UNIVERSITÄT LEIPZIG

Introduction to Combinatorics

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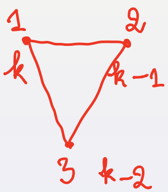
SoSe 2020 (Day 12 - 23/06/2020)

Contents

Given a graph G. Denote by $P_G(k)$ the number of ways to color G with k colors.

Example

1. If $G = K_n$ then $P_G(k) = P_{n,k} = k(k-1)\dots(k-n+1)$.
2. If $G = (V = \{v_1, \dots, v_n\}, E = \emptyset)$ then $P_G(k) = k^n$.



n vertices.

n (k−n+1)

Theorem

Given a graph G on n vertices. Then P_G is a polynomial of degree n, called the chromatic polynomial of G.

Proof.

□

$$* \quad \boxed{P_G(k) = P_{G-\{u,v\}}(k) - P_{G/\{u,v\}}(k)} \quad \text{for every } \{u,v\} \in E(G)$$

$$P_n(k) = \sum_{i=0}^{n} a_{ni} k^i$$

a_{n0}

$P_{G'}(k)$ $\quad G' = G - \{u,v\}$

$\parallel$

\# ways to coloring G' = \# ways to coloring G' with color(u) = color(v)

+ \# ways to coloring G' with color$(u) \neq$ color(v)

= \# ways to coloring $G/\{u,v\}$ = $P_{G/\{u,v\}}(k)$

+ \# ways to coloring G = $P_G(k)$

u. $\quad$ .v

$c: G' \to \mathbb{N}$ $\quad c(u) \neq c(v)$

$\updownarrow$ $c(u_1) \neq c(u_2) \quad \forall \{u_1,u_2\} \in E(G')$

$\tilde{c}: G \to \mathbb{N}, \quad \tilde{c}(u_1) = c(u_1) \quad \forall u_1 \in V(G)$

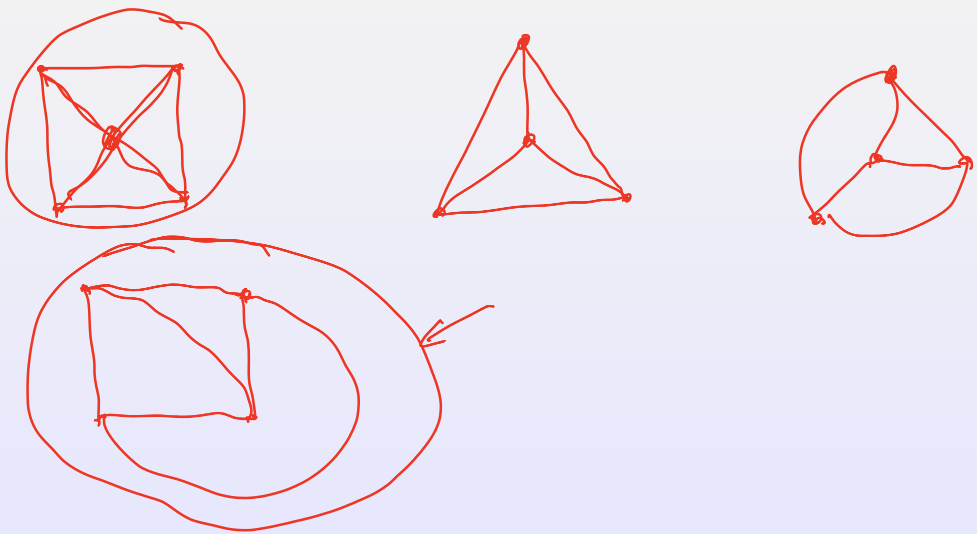
$$\left.\begin{array}{l} \tilde{c}(u_1) \neq \tilde{c}(u_2) \quad \forall \{u_1,u_2\} \in E(G') \\ \tilde{c}(u) = c(u) \neq c(v) = \tilde{c}(v) \quad \{u,v\} \in E(G) \end{array}\right\} \Rightarrow \tilde{c}(u_1) \neq \tilde{c}(u_2) \quad \forall \{u_1,u_2\} \in E(G)$$

Definition

A graph G is planar if it can be represented by a drawing in the plane so that no edges cross.

Example

1. K_4 is planar.



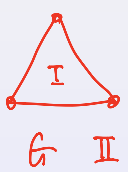
Theorem (Euler's Formula)

Suppose G is a connected planar graph, drawn so that no edges cross, with n vertices and m edges, and that the graph divides the plane into r regions. Then

$$r = m - n + 2.$$

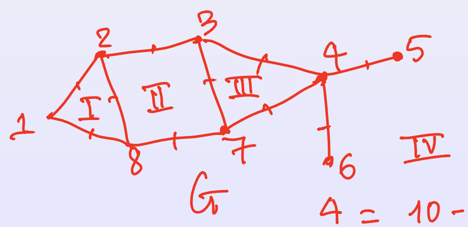
Proof.

$\square$



$m = 3$
$n = 3$
$r = 2$

- $r = m - n + 2$
  $2 = 3 - 3 + 2$ ✓



$n = 8$
$m = 10$
$r = 4$

$4 = 10 - 8 + 2$ ✓

Induction on m:

$\boxed{m \geq n-1}$ because G is connected

- $m = n-1 \Rightarrow G$ is a tree $\Rightarrow r = 1$

  $\Rightarrow 1 = (n-1) - n + 2$ ✓

- true for $m \geq n-1$ check $m+1 \geq n$

  $\Rightarrow G$ is not a tree $\Rightarrow$ has a cycle C.

  choose $e \in C$, $\boxed{G - e}$: connected planar

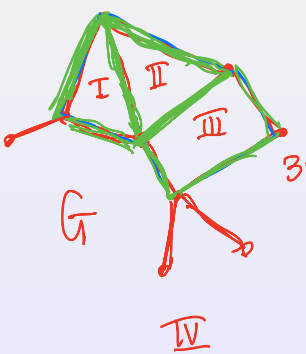
  $r - 1 = m - n + 2 \Rightarrow r = (m+1) - n + 2$

Lemma

Suppose G is a simple connected planar graph, drawn so that no edges cross, with $n \geq 3$ vertices and m edges, and that the graph divides the plane into r regions. Then $\boxed{m \leq 3n - 6}$.

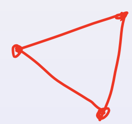
Proof.

□



f_i = # edges on the boundary of region i

$f_1 = 3$
$f_2 = 3$
$f_3 = 4$
$f_4 = 6$



$$3r \leq \sum_{i=1}^{r} f_i \leq 2m$$

$$\boxed{r = m - n + 2} \Rightarrow \boxed{3r = 3m - 3n + 6} \leq 2m$$

$$\Rightarrow m \leq 3n - 6.$$

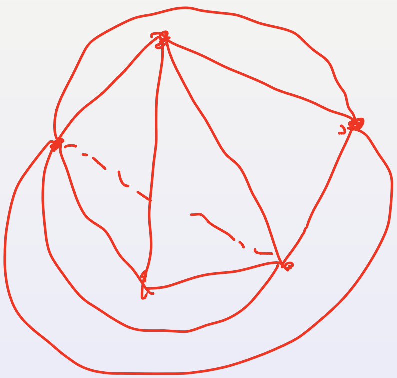
□

Theorem

K_5 is not planar.

Proof.

□



$n = 5$

$m = 10$

$$\begin{array}{ccc} m & \not\leq & 3n-6 \\ \| & & \| \\ 10 & & 15 \\ & & 9 \end{array}$$

Lemma

If G is planar then G has a vertex of degree at most 5.

Proof.

□

Assume G connected.

If otherwise $d_i \geq 6 \ \forall i \Rightarrow 2m = \sum_{i=1}^{n} d_i \geq 6n$

$\boxed{3n - 6 \geq m} \Rightarrow 6n - 12 \geq 2m \geq 6n$. ><

□

Theorem (Five Color Theorem)

Every planar graph can be colored with 5 colors.

Proof.

□

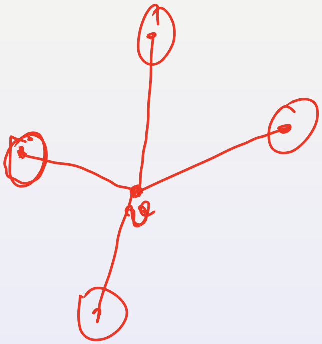
* Induction on n :

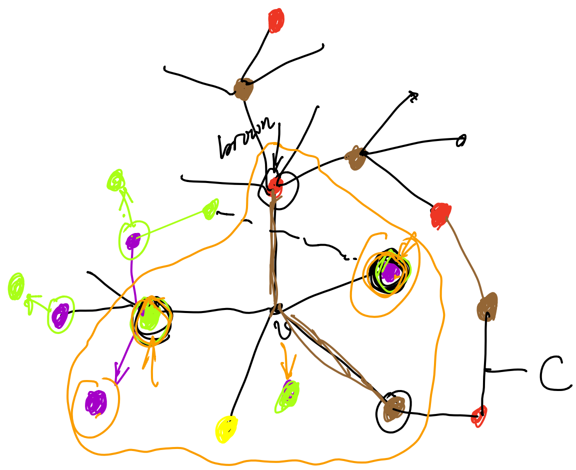
- $n \leq 5$. ✓

- $n \geqslant 6$. $\exists v \in G : d(v) \leq 5$.

  ⊕ $d(v) \leq 4$: ✓

  ⊕ $d(v) = 5$:



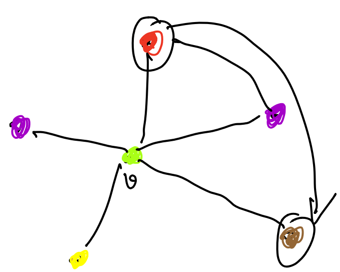


<u>Case 1</u>: $\exists$ an alternative red-brown path $P \Rightarrow \exists\, C = P \cup \{\bullet - v - \bullet\}$

$\Rightarrow \exists$ e.g. $\bullet$ inside C and $\bullet$ outside C.

We use the same technique to construct alternative green-purple path from the green one. But this path can not end at the $\bullet$ one.

Change green to purple and purple to green for all ~~pa~~ possible paths. $\Rightarrow$ Aftere recoloring we end up with



<u>Case 2</u>: $\nexists$ path $\Rightarrow$ apply the same for red-brown

$\Rightarrow$ we can color v by red after recoloring. $\square$