FAST DETERMINISTIC ALGORITHMS FOR MATRIX COMPLETION PROBLEMS *

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Abstract. Ivanoys, Karpinski and Saxena [*Deterministic polynomial time algorithms for matrix completion problems*, SIAM Journal on Computing, 39 (2010), pp. 3736–3751] have developed a deterministic polynomial time algorithm for finding scalars $x_1, \ldots, x_n$ that maximize the rank of the matrix $B_0 + x_1B_1 + \cdots + x_nB_n$ for given matrices $B_0, B_1, \ldots, B_n$, where $B_1, \ldots, B_n$ are of rank one. Their algorithm runs in $O(m^{4.37}n)$ time, where m is the larger of the row size and the column size of the input matrices.

In this paper, we present a new deterministic algorithm that runs in $O((m + n)^{2.77})$ time, which is faster than the previous one unless n is much larger than m. Our algorithm makes use of an efficient completion method for mixed matrices. As an application of our completion algorithm, we devise a deterministic algorithm for the multicast problem with linearly correlated sources.

We also consider a skew-symmetric version: maximize the rank of the matrix $B_0 + x_1B_1 + \cdots + x_nB_n$ for given skew-symmetric matrices $B_0, B_1, \ldots, B_n$, where $B_1, \ldots, B_n$ are of rank two. We design the first deterministic polynomial time algorithm for this problem based on the concept of mixed skew-symmetric matrices and a linear delta-covering problem.

Key words. matrix completion, mixed matrix, mixed skew-symmetric matrix, independent matching, delta-matroid, network coding

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1. Introduction.

In a *max-rank matrix completion problems*, or *matrix completion problems* for short, we are given a matrix whose entries may contain indeterminates, and we are to substitute appropriate values to the indeterminates so that the rank of the resulting matrix be maximized. A solution for the matrix completion problem is called a *max-rank completion,* or just a *completion.* Matrices with indeterminates and its completion demonstrate rich properties and applications in various areas: computing the size of maximum matching [8, 15], construction of network codes for multicast problems [10], computing all pairs edge connectivity of a directed graph [2], system analysis for electrical networks [18] and structural rigidity [17].

In this paper, we consider the following three kinds of matrix completion problems. Throughout this paper, we consider the underlying field as an arbitrary field, i.e., it may be finite or infinite, unless we state an explicit condition on the field size.

Matrix completion by rank-one matrices: Max-rank completion for a matrix in the form of $B_0 + x_1B_1 + \cdots + x_nB_n$, where B_0 is a matrix of arbitrary rank, $B_1, \ldots, B_n$ are matrices of rank one and $x_1, \ldots, x_n$ are indeterminates.

Mixed skew-symmetric matrix completion: Max-rank completion for a skew-symmetric matrix in which each indeterminate appears once (twice, if we count the symmetric counterpart).

Skew-symmetric matrix completion by rank-two skew-symmetric matrices: Max-rank completion for a matrix in the form of $B_0+x_1B_1+\cdots+x_nB_n$, where B_0 is a skew-symmetric matrix of arbitrary rank, $B_1, \ldots, B_n$ are skew-symmetric matrices of rank two and $x_1, \ldots, x_n$ are indeterminates.

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For the matrix completion by rank-one matrices, Lovász [16] has solved the special case of $B_0 = 0$ with matroid intersection. For the general case, Ivanoys, Karpinski and Saxena [13] have provided an algebraic approach that yields the first deterministic polynomial time algorithm. The running time is $O(m^{4.37}n)$, where m is the larger of the row size and the column size of given matrices and n is the number of indeterminates.

Mixed skew-symmetric matrices are studied in the works of Geelen, Iwata and Murota [9] and Geelen and Iwata [6]. The former paper provides a deterministic algorithm to compute the rank of mixed skew-symmetric matrices based on the linear delta-matroid parity problem. However, a matrix completion algorithm for mixed skew-symmetric matrices has been unknown.

For the skew-symmetric matrix completion by rank-two skew-symmetric matrices, Lovász [16] has shown that a completion can be found by solving the linear matroid matching problem if $B_0 = 0$. The general case with B_0 being an arbitrary skew-symmetric matrix has been unsolved.

A general matrix completion problem has been shown to be NP-hard by Harvey, Karger and Yekhanin [11], if we allow each indeterminate appears more than once.

1.1. Our Contribution. In this paper, we present new deterministic algorithms for the three matrix completion problems described above. Our approach builds on mixed matrices and mixed skew-symmetric matrices.

First, we prove that the matrix completion by rank-one matrices can be done in $O((m+n)^{2.77})$ time. It is faster than the previous algorithm of Ivanoys, Karpinski and Saxena [13] when $n = O(m^{2.46})$. Our method is based on a reduction to the mixed matrix completion. Furthermore, we provide a min-max theorem for the matrix completion by rank-one matrices. This theorem is a generalization of the result of Lovász [16] for the case $B_0 = 0$. As an application of the matrix completion by rank-one matrices, we devise a deterministic algorithm for the *multicast problem with linearly correlated sources* in *network coding*.

Second, we provide an algorithm for the mixed skew-symmetric matrix completion problem which runs in $O(m^4)$ time. This is the first deterministic polynomial time algorithm for the problem. Our method employs an algorithm for the delta-covering problem, and it can be regarded as a skew-symmetric version of mixed matrix completion algorithms of Geelen [7] and Harvey, Karger and Murota [10].

Finally, we show that the skew-symmetric matrix completion by rank-two skew-symmetric matrices can be reduced to the mixed skew-symmetric matrix completion. Using this reduction, we design a deterministic polynomial time algorithm, which runs in $O((m+n)^4)$ time.

1.2. Related Works. The beginning of studies for the matrix completion problem dates back to the works of Edmonds [4] and Lovász [15]. Lovász [15] has showed that random assignment to each indeterminate from a sufficiently large field achieves a max-rank completion with high probability. This randomized completion approach is useful both theoretically and practically. Cheung, Lau and Leung [2] have devised a randomized algorithm to compute edge connectivities for all pairs in a directed graph by the random completion for some matrix constructed from the graph.

While a randomized completion algorithm emerged in the early period of the studies, a satisfactory deterministic algorithm of matrix completion had been open until the end of the twentieth century. Lovász [16] has demonstrated that various completion problems of matrices without constant part admit essential relation to combinatorial optimization problems, along with polynomial time algorithms. Geelen [7] has described the first deterministic polynomial time algorithm for matrices with constant part such that each indeterminate appears only once. Geelen's algorithm takes $O(m^9)$ time and it works only over a field of size at least m. Later, Harvey, Karger and Murota [10] have devised an efficient algorithm for the same setting based on the independent matching problem. Their algorithm runs in $O(m^{2.77})$ time and works over an arbitrary field.

A matrix with indeterminates is called a *mixed matrix* if each indeterminate appears at most once. Mixed matrices are well-studied objects and have a deep connection to linear matroids and bipartite matchings. Murota [18] has presented efficient algorithms to compute the rank of a given mixed matrix and the combinatorial canonical form based on combinatorial properties of mixed matrices. A skew-symmetric version of a mixed matrix is called a *mixed skew-symmetric matrix*. Geelen, Iwata and Murota [9] have designed an efficient deterministic algorithm to compute the rank of mixed skew-symmetric matrices with the linear delta-matroid parity problem.

One of the most fruitful application areas of matrix completion is *network coding*, which is a new network communication framework proposed by Ahlswede et al. [1]. They have shown that network coding can achieve the best possible efficiency for the *multicast problem*, in which we have one information source and all sink nodes demand all information of the source. The *multicast problem with linearly correlated sources* is a generalization of the multicast problems. This problem is first considered by Ho et al. [12] and they have shown that *random network coding* finds a solution with high probability if the field size is sufficiently large. Harvey, Karger and Murota [10] have proposed another variation of multicast problems called the *anysource multicast problem*. They have designed a deterministic polynomial time algorithm for this problem with the matrix completion technique called the *simultaneous matrix completion*, under some condition for the field size. The linearly correlated multicast can be regarded as a natural generalization of the anysource multicast problem. For further information of network coding, the reader is referred to Yeung [20].

2. Preliminaries.

In this section, we introduce the concept and basic facts of mixed matrices and mixed skew-symmetric matrices, along with the corresponding combinatorial optimization problems. For further details, the reader is referred to Murota [18].

2.1. Mixed Matrix.

Let $\mathbf{K}$ be a subfield of a field $\mathbf{F}$. A matrix A over $\mathbf{F}$ is called a *mixed matrix* if $A = Q + T$, where Q is a matrix over $\mathbf{K}$ and T is a matrix over $\mathbf{F}$ such that the set of its nonzero entries is algebraically independent over $\mathbf{K}$.

A *layered mixed matrix* (*LM-matrix*) is a mixed matrix whose nonzero rows of Q are disjoint from its nonzero rows of T, i.e., a mixed matrix of the form $\left[\begin{smallmatrix} Q \\ T \end{smallmatrix}\right]$. Given an $m \times m$ mixed matrix $A = Q + T$, we associate an LM-matrix

$$\tilde{A} := \begin{bmatrix} I_m & Q \\ -Z & T' \end{bmatrix}, \tag{2.1}$$

where I_m is the identity matrix of size m, $Z := \operatorname{diag}[z_1, \ldots, z_m]$ is a diagonal matrix whose diagonal entries are new indeterminates $z_1, \ldots, z_m$, and $T' := ZT$. We can easily verify that $\operatorname{rank} \tilde{A} = m + \operatorname{rank} A$.

2.2. Independent Matching.

We now introduce the *independent matching problem*, which is an equivalent variation of the matroid intersection problem. Let $G = (V^+, V^-; E)$ be a bipartite graph with vertex set $V^+ \dot\cup V^-$ and edge set E. Let

$\mathbf{M}^+$ and $\mathbf{M}^-$ be matroids on V^+ and V^-, respectively. A matching M in G is said to be *independent* if the sets of vertices in V^+ and V^- incident to M are independent in $\mathbf{M}^+$ and $\mathbf{M}^-$, respectively. The independent matching problem is to find an independent matching of maximum size. The independent matching problem admits a min-max theorem, which is a generalization of the classical König-Egerváry theorem.

THEOREM 2.1 (Welsh [19]). *Let $G = (V^+, V^-; E)$ be a bipartite graph, $\mathbf{M}^+$ and $\mathbf{M}^-$ be matroids on V^+ and V^-, respectively. Then, we have*

$$\begin{aligned} &\max\{|M| : M \text{ is an independent matching}\} \\ &\quad = \min\{r^+(X^+) + r^-(X^-) : (X^+, X^-) \text{ is a cover of } G\}, \end{aligned} \tag{2.2}$$

where r^+ and r^- are the rank functions of $\mathbf{M}^+$ and $\mathbf{M}^-$, respectively.

2.3. Computing the Rank of Mixed Matrix. The rank of an LM-matrix, and therefore the rank of a general mixed matrix, can be computed by finding an independent matching of maximum size.

For an LM-matrix $A = \left[\begin{smallmatrix} Q \\ T \end{smallmatrix}\right]$, define a bipartite graph $G = (V^+, V^-; E)$ as follows. Put $V^+ := C_Q \cup R_T$ and $V^- := C$, where C and C_Q are the set of column indices and its copy, respectively, and R_T is the set of row indices of T. Let E be the set $\{j_Q j : j_Q \in C_Q \text{ and } j_Q \text{ is the copy of } j\} \cup \{ij : T_{ij} \neq 0\}$. Then, define a matroid $\mathbf{M}^+$ on V^+ as the direct sum of the linear matroid $\mathbf{M}[Q]$ and the free matroid on R_T. Finally, let $\mathbf{M}^-$ be the free matroid on V^-.

We are now ready to state a theorem that reveals a relationship between LM-matrices and independent matchings.

THEOREM 2.2 (Murota [18]). *For an LM matrix $A = \left[\begin{smallmatrix} Q \\ T \end{smallmatrix}\right]$, we have*

$$\operatorname{rank} A = \max\{|M| : M \text{ is an independent matching in } G\}.$$

2.4. Mixed Matrix Completion. In the *mixed matrix completion problem*, we are given a mixed matrix A and we are to maximize the rank of the matrix obtained by substituting values to the indeterminates of A. Throughout this paper, when we consider the mixed matrix completion problem, we assume that each indeterminate appears only *once* in a mixed matrix. This restricted class of mixed matrices is enough for our purpose.

Harvey, Karger and Murota [10] have developed an elegant algorithm for this problem by constructing an instance of the independent matching problem. Let $\tilde{A} = \left[\begin{smallmatrix} I & Q \\ -Z & T' \end{smallmatrix}\right]$ be the corresponding LM-matrix (2.1) of A. Construct G, $\mathbf{M}^+$ and $\mathbf{M}^-$ from $\tilde{A}$ as in the previous section. For an independent matching M of G, put $\mathcal{X}_M := \{T_{ij} :$ the edge corresponding to T'_{ij} is contained in $M\}$.

THEOREM 2.3 (Harvey, Karger and Murota [10]). *Let M be a maximum independent matching of the independent matching problem that minimizes $|\mathcal{X}_M|$. Then, substituting* 1 *to indeterminates in $\mathcal{X}_M$ and substituting* 0 *to the others yields a max-rank completion.*

Theorem 2.3 offers a simple algorithm that requires a subroutine to solve the weighted independent matching problem. Using a standard algorithm for the weighted matroid intersection [3], we can find a max-rank completion in $O(m^3 \log m)$ time, where m is the larger of the column size and the row size of the input mixed matrix. With the aid of fast matrix multiplication, the algorithm can be implemented to run in $O(m^{2.77})$ time [5].

2.5. Simultaneous Mixed Matrix Completion. The *simultaneous mixed matrix completion problem* is a more general completion problem: given a collection of mixed matrices that may share indeterminates, we are to maximize the rank of every matrix in the collection by substitutions. Harvey, Karger and Murota [10] showed that a simultaneous mixed matrix completion can be found in polynomial time under some condition of the field size.

THEOREM 2.4 (Harvey, Karger and Murota [10]). *Let $\mathcal{A}$ be a collection of mixed matrices and $\mathcal{X}$ be the set of indeterminates appearing in $\mathcal{A}$. A simultaneous completion for the collection $\mathcal{A}$ can be found deterministically in $O(|\mathcal{A}|(m^3 \log m + |\mathcal{X}|m^2))$ time if the field size is larger than $|\mathcal{A}|$, where m is the maximum of the row size and the column size of matrices in $\mathcal{A}$.*

2.6. Support Graph and Pffafian of Skew-Symmetric Matrix. A matrix A is said to be *skew-symmetric* if A is a square matrix such that $A_{ii} = 0$ for each i and $A_{ij} = -A_{ji}$ for each pair of distinct i and j. The *support graph* of an $m \times m$ skew-symmetric matrix A is an undirected graph $G = (V, E)$ with vertex set $V := \{1, \ldots, m\}$ and edge set $E := \{ij : A_{ij} \neq 0 \text{ and } i < j\}$. The *Pfaffian* of a skew-symmetric matrix A, denoted by $\operatorname{pf} A$, is a similar concept of determinants of matrices defined as follows:

$$\operatorname{pf} A := \sum_{M} \sigma_M \prod_{ij \in M} A_{ij}, \tag{2.3}$$

where the sum is taken over all perfect matchings M in the support graph of A and σ_M takes ± 1 in an appropriate manner (see Murota [18]). For each subset I of V, let $A[I]$ denote the submatrix of A whose rows and columns indexed by I. Such submatrices are called *principal submatrices* of A. The following is basic for skew-symmetric matrices.

LEMMA 2.5. *For a skew-symmetric matrix A, the rank of A equals the maximum size of I such that $A[I]$ is nonsingular, and $\det A = (\operatorname{pf} A)^2$ holds.*

2.7. Delta-matroid. A *delta-matroid* is a generalization of a matroid. Let V be a finite set and $\mathcal{F}$ be a non-empty family of subsets of V. The pair $(V, \mathcal{F})$ is called a delta-matroid if it satisfies the following condition:

For $F, F' \in \mathcal{F}$ and $i \in F \triangle F'$, there exists $j \in F \triangle F'$ such that $F \triangle \{i, j\} \in \mathcal{F}$,

where $F \triangle F' := (F \setminus F') \cup (F' \setminus F)$ denotes the symmetric difference of F and F'. Each member of $\mathcal{F}$ is called a *feasible set* of the delta-matroid $(V, \mathcal{F})$.

We can construct a delta-matroid from an $m \times m$ skew-symmetric matrix A. Let $V := \{1, \ldots, m\}$ and $\mathcal{F}_A := \{I : A[I] \text{ is nonsingular}\}$. Then it is known that $(V, \mathcal{F}_A)$ is a delta-matroid, and we denote this delta-matroid by $\mathbf{M}(A)$.

2.8. Mixed Skew-Symmetric Matrix. Let $\mathbf{K}, \mathbf{F}$ be fields such that $\mathbf{K}$ is a subfield of $\mathbf{F}$. A matrix $A = Q + T \in \mathbf{F}^{m \times m}$ is called a *mixed skew-symmetric matrix* if $Q \in \mathbf{K}^{m \times m}$ and $T \in \mathbf{F}^{m \times m}$ are skew-symmetric and the set $\{T_{ij} : T_{ij} \neq 0 \text{ and } i < j\}$ is algebraically independent over $\mathbf{K}$.

The rank of a mixed skew-symmetric matrix $A = Q + T$ can be characterized in terms of the corresponding delta-matroids $\mathbf{M}(Q)$ and $\mathbf{M}(T)$.

THEOREM 2.6 (Murota [18]). *For a mixed skew-symmetric matrix $A = Q + T$, we have*

$$\operatorname{rank} A = \max\{|F_Q \triangle F_T| : F_Q \in \mathcal{F}_Q \text{ and } F_T \in \mathcal{F}_T\}, \tag{2.4}$$

where $\mathcal{F}_Q$ and $\mathcal{F}_T$ are the families of feasible sets of $\mathbf{M}(Q)$ *and* $\mathbf{M}(T)$*, respectively.*

The maximization that appears in the right-hand side of (2.4) is the so called *(linear) delta covering problem.* Geelen, Iwata and Murota [9] have devised an algorithm for finding an optimal solution in $O(m^4)$ time, where m is the size of A.

3. Matrix Completion by Rank-one Matrices.

In this section, we show a reduction of the matrix completion by rank-one matrices to the mixed matrix completion and devise a faster deterministic polynomial time algorithm. A min-max theorem for the problem is also established. Finally, we consider the multicast problem with linearly correlated sources as an application of the matrix completion by rank-one matrices.

3.1. Reduction to the Mixed Matrix Completion.

Let B_0 be a matrix of arbitrary rank and $B_i = u_i v_i^\top$ be a rank-one matrix for $i = 1, \dots, n$. We associate the matrix $A = B_0 + x_1 B_1 + \cdots + x_n B_n$ with the mixed matrix $\tilde{A}$ defined as follows:

$$\tilde{A} := \left[\begin{array}{ccc|ccc|c} 1 & & & & & & v_1^\top \\ & \ddots & & & 0 & & \vdots \\ & & 1 & & & & v_n^\top \\ \hline x_1 & & & 1 & & & \\ & \ddots & & & \ddots & & 0 \\ & & x_n & & & 1 & \\ \hline & 0 & & u_1 & \cdots & u_n & B_0 \end{array}\right]. \tag{3.1}$$

By simple linear algebraic consideration, we obtain the following lemma.

Lemma 3.1. *The rank of $\tilde{A}$ is equal to $2n + \operatorname{rank} A$.*

Therefore, a max-rank completion of mixed matrix (3.1) yields an optimal solution of the original completion problem. A pseudocode description of our algorithm is presented in Algorithm 1.

Algorithm 1 Computing a solution of the matrix completion problem by rank-one matrices

Let $\tilde{A}$ be the mixed matrix (3.1).
Compute a max-rank completion $x_1, \dots, x_n$ of $\tilde{A}$.
return $x_1, \dots, x_n$.

Obviously, the running time of this algorithm is dominated by that of finding a max-rank completion of $\tilde{A}$. By Theorem 2.3, this can be done in $O((m+n)^{2.77})$ time with fast matrix multiplication, where m is the larger of the row size and the column size of A. Therefore we obtain the following theorem.

Theorem 3.2. *An optimal solution of the matrix completion by rank-one matrices can be found in $O((m+n)^{2.77})$ time.*

Our approach can be generalized to a collection of matrices, which we call the *simultaneous matrix completion by rank-one matrices.*

Simultaneous matrix completion by rank-one matrices: Given a collection of matrices in the form of $B_0 + x_1 B_1 + \cdots + x_n B_n$, where $B_1, \dots, B_n$ are of rank one, we are to find values that maximize the rank of every resulting matrix obtained by substituting these values in the collection.

A solution for the simultaneous matrix completion by rank-one matrices is called a *simultaneous max-rank completion.* By the above reduction and Theorem 2.4, we have the following theorem.

THEOREM 3.3. *Let $\mathcal{A}$ be a collection of matrices in the form of $B_0+x_1B_1+\cdots+x_nB_n$, where $B_1,\ldots,B_n$ are rank-one. Let $\mathcal{X}$ be the set of indeterminates appearing in $\mathcal{A}$. Then, a simultaneous max-rank completion by rank-one matrices can be found in $O\left(|\mathcal{A}|\left((|\mathcal{X}|+m)^3\log(|\mathcal{X}|+m)+|\mathcal{X}|(|\mathcal{X}|+m)^2\right)\right)$ time if the field size is larger than $|\mathcal{A}|$, where m is the maximum of the row size and the column size of matrices in $\mathcal{A}$.*

3.2. A Min-Max Theorem. In this section, we establish a min-max theorem for the matrix completion by rank-one matrices.

THEOREM 3.4. *Let B_0 be a matrix of arbitrary rank and $B_i=u_iv_i^\top$ be a rank-one matrix for $i=1,\ldots,n$. For any subset $J=\{j_1,\ldots,j_k\}\subseteq\{1,\ldots,n\}$, let us denote the matrix $[u_{j_1},\ldots,u_{j_k}]$ by $[u_j : j\in J]$ and the matrix $[v_{j_{k+1}},\ldots,v_{j_n}]$ by $[v_j : j\notin J]$, where $\{1,\ldots n\}\setminus J=\{j_{k+1},\ldots,j_n\}$. For $A=B_0+x_1B_1+\cdots+x_nB_n$, we have*

$$\begin{aligned}&\max\{\operatorname{rank} A : x_1,\ldots,x_n\}\\&=\min\left\{\operatorname{rank}\begin{bmatrix}0 & [v_j : j\notin J]^\top\\ [u_j : j\in J] & B_0\end{bmatrix} : J\subseteq\{1,\ldots,n\}\right\}.\end{aligned}\tag{3.2}$$

Proof. Let $\tilde{A}$ be the mixed matrix (3.1). Multiplying the row containing indeterminate x_i by a new indeterminate z_i, we can transform the matrix $\tilde{A}$ to the following LM-matrix:

$$\tilde{\tilde{A}}=\left[\begin{array}{ccc|ccc|c}1 & & & & & & v_1^\top\\ & \ddots & & & 0 & & \vdots\\ & & 1 & & & & v_n^\top\\\hline z_1x_1 & & & z_1 & & & \\ & \ddots & & & \ddots & & 0\\ & & z_nx_n & & & z_n & \\\hline & 0 & & u_1 & \cdots & u_n & B_0\end{array}\right].\tag{3.3}$$

Then we have that $\operatorname{rank}\tilde{\tilde{A}}=\operatorname{rank}\tilde{A}$.

We consider the independent matching problem associated with the LM-matrix $\tilde{\tilde{A}}$. Let $G=(V^+,V^-;E)$, $\mathbf{M}^+$ and $\mathbf{M}^-$ be as described in Section 2.3: V^+ is the direct sum of R_T and C_Q, and $V^-:=C$, the set of column indices of $\tilde{\tilde{A}}$. In this case, we have $E=\{j_Qj : j_Q\in C_Q \text{ and } j_Q \text{ is the copy of } j\}\cup\{ij : i=n+1,\ldots,2n \text{ and } j=i,i+n\}$ because the indeterminates are placed diagonally. Recall that the matroid $\mathbf{M}^+$ is the direct sum of a linear matroid over C_Q and the free matroid over R_T, and that $\mathbf{M}^-$ is the free matroid over V^-. Define a partition $\{C_1,C_2,C_3\}$ of C_Q as $C_1:=\{1,\ldots,n\}$, $C_2:=\{n+1,\ldots,2n\}$ and $C_3:=\{2n+1,\ldots,2n+m\}$. Define a partition $\{C_1',C_3',C_3'\}$ of V^- in the same manner.

Let (X^+,X^-) be a minimum rank cover of G. Then we have the following propositions.

PROPOSITION 3.5. *We can assume that $C_3\subseteq X^+$ and $C_3'\cap X^-=\emptyset$.*

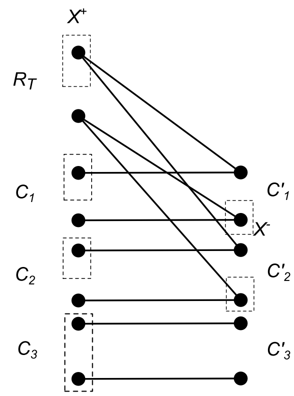

FIG. 3.1. *The corresponding independent matching problem of LM-matrix* (3.3).

Proof. For any $j \in C_3' \cap X^-$, $(X^+ \cup j, X^- \setminus j)$ is a cover because there is no indeterminate in C_3. Furthermore, $r^+(X^+ \cup j) + r^-(X^- \setminus j) \le r^+(X^+) + r^-(X^-)$ because $\mathbf{M}^-$ is a free matroid. Repeating this process completes the proof of the proposition. ∎

PROPOSITION 3.6. *We can assume that $j_T \in X^+ \cap R_T$ if and only if $j \in X^+ \cap C_1$, $j+n \in X^+ \cap C_2$ and $j', (j+n)' \notin X^-$, where j_T is the jth row index, j' is the copy of j.*

Proof. Let $j_T \in R_T$, $j \in C_1$ and $j' \in C_1'$. If $j_T \notin X^+$, then $j' \in X^-$ because the edge $j_T j'$ must be covered. Since the edge jj' is already covered, we can assume $j \notin X^+$. Conversely, if $j_T \in X^+$, then either $j \in X^+$ or $j' \in X^-$. However, applying the same argument in the proof of Proposition 3.5, we can delete j' from X^- and add j to X^+ without increasing the value of $r^+(X^+) + r^-(X^-)$. Since the exact same argument holds for j_T, $j+n$ and $(j+n)'$, this completes the proof of the proposition. ∎

By these propositions, we can assume that a minimum rank cover can be written in the form that $X^+ = J \cup J_1 \cup J_2 \cup C_3$ and $X^- = (C_1 \setminus J_1) \cup (C_2 \setminus J_2)$ for some $J \subseteq R_T$, where $J_1 := \{j \in C_1 : j_T \in J\}$, $J_2 := \{j+n \in C_2 : j_T \in J\}$. Thus, we obtain

$$
\begin{aligned}
&\max\{\operatorname{rank} \tilde{\tilde{A}} : x_1, \ldots, x_n\} \\
&\quad = \max\{|M| : M \text{ is an independent matching}\} \\
&\quad = \min\{r^+(X^+) + r^-(X^-) : (X^+, X^-) \text{ is a cover}\} \\
&\quad = \min\{r^+(J \cup J_1 \cup J_2 \cup C_3) + r^-((C_1 \setminus J_1) \cup (C_2 \setminus J_2)) : J \subseteq R_T\} \\
&\quad = \min\left\{2|J| + \operatorname{rank}\begin{bmatrix} 0 & [v_j : j \notin J]^\top \\ [u_j : j \in J] & B_0 \end{bmatrix} + 2(n - |J|) : J \subseteq \{1, \ldots, n\}\right\} \\
&\quad = 2n + \min\left\{\operatorname{rank}\begin{bmatrix} 0 & [v_j : j \notin J]^\top \\ [u_j : j \in J] & B_0 \end{bmatrix} : J \subseteq \{1, \ldots, n\}\right\},
\end{aligned}
$$

where the second equality follows from Theorem 2.1. Since $\operatorname{rank} \tilde{\tilde{A}} = 2n + \operatorname{rank} A$, this completes the proof of the theorem. ∎

The following min-max relation due to Lovász [16] is now immediate from Theorem 3.4.

THEOREM 3.7 (Lovász [16]). *Let $B_i = u_i v_i^\top$ be a rank-one matrix for $i = 1, \ldots, n$.*

Then, $A = x_1 B_1 + \cdots + x_n B_n$ *satisfies*

$$\begin{aligned}&\max\{\operatorname{rank} A : x_1, \ldots, x_n\}\\ &\quad = \min\{\dim\langle u_j : j \in J\rangle + \dim\langle v_j : j \in \{1, \ldots, n\} \setminus J\rangle : J \subseteq \{1, \ldots, n\}\},\end{aligned}$$

where $\langle \cdots \rangle$ *denotes the linear span.*

3.3. An Application to Network Coding. In this section we provide an application of the matrix completion by rank-one matrices: a deterministic algorithm for the multicast problem with linearly correlated sources. Note that if sources are independent, then this problem is equivalent to the *anysource multicast problem* [10]. In other words, this problem generalizes the anysource multicast problem to the case that sources are correlated each other.

As is often the case with studies of network coding, we concentrate on finding a *linear* solution, i.e., we assume that messages transmitted in the network are elements of a finite field and coding operations are restricted to be linear. The following algebraic framework is based on [12, 14].

Let $\mathbf{F}$ be a finite field. A row vector $x = [x_1 \cdots x_d] \in \mathbf{F}^d$ is called an *original message*. A *network* is a directed acyclic graph $G = (V, E)$ with node set V and edge set E. Let $S := \{s_1, \ldots, s_r\} \subseteq V$ and $T \subseteq V$ be the sets of *source nodes* and *sink nodes*, respectively. Each source node s_i has correlated messages xC_i, where C_i is a given matrix. Each edge e transmits a scalar message $y_e \in \mathbf{F}$ that is uniquely determined by the messages at its tail node. More precisely, for each edge e, y_e satisfies the following condition:

$$y_e = \begin{cases} \sum_{e' : e' \in \mathrm{In}(e)} k_{e',e} y_{e'} + xC_i A_{i,e} & \text{if the tail of } e \text{ is } s_i \in S,\\ \sum_{e' : e' \in \mathrm{In}(e)} k_{e',e} y_{e'} & \text{otherwise,}\end{cases} \tag{3.4}$$

where $\mathrm{In}(e) := \{e' \in E : e' = uv \text{ and } e = vw \text{ for some } u, v \text{ and } w\}$.

Each sink node t has to decode the original message x from the messages $\{y_e : e \in \mathrm{In}(t)\}$. This condition can be represented as follows:

$$x_j = \sum_{e : e \in \mathrm{In}(t)} p_{t,e,j} y_e \quad \text{for } j = 1, \ldots, d, \tag{3.5}$$

where $\mathrm{In}(t) := \{e \in E : \text{the head of } e \text{ is } t\}$.

Conditions (3.4) and (3.5) can be represented with a row vector y and matrices A, C, K and P_t for each $t \in T$, as the following linear equations:

$$y = yK + xCA, \tag{3.6}$$
$$x = yP_t \quad (t \in T). \tag{3.7}$$

Our goal is to find matrices A, K and P_t $(t \in T)$ satisfying conditions (3.6) and (3.7) for an *arbitrary* x. The next lemma is the key observation of our approach.

Lemma 3.8. *Matrices A, K and P_t $(t \in T)$ satisfy conditions* (3.6) *and* (3.7) *for an arbitrary x if and only if* $N_t := \left[\begin{smallmatrix} CA & 0 \\ I-K & P_t\end{smallmatrix}\right]$ *is nonsingular for each* $t \in T$.

Proof. The proof idea is due to Koetter et al. [14] and Ho et al. [12]. For sink $t \in T$, let M_t be the matrix $CA(I-K)^{-1}P_t$. The network G is acyclic and therefore K is nilpotent. Since the right-hand side of the identity $(I-K)^{-1} = I + K + K^2 + \cdots$ is a finite sum, $I - K$ is nonsingular and thus M_t is well-defined. One can easily check that matrices K, A and P_t ($t \in T$) satisfy conditions (3.6) and (3.7) for an arbitrary x if and only if M_t is nonsingular for every $t \in T$.

Next, we show that the singularities of M_t and N_t coincide. Recall that $N_t := \left[\begin{smallmatrix} CA & 0 \\ I-K & P_t \end{smallmatrix}\right]$. Consider the following identity:

$$\begin{bmatrix} CA & 0 \\ I-K & P_t \end{bmatrix} \begin{bmatrix} (I-K)^{-1}P_t & (I-K)^{-1} \\ -I & 0 \end{bmatrix} = \begin{bmatrix} M_t & CA(I-K)^{-1} \\ 0 & I \end{bmatrix}. \tag{3.8}$$

Taking the determinants of both sides, we can show that M_t is nonsingular if and only if N_t is nonsingular. This completes the proof of Lemma 3.8. ∎

By Lemma 3.8, finding a simultaneous max-rank completion for the collection $\mathcal{N} := \{N_t : t \in T\}$ is equivalent to finding a linear network code for the multicast problem with each nonzero entry of matrices A, K and P_t ($t \in T$) being regarded as indeterminates. Note that N_t is not a mixed matrix since a nonzero entry of A could appear in multiple entries. However, each nonzero entry of A must appear in the same column of N_t. Therefore, N_t can be written as $B_0 + \sum_z zB_z$, where the sum is taken over indeterminates appearing in N_t and B_z is a rank-one matrix for each variable z. Applying Theorem 3.3 to the collection $\mathcal{N}$, we have the following theorem.

THEOREM 3.9. *A linear code for the multicast problem with linearly correlated sources can be found in polynomial time.*

4. Mixed Skew-Symmetric Matrix Completion.

In this section, we give a deterministic algorithm for the mixed skew-symmetric matrix completion problem. Our method makes use of the linear delta-matroid covering problem.

Let $A = Q + T$ be a mixed skew-symmetric matrix of rank r. An outline of our algorithm is as follows. We set a 0-1 value to a carefully chosen indeterminate of T. This is equivalent to updating the constant part Q to a new constant matrix Q'. We argue that Q' has the larger rank than that of Q if we set an appropriate value. By repeating this process, the rank of the constant part Q' increases gradually and finally reaches r. Then Q' is a completed matrix of maximum rank and the algorithm returns Q'. A pseudocode description of the algorithm is in Algorithm 2.

Algorithm 2 Computing a max-rank completion for skew-symmetric matrix $A = Q + T$

Set $Q' := Q$.
Compute $F_Q \in \mathcal{F}_Q$ and $F_T \in \mathcal{F}_T$ that maximize $|F_Q \triangle F_T|$.
Compute a perfect matching M in the support graph of $T[F_T]$.
while $F_Q \cap F_T \neq \emptyset$ **do**
    Pick $i \in F_Q \cap F_T$ and let j be the vertex matched to i by M.
    Let $F_T := F_T \setminus \{i, j\}$ and $M := M \setminus \{ij\}$.
end while
for each $ij \in M$ **do**
    if $\mathrm{pf}\, Q'[F_Q \cup \{i, j\}] = 0$ **then**
        Let $Q'_{ij} := Q'_{ij} + 1$ and $Q'_{ji} := Q'_{ji} - 1$.
        Let $F_Q := F_Q \cup \{i, j\}$.
    end if
end for
return Q'

The rest of this section describes the details of our algorithm. Let $\mathcal{F}_Q$ and $\mathcal{F}_T$ denote the families of feasible sets of the delta-matroids $\mathbf{M}(Q)$ and $\mathbf{M}(T)$, respectively.

Let F_Q and F_T be members of $\mathcal{F}_Q$ and $\mathcal{F}_T$, respectively, with $|F_Q \triangle F_T|$ maximum. Note that $|F_Q \triangle F_T| = r$ from Theorem 2.6. Consider the support graph G of $T[F_T]$. Since $T[F_T]$ is nonsingular, G has a perfect matching M. We show that we can shrink F_T so that F_Q and F_T are disjoint without decreasing the value of $|F_T \triangle F_Q|$.

LEMMA 4.1. *Let j be the vertex of G matched to a vertex $i \in F_Q \cap F_T$ by M. Then $F_T \setminus \{i,j\}$ is a feasible set of $\mathbf{M}(T)$ and $|F_Q \triangle (F_T \setminus \{i,j\})| = |F_Q \triangle F_T|$.*

Proof. Since $G \setminus \{i,j\}$ has a perfect matching, namely $M \setminus \{ij\}$, $T[F_T \setminus \{i,j\}]$ is nonsingular. Thus $F_T \setminus \{i,j\}$ is feasible in $\mathbf{M}(T)$. Suppose that $j \in F_Q \cap F_T$. Then $|F_Q \triangle (F_T \setminus \{i,j\})| > |F_Q \triangle F_T|$, this contradicts the maximality of $|F_Q \triangle F_T|$. Therefore $j \in F_T \setminus F_Q$ and $|F_Q \triangle (F_T \setminus \{i,j\})| = |F_Q \triangle F_T|$. □

Thus we can assume that F_Q and F_T are disjoint without loss of generality. If F_T is empty, then $r = |F_Q| = \operatorname{rank} Q[F_Q]$ and therefore setting $T := 0$ achieves a max-rank completion. So we consider the case that F_T is nonempty. Let ij be an edge of M. Substituting a value α to T_{ij} (and $-\alpha$ to T_{ji}) is equivalent to adding α to Q_{ij} (and $-\alpha$ to Q_{ji}). Let Q' be the matrix obtained from Q by replacing Q_{ij} and Q_{ji} with $Q_{ij} + \alpha$ and $Q_{ji} - \alpha$, respectively. The following lemma offers the core of our completion algorithm.

LEMMA 4.2. *For any edge ij of M, there exists a value $\alpha \in \{0,1\}$ such that $Q'[F_Q \cup \{i,j\}]$ is nonsingular.*

Proof. From the definition of Pfaffian (2.3), we have the following identity:

$$\operatorname{pf} Q'[F_Q \cup \{i,j\}] = \operatorname{pf} Q[F_Q \cup \{i,j\}] + (-1)^k \alpha \cdot \operatorname{pf} Q[F_Q], \tag{4.1}$$

where k is some integer uniquely determined by i and j. Since F_Q is a feasible set of $\mathbf{M}(A)$, $Q[F_Q]$ is nonsingular. Thus $\operatorname{pf} Q[F_Q]$ is nonzero. Set $\alpha := 0$ if $\operatorname{pf} Q[F_Q \cup \{i,j\}] \neq 0$, and otherwise set $\alpha := 1$. This ensures that $\operatorname{pf} Q'[F_Q \cup \{i,j\}] \neq 0$. □

Note that $\operatorname{rank} Q'[F_Q \cup \{i,j\}] = \operatorname{rank} Q[F_Q] + 2$. After applying Lemma 4.2, $F_Q \cup \{i,j\}$ is feasible in $\mathbf{M}[Q']$ and $F_T \setminus \{i,j\}$ is feasible in $\mathbf{M}[T']$ where T' is the matrix obtained from T by setting $T_{ij} = T_{ji} = 0$. Also, $M \setminus \{i,j\}$ is a perfect matching in the support graph of $T'[F_T \setminus \{i,j\}]$. Thus we can apply Lemma 4.2 to Q', T', $F_Q \cup \{i,j\}$, $F_T \setminus \{i,j\}$ and $M \setminus \{i,j\}$. Repeating this until $M = \emptyset$, we obtain a skew-symmetric matrix Q' such that $\operatorname{rank} Q'[F_Q \cup F_T] = \operatorname{rank} Q[F_Q] + 2|M| = |F_Q| + |F_T| = r$. Therefore, Q' is a max-rank completion of A.

Now we analyze the running time for an $m \times m$ mixed skew-symmetric matrix. An optimal pair of F_Q and F_T can be found in $O(m^4)$ time by the algorithm of Geelen, Iwata and Murota [9]. A perfect matching M can be found in $O(m^3)$ time. Since the Pfaffian of a principal submatrix of Q' can be computed in $O(m^3)$ time and $|M| = m/2$, the iteration of setting values to indeterminates of T takes $O(m^4)$ time. Thus we obtain the following theorem.

THEOREM 4.3. *A max-rank completion for an $m \times m$ mixed skew-symmetric matrix can be found in $O(m^4)$ time.*

5. Skew-Symmetric Matrix Completion by Rank-two Skew-Symmetric Matrices. In this section, we consider the skew-symmetric matrix completion by rank-two skew-symmetric matrices. We show that this problem can be reduced to the mixed skew-symmetric matrix completion as in the case of the matrix completion by rank-one matrices. Let $A := B_0 + x_1 B_1 + \cdots + x_n B_n$, where B_0 is an $m \times m$ skew-symmetric matrix and $B_1, \ldots, B_n$ are skew-symmetric matrices of rank two. First, note that for $i = 1, \ldots, n$, there exists some vectors u_i and v_i such that $B_i = u_i v_i^\top - v_i u_i^\top$.

Let $\tilde{A}$ be the following mixed skew-symmetric matrix:

$$\tilde{A} := \begin{bmatrix}
\begin{matrix}0&1\\-1&0\end{matrix} & & & \begin{matrix}-v_1^\top\\ \mathbf{0}^\top\end{matrix} & \begin{matrix}0&0\\x_1&0\end{matrix} & & \\
& \ddots & & \vdots & & \ddots & \\
& & \begin{matrix}0&1\\-1&0\end{matrix} & \begin{matrix}-v_n^\top\\ \mathbf{0}^\top\end{matrix} & & & \begin{matrix}0&0\\x_n&0\end{matrix} \\
\begin{matrix}v_1&\mathbf{0}\end{matrix} & \cdots & \begin{matrix}v_n&\mathbf{0}\end{matrix} & B_0 & \begin{matrix}\mathbf{0}&u_1\end{matrix} & \cdots & \begin{matrix}\mathbf{0}&u_n\end{matrix} \\
\begin{matrix}0&-x_1\\0&0\end{matrix} & & & \begin{matrix}\mathbf{0}^\top\\ -u_1^\top\end{matrix} & \begin{matrix}0&1\\-1&0\end{matrix} & & \\
& \ddots & & \vdots & & \ddots & \\
& & \begin{matrix}0&-x_n\\0&0\end{matrix} & \begin{matrix}\mathbf{0}^\top\\ -u_n^\top\end{matrix} & & & \begin{matrix}0&1\\-1&0\end{matrix}
\end{bmatrix}. \tag{5.1}$$

By a sequence of basic operations for $\tilde{A}$, one can easily obtain the following lemma.

LEMMA 5.1. *The skew-symmetric matrices A and $\tilde{A}$ satisfy* $\operatorname{rank} \tilde{A} = 4n + \operatorname{rank} A$.

Thus a max-rank completion for $\tilde{A}$ yields a max-rank completion for A. Using the completion algorithm of Section 4, we can obtain a completion for A.

THEOREM 5.2. *A solution for the skew-symmetric matrix completion by rank-two skew-symmetric matrices can be found in $O((m+n)^4)$ time.*

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REFERENCES

[1] R. Ahlswede, N. Cai, S.-Y. R. Li, and R. W. Yeung, *Network information flow*, IEEE Transactions on Information Theory, 46 (2000), pp. 1204–1216.

[2] H. Y. Cheung, L. C. Lau, and K. M. Leung, *Graph connectivities, network coding, and expander graphs*, SIAM Journal on Computing, 42 (2013), pp. 733–751.

[3] W. H. Cunningham, *Improved bounds for matroid partition and intersection algorithms*, SIAM Journal on Computing, 15 (1986), pp. 948–957.

[4] J. Edmonds, *Systems of distinct representatives and linear algebra*, Journal of Research of the National Bureau of Standards, B71 (1967), pp. 241–245.

[5] H. N. Gabow and Y. Xu, *Efficient theoretic and practical algorithms for linear matroid intersection problems*, Journal of Computer and System Sciences, 53 (1996), pp. 129–147.

[6] J. Geelen and S. Iwata, *Matroid matching via mixed skew-symmetric matrices*, Combinatorica, 25 (2005), pp. 187–215.

[7] J. F. Geelen, *Maximum rank matrix completion*, Linear Algebra and Its Applications, 288 (1999), pp. 211–217.

[8] ———, *An algebraic matching algorithm*, Combinatorica, 20 (2000), pp. 61–70.

[9] J. F. Geelen, S. Iwata, and K. Murota, *The linear delta-matroid parity problem*, Journal of Combinatorial Theory, B88 (2003), pp. 377–398.

[10] N. J. A. Harvey, D. R. Karger, and K. Murota, *Deterministic network coding by matrix completion*, in Proceedings of the 16th Annual ACM-SIAM Symposium on Discrete Algorithms, 2005, pp. 489–498.

[11] N. J. A. Harvey, D. R. Karger, and S. Yekhanin, *The complexity of matrix completion*, in Proceedings of the 17th Annual ACM-SIAM Symposium on Discrete Algorithm, 2006, pp. 1103–1111.

[12] T. Ho, M. Médard, R. Koetter, D. R. Karger, M. Effros, J. Shi, and B. Leong, *A random linear network coding approach to multicast*, IEEE Transactions on Information Theory, 52 (2006), pp. 4413–4430.

[13] G. Ivanyos, M. Karpinski, and N. Saxena, *Deterministic polynomial time algorithms for matrix completion problems*, SIAM Journal on Computing, 39 (2010), pp. 3736–3751.
[14] R. Koetter and M. Médard, *An algebraic approach to network coding*, IEEE/ACM Transactions on Networking, 11 (2003), pp. 782–795.
[15] L. Lovász, *On determinants, matchings and random algorithms*, Fundamentals of Computation Theory, FCT, (1979), pp. 565–574.
[16] ———, *Singular spaces of matrices and their application in combinatorics*, Bulletin of the Brazilian Mathematical Society, 20 (1989), pp. 87–99.
[17] L. Lovász and Y. Yemini, *On generic rigidity in the plane*, SIAM Journal on Algebraic and Discrete Methods, 1 (1982), pp. 91–98.
[18] K. Murota, *Matrices and Matroids for System Analysis*, Springer-Verlag, Berlin, 2nd ed., 2009.
[19] D. Welsh, *On matroid theorems of Edmonds and Rado*, Journal of the London Mathematical Society, S2 (1970), pp. 251–256.
[20] R. W. Yeung, *Information Theory and Network Coding*, Springer-Verlag, Berlin, 2008.