UNIVERSITÄT
LEIPZIG

Introduction to Combinatorics

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SoSe 2020 (Day 1 - 07/04/2020)

Contents

Some definitions

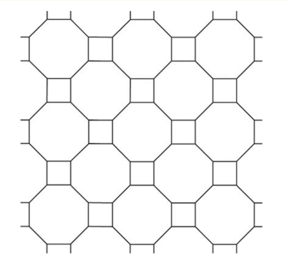
1. a branch of mathematics dealing with combinations and permutations. [Collin's dictionary]
2. the branch of mathematics dealing with combinations of objects belonging to a finite set in accordance with certain constraints, such as those of graph theory. [Lexico's dictionary]
3. an area of mathematics primarily concerned with counting, both as a means and an end in obtaining results, and certain properties of finite structures. [Wikipedia]
4. the branch of mathematics studying the enumeration, combination, and permutation of sets of elements and the mathematical relations that characterize their properties. [Wolfram MathWorld]

Types of problems

1. Existence problem: Does $\cdots$ exist?
2. Construction problem: If $\cdots$ exists, how can we construct it?
3. Enumeration problem: How many $\cdots$ are there?
4. Optimization problem: Which $\cdots$ is the best?

Motivative examples - Existence problems

Tiling problems:



This figure shows a tiled floor pattern constructed from squares and regular octagons that match up exactly without any gaps or overlaps. What other tiling patterns are possible? For example, does there exist a tiling pattern with both squares and regular hexagons?

Answer: No

If there existed a tiling with both squares and hexagons, then each meeting point would have a combination of 90^o angles (for the squares) and 120^o angles (for the hexagons) adding up to 360^o. But this is impossible, for if there were only one hexagon (120^o) then the remaining 240^o couldn't be made up of right angles, if there were two hexagons (240^o) the remaining 120^o couldn't be made up from right angles, and if there were three hexagons (360^o) there'd be no room for any squares. So no combination of both 90^o and 120^o angles can make up the desired 360^o and no such tiling can exist.

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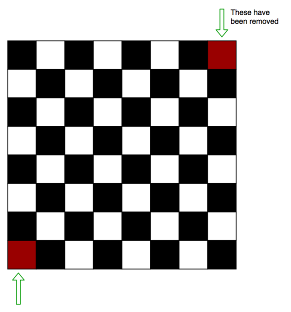
Putting dominoes on a chessboard:



It's easy to see that we can cover all the sixty-four squares of a chessboard with dominoes. If we remove a single corner square, then this is no longer possible, since an odd number of squares remains. But what happens if we remove two opposite corner squares? Does there exist a domino covering of the remaining sixty- two squares?

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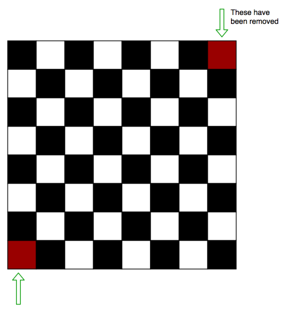

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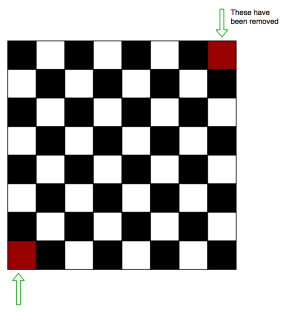

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Answer: No

At first sight it may seem as though much experimentation will be needed. But the answer is very simple once we notice that every domino must cover a black square and a white square. Since the two removed corner squares have the same colour, there are now thirty-two squares of one colour and thirty of the other colour, so no covering by dominoes can exist.

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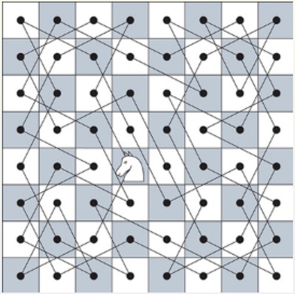
The knight's tour problem:



Does there exist a 'knight's tour' in which a knight visits all sixty-four squares just once and returns to its starting point? We can similarly ask whether knight's tours exist for chessboards of other sizes - such as a 4×4 or 5×5 chessboard. We answer this when we discuss Hamiltonian cycle.

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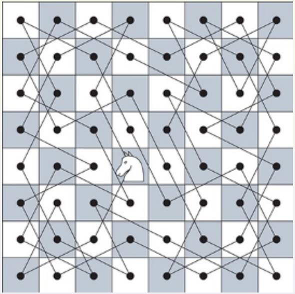

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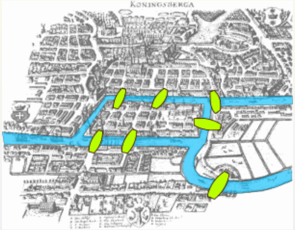
The Königsberg bridges problem:



It is said that the citizens of Königsberg wished to take a walk, crossing each bridge exactly once before returning to their starting point. Does such a route exist? This problem was answered by the Swiss mathematician Leonhard Euler, who extended his solution to any arrangement of regions connected by bridges. We present his answer when we discuss Eulerian cycle.

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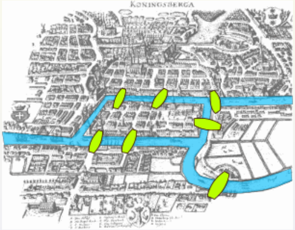

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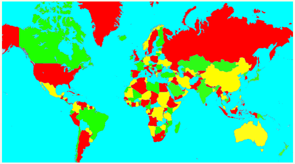
The map-colour problem:

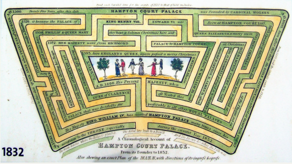


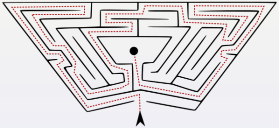
Can the world map be coloured with just four colours so that neighbouring countries receive different colours? do there exist maps that require five or more colours?

Motivative examples - Construction problems

One way to solve an existence problem is to construct a solution explicitly—indeed, this is sometimes the only way to do so. But for other problems we may be able to prove by theoretical means that a solution exists without our being able to construct one. Moreover, some construction problems are amenable to trial-and-error experimentation, while others will require a more systematic approach. Both methods have been used for the tracing of mazes:

The Hampton Court Maze:





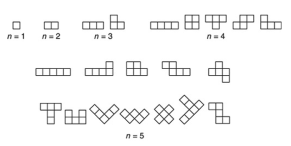
Suppose that we're stuck in the middle of a maze. We know that there must be a way of escaping from the maze, but we need to find a method for doing so?

Motivative examples - Enumeration problems

Ever since earliest times people have needed to count the objects around them. We're used to such questions as:

1. How many children have you got?
2. How many shopping days are there until Christmas?
3. How many solutions of the Hampton Court Maze problem?

Polyominoes:



Just as a domino is formed from two squares of equal size, so a tromino is formed from three squares in a straight line or an L-shape, and an $n-$omino is formed from n squares. How many n-ominoes are there for a given value of n?

No answer to our question is known in general!
Some calculations: $S_1 = S_2 = 1$, $S_3 = 2$, $S_4 = 5$, $S_5 = 12$, and $S_{28} = 153.511.100.594.603$.

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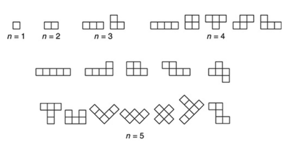

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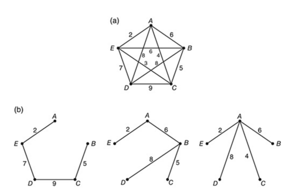
Motivative examples - Optimization problems

We wish to drive from Berlin to Munich in the shortest possible time. Which of the many available routes should we choose, given the travel times between pairs of neighbouring cities on the various routes?

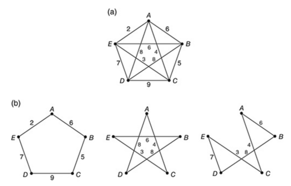
1. Minimum connector problem: We wish to connect a number of cities by links, such as canals, railway lines, or air routes, but the cost of the connections is high. How can we minimize the total cost while ensuring that we can still get from each city to every other one?
2. Travelling salesman problem: A travelling salesman wishes to visit a number of cities to sell his wares, and return to his starting point. If the costs of travelling between the cities are known, how should he plan his route so as to minimize the total cost?

The minimum connector problem:



This figure shows a minimum connector problem with five cities, A, B, C, D, and E, where the numbers refer to the connection costs between pairs of cities. Some experimentation gives the solutions with total costs 23, 21, and 20. But is 20 the best solution, or is there a smaller one?

The travelling salesman problem:



This figure gives an example of the travelling salesman problem. Again, some experimentation gives the solutions with total costs 29, 29, and 28. But is 28 the best solution, or is there a smaller one - and is there an efficient algorithm that always produces an optimal solution?

Addition Principle

Proposition

We say a finite S is partitioned into parts $S_1, \ldots, S_k$ if the parts are disjoint and their union is S. Then, we have

$$|S| = |S_1| + \cdots + |S_k|.$$

Example

Let S be the set of students attending lecture Introduction to Combinatorics. It can be partitioned into two parts S_1 and S_2 where S_1 is the set of students that like easy examples and S_2 is the set of students that don't like easy examples. If $|S_1| = 12$ and $|S_2| = 5$ then we can conclude $|S| = 17$.

Multiplication Principle

Proposition

If S is a finite set that is the product of $S_1, \ldots, S_k$, i.e., $S = S_1 \times \cdots \times S_k$, then

$$|S| = |S_1| \times \cdots \times |S_k|.$$

Example

You are eating at a mensa of Leipzig University and recognize that you have

(a) two choices for appetizers: soup or juice;

(b) three for the maincourse: a meat, fish, or vegetable dish;

(c) two for dessert: ice cream or cake.

How many possible choices do you have for your complete meal?

Pigeonhole Principle

Proposition

Let $S_1, \dots, S_k$ be finite sets that are pairwise disjoint and $|S_1| + \cdots + |S_k| = n$, then

1. $\exists i \in \{1, \dots, k\} : |S_i| \geq \lceil \frac{n}{k} \rceil$;
2. $\exists j \in \{1, \dots, k\} : |S_j| \leq \lfloor \frac{n}{k} \rfloor$.

Example

Assume there are 5 holes in the wall where pigeons nest. Say there is a set S_i of pigeons nesting in hole i. Assume there are $n = 17$ pigeons in total. Then we know:

1. There is some hole with at least $\lceil 17/5 \rceil = 4$ pigeons;
2. There is some hole with at most $\lfloor 17/5 \rfloor = 3$ pigeons.

Double counting

Proposition

If we count the same quantity in two different ways, then this gives us an identity.

Example (Handshaking Lemma)

Assume there are n people at a party and everybody will shake hands with everybody else. How many handshakes will occur?

The way to count:
Every person shakes $n-1$ hands and there are n people. However, person i shakes hands of person j is the same as person j shakes hands of person i, i.e., every handshake is counted twice. Therefore, the total number of handshakes is $\frac{n(n-1)}{2}$.

Permutations

Definition

Let S be any finite set. A permutation of S is a one-to-one mapping of S onto itself.

Example

Given $S = \{a_1, a_2, a_3, a_4\}$. A possible permutation σ would be

$$\sigma = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ a_2 & a_1 & a_4 & a_3 \end{pmatrix}$$

which could be also written in the form

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$$

indicating that a_1 went to a_2, a_2 to a_1, a_3 to a_4, and a_4 to a_3.
We also sometimes write $\sigma = 2143$ and call it a rearrangement of S.

Theorem

The total number of permutations of a set S of n elements is given by $n! = n \cdot (n-1) \cdot \ldots \cdot 1$.

Proof.

There are n ways to assign the first element, for each of these we have $n-1$ ways to assign the second object, $n-2$ for the third, and so forth. This proves the theorem. □

k-Permutations

Definition

Let S be an n-element set and let k be an integer between 0 and n. Then a k-permutation of S is an ordered listing of a subset of S of size k.

Theorem

The total number of k-permutations of a set S of n elements is given by

$$A_{n,k} = n \cdot (n-1) \cdot \ldots \cdot (n-k+1) = \frac{n!}{(n-k)!}.$$

k-combinations

Definition

Let S be an n-element set and let k be an integer between 0 and n. Then a k-combination of S is an unordered listing of a subset of S of size k.

Theorem

The total number of k-combinations of a set S of n elements is given by

$$S_{n,k} = \frac{A_{n,k}}{k!} = \frac{n!}{(n-k)!k!} = \binom{n}{k}.$$

Theorem

Prove that, for $1 \leq k \leq n-1$,

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$