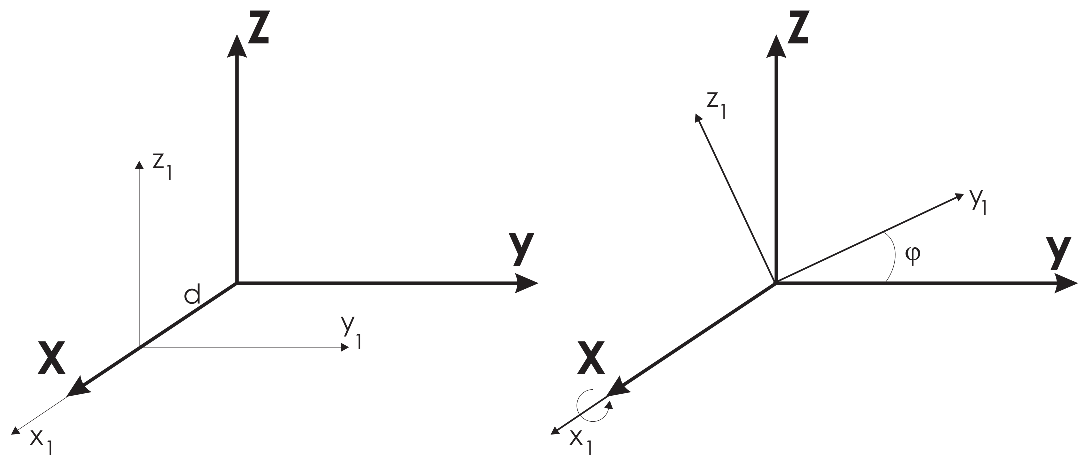


Figure 1.2: Cartesian coordinate system. The positive direction for each axis

The position of a point can be described using the Cartesian coordinates of an object expressed relative to the origin of a coordinate system (x_1, y_1, z_1) or through a vector of position p_1. For example, the position of a parallelepiped in a coordinate system can be described using 8 position vectors, one for each one of the vertices. In the case that the object is moving, the calculation of the new position assumes the calculation of a new set of 8 vectors of position. This situation can become even more complicated if we are talking about irregular objects (a particular case, when for each position of the object it might be necessary to calculate more than 8 vector of position), or in the case in which we have more objects that move independently in relation to each other. An alternative and more efficient solution is to attach a different coordinate system on each object, which will displace together with the object itself. If the object is rigid (i.e. it does not deform), the position of every point belonging to the object remains the same in respect to the coordinate system attached on the object, independent of its the displacement. In this way, the problem of calculating the motion of an object is reduced to the calculation of the relation between two



(a) Translations (b) Rotations

Figure 1.3: Elementary transformations

coordinate systems (reference system and the coordinate system attached on the object). Moreover, this relationship allows the calculation of the new position of any point belonging to the object of interest. The position of point P is described by its Cartesian coordinates: $P(x, y, z)$.

1.2 Elementary Transformations. Homogeneous Transformations.

A rigid solid, and an its inertial coordinate system, have 6 degrees of freedom (DOF), or 6 independent ways in which the object can move. These elementary transformations are:

1. Translation on X axis. If the translation takes place with a distance 'd', this is denoted as: $Trans(X, d)$. (see figure 1.3a)

2. Translation on Y axis, denoted $Trans(Y, d)$.

3. Translation on Z axis, denoted $Trans(Z, d)$.

4. Rotation around X axis, denoted $Rot(X, \varphi)$. The positive direction of the rotation φ is given by the rule of the right hand. (see figure 1.3b)

5. Rotation around Y axis, denoted $Rot(Y, \varphi)$.

6. Rotation around Z axis, denoted $Rot(Z, \varphi)$.

The motion of a rigid object in respect to a reference coordinate system can be described as a succession of elementary rotations and translations applied on the coordinate system that is attached on the object. For the description of any translation in the three dimensional space, we need only three parameters. Therefore the matrix representation of a translation can be made using a vector with three elements $w = [w_x, w_y, w_z]^T$. This transformation is finite, but it is not linear.

For the rotation, there are more than one method of representation. One of the most popular methods is by using rotation matrices. Considering a Cartesian coordinate system on which we apply an arbitrary rotation, we express the orientation of the set of axes (represented by the vectors u, v, w) in respect to the original coordinate system (represented by the vectors x, z, y). This representation describes completely the rotation. The three vectors u,v și w, that form the expression of the new coordinate system, are each represented of the three components, resulting therefore in a total of 9 parameters that represent a rotation.

$$A = \begin{bmatrix} \hat{u}_x & \hat{v}_x & \hat{w}_x \\ \hat{u}_y & \hat{v}_y & \hat{w}_y \\ \hat{u}_z & \hat{v}_z & \hat{w}_z \end{bmatrix} \tag{1.1}$$

Each one of the elements of the matrix represents the cosine of the angle between an axis of the new and one of the reference coordinate system (x, z or y), that is why the matrix of 1.1 is called as well *direction cosine matrix*[1]. The rotation matrix is an orthogonal matrix, with real elements and a determinant of $+1$. The eigenvalues of the matrix are $\left\{1, e^{\pm i \cdot \theta}\right\}$ where e is the unit vector corresponding to the direction around which the rotation is made and θ is the angle of the rotation.

In the case that the rotation is happening with an angle φ around one of the elementary axes (x, y or z), the direction cosine matrix has the following form:

- Rotation around x axis: $A_x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi \\ 0 & \sin\varphi & \cos\varphi \end{pmatrix}$
- Rotation around y axis: $A_y = \begin{pmatrix} \cos\varphi & 0 & \sin\varphi \\ 0 & 1 & 0 \\ -\sin\varphi & 0 & \cos\varphi \end{pmatrix}$
- Rotation around z axis: $A_z = \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Therefore, a 3×3 matrix can be used for describing a rotation, but not for a translation.

The *homogeneous coordinates* were introduced by Möbius to allow the description of finite transformations using matrices. Furthermore, homogeneous coordinates allow to work the same way for rotational transformations as well as translational transformations.

[1] Direction Cosine Matrix – DCM

In principle, the coordinates of a point in n-dimensional space can be represented through a vector of $n+1$ dimensions, by adding a non-zero scaling factor. The homogeneous coordinates of a point in three dimensional space are obtained through its Cartesian coordinates by adding a scaling factor, which for the robotics applications equals 1. Therefore, a point $P(x, y, z)$ has homogeneous coordinates:

$$\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Geometrical transformations are represented in this case through a 4×4 matrix:

$$T = \left[\begin{array}{ccc|c} 3 & \times & 3 & 3 \times 1 \\ \hline 1 & \times & 3 & 1 \times 1 \end{array}\right] = \left[\begin{array}{ccc|c} & rotatie & & \begin{matrix} trans- \\ la- \\ tie \end{matrix} \\ \hline 0 & 0 & 0 & 1 \end{array}\right] = \begin{bmatrix} X & Y & Z & P \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{1.2}$$

where $X = \begin{bmatrix} X_X \\ X_Y \\ X_Z \end{bmatrix}$ represents the direction of X axis of the new coordinate system,

$Y = \begin{bmatrix} Y_X \\ Y_Y \\ Y_Z \end{bmatrix}$ represents the direction of Y axis of the new coordinate system,

$Z = \begin{bmatrix} Z_X \\ Z_Y \\ Z_Z \end{bmatrix}$ represents the direction of Z axis of the new coordinate system, while

$P = \begin{bmatrix} P_X \\ P_Y \\ P_Z \end{bmatrix}$ represents the position of the origin of the new coordinate system.

The matrix T therefore describes the position (through vector P) and the orientation (through vectors X, Y and Z) of the new coordinate system in respect to the reference system. Through the multiplication of two homogeneous transformation matrices, the result is still a homogeneous transformation matrix.

The following homogeneous transformation matrices are associated to elementary geometrical transformations:

$$Trans(X,a) = \begin{bmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (1.3)$$

$$Trans(Y,a) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & a \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (1.4)$$

$$Trans(Z,a) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & a \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (1.5)$$

$$Rot(X,\varphi) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi & 0 \\ 0 & \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (1.6)$$

$$Rot(Y,\varphi) = \begin{bmatrix} \cos\varphi & 0 & \sin\varphi & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\varphi & 0 & \cos\varphi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (1.7)$$

$$Rot(Z,\varphi) = \begin{bmatrix} \cos\varphi & -\sin\varphi & 0 & 0 \\ \sin\varphi & \cos\varphi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (1.8)$$

A matrix of a transformation can be understood in different ways::

- as a transformation from one coordinate system to another one;
- as a description of the origin and orientation of the new coordinate system in respect to the reference coordinate system.;
- as a description of the displacement of an object from a position (reference system) to another one (new coordinate system);
- as a method that allows the calculation of the position of a point, that is part of an object, in respect to a system of reference, knowing its position in respect to the new coordinate system (Charles theorem, see figure 1.4):

$$^{R}q =^{R} T_N \cdot^{N} q \quad (1.9)$$

where:

${}^{R}q$ is the vector of position of a point in the system R (reference system);

${}^{R}T_N$ is the transformation of the new coordinate system in respect to the reference system, known as *direct transformation.* In other words, it is the transformation that is applied on system R so that it arrives at the position of system N.

${}^{N}q$ is the vector of position of a point in the coordinate system N(new coordinate system).

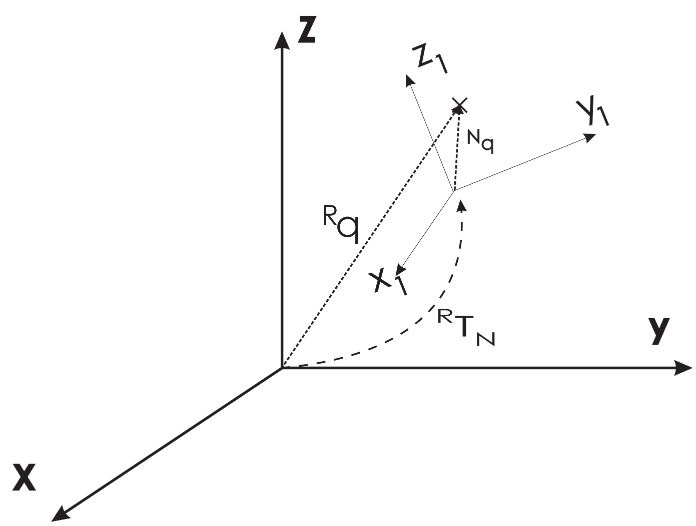


Figure 1.4: Calculation of the position of a point in the reference coordinate system, using direct transformation and the coordinates of the point in it's own coordinate system

1.3 Combination of transformations

As it was shown, a geometrical transformation can be decomposed in a succession of elementary rotations and translations. The combination of a succession of elementary geometrical transformations in a general transformation, can be therefore done:

- Using **left multiplication** of homogeneous transformation matrices if the transformations are made in respect to a reference coordinate system. (**Absolute transformations**). In this conditions,if we have a succession of n absolute transformations(T_i, $i = \overline{1 \ldots n}$), the transformation matrix can be written as:

$$T_A = \prod_{i=n}^{1} T_i = T_n \cdot T_{n-1} \cdot \ldots \cdot T_1 \tag{1.10}$$

- Using **right multiplication** of homogeneous transformation matrices if transformations are made in respect to a new coordinate system (the system that is obtained as a result of the last transformation) (**Relative transformations**). In this conditions, if we have a succession of n relative transformations ($T_i, \quad i = \overline{1 \ldots n}$), the transformation matrix can be written as:

$$T_r = \prod_{i=1}^{n} T_i = T_1 \cdot T_2 \cdot \ldots \cdot T_n \tag{1.11}$$

You can see a visualisation of left and right multiplications on this series of videos: `https://www.youtube.com/playlist?list=PLhgMOMe0tLWhx4oO2fLJUe2jf0sp0vHXc`

With the help of direct geometrical transformations it is possible to determine the position of a point in the reference system ${}^{R}q$ if we know its position in another coordinate system ${}^{N}q$ with direct geometrical transformation ${}^{R}T_N$. If we want to find ${}^{N}q$, with given ${}^{R}q$ and ${}^{R}T_N$, we must proceed as following:

$${}^{R}q = T \cdot {}^{N}q \rightarrow T^{-1} \cdot {}^{R}q = T^{-1} \cdot T \cdot {}^{N}q \rightarrow {}^{N}q = T^{-1} \cdot {}^{R}q \tag{1.12}$$

Therefore, the inverse geometric transformation is described with the inverse matrix of the direct transformation.

Denoted:

$$T = \begin{bmatrix} X_X & Y_X & Z_X & P_X \\ X_Y & Y_Y & Z_Y & P_Y \\ X_Z & Y_Z & Z_Z & P_Z \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{1.13}$$

it can be demonstrated that the inverse of a homogeneous transformation matrix, in form of 1.13, is:

$$T^{-1} = \begin{bmatrix} X_X & X_Y & X_Z & -P_X \cdot X_X - P_Y \cdot X_Y - P_Z \cdot X_Z \\ Y_X & Y_Y & Y_Z & -P_X \cdot Y_X - P_Y \cdot Y_Y - P_Z \cdot Y_Z \\ Z_X & Z_Y & Z_Z & -P_X \cdot Z_X - P_Y \cdot Z_Y - P_Z \cdot Z_Z \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{1.14}$$

1.4 Proposed problems

1. A point with coordinates $P(1, 2, 3)$ is given. We apply the following transformations on the point: $Trans(X, 4)$, $Trans(Y, -4)$, $Rot(X, 90°)$, $Trans(Z, 4)$, $Rot(Y, 90°)$. Determine the new coordinates of the point if we consider:

a) Absolute transformations;

b) Relative transformations;

2. Consider the location of objects **A** and **B** of figure 1.5. We would like to locate object **B** on top of object **A** so that line $\boldsymbol{rs}$ coincides with line $\boldsymbol{pq}$ and the coordinate systems of the two objects to be totally coinciding:

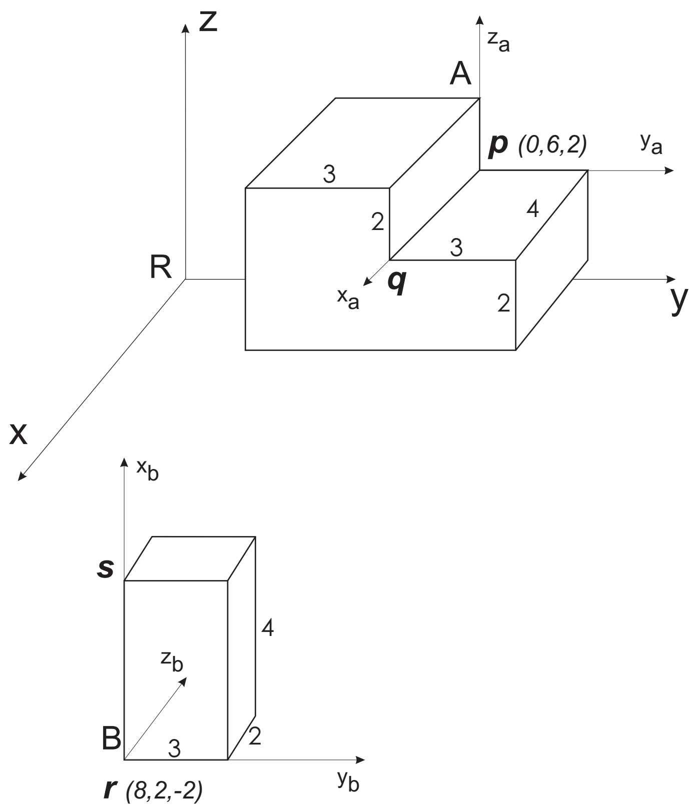


Figure 1.5: Object **B** should be located on top of object **A** so that line $\boldsymbol{rs}$ coincides with line $\boldsymbol{pq}$

a) What transformations should be applied on object **B** so that it arrives at the desired position?

b) What is the position of point $\boldsymbol{s}$ in the reference system, in its own system and in the system of object **A**, before the transformation?

c) What is the position of point $\boldsymbol{s}$ in the reference system after the transformation?

1.4.1 MATLAB script

To visualise the results, you can use a MATLAB function that is available on your Desktop. Open MATLAB and navigate to the folder 'RSC/Lab1'. Then right click on the 'Objects.mat' file and load its contents into the MATLAB workspace.

This file, contains four arrays, which represent the position of the two objects in the example above. The arrays V1 and V2 contain the position of the edges of the objects, and they should be transformed (the arrays F1 and F2 just contain the information of how the points are creating the faces).

V1 and V2 are of size 8×4 & 12×4 respectively, each row representing a different point and each column a coordinate (X, Y, Z and 1).

You can transform the objects by transforming the arrays V1 and V2, and calling the 'Draw_Objects' function to see the result of the transformation. If you want to start over from the initial position, just load again the 'Objects.mat' file.

1.4.2 Robotics toolbox

The robotics toolbox is a set of functions that are relevant for robotic applications. You can download it (and its documentation)for free from this website: `http://petercorke.com/wordpress/toolboxes/robotics-toolbox`

Writing transformation matrices by hand is a tedious task which can be automated, and the robotics toolbox has a set of functions for generating quickly the basic transformation matrices. These functions are the following:

trotx, troty, trotz (for rotational transformation matrices) and transl (for translation transformation matrices).

To use them, simply type e.g. $trotx(\theta)$, and MATLAB will spit out a homogeneous transformation matrix for a rotation θ around axis X. Pay attention that θ should be in radians. If you want the angle to be expressed in degrees instead, you need to give 'deg' as an argument:

$trotx(\theta,'deg')$

For the translation transformation matrices, we only have one function for all three axis. The function gets either one array with three elements as an argument (representing the translation on X, Y, and Z axes), or three arguments one for each translation:

$transl(a, b, c)$ or $transl([a, b, c])$