Introduction to Combinatorics

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Contents

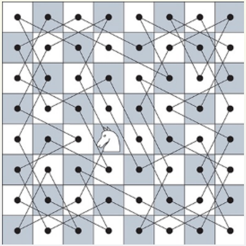
Problem:

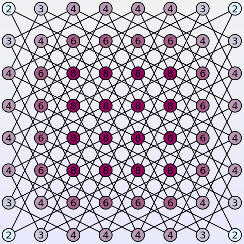
Suppose a number of cities are connected by a network of roads. Is it possible to visit all the cities exactly once, without traveling any road twice? We assume that these roads do not intersect except at the cities. Again there are two versions of this problem, depending on whether we want to end at the same city in which we started.

In graph theory:
This problem can be represented by a graph: the vertices represent cities, the edges represent the roads. We want to know if this graph has a cycle, or path, that uses every vertex exactly once.

Remark: loops can never be used in a Hamilton cycle or path (except in the trivial case of a graph with a single vertex), and at most one of the edges between two vertices can be used. So we assume that all graphs are simple.

The knight's tour problem:





Does there exist a 'knight's tour' in which a knight visits all sixty-four squares just once and returns to its starting point?

Question: We can similarly ask whether knight's tours exist for chessboards of other sizes - such as a 4×4 **or** 5×5 **chessboard?**

Answer: The above question is equivalent to the question: "Is there a Hamilton cycle for the knight-tour graph?"

Definition

- A path in G is a walk in which all edges are distinct and all vertices are distinct. Notation P_n
- A cycle in G is a path with an extra edge by joining the first and last vertices. Notation C_n

Example

Definition

A cycle that uses every vertex in a graph exactly once is called a Hamilton cycle. A path that uses every vertex in a graph exactly once is called a Hamilton path.

Remark:
-There is no good characterization of graphs with Hamilton paths and cycles.
-If a graph has a Hamilton cycle then it also has a Hamilton path but the inverse way is not true.

Example

Theorem (Ore'1960)

Given $n \geq 3$. If G is a simple graph on n vertices satisfying the Ore property, i.e., $d(v) + d(w) \geq n$ whenever v and w are not adjacent, then G has a Hamilton cycle.

Proof.

$\square$

Theorem

Given $n \geq 3$. If G is a simple graph on n vertices satisfying $d(v) + d(w) \geq n - 1$ whenever v and w are not adjacent, then G has a Hamilton path.

Proof.

$\square$

Definition

The distance between vertices v and w is the length of a shortest walk between them, denoted by $d(v, w)$. If there is no walk between v and w, the distance is undefined.

Example

Theorem

A graph G is bipartite if and only if all closed walks in G are of even length.

Proof.

$\square$

Corollary

A graph G is bipartite if and only if all cycles in G are of even length.

Proof.

$\square$

Definition

A complete bipartite graph $G = (V, W, E) = K_{r,s}$ is a bipartite graph with $|V| = r, |W| = s$, and every vertex of V is adjacent to every vertex of W.

Example

Theorem

A complete bipartite graph $K_{r,s}$ has a Hamilton cycle if and only if $r = s$.

Proof.

$\square$