UNIVERSITÄT
LEIPZIG

Introduction to Combinatorics

Dat Tran (FMI, Leipzig University)

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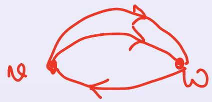
Contents

1. Introduction to Graph theory
   - Directed graphs

Definition

A directed graph, also called a digraph, is a graph in which the edges have a direction, indicated with an arrow on the edge; more formally, if v and w are vertices, an edge is an unordered pair $\{v, w\}$, while a directed edge, called an arc, is an ordered pair (v, w) or (w, v). The arc (v, w) is drawn as an arrow from v to w. It is possible to have multiple arcs, namely, an arc (v, w) may be included multiple times in the multiset of arcs. As before, a digraph is called simple if there are no loops or multiple arcs.

Example

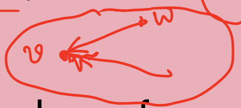


$G = (V, E)$

$\vec{G} = (V, \vec{E})$

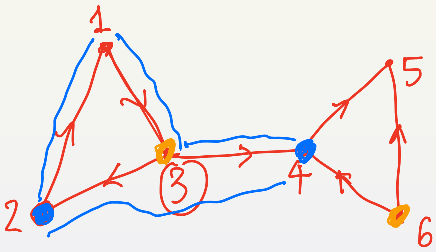
Definition

We denote by E_v^- the set of all arcs of the form (w, v), and by E_v^+ the set of arcs of the form (v, w).



- The indegree of v, denoted by $d^-(v)$, is the number of arcs in E_v^-; The outdegree, $d^+(v)$, is the number of arcs in E_v^+. If the vertices are $v_1, v_2, \ldots, v_n$, the degrees are usually denoted $d_1^-, d_2^-, \ldots, d_n^-$ and $d_1^+, d_2^+, \ldots, d_n^+$. Note that both $\sum_{i=0}^n d_i^-$ and $\sum_{i=0}^n d_i^+$ count the number of arcs exactly once, and of course $\sum_{i=0}^n d_i^- = \sum_{i=0}^n d_i^+$.
- A walk in a digraph is a sequence $v_1, e_1, v_2, e_2, \ldots, v_{k-1}, e_{k-1}, v_k$ such that $e_k = (v_i, v_{i+1})$; if $v_1 = v_k$, it is a closed walk or a circuit.
- A path in a digraph is a walk in which all vertices are distinct. It is not hard to show that, as for graphs, if there is a walk from v to w then there is a path from v to w.

Example



$E_3^- = \{(1,3)\} \Rightarrow d_3^- = 1$

$E_3^+ = \{(3,2),(3,4)\} \Rightarrow d_3^+ =$

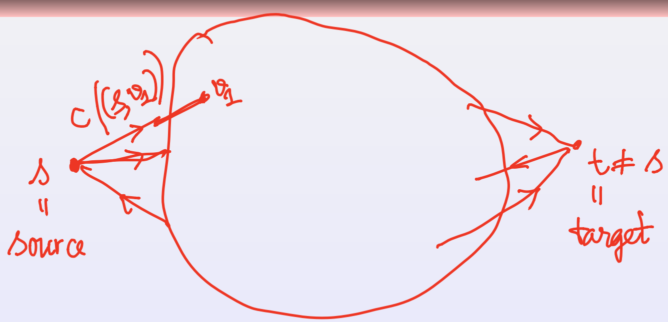
No path from 3 to 6.

Definition

A network is a digraph with a designated source s and target $t \neq s$. In addition, each arc e has a positive capacity, $c(e)$.

Example

Networks can be used to model transport through a physical network, of a physical quantity like oil or electricity, or of something more abstract, like information.



Definition

A flow in a network is a function f from the arcs of the digraph to $\mathbb{R}$, with $0 \leq f(e) \leq c(e)$ for all e, and such that

$f: \vec{E} \to \mathbb{R}$

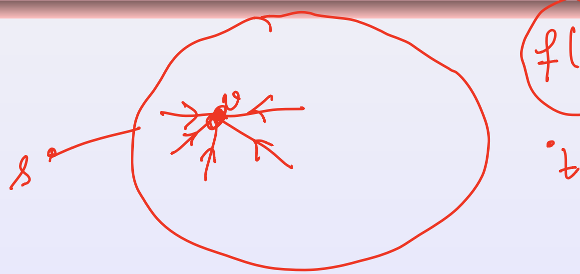
$$\sum_{e \in E_v^+} f(e) = \sum_{e \in E_v^-} f(e),$$

in-infos   out-infos

for all v other than s and t.

Example

$f(E_v^+) = f(E_v^-)$



Suppose that U is a set of vertices in a network with $s \in U$ and $t \notin U$. Let $\overrightarrow{U}$ be the set of arcs (v, w) with $v \in U, w \notin U$, and $\overleftarrow{U}$ be the set of arcs (v, w) with $v \notin U, w \in U$.

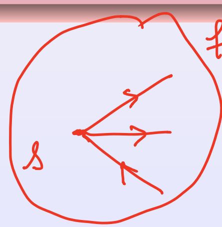
Theorem

For any flow f in a network, the net flow out of the source is equal to the net flow into the target, namely,

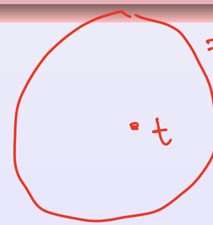
$$\sum_{e \in E_s^+} f(e) - \sum_{e \in E_s^-} f(e) = \sum_{e \in E_t^-} f(e) - \sum_{e \in E_t^+} f(e),$$

Proof.

□



$f(E_s^+) - f(E_s^-)$



$f(E_t^-) - f(E_t^+)$

Definition

The value of a flow, denoted $val(f)$, is $\sum_{e \in E_s^+} f(e) - \sum_{e \in E_s^-} f(e)$. A maximum flow in a network is any flow f whose value is the maximum among all flows.

Example

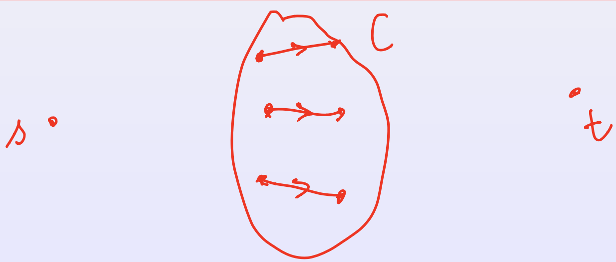
* Maximum flow

Definition

A cut in a network is a set C of arcs with the property that every path from s to t uses an arc in C, that is, if the arcs in C are removed from the network there is no path from s to t. The capacity of a cut, denoted $c(C)$, is $\sum_{e \in C} c(e)$. A minimum cut is one with minimum capacity. A cut C is minimal if no cut is properly contained in C.

Example

- A minimum cut is a minimal cut.
- If U is a set of vertices containing s but not t, then $\overrightarrow{U}$ is a cut.



$\forall P$ path from s to t then $P \cap C = \phi$.

$c(e) \equiv 1$.

$\vec{U} = \{(v,w) : v \in U, w \notin U\}$

Lemma

Suppose C is a minimal cut. Then there is a set U containing s but not t such that $C = \overrightarrow{U}$.

Proof.

□

$U = \{$ ~~vertex~~ v s.t. $\exists P$ a path from s to v ~~but~~ using no arc in $C \} \cup \{s\}$

$= \{ v \in V : \exists$ a path P from s to v : $P \cap C = \emptyset \} \cup \{s\}$

$\Rightarrow t \notin U$

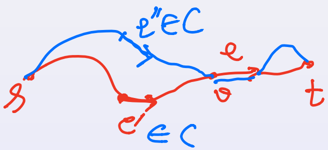
$(\Rightarrow)$ $C \subseteq \overrightarrow{U}$: $\forall e = (v,w) \in C \overset{?}{\Rightarrow} e \in \overrightarrow{U} \Leftrightarrow v \in U$

minimal

Ass. $\forall P$ a path from s to ~~t~~ using e and another acr of C

$\Rightarrow \exists$ a path P from s to t using e but no other arc in C.

$\Rightarrow P_{[s,v]} \cap C = \emptyset \Rightarrow v \in U$.



$C' = C - e$

Theorem (Max-flow, min-cut theorem)

Suppose in a network all arc capacities are integers. Then the value of a maximum flow is equal to the capacity of a minimum cut. Moreover, there is a maximum flow f for which all $f(e)$ are integers.

Proof.

□

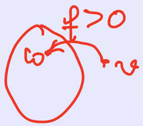
$c(e) \in \mathbb{N} \quad \forall e \in \vec{E}$

$$\max_{f:\text{ flow}} \operatorname{val}(f) = \min_{C:\text{ cut}} c(C)$$

$$\exists f_{max} : \begin{cases} \operatorname{val}(f_{max}) = \max_{f} \operatorname{val}(f) \\ f_{max}(e) \in \mathbb{N} \quad \forall e \in \vec{E} \end{cases}$$

Algorithm:

- $f \equiv 0$
- $\boxed{U = \{s\}}$. Repeat:
  - If $\exists\, e = (v,w) : v \in U,\ w \notin U,\ f(e) < c(e)$ then add w to U; $f(e) := f(e) + 1$
  - If $\exists\, e = (v,w) : v \notin U,\ w \in U,\ f(e) > 0$ then add v to U. $f(e) := f(e) - 1$
  - $t \in U \to$ continue the algorithm.
  - $\boxed{t \notin U}$: Stop



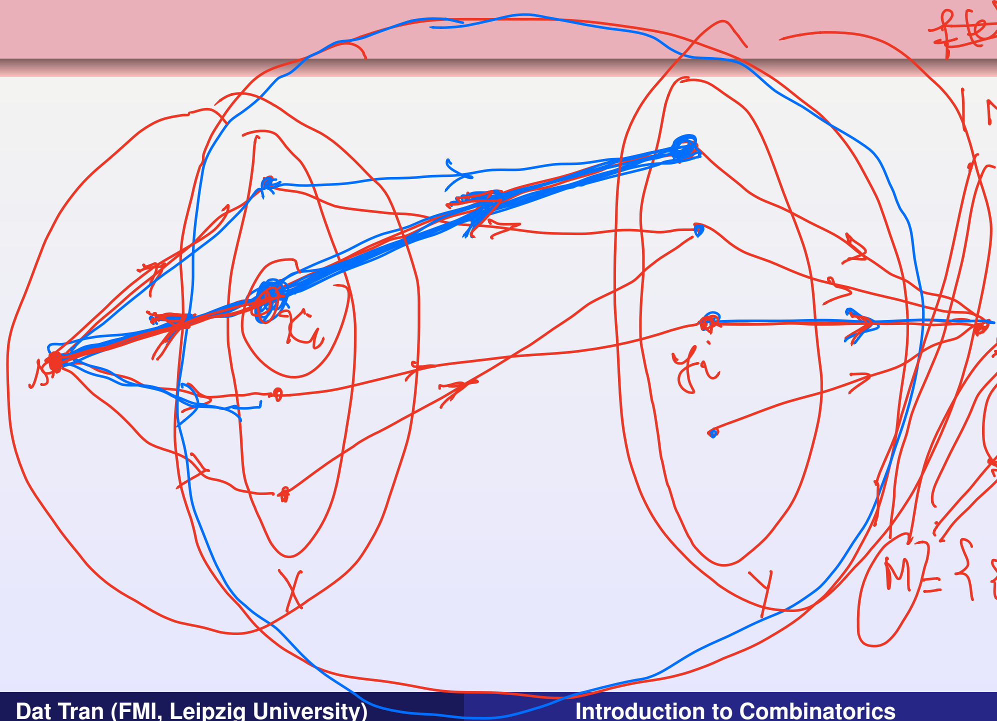
Corollary

In a bipartite graph G, the size of a maximum matching is the same as the size of a minimum vertex cover.

Proof.

~~f(e)~~ = $\boxed{c(e) = 1}$

□



$|M_{max}| = |C_{min}|$ → vertex cover

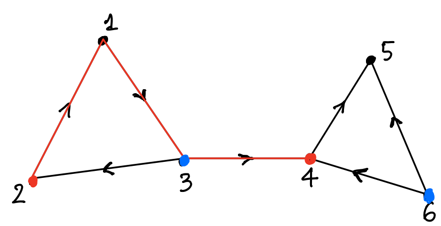
$\|$

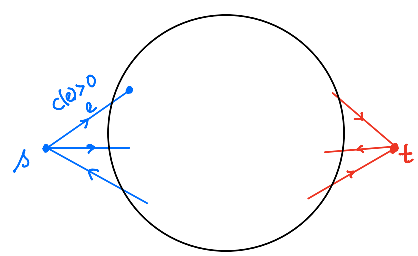
$|val(f_{max})| = |$minimum cut$|$

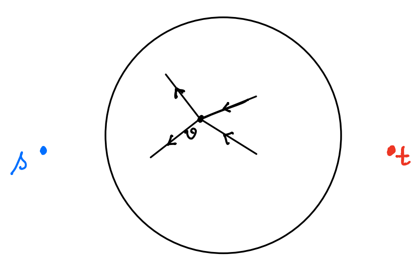
$K = \{x_i : (s, x_i) \in C\} \cup \{y_j : (y_j, t) \in C\}$

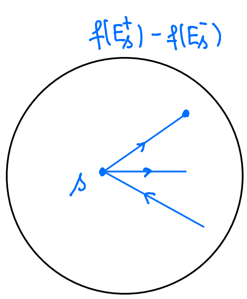
$M = \{\{x_i, y_j\} : f(x_i, y_j) = 1\}$

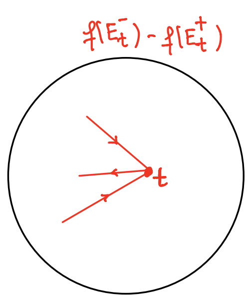
□.

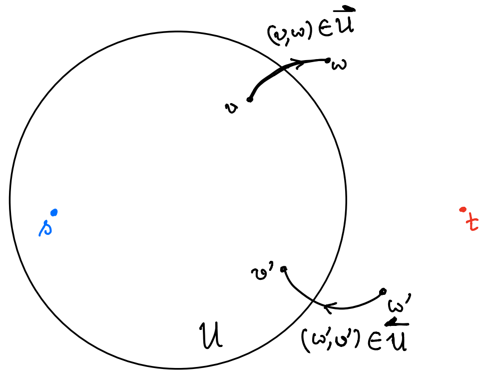

1
2
3
4
5
6


c(e)>0
e
s
t


s
t


$f(E_s^+) - f(E_s^-)$
s


$f(E_t^-) - f(E_t^+)$
t



$(v,w) \in \vec{\mathcal{U}}$
w
v
s
t
v'
w'
$(w',v') \in \overleftarrow{\mathcal{U}}$
$\mathcal{U}$