Introduction to Combinatorics

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Contents

Definition

The degree of a vertex v is the number of edges incident with v, denoted $d(v)$ (where loops are counted twice).

Example

Definition

The degree sequence of a graph is a list of its degree; the order does not matter, but usually we list the degrees in increasing or decreasing order.

Example

Theorem

In any graph $G = (V, E)$, the sum of the degree sequence is equal to twice the number of edges, that is,

$$\sum_{v \in V} d(v) = 2|E|.$$

Proof.

$\square$

Corollary

The number of odd numbers in a degree sequence is even.

Proof.

□

Theorem

A finite sequence of non-negative integers is a degree sequence of a graph (allow with loops and multiple edges) if and only if the sum of the sequence is even.

Proof.

$\square$

Definition

A sequence that is the degree sequence of a simple graph is said to be graphical.

Example

Theorem (Erdös-Gallai's 1960)

A finite sequence of non-negative integers $d_1 \geq d_2 \geq \cdots \geq d_n$ is graphical if and only if $\sum_{i=1}^{n} d_i$ is even and for all $k \in [n]$,

$$\sum_{i=1}^{k} d_i \leq k(k-1) + \sum_{i=k+1}^{n} \min\{d_i, k\}.$$

Proof.

$\square$

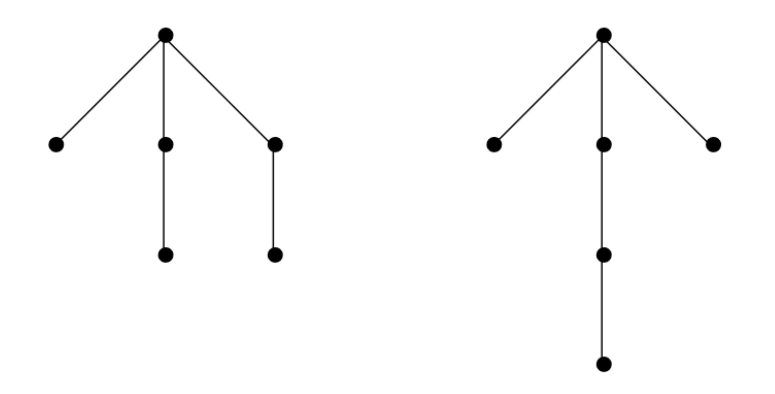
Definition

Given two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$. G_1 and G_2 are isomorphic, denoted by $G_1 \cong G_2$, if there is a bijection $f : V_1 \to V_2$ such that $\{v_1, v_2\} \in E_1$ if and only if $\{f(v_1), f(v_2)\} \in E_2$. In addition, the repetition numbers of $\{v_1, v_2\}$ and $\{f(v_1), f(v_2)\}$ are the same if multiple edges or loops are allowed. This bijection f is called a graph isomorphism.

Example

Remark

Clearly, if two graphs are isomorphic, their degree sequences are the same. The converse is not true. For example, two following graphs have the same degree sequence $1, 1, 1, 2, 2, 3$ but not isomorphic:



Definition

A graph $H = (W, F)$ is a subgraph of the graph $G = (V, E)$ if $W \subseteq V$ and $F \subseteq E$. H is an induced subgraph if F consists of all edges in E with endpoints in W. When $U \subseteq V$, we denote the induced subgraph of G on vertices U as $G[U]$.

Example

Definition

If a graph G is not connected, define $v \sim_G w$ if and only if there is a walk connecting v and w. This is an equivalence relation. Each equivalence class corresponds to an induced subgraph of G, called the connected components of G.

Example

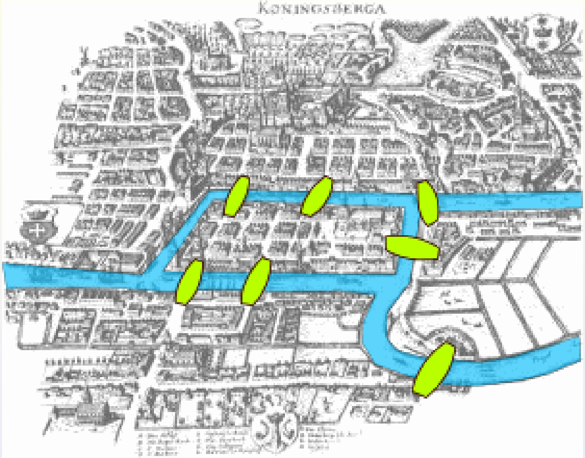
Definition

- Given two vertices v, w in G. A walk in G from v to w is a sequence of vertices and edges,

$$v = v_1, e_1, v_2, e_2, \ldots, v_k, e_k, v_{k+1} = w$$

  such that the endpoints of edge e_i are v_i and v_{i+1}, $\forall i = 1, \ldots, k$.
  If $v = w$, the walk is a closed walk or a circuit.
- A circuit in G in which every edge of G is used exactly once is called an Euler circuit.
- A walk in G in which every edge of G is used exactly once but is not an Euler circuit is called an Euler walk.

Euler walks/circuits



Theorem

If G is a connected graph, then G contains an Euler circuit if and only if every vertex has even degree.

Proof.

□

Theorem

A connected graph G has an Euler walk if and only if exactly two vertices have odd degree.

Proof.

$\square$