Introduction to Combinatorics

Dat Tran (FMI, Leipzig University)

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Contents

Binomial coefficients

Definition

Let $n, k \in \mathbb{N}_0$ be natural numbers with $0 \leq k \leq n$. The binomial coefficient $C(n, k)$ is the coefficient of the monomial X^k in the expansion of $(1+X)^n$, i.e.,

$$(1+X)^n = \sum_{k=0}^{n} C(n, k) X^k.$$

Theorem

$$C(n, k) = \binom{n}{k}.$$

Proof.

We prove this by induction on n. □

Proposition

Let $n, k \in \mathbb{N}_0$ be natural numbers with $0 \leq k \leq n$. Then,

1. $$C(n,k) = C(n,n-k)$$
2. $$\sum_{k=0}^{n} C(n,k) = 2^n$$
3. $$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} C(n,2k) = 2^{n-1}$$
4. *For $n \geq 2$ and $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$,*
   $$C(n,i) > C(n,i-1).$$

Pascal's triangle

Pascal's triangle is a triangular array of the binomial coefficients which is named after the French mathematician Blaise Pascal. The triangle may be constructed in the following manner:

- In row 0 (the topmost row), there is a unique nonzero entry 1.
- Each entry of each subsequent row is constructed by adding the number above and to the left with the number above and to the right, treating blank entries as 0.

Definition

A partition of a set S is a finite collection of non-empty subsets $A_i \subseteq S, 1 \leq i \leq k$ such that $\cup_{i=1}^{k} A_i = S$ and for every $i \neq j, A_i \cap A_j = \emptyset$.

Example

All partitions of the set $\{1, 2, 3\}$ are $\{\{1\}, \{2\}, \{3\}\}$, $\{\{1, 2\}, \{3\}\}$, $\{\{1, 3\}, \{2\}\}$, $\{\{2, 3\}, \{1\}\}$, and $\{\{1, 2, 3\}\}$

Definition (Bell numbers)

Let $[n] = \{1, 2, 3, \ldots, n\}$. The n^{th} Bell number is the number of all partitions of $[n]$, named after Eric Temple Bell.

Theorem

The Bell numbers satisfy a recurrence relation

$$B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B_k.$$

Proof.

$\square$

Multiset

Definition

A multiset is like a set, except that elements may appear more than once.

Remark

To distinguish multisets from sets, and to shorten the expression in most cases, we use a repetition number with each element. For example, $\{1 \cdot a, 2 \cdot b, 3 \cdot c\}$. We also allow elements to be included an infinite number of times, indicated with ∞ for the repetition number, like $\{\infty \cdot a, 2 \cdot b, 3 \cdot c\}$.

Definition

We say that a multiset A is a submultiset of B if the repetition number of every element of A is less than or equal to its repetition number in B.
A multiset is finite if it contains only a finite number of distinct elements, and the repetition numbers are all finite.

Example

$\{20 \cdot a, 5 \cdot b, 2 \cdot c\}$ is a submultiset of $\{\infty \cdot a, 5 \cdot b, 3 \cdot c\}$.

Theorem

Let $A = \{\infty \cdot a_1, \infty \cdot a_2, \ldots, \infty \cdot a_n\}$. Then the number of submultisets of size k is $\binom{k+n-1}{n-1}$.

Proof.

$\square$

Example

How many solutions does $x_1 + x_2 + x_3 + x_4 = 20$ have in non-negative integers? That is, how many 4-tuples $(m_1; m_2; m_3; m_4)$ of non-negative integers are solutions to the equation?

Solve.

The above question is equivalent to how many submultisets of size 20 are there of the multiset $\{\infty \cdot a_1; \infty \cdot a_2; \infty \cdot a_3; \infty \cdot a_4\}$? A submultiset of size 20 is of the form $\{m_1 \cdot a_1; m_2 \cdot a_2; m_3 \cdot a_3; m_4 \cdot a_4\}$ where $\sum_{i=1}^{4} m_i = 20$, and these are in $1-1$ correspondence with the set of 4-tuples $(m_1; m_2; m_3; m_4)$ of non-negative integers such that $\sum_{i=1}^{4} m_i = 20$. Thus, the number of solutions is $\binom{20+4-1}{20}$.

$\square$

With vs without replacement

- The number of permutations of n things taken k at a time without replacement is $A_{n,k} = \frac{n!}{(n-k)!}$;
- the number of permutations of n things taken k at a time with replacement is n^k.
- The number of combinations of n things taken k at a time without replacement is $\binom{n}{k}$;
- the number of combinations of n things taken k at a time with replacement is $\binom{k+n-1}{k}$.

Definition

Denotes $2^{[n]}$ the set of all subsets of $[n]$, and $\begin{bmatrix} n \\ k \end{bmatrix}$ denotes the set of subsets of $[n]$ of size k.

Example

$$\begin{bmatrix} 3 \\ 0 \end{bmatrix} = \{\emptyset\}; \quad \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \{\{1\},\{2\},\{3\}\};$$

$$\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \{\{1,2\},\{1,3\},\{2,3\}\}; \quad \begin{bmatrix} 3 \\ 3 \end{bmatrix} = \{\{1,2,3\}\};$$

$$2^{[3]} = \begin{bmatrix} 3 \\ 0 \end{bmatrix} \cup \begin{bmatrix} 3 \\ 1 \end{bmatrix} \cup \begin{bmatrix} 3 \\ 2 \end{bmatrix} \cup \begin{bmatrix} 3 \\ 3 \end{bmatrix}.$$

Definition

A chain in $2^{[n]}$ is a sequence of subsets of $[n]$ that are linearly ordered by inclusion. An anti-chain in $2^{[n]}$ is a sequence of subsets of $[n]$ that are pairwise incomparable.

Example

In $2^{[3]}$

- $\{\emptyset, \{1\}, \{1,2,3\}\}$ is a chain;
- $\{\{1\}, \{2\}, \{3\}\}$ is an anti-chain;
- $\{\{1\}, \{1,3\}, \{2,3\}\}$ is neither a chain nor an anti-chain.

Sperner's theorem

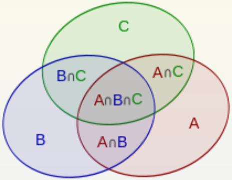
Theorem

The only anti-chains of largest size are $\begin{bmatrix} n \\ \lfloor \frac{n}{2} \rfloor \end{bmatrix}$ *and* $\begin{bmatrix} n \\ \lceil \frac{n}{2} \rceil \end{bmatrix}$.

Proof.

See [1], pp. 36-38. $\square$

The Inclusion-Exclusion Formula



The inclusion-exclusion formula shows the number of elements in the union of many finite sets.

$$\begin{aligned}|A \cup B \cup C| = &|A| + |B| + |C| \\ &- |A \cap B| - |A \cap C| - |B \cap C| \\ &+ |A \cap B \cap C|.\end{aligned}$$

Proposition

Let $A_1, \ldots, A_n$ be finite sets. Then

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{k=1}^{n} (-1)^{k+1} \left(\sum_{1 \le i_1 < \cdots < i_k \le n} |A_{i_1} \cap \cdots \cap A_{i_k}| \right).$$

Proof.

By induction on n. □

Corollary

Let $A_1, \dots, A_n \subset S$ be finite sets. Denote by $\bar{A} = S \setminus A$. Then

$$\left|\bigcup_{i=1}^{n} \bar{A}_i\right| = |S| + \sum_{k=1}^{n}(-1)^k \left(\sum_{1 \le i_1 < \cdots < i_k \le n} |A_{i_1} \cap \cdots \cap A_{i_k}| \right).$$

Example

Find the number of non-negative integer solutions to $x_1 + x_2 + x_3 = 7$ with $x_1 \le 2$, $x_2 \le 4$, and $x_3 \le 3$.

Denote by $S = \{(x_1, x_2, x_3) \in \mathbb{N}_0^3 : x_1 + x_2 + x_3 = 7\}$;
$A_1 = \{(x_1, x_2, x_3) \in S : x_1 \ge 3\}$, $A_2 = \{(x_1, x_2, x_3) \in S : x_2 \ge 5\}$,
$A_3 = \{(x_1, x_2, x_3) \in S : x_3 \ge 4\}$; $A \cap B = AB$. Then

$$\begin{aligned}|\bar{A}_1 \bar{A}_2 \bar{A}_3| &= |S| - |A_1| - |A_2| - |A_3| + |A_1 A_2| + |A_1 A_3| + |A_2 A_3| - |A_1 A_2 A_3| \\ &= \binom{9}{2} - \binom{6}{2} - \binom{4}{2} - \binom{5}{2} + 0 + 1 + 0 - 0 = 6.\end{aligned}$$

Derangement

Definition

Given a natural number $n \in \mathbb{N}_0$. A derangement of $[n]$ is a permutation of $[n]$ have none of the integers in their correct locations, i.e., 1 is not the first, 2 is not the second, and so on. Denote by D_n is the numbers of derangements of $[n]$.

Theorem

$$D_n = n! \sum_{k=0}^{n} (-1)^k \frac{1}{k!}.$$

Proof.

$\square$

Theorem

D_n satisfies the recurrence relations

1. $D_n = (n-1)(D_{n-1} + D_{n-2})$;
2. $D_n = nD_{n-1} + (-1)^n$.

Proof.

$\square$