Introduction to Combinatorics

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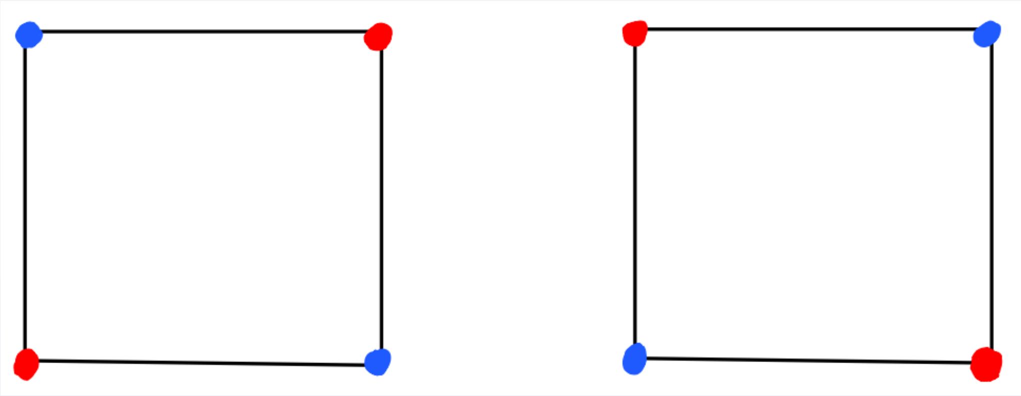
SoSe 2020 (Day 14 - 07/07/2020)

Contents

Motivation: We have talked about the number of ways to properly color a graph with k colors, given by the chromatic polynomial. For example, the chromatic polynomial for the graph in figure below is $P_G(k) = k^4 - 4k^3 + 6k^2 - 3k$, and $P_G(2) = 2$. The two colorings are shown in the figure, but in an obvious sense they are the same coloring, since one can be turned into the other by simply rotating the graph. We will consider a slightly different sort of coloring problem, in which we count the “truly different” colorings of objects.

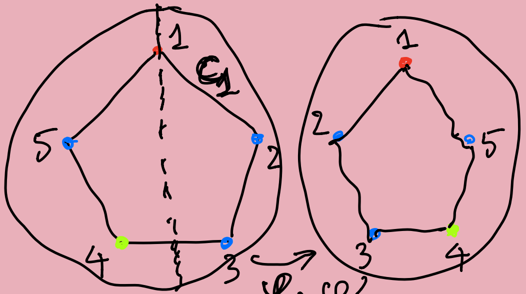


C_4

Definition

A set of permutations G is called a group of permutations if G has the three properties:

1. If σ_1 and σ_2 are in G then so is $\sigma_1 \circ \sigma_2$;
2. The identity permutation *id* is in G;
3. If $\sigma \in G$ then $\sigma^{-1} \in G$.



$\varphi \circ \varphi'$

$\varphi(c_1) \neq c_1$

Example

The group of all permutations of $\{1, 2, \ldots, n\}$ is denoted S_n, the symmetric group on n items. It satisfies the three required conditions by simple properties of bijections.

$\left\{ \begin{array}{l} \text{rotation} \\ \text{reflexation} \end{array} \right.$ $\Rightarrow$ the same coloring.

$\varphi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 5 & 4 & 3 & 2 \end{pmatrix}$

$c_1 \sim_G c_2$ if $\boxed{\exists \varphi \in G : \varphi(c_1) = c_2}$

Definition

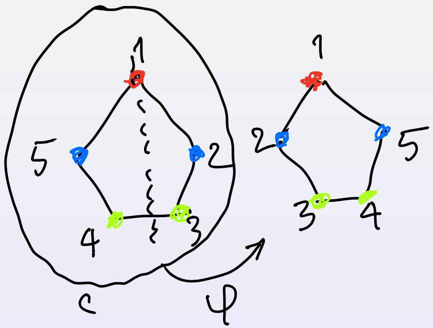
If c is a coloring, $[c]$ is the orbit of c, that is, the equivalence class of c. Let $G(c)$ be the set of permutations in G that fix c, that is, those φ such that $\varphi(c) = c$.

Example

$c_1 \sim_G c_2$ $\qquad \sim_G$: equivalence relation.

$[c] = \{c_1 : c_1 \sim_G c\}$: equivalence class of c.

$G(c) = \{\varphi \in G : \varphi(c) = c\}$



$\varphi(c) = c$

Lemma

$G(c)$ is a group of permutations. $\leq G$

Proof.

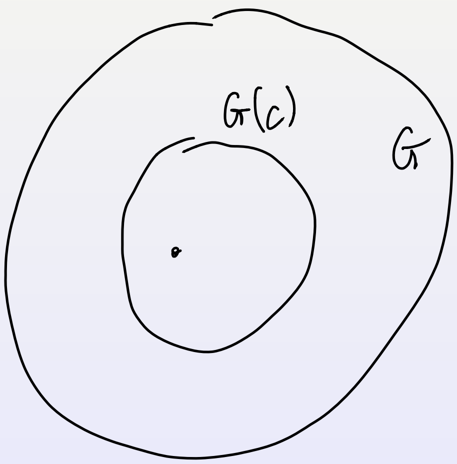
□

Lemma

$|G| = |[c]||G(c)|.$

Proof.

$\square$



$$\frac{|G|}{|G(c)|} = |\underbrace{[c]}_{\text{orbit go through } c}|$$

Corollary

If $c \sim_G d$ then $|G(c)| = |G(d)|$.

Proof.

$\square$

Definition

If group G acts on the colorings of an object and $\sigma \in G$, $fix(\sigma)$ is the set of colorings that are fixed by σ.

Example

$$fix(\sigma) = \{ c \in \{\text{colorings}\} : \sigma(c) = c \}$$

$$G(c) = \{ \varphi \in G : \varphi(c) = c \}$$

Theorem (Burnside's theorem)

If group G acts on the colorings of an object, the number of distinct colorings modulo G is

$$\frac{1}{|G|}\sum_{\sigma\in G}|fix(\sigma)|.$$

Proof.

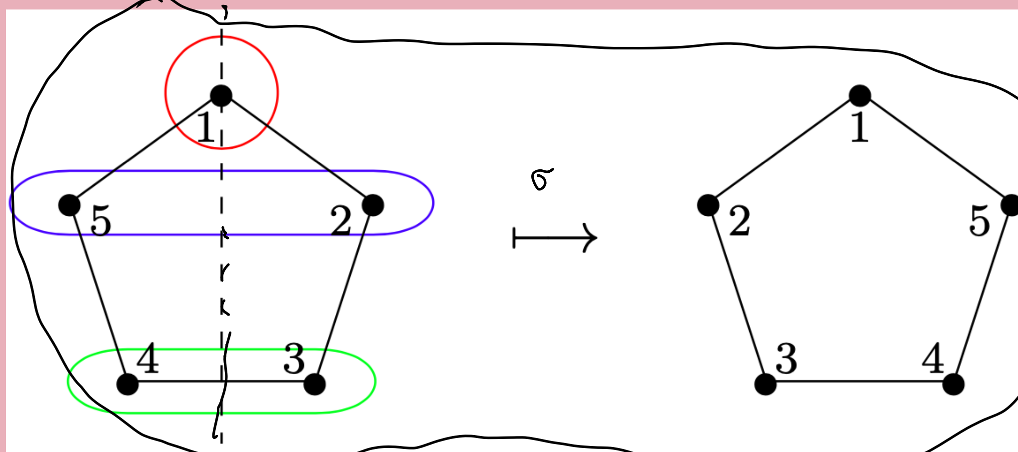
∎

Definition

Every permutation can be written in cycle form: The permutation

$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 5 & 4 & 3 & 2 \end{pmatrix}$

$= (1)(2,5)(3,4)$



$c \in fix(\sigma) \Leftrightarrow \sigma(c) = c$

is $(1)(2,5)(3,4)$. A cycle in this context is a sequence $(x_1, x_2, \ldots, x_k)$, meaning that $\sigma(x_1) = x_2, \sigma(x_2) = x_3$, and so on until $\sigma(x_k) = x_1$. Following our reasoning above, the vertices in each cycle must be colored the same color, and the total number of colors fixed by σ is k^m, where m is the number of cycles.

$\#\sigma$ = the number of cycles in the cycle form of σ

Corollary

If group G acts on the colorings of an object, the number of distinct colorings modulo G with k colors is

$$\frac{1}{|G|}\sum_{\sigma\in G} k^{\#\sigma}.$$

$|O_5| = \frac{1}{|D_5|} \sum_{\sigma\in D_5} |fix(\sigma)|$

$|fix(\mu_1)| = k^3 = \dots |fix(\mu_5)|$

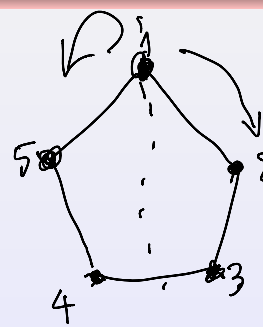
$id = (1)(2)(3)(4)(5)$

$|fix(\sigma_1)| = k = |fix(\sigma_2)| \dots = |fix(\sigma_4)| \Rightarrow |fix(id)| = k^5$

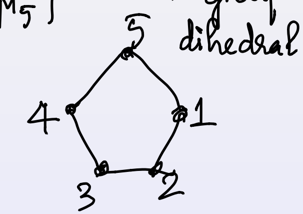
Proof.

$\sigma_1 = (2,3,4,5,1) \Rightarrow c$ has the same color if $c \in fix(\sigma_1)$

□



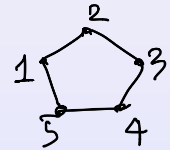
$D_5 = \{ id, \sigma_1, \sigma_2, \sigma_3, \sigma_4, \mu_1, \mu_2, \mu_3, \mu_4, \mu_5 \} \longrightarrow$ group dihedral



$\sigma_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \end{pmatrix}$

$\vdots$

$\sigma_4 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 1 & 2 & 3 & 4 \end{pmatrix}$

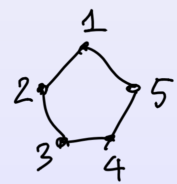


$|O_5|$

Q: # distinct colorings under D_5.

$|D_5| = 10$

$\mu_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 5 & 4 & 3 & 2 \end{pmatrix}$



$\mu_5 = \dots = (1)(2,5)(3,4)$

Motivation

Suppose we are interested in a more detailed inventory of the colorings of an object, namely, instead of the total number of colorings we seek the number of colorings with a given number of each color.

Example

How many distinct ways are there to color the vertices of a regular pentagon modulo D_5 so that one vertex is red, two are blue, and two are green?

We can approach this as before, that is, the answer is

$$\frac{1}{|D_5|}\sum_{\sigma\in D_5}|fix(\sigma)|,$$

$\sigma_1 = (2,3,4,5,1)$

$\mu_1 = (1)(2,5)(3,4)$

where $fix(\sigma)$ now means the colorings with one red, two blues, and two greens that are fixed by σ.

k^5

$|fix(id)| = \binom{5}{2}\binom{3}{2}$

$|fix(\mu_1)| = 2 = \dots |fix(\mu_5)|$

$|fix(\sigma_1)| = 0 = \dots fix(\sigma_4)|$

$\sigma_1(c) = c \Rightarrow c(\text{vertex})$ the same

Definition

The type of a permutation $\sigma \in S_n$ is $\tau(\sigma) = (\tau_1(\sigma), \tau_2(\sigma), \ldots, \tau_n(\sigma))$, where $\tau_i(\sigma)$ is the number of *i*-cycles in the cycle form of σ.

Definition

The cycle index of G is

$$\mathcal{P}_G = \frac{1}{|G|} \sum_{\sigma \in G} \prod_{i=1}^{n} x_i^{\tau_i(\sigma)}.$$

$\to |fix(\sigma)|$

$\sigma_1 = (2,3,4,5,1)$

$\tau(\sigma_1) = (0,0,0,0,1)$

$\mu_1 = (1)(2,5)(3,4)$

$\tau(\mu_1) = (\tau_1(\mu_1), \tau_2(\mu_1), \tau_3(\mu_1), \tau_4(\mu_1), \tau_5(\mu_1))$

$\tau_1(\mu_1) = 1$, $\tau_2(\mu_1) = 2$, $\tau_3(\mu_1) = 0$, $\tau_4(\mu_1) = 0$, $\tau_5(\mu_1) = 0$

$D_4 = \{id, \sigma_1, \sigma_2, \sigma_3, \mu_1, \mu_2, \mu_3, \mu_4\} \Rightarrow |D_4| = 8$

Example

The cycle index of D_4 is

$\tau(id) = (4,0,0,0)$

$id = (1)(2)(3)(4)$

$\frac{1}{|D_4|} \sum_{\sigma \in D_4} \prod_{i=1}^{4} x_i^{\tau_i(\sigma)}$

$$\frac{1}{8}(x_1^4 + x_4^1 + x_2^2 + x_4^1 + x_2^2 + x_2^2 + x_1^2 x_2 + x_1^2 x_2) = \boxed{\frac{1}{8}x_1^4 + \frac{1}{4}x_1^2 x_2 + \frac{3}{8}x_2^2 + \frac{1}{4}x_4.}$$

Substituting as above gives

$$\frac{1}{8}(r+b)^4 + \frac{1}{4}(r+b)^2(r^2+b^2) + \frac{3}{8}(r^2+b^2)^2 + \frac{1}{4}(r^4+b^4) = r^4 + r^3 b + 2r^2 b^2 + rb^3 + b^4.$$

Thus there is one all red coloring, one with three reds and one blue, and so on, as shown below: