UNIVERSITÄT
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Introduction to Combinatorics

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Contents

Definition

If v and w are non-adjacent vertices in G, $\kappa_G(v, w)$ is the smallest number of vertices whose removal separates v from w, that is, disconnects G leaving v and w in different components. A cutset that separates v and w is called a separating set for v and w. $p_G(v, w)$ is the maximum number of internally disjoint paths between v and w.

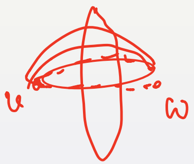
Example

Theorem

If v and w are non-adjacent vertices in G then $\boxed{\kappa_G(v, w) = p_G(v, w)}$.

Proof.

□



$v \nsim w$

S : cutset

$|S|$

Theorem (Menger's Theorem)

If G has at least $k + 1$ vertices, then G is k-connected if and only if between every two vertices u and v there are k pairwise internally disjoint paths.

Proof.

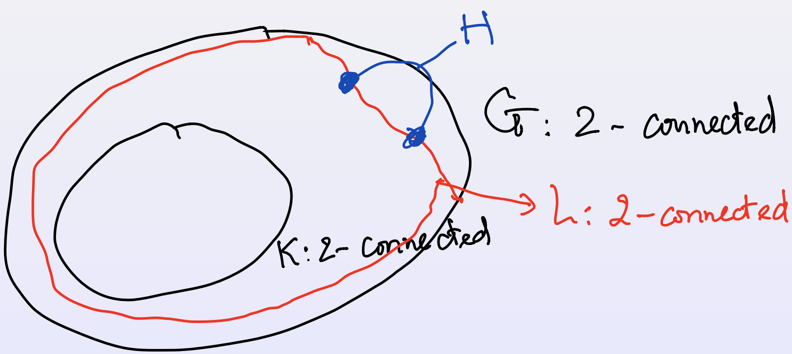
□

Theorem (The Handle Theorem)

Suppose G is 2-connected and K is a 2-connected proper subgraph of G. Then there are subgraphs L and H (the handle) of G such that L is 2-connected, L contains K, H is a simple path, L and H share exactly the endpoints of H, and G is the union of L and H.

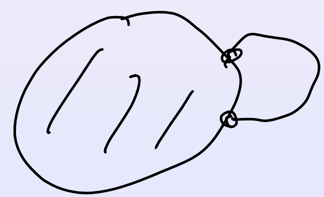
Proof.

□



$\boxed{K < G}$

$L \cup H = G$

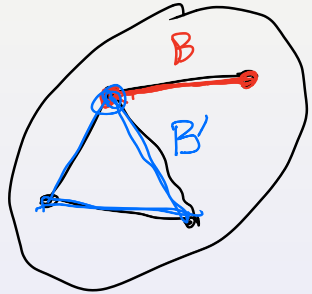


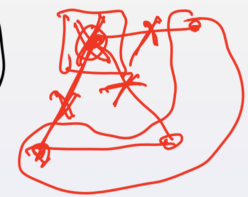
Definition

A block in a graph G is a maximal induced subgraph on at least two vertices without a cutpoint.

Example

B :
- maximal induced subgraph
- $|V(B)| \geq 2$ ✓ ✓
- without cutpoint. ✓ ✓





2 - connected subgraphs

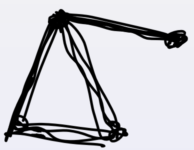
Theorem

The blocks of G partition the edges.

Proof.

□

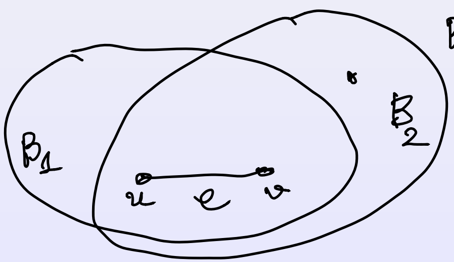
$\forall \underline{e} \in E(G)$ $\exists!$ block contains e:



- $\forall e \in E(G)$: If e is <u>not inside any 2-connected induced graph</u>

  $\Rightarrow$ <u>$\{e,$ endpoints$(e)\}$</u> is a block. $\exists$ e

- $\forall e \in E(G)$. Assume that $\exists B_1, B_2$ blocks contain e.



Prove: $G[V(B_1) \cup V(B_2)]$ has no cutpoint.

In fact, if $\exists w \in V(B_1) \cup V(B_2)$: B − w is disconnected.

$\Rightarrow \left[\begin{array}{l} B_1 - w \text{ is disconnected} \Rightarrow w \text{ is a cutpoint in } B_1 \\ B_2 - w \text{ is disconnected} \Rightarrow \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_} \ B_2. \end{array}\right.$ □

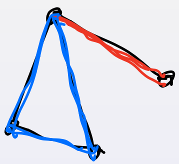
Theorem

If G is connected but not 2-connected, then every vertex that is in two blocks is a cutpoint of G.

Proof.

□

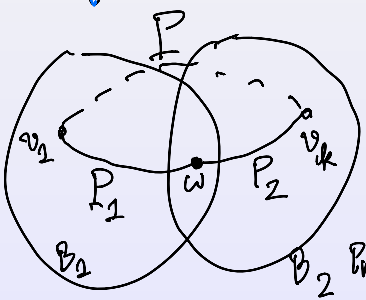
$w \in B_1 \cap B_2$ $\quad$ B_1, B_2 : blocks. $\quad$ maximal induced graph. without cutpoint.



Assume $G - w$ is connected. $\Rightarrow$

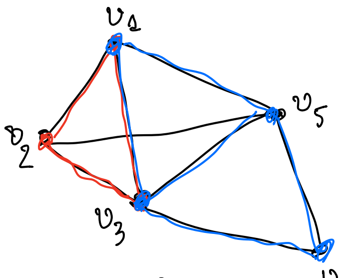
B_1 block $\Rightarrow \exists\, v_1 \in B_1 : v_1 \neq w \Rightarrow v_1, v_k \in G - w$

B_2 block $\Rightarrow \exists\, v_k \in B_2 : v_k \neq w$



$\Rightarrow \exists$ a path $\{v_1, v_2, \dots, v_k\} \subseteq G - w$

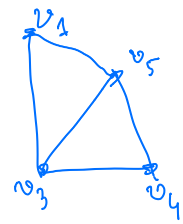
Prove $\boxed{G[V(B_1) \cup V(B_2) \cup \{v_1, \dots, v_k\}]}$ is 2-connected. $\Rightarrow$ ><.



G

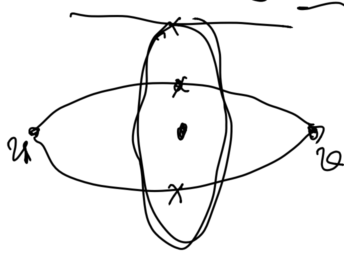
$G[\{v_1, v_2, v_3\}]$

$G[\{v_1, v_3, v_4, v_5\}]$



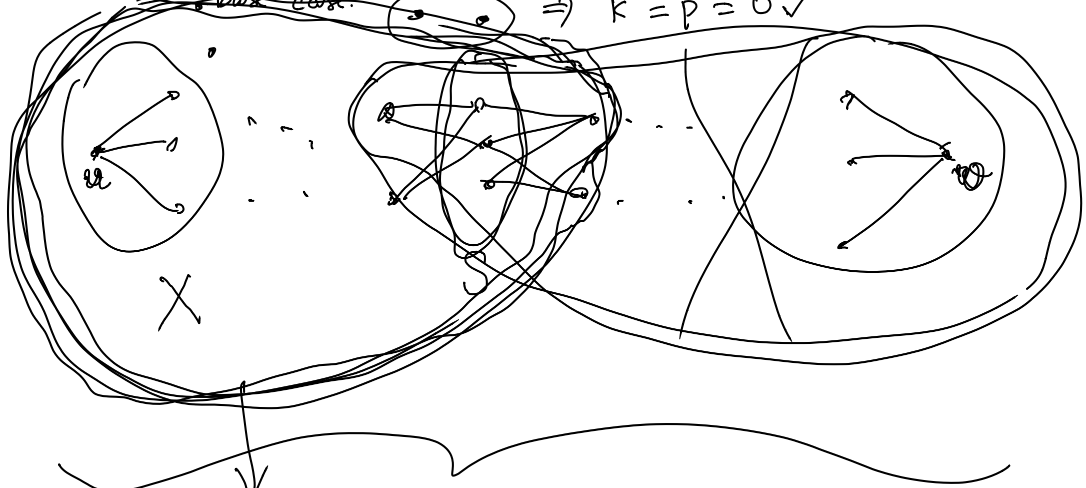
* Thm: $\forall u \neq v \Rightarrow \kappa_G(u,v) = p_G(u,v)$

* Proof. • $\kappa_G(u,v) \geq \underbrace{p_G(u,v)}_{= k}$



• $\kappa_G(u,v) \leq p_G(u,v)$. Induction on $n = |V(G)|$

~~• base case:~~ $\Rightarrow \kappa = p = 0$ ✓



$|V(G_1)| < n \qquad |V(G)| = n$

$\Downarrow$

$\kappa_G(u,v) \leq p_G(u,v)$

$K_{G_1}(u,v) = p_{G_1}(u,v)$

$K_G(u,v) \qquad p_G(u,v)$