UNIVERSITÄT LEIPZIG

Introduction to Combinatorics

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Contents

Generating function

Definition

$f(x)$ is a generating function for the sequence $\{a_n\}_{n\geq 0}$ if

$$f(x) = \sum_{n=0}^{\infty} a_n x^n.$$

Example

$f(x) = e^{2x}$ is the generating function for the sequence $\{a_n = \frac{2^n}{n!}\}_{n\geq 0}$.

Definition

Given $r \in \mathbb{R}$ and $k \in \mathbb{N}_0$, define the generalized binomial coefficient by

$$\binom{r}{k} = \frac{r(r-1)\cdots(r-k+1)}{k!}.$$

Theorem (Newton's binomial theorem)

For any real number r, when $-1 < x < 1$ we have

$$(x+1)^r = \sum_{k=0}^{\infty} \binom{r}{k} x^k.$$

Proof.

$\square$

Example

For n is a positive integer, then $(1-x)^{-n}$ is the generating function for $\{a_k\}_{k\geq 0}$ where $a_k = \binom{n+k-1}{n-1}$, the number of submultisets of $\{\infty \cdot 1, \infty \cdot 2, \ldots, \infty \cdot n\}$ of size k.

In many cases it is possible to directly construct the generating function whose coefficients solve a counting problem. For example,

Example

Find the number of solutions to $x_1 + x_2 + x_3 + x_4 = 17$, where $0 \le x_1 \le 2, 0 \le x_2 \le 5, 0 \le x_3 \le 5, 2 \le x_4 \le 6$.

Solve.

Consider the function

$$f(x) = (1 + x + x^2)(1 + x + x^2 + x^3 + x^4 + x^5)^2(x^2 + x^3 + x^4 + x^5 + x^6).$$

The number of solutions to the above problem will be the coefficient of x^{17}.

$$\begin{aligned} f(x) = & x^{18} + 4x^{17} + 10x^{16} + 19x^{15} + 31x^{14} + 45x^{13} + 58x^{12} + 67x^{11} \\ & + 70x^{10} + 67x^9 + 58x^8 + 45x^7 + 31x^6 + 19x^5 + 10x^4 + 4x^3 + x^2 \end{aligned}$$

Therefore the number of solutions is 4. $\square$

Example

Find the generating function for $\{a_k\}_{k\geq 0}$, where a_k is the number of solutions to $x_1 + x_2 + x_3 + x_4 = k$, where $0 \leq x_1, 0 \leq x_2 \leq 5, 0 \leq x_3 \leq 5, 2 \leq x_4 \leq 6$.

Solve.

The generating function is

$$f(x) = (1 + x + x^2 + \cdots)(1 + x + x^2 + x^3 + x^4 + x^5)^2(x^2 + x^3 + x^4 + x^5 + x^6)$$
$$= \frac{(1 + x + x^2 + x^3 + x^4 + x^5)^2(x^2 + x^3 + x^4 + x^5 + x^6)}{1 - x}.$$

$\square$

Definition

$f(x)$ is an exponential generating function for the sequence $\{a_n\}_{n\geq 0}$ if

$$f(x) = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!}.$$

Example

Find an exponential generating function for the number of permutations with repetition of length n of the set $\{a; b; c\}$, in which there are an odd number of $a's$, an even number of $b's$, and any number of $c's$.

Solve.

The exponential generating function is

$$f(x) = \sum_{i=0}^{\infty} \frac{x^{2i+1}}{(2i+1)!} \sum_{j=0}^{\infty} \frac{x^{2j}}{(2j)!} \sum_{k=0}^{\infty} \frac{x^k}{k!} = \frac{e^x - e^{-x}}{2} \frac{e^x + e^{-x}}{2} e^x = \frac{1}{4}(e^{3x} - e^{-x}).$$

Definition

A partition of a positive integer n is a multiset of positive integers that sum to n. We denote the number of partitions of n by p_n.

Example

$p_5 = 7$ with partitions

$$\begin{aligned} &5 \\ &4+1 \\ &3+2 \\ &3+1+1 \\ &2+2+1 \\ &2+1+1+1 \\ &1+1+1+1+1. \end{aligned}$$

The generating function for $\{p_n\}_{n\geq 0}$ is

$$f(x) = (1 + x + x^2 + \cdots)(1 + x^2 + x^4 + \cdots)\cdots(1 + x^k + x^{2k} + \cdots)\cdots$$
$$= \prod_{k=1}^{\infty}\sum_{i=0}^{\infty} x^{ik} = \prod_{k=1}^{\infty}\frac{1}{1 - x^k}$$

Example

Find p_8.

Solve.

$$(1 + x + \cdots + x^8)(1 + x^2 + x^4 + x^6 + x^8)(1 + x^3 + x^6)(1 + x^4 + x^8)(1 + x^5)$$
$$\times (1 + x^6)(1 + x^7)(1 + x^8)$$
$$= 1 + x + 2x^2 + 3x^3 + 5x^4 + 7x^5 + 11x^6 + 15x^7 + 22x^8 + \cdots$$

so $p_8 = 22$. $\square$

Definition

Denote by $p_d(n)$ the number of partitions into distinct parts and $p_o(n)$ the number of partitions into odd parts.

Example

- The partitions into distinct parts of $n = 6$ are
$$6, 5+1, 4+2, 3+2+1,$$
so $p_d(6) = 4$;
- The partitions into odd parts of $n = 6$ are
$$5+1, 3+3, 3+1+1+1, 1+1+1+1+1+1,$$
so $p_o(6) = 4$.

Theorem

For every n, we have $p_d(n) = p_o(n)$.

Proof.

$\square$

Theorem

Denote by $p_k(n)$ *the number of partitions of n into exactly k parts for* $1 \leq k \leq n$. *Then*

$$p_k(n) = p_k(n-k) + p_{k-1}(n-1).$$

Proof.

$\square$

Definition

A recurrence relation defines a sequence $\{a_n\}_{n\geq 0}$ by expressing a typical term a_n in terms of earlier terms, a_i for $i < n$. For example,

Example

- the Fibonacci sequence is defined by

$$F_0 = 0, F_1 = 1, F_n = F_{n-1} + F_{n-2}.$$

- the number of derangements of $[n]$

$$D_1 = 0, D_n = nD_{n-1} + (-1)^n.$$

We can use the generating function to solve recurrence relations.

Example

- What is the generating function for the Fibonacci sequence $\{F_n\}_{n\geq 0}$?
- Find F_n

Solve.

$f(x) = \sum_{n\geq 0} F_n x^n$. It implies that
$f(x) - xf(x) - x^2 f(x) = x + (F_2 - 1)x^2 + \sum_{k=3}^{\infty}(F_k - F_{k-1} - F_{k-2})x^k = x$.
Therefore $f(x) = \frac{-x}{x^2+x-1}$. Rewrite $f(x) = \frac{c_1}{x-x_1} + \frac{c_2}{x-x_2}$ with $c_{1,2} = \frac{\pm 1 - \sqrt{5}}{2\sqrt{5}}$ and $x_{1,2} = \frac{-1\pm\sqrt{5}}{2}$. Then,

$$f(x) = -\frac{c_1}{x_1}\frac{1}{1 - x/x_1} - \frac{c_2}{x_2}\frac{1}{1 - x/x_2} = \sum_{n=0}^{\infty}\left(-\frac{c_1}{x_1}\left(\frac{1}{x_1}\right)^n - \frac{c_2}{x_2}\left(\frac{1}{x_2}\right)^n\right) x^n.$$

Therefore $F_n = -\frac{c_1}{x_1}\left(\frac{1}{x_1}\right)^n - \frac{c_2}{x_2}\left(\frac{1}{x_2}\right)^n$. $\square$