Introduction to combinatorics

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(Version July 18, 2020)

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Preface

This script is intended for my course at Leipzig University in summer semester 2020. It contains some fundamental concepts of Combinatorics such as basic counting principles, permutations and combinations, the Inclusion-Exclusion principle, generating functions, systems of distinct representatives, and many topics in graph theory. It mainly follows the reference [3] except almost proofs and solutions are self-contained. Some other material have been chosen from [5], [2], and [6].

1. What is Combinatorics?

Some definitions:

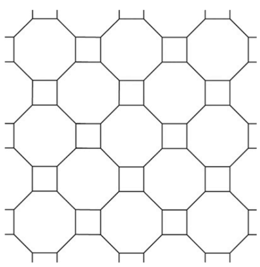
1. a branch of mathematics dealing with combinations and permutations. [Collin's dictionary]
2. the branch of mathematics dealing with combinations of objects belonging to a finite set in accordance with certain constraints, such as those of graph theory. [Lexico's dictionary]
3. an area of mathematics primarily concerned with counting, both as a means and an end in obtaining results, and certain properties of finite structures. [Wikipedia]
4. the branch of mathematics studying the enumeration, combination, and permutation of sets of elements and the mathematical relations that characterize their properties. [Wolfram MathWorld]

Types of problems:

1. Existence problem: Does $\cdots$ exist?
2. Construction problem: If $\cdots$ exists, how can we construct it?
3. Enumeration problem: How many $\cdots$ are there?
4. Optimization problem: Which $\cdots$ is the best?

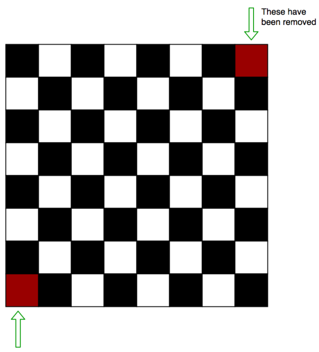
Motivative examples:

1. **Existence problems:**
   - *Tiling problems:*



This figure shows a tiled floor pattern constructed from squares and regular octagons that match up exactly without any gaps or overlaps. What other tiling patterns are possible?
Question: For example, does there exist a tiling pattern with both squares and regular hexagons?
Answer: No.
If there existed a tiling with both squares and hexagons, then each meeting point would have a combination of 90^o angles (for the squares) and 120^o angles (for the hexagons) adding up to 360^o. But this is impossible, for if there were only one hexagon (120^o) then the remaining 240^o couldn't be made up of right angles, if there were two hexagons (240^o) the remaining 120^o couldn't be made up from right angles, and if there were three hexagons (360^o) there'd be no room for any squares. So no combination of both 90^o and 120^o angles can make up the desired 360^o and no such tiling can exist.

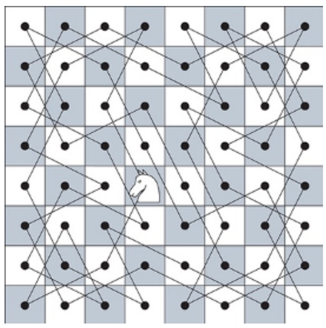
- *Putting dominoes on a chessboard:*



It's easy to see that we can cover all the sixty-four squares of a chessboard with dominoes.
If we remove a single corner square, then this is no longer possible, since an odd number of squares remains.

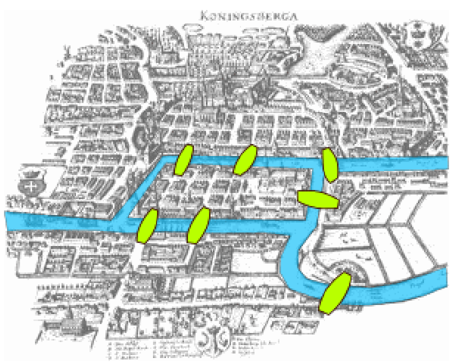
Question: But what happens if we remove two opposite corner squares? Does there exist a domino covering of the remaining sixty-two squares?
Answer: No.
We notice that every domino must cover a black square and a white square. Since the two removed corner squares have the same colour, there are now thirty-two squares of one colour and thirty of the other colour, so no covering by dominoes can exist.

- *The knight's tour problem:*



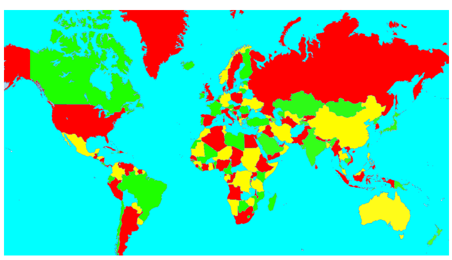
Does there exist a 'knight's tour' in which a knight visits all sixty-four squares just once and returns to its starting point?
Question: We can similarly ask whether knight's tours exist for chessboards of other sizes - such as a 4×4 or 5×5 chessboard?
Answer: We answer this when we discuss Hamiltonian cycle 6.6.

- *The Königsberg bridges problem:*



Question: It is said that the citizens of Königsberg wished to take a walk, crossing each bridge exactly once before returning to their starting point. Does such a route exist?
Answer: This problem was answered by the Swiss mathematician Leonhard Euler, who extended his solution to any arrangement of regions connected by bridges. We present his answer when we discuss Eulerian cycle 6.5.

- *The map-colour problem:*

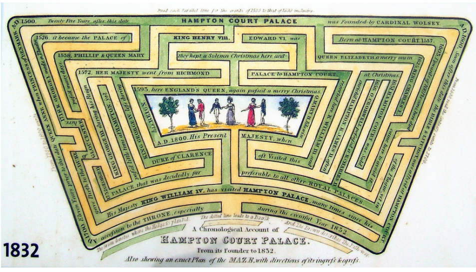


Question: Can the world map be coloured with just four colours so that neighbouring countries receive different colours? do there exist maps that require five or more colours?
Answer: We will answer this when we discuss about colouring planar graphs 6.12.

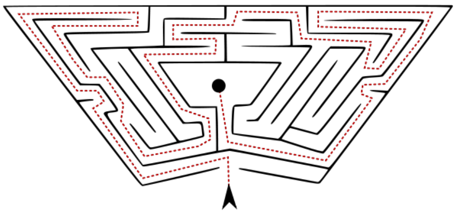
2. **Construction problems:**

   One way to solve an existence problem is to construct a solution explicitly-indeed, this is sometimes the only way to do so. But for other problems we may be able to prove by theoretical means that a solution exists without our being able to construct one. Moreover, some construction problems are amenable to trial-and-error experimentation, while others will require a more systematic approach. Both methods have been used for the tracing of mazes:

   The Hampton Court Maze:



   Suppose that we're stuck in the middle of a maze. We know that there must be a way of escaping from the maze, but we need to find a method for doing so?



3. **Enumeration problems:**

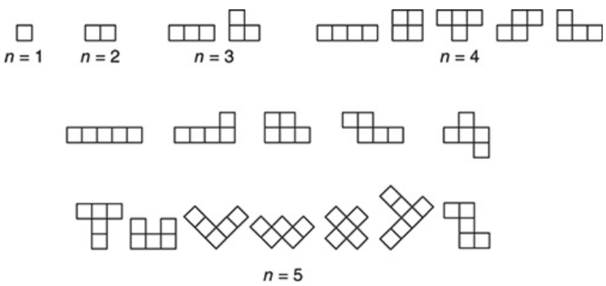
   Ever since earliest times people have needed to count the objects around them. We're used to such questions as:

   a) How many children have you got?

   b) How many shopping days are there until Christmas?

   c) How many solutions of the Hampton Court Maze problem?

   Polyominoes:



   Just as a domino is formed from two squares of equal size, so a tromino is formed from three squares in a straight line or an L-shape, and an $n-$omino is formed from n squares.

   Question: How many n-ominoes are there for a given value of n?

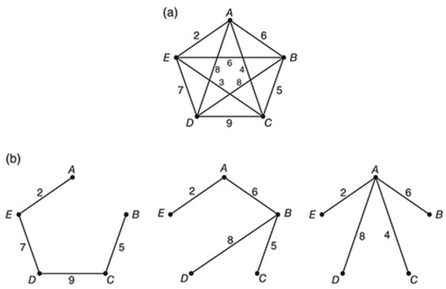
   Answer: No answer to our question is known in general!
   Some calculations: $S_1 = S_2 = 1$, $S_3 = 2$, $S_4 = 5$, $S_5 = 12$, and $S_{28} = 153.511.100.594.603$.

4. **Optimization problems:** We wish to drive from Berlin to Munich in the shortest possible time. Which of the many available routes should we choose, given the travel times between pairs of neighbouring cities on the various routes?

a) Minimum connector problem: We wish to connect a number of cities by links, such as canals, railway lines, or air routes, but the cost of the connections is high. How can we minimize the total cost while ensuring that we can still get from each city to every other one?

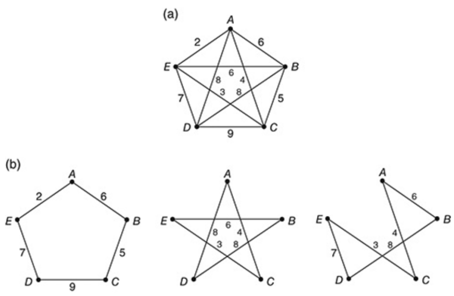
b) Travelling salesman problem: A travelling salesman wishes to visit a number of cities to sell his wares, and return to his starting point. If the costs of travelling between the cities are known, how should he plan his route so as to minimize the total cost?

- *The minimum connector problem:*



This figure shows a minimum connector problem with five cities, A, B, C, D, and E, where the numbers refer to the connection costs between pairs of cities. Some experimentation gives the solutions with total costs 23, 21, and 20. But is 20 the best solution, or is there a smaller one?

- *The travelling salesman problem:*



This figure gives an example of the travelling salesman problem. Again, some experimentation gives the solutions with total costs 29, 29, and 28. But is 28 the best solution, or is there a smaller one - and is there an efficient algorithm that always produces an optimal solution?

2. Permutations and Combinations

2.1. Basic counting principles

2.1.1. Addition Principle

Proposition 2.1.1. *We say a finite S is partitioned into parts $S_1, \ldots, S_k$ if the parts are disjoint and their union is S. Then, we have*

$$|S| = |S_1| + \cdots + |S_k|.$$

Example 2.1.1. Let S be the set of students attending lecture Introduction to Combinatorics. It can be partitioned into two parts S_1 and S_2 where S_1 is the set of students that like easy examples and S_2 is the set of students that don't like easy examples. If $|S_1| = 12$ and $|S_2| = 5$ then we can conclude $|S| = 17$.

2.1.2. Multiplication Principle

Proposition 2.1.2. *If S is a finite set that is the product of $S_1, \ldots, S_k$, i.e., $S = S_1 \times \cdots \times S_k$, then*

$$|S| = |S_1| \times \cdots \times |S_k|.$$

Example 2.1.2. You are eating at a mensa of Leipzig University and recognize that you have

(a) two choices for appetizers: soup or juice;

(b) three for the maincourse: a meat, fish, or vegetable dish;

(c) two for dessert: ice cream or cake.

How many possible choices do you have for your complete meal?

Solve. Put $S_1 = \{\text{soup, juice}\}$; $S_2 = \{\text{meat, fish, vegetable dish}\}$; $S_3 = \{\text{ice cream, cake}\}$. Then a complete meal is an element of $S = S_1 \times S_2 \times S_3$. Therefore there are $|S| = 2 \times 3 \times 2 = 12$ complete meals for your choice. □

2.1.3. Pigeonhole Principle

Proposition 2.1.3. *Let $S_1, \ldots, S_k$ be finite sets that are pairwise disjoint and $|S_1| + \cdots + |S_k| = n$, then*

1. $\exists i \in \{1, \ldots, k\} : |S_i| \geq \lceil \frac{n}{k} \rceil$;
2. $\exists j \in \{1, \ldots, k\} : |S_j| \leq \lfloor \frac{n}{k} \rfloor$.

Example 2.1.3. Assume there are 5 holes in the wall where pigeons nest. Say there is a set S_i of pigeons nesting in hole i. Assume there are $n = 17$ pigeons in total. Then we know:

1. There is some hole with at least $\lceil 17/5 \rceil = 4$ pigeons;
2. There is some hole with at most $\lfloor 17/5 \rfloor = 3$ pigeons.

2.1.4. Double counting

Proposition 2.1.4. *If we count the same quantity in two different ways, then this gives us an identity.*

Example 2.1.4 (Handshaking Lemma). (i) Assume there are n people at a party and everybody will shake hands with everybody else. How many handshakes will occur?

(ii) In a party of people some of whom shake hands, there is an even number of people must have shaken an odd number of other people's hands.

Solve. (i) Every person shakes $n-1$ hands and there are n people. However, person i shakes hands of person j is the same as person j shakes hands of person i, i.e., every handshake is counted twice. Therefore, the total number of handshakes is $\frac{n(n-1)}{2}$.

(ii) See details in Corollary 6.3.1.

□

2.2. Permutations and Combinations

2.2.1. Permutations

Definition 2.2.1. Let S be any finite set. A permutation of S is a one-to-one mapping of S onto itself.

Example 2.2.1. Given $S = \{a_1, a_2, a_3, a_4\}$. A possible permutation σ would be

$$\sigma = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ a_2 & a_1 & a_4 & a_3 \end{pmatrix}$$

which could be also written in the form

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$$

indicating that a_1 went to a_2, a_2 to a_1, a_3 to a_4, and a_4 to a_3.
We also sometimes write $\sigma = 2143$ and call it a rearrangement of S and denote by S_n the set of all permutations of a set of n elements.

Theorem 2.2.1. *The total number of permutations of a set S of n elements is given by*

$$|S_n| = n! = n \cdot (n-1) \cdot \ldots \cdot 1.$$

Proof. There are n ways to assign the first element, for each of these we have $n-1$ ways to assign the second object, $n-2$ for the third, and so forth. This proves the theorem. □

2.2.2. k-Permutations

Definition 2.2.2. Let S be an n-element set and let k be an integer between 0 and n. Then a k-permutation of S is an ordered listing of a subset of S of size k.

Theorem 2.2.2. *The total number of k-permutations of a set S of n elements is given by*

$$P_{n,k} = n \cdot (n-1) \cdot \ldots \cdot (n-k+1) = \frac{n!}{(n-k)!}.$$

Proof. Similar to the previous theorem 2.2.1, there are n ways to assign the first element, for each of these we have $n-1$ ways to assign the second object, $n-2$ for the third, and so forth until we have $(n-k+1)$ ways to assign the k^{th} object. This proves the theorem. □

2.2.3. k-combinations

Definition 2.2.3. Let S be an n-element set and let k be an integer between 0 and n. Then a k-combination of S is an unordered listing of a subset of S of size k.

Theorem 2.2.3. *The total number of k-combinations of a set S of n elements is given by*

$$C_{n,k} = \frac{P_{n,k}}{k!} = \frac{n!}{(n-k)!k!} = \binom{n}{k}.$$

Proof. It follows from the previous theorem with note that the number of ordered k-sets in an unordered k-set are $k!$. □

Remark. We call $\binom{n}{k}$ as "n choose k".

Theorem 2.2.4. *Prove that, for $1 \leq k \leq n-1$,*

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

Proof. In fact, for $1 \leq k \leq n-1$ we have

$$\begin{aligned}\binom{n-1}{k-1} + \binom{n-1}{k} &= \frac{(n-1)!}{(k-1)!(n-k)!} + \frac{(n-1)!}{k!(n-1-k)!} \\ &= \frac{(n-1)!}{k!(n-k)!}\Big(k + (n-k)\Big) = \frac{n!}{k!(n-k)!} = \binom{n}{k}.\end{aligned}$$

□

Exercise 2.2.1 (E1.1: Dance partners)**.** *How many pairs of dance partners can be selected from a group of* 12 *women and* 20 *men?*

Exercise 2.2.2 (E1.2: Poker hand)**.** *A poker hand consists of* 5 *cards from a standard* 52 *card deck with four suits and thirteen values in each suit; the order of the cards in a hand is irrelevant. How many hands consist of* 2 *cards with one value and* 3 *cards of another value (a full house)? How many consist of* 5 *cards from the same suit (a flush)?*

Exercise 2.2.3 (E1.3: Boarding)**.** *In light of the Corona crisis, for the return of a group of German from a foreign country, an aircraft was sent that could carry exactly 400 passengers. The plane is waiting at the foreign airport for passenger boarding, and there is a long queue of exactly 400 passengers waiting for boarding.*

The rules are: Every passenger is given a ticket with an indicated seat (i.e. 14E, 31A and so). Every passenger on the aircraft must sit in the designated place. But, if for some reason it is occupied, they will sit in a random seat.

Passengers are asked to start boarding, but the first passenger, who is very confused by the situation, has lost his ticket and cannot remember his seat. This passenger has now settled in a random seat. Given that the rest of the passengers know what their seat is and follow the instructions.

The question is:
i) what is the probability that the last passenger in line for the plane will be seated in the designated seat?
ii) what is the probability that the k^{th} $(2 \leq k \leq n)$ passenger in line for the plane will be seated in the designated seat?

2.3. Other fundamental concepts

2.3.1. Binomial coefficients

Definition 2.3.1. Let $n, k \in \mathbb{N}_0$ be natural numbers with $0 \le k \le n$. The binomial coefficient $C(n,k)$ is the coefficient of the monomial X^k in the expansion of $(1+X)^n$, i.e.,

$$(1+X)^n = \sum_{k=0}^{n} C(n,k)X^k. \tag{2.3.1}$$

Theorem 2.3.1. *Let $n, k \in \mathbb{N}_0$ be natural numbers with $0 \le k \le n$. Then*

$$C(n,k) = \binom{n}{k}.$$

Proof. We prove this by induction on n as follows:

- Base case:
  - $n = 0$: $C(0,0) = (1+X)^0 = 1$. Therefore $C(0,0) = 1 = \binom{0}{0}$;
  - $n = 1$: $C(1,0) + C(1,1)X = (1+X)^1 = 1 + X$. Therefore $C(1,0) = 1 = \binom{1}{0}$ and $C(1,1) = 1 = \binom{1}{1}$;
- Inductive step: assume that this is true for $n \ge 1$, i.e., $C(n,k) = \binom{n}{k}$ for all $0 \le k \le n$, we will prove this is also true for $n+1$. In fact, we have

$$\begin{aligned}
(1+X)^{n+1} &= (1+X)(1+X)^n = (1+X)\sum_{k=0}^{n} C(n,k)X^k \\
&= \sum_{k=0}^{n} C(n,k)X^k + \sum_{k=0}^{n} C(n,k)X^{k+1} \\
&= C(n,0) + \sum_{k=1}^{n} C(n,k)X^k + \sum_{k=1}^{n} C(n,k-1)X^k + C(n,n)X^{n+1} \\
&= \binom{n}{0} + \sum_{k=1}^{n}\left(\binom{n}{k} + \binom{n}{k-1}\right)X^k + \binom{n}{n}X^{n+1} \text{ (due to the inductive hypothesis)} \\
&= 1 + \sum_{k=1}^{n}\binom{n+1}{k}X^k + X^{n+1} \text{ (due to Theorem 2.2.4).}
\end{aligned}$$

  Therefore $C(n+1,0) = 1 = \binom{n+1}{0}$, $C(n+1,k) = \binom{n+1}{k}$, and $C(n+1,n+1) = 1 = \binom{n+1}{n+1}$, which completes the proof.

□

Proposition 2.3.1. *Let $n, k \in \mathbb{N}_0$ be natural numbers with $0 \le k \le n$. Then,*

1. $$C(n,k) = C(n,n-k);$$

2. $$\sum_{k=0}^{n} C(n,k) = 2^n;$$

3. $$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} C(n,2k) = 2^{n-1};$$

4. For $n \geq 2$ *and* $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$,

$$C(n,i) > C(n,i-1).$$

Proof. 1. It follows from Theorem 2.3.1

$$C(n,k) = \binom{n}{k} = \frac{n!}{k!(n-k)!} = \binom{n}{n-k} = C(n,n-k);$$

2. Set $X = 1$ in (2.3.1) we have

$$\sum_{k=0}^{n} C(n,k)1^k = (1+1)^n = 2^n;$$

3. Set $X = -1$ in (2.3.1) we have $\sum_{0\leq k\leq n,\, k \text{ even}} C(n,k) = \sum_{0\leq k\leq n,\, k \text{ odd}} C(n,k)$ therefore

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} C(n,2k) = \frac{1}{2}\Bigg(\sum_{0\leq k\leq n,\, k \text{ even}} C(n,k) + \sum_{0\leq k\leq n,\, k \text{ odd}} C(n,k) \Bigg) = \frac{1}{2}\sum_{k=0}^{n} C(n,k) = 2^{n-1}$$

4. For $n \geq 2$ and $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$, we have $2i \leq n < n+1$ therefore

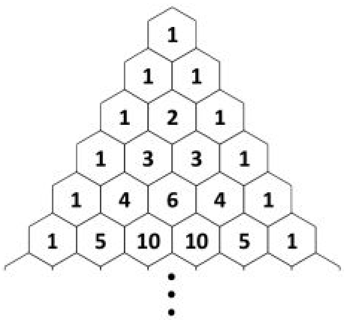
$$\frac{C(n,i)}{C(n,i-1)} = \frac{\binom{n}{i}}{\binom{n}{i-1}} = \frac{n+1-i}{i} > 1.$$

$\square$

Corollary 2.3.1. *From Proposition 2.3.1 1. and 4. we have for every* $n \geq 2$*:*

$$C(n,0) < C(n,1) < \ldots < C\Big(n, \Big\lfloor \frac{n}{2} \Big\rfloor\Big) = C\Big(n, \Big\lceil \frac{n}{2} \Big\rceil\Big) > \ldots > C(n,n-1) > C(n,n).$$

2.3.2. Pascal's triangle



Pascal's triangle is a triangular array of the binomial coefficients which is named after the French mathematician Blaise Pascal. The triangle may be constructed in the following manner:

- In row 0 (the topmost row), there is a unique nonzero entry 1.
- Each entry of each subsequent row is constructed by adding the number above and to the left with the number above and to the right, treating blank entries as 0.

2.3.3. Bell numbers

Definition 2.3.2. A partition of a set S is a finite collection of non-empty subsets $A_i \subseteq S, 1 \leq i \leq k$ such that $\cup_{i=1}^{k} A_i = S$ and for every $i \neq j, A_i \cap A_j = \emptyset$.

Example 2.3.1. All partitions of the set $\{1,2,3\}$ are $\{\{1\},\{2\},\{3\}\}$, $\{\{1,2\},\{3\}\}$, $\{\{1,3\},\{2\}\}$, $\{\{2,3\},\{1\}\}$, and $\{\{1,2,3\}\}$

Definition 2.3.3 (Bell numbers)**.** Let $[n] = \{1,2,3,\ldots,n\}$. The n^{th} Bell number is the number of all partitions of $[n]$, named after Eric Temple Bell.

Theorem 2.3.2. *The Bell numbers satisfy a recurrence relation*

$$B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B_k.$$

Proof. Let $S = [n+1] := \{1, 2, \ldots, n+1\}$ and $\{A_1, \ldots, A_m\}$ is a partition of S. Then there is a set, say A_1, contains $n+1$. It means that $|A_1| = k+1$ with $0 \leq k \leq n$. Therefore $\{A_2, \ldots, A_m\}$ is a partition of $S - A_1$ with $|S - A_1| = n - k$. The number of sets of size $k+1$ containing $n+1$ is nothing but the number of k-sets in n-set $\{1, \ldots, n\}$ and therefore is $\binom{n}{k}$. Therefore the total number of partitions of S in which $n+1$ is in a set of size $k+1$ is $\binom{n}{k} B_{n-k}$. It implies that

$$\begin{aligned} B_{n+1} &= \sum_{k=0}^{n} \binom{n}{k} B_{n-k} \\ &= \sum_{i=0}^{n} \binom{n}{i} B_i \quad \text{by changing } i = n-k \text{ and using } \binom{n}{k} = \binom{n}{n-k}. \end{aligned}$$

□

2.3.4. Choice with replacement

Multiset

Definition 2.3.4. A multiset is like a set, except that elements may appear more than once.

Remark. To distinguish multisets from sets, and to shorten the expression in most cases, we use a repetition number with each element. For example, $\{1 \cdot a, 2 \cdot b, 3 \cdot c\}$. We also allow elements to be included an infinite number of times, indicated with ∞ for the repetition number, like $\{\infty \cdot a, 2 \cdot b, 3 \cdot c\}$.

Definition 2.3.5. We say that a multiset A is a submultiset of B if the repetition number of every element of A is less than or equal to its repetition number in B.
A multiset is finite if it contains only a finite number of distinct elements, and the repetition numbers are all finite.

Example 2.3.2. $\{20 \cdot a, 5 \cdot b, 2 \cdot c\}$ is a submultiset of $\{\infty \cdot a, 5 \cdot b, 3 \cdot c\}$.

Theorem 2.3.3. *Let $A = \{\infty \cdot a_1, \infty \cdot a_2, \ldots, \infty \cdot a_n\}$. Then the number of submultisets of size k is $\binom{k+n-1}{n-1}$.*

Proof. Consider $k+n-1$ empty positions labelled $\{1, 2, \ldots, k+n-1\}$ and put $n-1$ makers $M_1, \ldots, M_{n-1}$ into these positions so that each maker occupy exactly one position ($1 \leq M_1 < M_2 < \cdots < M_{n-1} \leq k+n-1$). Now we fill all positions up to the first maker M_1 with a_1, up to the second maker M_2 with a_2, and so on. This way uniquely identifies a submultiset of size $(k+n-1) - (n-1) = k$. Therefore the number of submultisets of size k is exactly the number of ways we put $n-1$ makers which is $\binom{k+n-1}{n-1}$. □

Example 2.3.3. How many solutions does $x_1 + x_2 + x_3 + x_4 = 20$ have in non-negative integers? That is, how many 4-tuples $(m_1; m_2; m_3; m_4)$ of non-negative integers are solutions to the equation?

Solve. The above question is equivalent to how many submultisets of size 20 are there of the multiset $\{\infty \cdot a_1; \infty \cdot a_2; \infty \cdot a_3; \infty \cdot a_4\}$? A submultiset of size 20 is of the form $\{m_1 \cdot a_1; m_2 \cdot a_2; m_3 \cdot a_3; m_4 \cdot a_4\}$ where $\sum_{i=1}^{4} m_i = 20$, and these are in $1-1$ correspondence with the set of 4-tuples $(m_1; m_2; m_3; m_4)$ of non-negative integers such that $\sum_{i=1}^{4} m_i = 20$. Thus, the number of solutions is $\binom{20+4-1}{20}$.

□

With vs without replacement

- The number of permutations of n things taken k at a time without replacement is $A_{n,k} = \frac{n!}{(n-k)!}$;
- the number of permutations of n things taken k at a time with replacement is n^k.
- The number of combinations of n things taken k at a time without replacement is $\binom{n}{k}$;
- the number of combinations of n things taken k at a time with replacement is $\binom{k+n-1}{k}$.

2.3.5. Sperner's Theorem

Definition 2.3.6. Denotes $2^{[n]}$ the set of all subsets of $[n]$, and $\begin{bmatrix} n \\ k \end{bmatrix}$ denotes the set of subsets of $[n]$ of size k.

Example 2.3.4.

$$\begin{bmatrix} 3 \\ 0 \end{bmatrix} = \{\emptyset\}; \quad \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \{\{1\},\{2\},\{3\}\};$$

$$\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \{\{1,2\},\{1,3\},\{2,3\}\}; \quad \begin{bmatrix} 3 \\ 3 \end{bmatrix} = \{\{1,2,3\}\};$$

$$2^{[3]} = \begin{bmatrix} 3 \\ 0 \end{bmatrix} \cup \begin{bmatrix} 3 \\ 1 \end{bmatrix} \cup \begin{bmatrix} 3 \\ 2 \end{bmatrix} \cup \begin{bmatrix} 3 \\ 3 \end{bmatrix}.$$

Definition 2.3.7. A chain in $2^{[n]}$ is a sequence of subsets of $[n]$ that are linearly ordered by inclusion. An anti-chain in $2^{[n]}$ is a sequence of subsets of $[n]$ that are pairwise incomparable.

Example 2.3.5. In $2^{[3]}$

- $\{\emptyset,\{1\},\{1,2,3\}\}$ is a chain;
- $\{\{1\},\{2\},\{3\}\}$ is an anti-chain;
- $\{\{1\},\{1,3\},\{2,3\}\}$ is neither a chain nor an anti-chain.

Example 2.3.6. Every $\begin{bmatrix} n \\ k \end{bmatrix}$ is an anti-chain in $2^{[n]}$ because if otherwise there are two different elements A and $B \in \begin{bmatrix} n \\ k \end{bmatrix}$ so that $A \subseteq B$. Since $|A| = |B| = k$ it implies that $A = B$ and then is a contradiction to $A \neq B$.

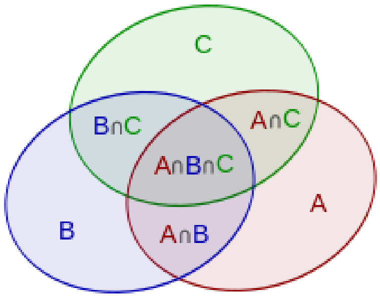
Theorem 2.3.4 (Sperner's theorem)**.** *The only anti-chains of largest size are* $\begin{bmatrix} n \\ \lfloor \frac{n}{2} \rfloor \end{bmatrix}$ *and* $\begin{bmatrix} n \\ \lceil \frac{n}{2} \rceil \end{bmatrix}$.

Proof. See [3], pp. 36-38 or the proof of David Lubell (see, [1], pp. 213-214). □

3. Inclusion-Exclusion principle

3.1. The Inclusion-Exclusion Formula

The Inclusion-Exclusion Formula



The inclusion-exclusion formula shows the number of elements in the union of many finite sets.

$$\begin{aligned}|A \cup B \cup C| =& |A| + |B| + |C| \\ & - |A \cap B| - |A \cap C| - |B \cap C| \\ & + |A \cap B \cap C|.\end{aligned}$$

Proposition 3.1.1. *Let $A_1, \dots, A_n$ be finite sets. Then*

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{k=1}^{n} (-1)^{k+1} \left(\sum_{1 \le i_1 < \dots < i_k \le n} |A_{i_1} \cap \dots \cap A_{i_k}| \right).$$

Proof. We use notation AB for $A \cap B$ and will prove this proposition by induction on n as follows

- Base case: $n = 2$: $|A_1 \cup A_2| = |A_1| + |A_2 \setminus A_1| = |A_1| + |A_2| - |A_1 A_2|$.
- Inductive step: Assume that this is true for n, we prove this is also true for $n+1$. In fact, assume that $A_1, \dots, A_{n+1}$ are finite sets. Then

$$\begin{aligned}
\left|\bigcup_{i=1}^{n+1} A_i\right| &= \left|\bigcup_{i=1}^{n} A_i\right| + |A_{n+1}| - \left|\left(\bigcup_{i=1}^{n} A_i\right) A_{n+1}\right| \quad (\text{apply for } A = \bigcup_{i=1}^{n} A_i, B = A_{n+1}) \\
&= \sum_{k=1}^{n} (-1)^{k+1} \left(\sum_{1 \le i_1 < \dots < i_k \le n} |A_{i_1} \cdots A_{i_k}| \right) + |A_{n+1}| - \left|\left(\bigcup_{i=1}^{n} \left(A_i A_{n+1}\right)\right)\right| \\
&\quad \text{(due to the inductive hypothesis and the distribution property of } \cap \text{ and } \cup) \\
&= \sum_{k=1}^{n} (-1)^{k+1} \left(\sum_{1 \le i_1 < \dots < i_k \le n} |A_{i_1} \cdots A_{i_k}| \right) + |A_{n+1}| \\
&\quad - \sum_{k=1}^{n} (-1)^{k+1} \left(\sum_{1 \le i_1 < \dots < i_k \le n} |(A_{i_1} A_{n+1}) \cdots (A_{i_k} A_{n+1})| \right) \\
&\quad \text{(due to the inductive hypothesis for } \{A_i A_{n+1}\}_{i=1}^{n}) \\
&= \sum_{1 \le i_1 \le n} |A_{i_1}| - \sum_{1 \le i_1 < i_2 \le n} |A_{i_1} A_{i_2}| + \dots + (-1)^{n+1} |A_1 \cdots A_n| + |A_{n+1}| \\
&\quad - \left(\sum_{1 \le i_1 \le n} |A_{i_1} A_{n+1}| - \sum_{1 \le i_1 < i_2 \le n} |A_{i_1} A_{i_2} A_{n+1}| + \dots + (-1)^{n+1} |A_1 \cdots A_n A_{n+1}| \right) \\
&= \sum_{1 \le i_1 \le n+1} |A_{i_1}| - \sum_{1 \le i_1 < i_2 \le n+1} |A_{i_1} A_{i_2}| + \dots + (-1)^{n+2} |A_1 \cdots A_{n+1}|.
\end{aligned}$$

This completes the proof. □

Corollary 3.1.1. *Let $A_1, \dots, A_n \subset S$ be finite sets. Denote by $\bar{A} = S \setminus A$. Then*

$$\left|\bigcap_{i=1}^n \bar{A}_i\right| = |S| + \sum_{k=1}^n (-1)^k \left(\sum_{1 \le i_1 < \cdots < i_k \le n} |A_{i_1} \cap \cdots \cap A_{i_k}| \right).$$

Proof. In fact, it follows from the above proposition 3.1.1 and

$$\left|\bigcap_{i=1}^n \bar{A}_i\right| = \left|\bigcap_{i=1}^n \left(S \setminus A_i\right)\right| = \left|S \setminus \bigcup_{i=1}^n A_i\right| = |S| - \left|\bigcup_{i=1}^n A_i\right|.$$

□

Example 3.1.1. Find the number of non-negative integer solutions to $x_1 + x_2 + x_3 = 7$ with $x_1 \le 2$, $x_2 \le 4$, and $x_3 \le 3$.

Denote by $S = \{(x_1, x_2, x_3) \in \mathbb{N}_0^3 : x_1 + x_2 + x_3 = 7\}$; $A_1 = \{(x_1, x_2, x_3) \in S : x_1 \ge 3\}$, $A_2 = \{(x_1, x_2, x_3) \in S : x_2 \ge 5\}$, $A_3 = \{(x_1, x_2, x_3) \in S : x_3 \ge 4\}$; $A \cap B = AB$. Then

$$\begin{aligned} |\bar{A}_1\bar{A}_2\bar{A}_3| &= |S| - |A_1| - |A_2| - |A_3| + |A_1A_2| + |A_1A_3| + |A_2A_3| - |A_1A_2A_3| \\ &= \binom{9}{2} - \binom{6}{2} - \binom{4}{2} - \binom{5}{2} + 0 + 1 + 0 - 0 = 6. \end{aligned}$$

3.2. Forbidden Position Permutations

3.2.1. Derangement

Definition 3.2.1. Given a natural number $n \in \mathbb{N}$. A derangement of $[n]$ is a permutation of $[n]$ have none of the integers in their correct locations, i.e., 1 is not the first, 2 is not the second, and so on. Denote by $\mathcal{D}_n = \{\sigma \in S_n : \sigma(i) \ne i, \forall i \in [n]\}$ the set of all derangements of $[n]$ and $D_n = |\mathcal{D}_n|$.

Theorem 3.2.1.

$$D_n = n! \sum_{k=0}^n (-1)^k \frac{1}{k!}.$$

Proof. Denote by A_i the set of permutations of $[n]$ in which i is in the correct place. Then $D_n = \left|\bigcap\limits_{i=1}^n \bar{A}_i\right|$. Note that A_i is nothing but the set of permutations of $n-1$ elements different from i therefore $|A_i| = (n-1)!$. Similarly, $A_{i_1} \cap \cdots \cap A_{i_k}$ is nothing but the set of permutations of $n-k$ elements different from $i_1, \dots, i_k$ therefore $|A_{i_1} \cap \cdots \cap A_{i_k}| = (n-k)!$ for $2 \le j \le n$. Thus, by the Corollary 3.1.1 we have

$$\begin{aligned} \left|\bigcap_{i=1}^n \bar{A}_i\right| &= |S| + \sum_{k=1}^n (-1)^k \binom{n}{k} (n-k)! \\ &= n! + \sum_{k=1}^n (-1)^k \frac{n!}{k!} \\ &= n! \sum_{k=0}^n (-1)^k \frac{1}{k!}. \end{aligned}$$

This completes the proof. □

Remark. We use the convention that $D_0 = 1$ which is still satisfied this formula.

Theorem 3.2.2. *D_n satisfies the recurrence relations*

1. $D_n = (n-1)(D_{n-1} + D_{n-2})$ *for* $n \geq 2$;

2. $D_n = nD_{n-1} + (-1)^n$ *for* $n \geq 1$.

Proof. 1. For each $i \in \{2, \ldots, n\}$, we denote by $A_i = \{\sigma \in \mathcal{D}_n : \sigma(1) = i, \sigma(i) = 1\}$ and $B_i = \{\sigma \in \mathcal{D}_n : \sigma(1) \neq i, \sigma(i) = 1\}$. Then $\mathcal{D}_n = \dot\bigcup_{i=2}^n (A_i \dot\cup B_i)$ and therefore $D_n = \sum_{i=2}^n (|A_i| + |B_i|)$. Note that for each $\sigma \in A_i$, we define $\sigma' : \{2, \ldots, i-1, i+1, \ldots, n\} \to \{2, \ldots, i-1, i+1, \ldots, n\}, \sigma'(k) = \sigma(k), \forall k \in \{2, \ldots, i-1, i+1, \ldots, n\}$. Then $\sigma \mapsto \sigma'$ is bijective from A_i to the set of all derangements on $\{2, \ldots, i-1, i+1, \ldots, n\}$. Therefore $|A_i| = D_{n-2}$. On the other hand, for $\sigma \in B_i$, we define $\sigma'' : \{i, 2, \ldots, i-1, i+1, \ldots, n\} \to \{i, 2, \ldots, i-1, i+1, \ldots, n\}$ with $\sigma''(i) = \sigma(1), \sigma''(k) = \sigma(k), \forall k \in \{2, \ldots, i-1, i+1, \ldots, n\}$. Then $\sigma \mapsto \sigma''$ is bijective from B_i to the set of all derangements on $\{i, 2, \ldots, i-1, i+1, \ldots, n\}$. Therefore $|B_i| = D_{n-1}$. It implies that $D_n = (n-1)(D_{n-2} + D_{n-1})$.

2. We prove this by induction on n based on the first result and Theorem 3.2.1 as follows:
   - Base case: $n = 1$: $D_1 = 1D_0 + (-1)^1$ is true because $D_1 = 0$ and $D_0 = 1$ from Theorem 3.2.1.
   - Inductive step: Assume that this is true for $n \geq 1$, we will prove it is also true for $n+1$. In fact,

$$D_{n+1} \overset{1.}{=} nD_n + nD_{n-1} \overset{\text{inductive hypothesis}}{=} nD_n + \Big(D_n - (-1)^n\Big) = (n+1)D_n + (-1)^{n+1}.$$

It completes the proof.

□

Exercise 3.2.1 (E2.1: Binomial coefficients)**.** *1. Given two natural numbers* $n, k \in \mathbb{N}_0$ *with* $k \leq n$. *Find a simple expression for* $\sum\limits_{k=0}^{n} \frac{(-1)^k}{k+1}\binom{n}{k}$;

2. Given three natural numbers $m, n, k \in \mathbb{N}_0$ *with* $k \leq \min\{m, n\}$. *Prove that* $\sum_{i=0}^k \binom{m}{i}\binom{n}{k-i} = \binom{m+n}{k}$.

Exercise 3.2.2 (E2.2: Number of integer solutions)**.** *Find the number of integer solutions to*

1. $x_1 + x_2 + x_3 + x_4 + x_5 = 2020, \quad x_1 \geq -3, x_2 \geq -2, x_3 \geq 4, x_4 \geq 3, x_5 \geq 10$;

2. $x_1 + x_2 + x_3 + x_4 = 30, \quad 1 \leq x_1 \leq 6, 2 \leq x_2 \leq 8, 0 \leq x_3 \leq 10, 3 \leq x_4 \leq 12$.

Exercise 3.2.3 (E2.3: Derangements)**.** *Let* D_n *be the number of derangements of* $[n]$. *Prove that*

$$n! = \sum_{k=0}^{n} \binom{n}{k} D_{n-k}.$$

4. Generating Functions

4.1. Newton's Binomial Theorem

Generating function

Definition 4.1.1. $f(x)$ is a generating function for the sequence $\{a_n\}_{n\geq 0}$ if

$$f(x) = \sum_{n=0}^{\infty} a_n x^n.$$

Example 4.1.1. $f(x) = e^{2x}$ is the generating function for the sequence $\{a_n = \frac{2^n}{n!}\}_{n\geq 0}$.

Definition 4.1.2. Given $r \in \mathbb{R}$ and $k \in \mathbb{N}_0$, define the generalized binomial coefficient by

$$\binom{r}{k} = \frac{r(r-1)\cdots(r-k+1)}{k!}.$$

Theorem 4.1.1 (Newton's binomial theorem)**.** *For any real number r, when $-1 < x < 1$ we have*

$$(x+1)^r = \sum_{k=0}^{\infty} \binom{r}{k} x^k.$$

Proof. Consider the function $f(x) = \sum_{n=0}^{\infty} \binom{r}{n} x^n$ on $(-1,1)$. This function is well-defined because the series on the right hand side is uniformly absolutely convergent on $(-1,1)$:

$$\begin{aligned}
\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\binom{r}{n+1}x^{n+1}}{\binom{r}{n}x^n}\right| &= \left|\frac{\frac{r(r-1)\cdots(r-n)}{(n+1)!}}{\frac{r(r-1)\cdots(r-n+1)}{n!}}\right| |x| \\
&= \frac{|r-n|}{n+1}|x| \to |x| \leq \delta < 1 \text{ as } n \to \infty.
\end{aligned}$$

Similarly, $\sum_{n=1}^{\infty} \binom{r}{n} n x^{n-1}$ is uniformly absolutely convergent on $(-1,1)$. Therefore

$$\begin{aligned}
(1+x)f'(x) &= (1+x)\sum_{n=1}^{\infty}\binom{r}{n} n x^{n-1} \\
&= \sum_{n=1}^{\infty}\binom{r}{n} n x^{n-1} + \sum_{n=1}^{\infty}\binom{r}{n} n x^{n} \\
&= \sum_{n=0}^{\infty}\binom{r}{n+1}(n+1)x^{n} + \sum_{n=1}^{\infty}\binom{r}{n} n x^{n} \\
&= r + \sum_{n=1}^{\infty}\left(\binom{r}{n+1}(n+1) + \binom{r}{n} n\right)x^n \\
&= r + \sum_{n=1}^{\infty}\left(\frac{r(r-1)\cdots(r-n)}{n!} + \frac{r(r-1)\cdots(r-n+1)}{n!} n\right)x^n \\
&= r\left(1 + \sum_{n=1}^{\infty}\binom{r}{n} x^n\right) \\
&= rf(x) > 0.
\end{aligned}$$

It implies that $\frac{f'(x)}{f(x)} = \frac{r}{1+x}$ or $(\ln f(x))' = r\ln(1+x)'$. Thus, $f(x) = e^{C_1}(1+x)^r$. Because $f(0) = \sum_{n=0}^{\infty}\binom{r}{n}0^n = 1$, it implies $C_1 = 0$ and therefore $f(x) = (1+x)^r$. This completes the proof.

□

Example 4.1.2. For n is a positive integer, then $(1-x)^{-n}$ is the generating function for $\{a_k\}_{k\geq 0}$ where $a_k = \binom{n+k-1}{n-1}$, the number of submultisets of $\{\infty\cdot 1, \infty\cdot 2, \ldots, \infty\cdot n\}$ of size k.

In many cases it is possible to directly construct the generating function whose coefficients solve a counting problem. For example,

Example 4.1.3. Find the number of integer solutions to $x_1+x_2+x_3+x_4 = 17$, where $0\leq x_1\leq 2, 0\leq x_2\leq 5, 0\leq x_3\leq 5, 2\leq x_4\leq 6$.

Solve. Restate the problem: We would like to find the number of solutions for the system

$$m_1+m_2+m_3+m_4 = 17$$

where

$$\begin{cases} m_1\in I_1 = \{0,1,2\}, \\ m_2\in I_2 = \{0,1,2,3,4,5\}, \\ m_3\in I_3 = \{0,1,2,3,4,5\}, \\ m_4\in I_4 = \{2,3,4,5,6\}. \end{cases}$$

Note that we have the equality

$$\left(\sum_{m_1\in I_1} x^{m_1}\right)\left(\sum_{m_2\in I_2} x^{m_2}\right)\left(\sum_{m_3\in I_3} x^{m_3}\right)\left(\sum_{m_4\in I_4} x^{m_4}\right) = \sum_{n=0}^{\infty}\left(\sum_{\substack{m_1\in I_1, m_2\in I_2, m_3\in I_3, m_4\in I_4: \\ m_1+m_2+m_3+m_4=n}} 1\right)x^n = \sum_{n=0}^{\infty} a_n x^n$$

Therefore, the number of solutions for the system is nothing but the coefficient a_{17} of the generating function

$$\begin{aligned} f(x) &= \left(\sum_{m_1\in I_1} x^{m_1}\right)\left(\sum_{m_2\in I_2} x^{m_2}\right)\left(\sum_{m_3\in I_3} x^{m_3}\right)\left(\sum_{m_4\in I_4} x^{m_4}\right) \\ &= (1+x+x^2)(1+x+x^2+x^3+x^4+x^5)^2(x^2+x^3+x^4+x^5+x^6) \\ &= x^{18}+4x^{17}+10x^{16}+19x^{15}+31x^{14}+45x^{13}+58x^{12}+67x^{11} \\ &\quad +70x^{10}+67x^9+58x^8+45x^7+31x^6+19x^5+10x^4+4x^3+x^2. \end{aligned}$$

Therefore the number of solutions is 4. □

Example 4.1.4. Find the generating function for $\{a_k\}_{k\geq 0}$, where a_k is the number of solutions to $x_1+x_2+x_3+x_4 = k$, where $0\leq x_1, 0\leq x_2\leq 5, 0\leq x_3\leq 5, 2\leq x_4\leq 6$.

Solve. The generating function is

$$\begin{aligned} f(x) &= (1+x+x^2+\cdots)(1+x+x^2+x^3+x^4+x^5)^2(x^2+x^3+x^4+x^5+x^6) \\ &= \frac{(1+x+x^2+x^3+x^4+x^5)^2(x^2+x^3+x^4+x^5+x^6)}{1-x}. \end{aligned}$$

□

4.2. Exponential Generating Functions

Definition 4.2.1. $f(x)$ is an exponential generating function for the sequence $\{a_n\}_{n\geq 0}$ if

$$f(x) = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!}.$$

Example 4.2.1. Find an exponential generating function for the number of permutations with repetition of length n of the set $\{a; b; c\}$, in which there are an odd number of $a's$, an even number of $b's$, and any number of $c's$.

Solve. The exponential generating function is

$$f(x) = \sum_{i=0}^{\infty} \frac{x^{2i+1}}{(2i+1)!} \sum_{j=0}^{\infty} \frac{x^{2j}}{(2j)!} \sum_{k=0}^{\infty} \frac{x^k}{k!} = \frac{e^x - e^{-x}}{2} \frac{e^x + e^{-x}}{2} e^x = \frac{1}{4}(e^{3x} - e^{-x}).$$

□

4.3. Partitions of Integers

Definition 4.3.1. A partition of a positive integer n is a multiset of positive integers that sum to n. We denote the number of partitions of n by p_n.

Example 4.3.1. $p_5 = 7$ with partitions

$$\begin{aligned}
&5\\
&4+1\\
&3+2\\
&3+1+1\\
&2+2+1\\
&2+1+1+1\\
&1+1+1+1+1.
\end{aligned}$$

The generating function for $\{p_n\}_{n\geq 0}$ is

$$\begin{aligned}
f(x) &= (1 + x + x^2 + \cdots)(1 + x^2 + x^4 + \cdots)\cdots(1 + x^k + x^{2k} + \cdots)\cdots \\
&= \prod_{k=1}^{\infty} \sum_{i=0}^{\infty} x^{ik} = \prod_{k=1}^{\infty} \frac{1}{1 - x^k}
\end{aligned}$$

Example 4.3.2. Find p_8.

Solve.

$$\begin{aligned}
&(1 + x + \cdots + x^8)(1 + x^2 + x^4 + x^6 + x^8)(1 + x^3 + x^6)(1 + x^4 + x^8)(1 + x^5) \\
&\times (1 + x^6)(1 + x^7)(1 + x^8) \\
&= 1 + x + 2x^2 + 3x^3 + 5x^4 + 7x^5 + 11x^6 + 15x^7 + 22x^8 + \cdots
\end{aligned}$$

so $p_8 = 22$. □

Definition 4.3.2. Denote by $p_d(n)$ the number of partitions into distinct parts and $p_o(n)$ the number of partitions into odd parts.

Example 4.3.3. • The partitions into distinct parts of $n = 6$ are

$$6, 5+1, 4+2, 3+2+1,$$

so $p_d(6) = 4$;

• The partitions into odd parts of $n = 6$ are

$$5+1, 3+3, 3+1+1+1, 1+1+1+1+1+1,$$

so $p_o(6) = 4$.

Theorem 4.3.1. *For every n, we have $p_d(n) = p_o(n)$.*

Proof. Given two subsets $\mathcal{I}, \mathcal{N} \subseteq \mathbb{N}_0$ and denote by $\mathcal{I}^* = \mathcal{I} - \{0\}$. We have

$$\prod_{i\in\mathcal{I}^*}\left(\sum_{n_i\in\mathcal{N}} x^{in_i}\right) = \sum_{n=0}^{\infty}\left(\sum_{\substack{i_1n_{i_1}+\cdots+i_kn_{i_k}=n\\ i_1,\ldots,i_k\in\mathcal{I}^*\\ n_{i_1},\ldots,n_{i_k}\in\mathcal{N}}} 1\right)x^n.$$

This means that $\prod_{i\in\mathcal{I}^*}\left(\sum_{n_i\in\mathcal{N}} x^{in_i}\right)$ is the generating function for $\{a_n\}_{n\in\mathbb{N}_0}$ with a_n is the number of partitions of n into parts in $\mathcal{I}^*$ with repetitions in $\mathcal{N}$. For distinct parts, we have $\mathcal{N} = \{0, 1\}$ and $\mathcal{I}^* = \mathbb{N}$. The generating function for the number of partitions of an integer into distinct parts $\{p_d(n)\}_{n=0}^{\infty}$ is

$$f_d(x) = \prod_{i\in\mathbb{N}}\left(\sum_{n_i\in\{0,1\}} x^{in_i}\right) = (1+x)(1+x^2)(1+x^3)(1+x^4)\cdots = \prod_{k=1}^{\infty}(1+x^k).$$

For odd parts, we have $\mathcal{N} = \mathbb{N}_0$ and $\mathcal{I}^* = \{1, 3, 5, \ldots\}$. The generating function for the number of partitions of an integer into odd parts $\{p_o(n)\}_{n=0}^{\infty}$ is

$$\begin{aligned} f_o(x) &= \prod_{i\in\{1,3,5,\ldots\}}\left(\sum_{n_i\in\mathbb{N}_0} x^{in_i}\right) = (1+x+x^2+\cdots)(1+x^3+x^6+\cdots)(1+x^5+x^{10}+\cdots)\cdots \\ &= \frac{1}{1-x}\frac{1}{1-x^3}\frac{1}{1-x^5}\cdots \end{aligned}$$

Therefore, we have

$$\begin{aligned} f_d(x) &= (1+x)(1+x^2)(1+x^3)(1+x^4)\cdots \\ &= \frac{1-x^2}{1-x}\frac{1-x^4}{1-x^2}\frac{1-x^6}{1-x^3}\frac{1-x^8}{1-x^4}\cdots \\ &= \frac{1}{1-x}\frac{1}{1-x^3}\frac{1}{1-x^5}\cdots \qquad \text{(by removing the common factors of the form } (1-x^{2k})\text{)} \\ &= f_o(x). \end{aligned}$$

It implies the proof.

□

Theorem 4.3.2. *Denote by $p_k(n)$ the number of partitions of n into exactly k parts for $1 \le k \le n$. Then*

$$p_k(n) = p_k(n-k) + p_{k-1}(n-1).$$

Proof. Consider $\{n_1, \ldots, n_k\}$ as an arbitrary partition of n with size k. There are two cases:

- Case 1: If $n_i \geq 2$ for all $i \in [k]$ then $\{n_1, \dots, n_k\}$ is 1-1 corresponding to a partition $\{n_1 - 1, \dots, n_k - 1\}$ of $n - k$ of size k.
- Case 2: If there exists $i \in [k]$ such that $n_i = 1$ then $\{n_1, \dots, n_k\}$ is 1-1 corresponding to a partition $\{n_1, \dots, n_{i-1}, n_{i+1}, \dots, n_k\}$ of $n - 1$ with size $k - 1$.

This implies the proof. □

4.4. Recurrence Relations

Definition 4.4.1. A recurrence relation defines a sequence $\{a_n\}_{n\geq 0}$ by expressing a typical term a_n in terms of earlier terms, a_i for $i < n$. For example,

Example 4.4.1.
- the Fibonacci sequence is defined by

$$F_0 = 0, F_1 = 1, F_n = F_{n-1} + F_{n-2}.$$

- the number of derangements of $[n]$

$$D_1 = 0, D_n = nD_{n-1} + (-1)^n.$$

We can use the generating function to solve recurrence relations.

Example 4.4.2.
- What is the generating function for the Fibonacci sequence $\{F_n\}_{n\geq 0}$?
- Find F_n

Solve. $f(x) = \sum_{n\geq 0} F_n x^n$. It implies that $f(x) - xf(x) - x^2 f(x) = x + (F_2 - 1)x^2 + \sum_{k=3}^{\infty}(F_k - F_{k-1} - F_{k-2})x^k = x$. Therefore $f(x) = \frac{-x}{x^2+x-1}$. Rewrite $f(x) = \frac{c_1}{x-x_1} + \frac{c_2}{x-x_2}$ with $c_{1,2} = \frac{\pm 1 - \sqrt{5}}{2\sqrt{5}}$ and $x_{1,2} = \frac{-1\pm\sqrt{5}}{2}$. Then,

$$f(x) = -\frac{c_1}{x_1}\frac{1}{1 - x/x_1} - \frac{c_2}{x_2}\frac{1}{1 - x/x_2} = \sum_{n=0}^{\infty}\left(-\frac{c_1}{x_1}\left(\frac{1}{x_1}\right)^n - \frac{c_2}{x_2}\left(\frac{1}{x_2}\right)^n\right)x^n.$$

Therefore $F_n = -\frac{c_1}{x_1}\left(\frac{1}{x_1}\right)^n - \frac{c_2}{x_2}\left(\frac{1}{x_2}\right)^n$. □

Exercise 4.4.1 (E3.1: Exponential generating functions). *(i) Find an exponential generating function for the number of permutations with repetition of length n of the set $\{a, b, c\}$, in which there are an odd number of $a's$, an even number of $b's$, and an even number of $c's$;*

(ii) Find an exponential generating function for the number of permutations with repetition of length n of the set $\{a, b, c\}$, in which the number of $a's$ is even and at least 2, the number of $b's$ is even and at most 6, and the number of $c's$ is at least 3.

Exercise 4.4.2 (E3.2: Partition of integers). *(i) Find the generating function for the number of partitions of an integer into distinct even parts.*

(ii) Find the number of such partitions of $n = 30$.

Exercise 4.4.3 (E3.3: Recurrence relations). *Find the generating functions for below recurrence relations and then find formula for h_n*

(i)

$$h_n = 4h_{n-1} - 3h_{n-2}, \quad \text{with } h_0 = 2, h_1 = 5$$

(ii)

$$h_n = 3h_{n-1} + 4h_{n-2}, \quad \text{with } h_0 = 0, h_1 = 1.$$

5. Systems of Distinct Representatives

Definition 5.0.1. Given a collection $\mathcal{A}$ of n sets $\mathcal{A} = \{A_1, \ldots, A_n\}$. A system of distinct representatives (abbreviated as SDR) for $\mathcal{A}$ is a set of n distinct elements $\{x_1, \ldots, x_n\}$ with $x_i \in A_i, \forall i \in [n]$.

Example 5.0.1. Given $\mathcal{A} = \{A_1, A_2\}$ with $A_1 = \{a, b\}$, $A_2 = \{a, c\}$. Then there are totally three SDRs for $\mathcal{A}$ which are $\{a, c\}$, $\{b, a\}$, and $\{b, c\}$.

5.1. Existence of SDRs

There are collections $\mathcal{A}$ which has no SDR. For example,

- $\mathcal{A} = \{A_1, A_2, A_3\}$ with
$$A_1 = A_2 = A_3 = \{a, b\}$$
- $\mathcal{A} = \{A_1, A_2, A_3, A_4\}$ with
$$A_1 = A_2 = A_3 = \{a, b\}; A_4 = \{b, c, d, e\}.$$

Theorem 5.1.1 (Hall's theorem)**.** *A collection $\mathcal{A} = \{A_1, \ldots, A_n\}$ has an SDR if and only if $\mathcal{A}$ satisfies the Hall's condition, i.e., for every subset $\mathcal{I} \subseteq [n]$,*

$$| \cup_{i \in \mathcal{I}} A_i| \geq |\mathcal{I}|$$

(from now on we use the convention that $|\emptyset| = 0$ and $\cup_{i \in \emptyset} A_i = \emptyset$).

Proof. ($\Rightarrow$:) Assume that $\mathcal{A} = \{A_1, \ldots, A_n\}$ has an SDR $\{x_1, \ldots, x_n\}$ and there exists $\mathcal{I} \subseteq [n]$: $| \cup_{i \in \mathcal{I}} A_i| < |\mathcal{I}|$. Then $\{x_i\}_{i \in \mathcal{I}}$ have $|\mathcal{I}|$ elements but have $| \cup_{i \in \mathcal{I}} \{x_i\}| \leq | \cup_{i \in \mathcal{I}} A_i| < |\mathcal{I}|$ values which is a contradiction to the pigeon-hole principle. Therefore $| \cup_{i \in \mathcal{I}} A_i| \geq |\mathcal{I}|$ for every subset $\mathcal{I} \subseteq [n]$.
($\Leftarrow$:) Suppose that $\mathcal{A}$ satisfies the Hall's condition, we prove that $\mathcal{A}$ has an SDR $\{x_1, \ldots, x_n\}$ by induction on n. In fact,

- Base case: for $n = 1$, $\mathcal{A} = \{A_1\}$ satisfies the Hall's condition implies that $|A_1| \geq 1$ (by choosing $\mathcal{I} = \{1\}$). Therefore there exists $x_1 \in A_1$ which is an SDR of $\mathcal{A}$.
- Inductive step: Assume that it is true until $\leq n - 1$, we prove that it is also true for n. Indeed, consider $\mathcal{A} = \{A_1, \ldots, A_n\}$ satisfying Hall's condition. There are two cases:
  - Case 1: $| \cup_{i \in \mathcal{I}} A_i| \geq |\mathcal{I}| + 1, \quad \forall \emptyset \neq \mathcal{I} \subsetneq [n]$. Because $|A_n| \geq 1$ (by choosing $\mathcal{I} = \{n\}$), there exists $x_n \in A_n$. Set $B_j = A_j \setminus \{x_n\}$ for all $j \in [n-1]$. Then $\{B_j\}_{j=1}^{n-1}$ satisfies Hall's condition because for all $\mathcal{J} \subseteq [n-1]$ we have
  $$\begin{aligned} | \cup_{j \in \mathcal{J}} B_j| &= | \cup_{j \in \mathcal{J}} (A_j \setminus \{x_n\})| \\ &= | \cup_{j \in \mathcal{J}} A_j \setminus \{x_n\}| \geq | \cup_{j \in \mathcal{J}} A_j| - 1 \geq (|\mathcal{J}| + 1) - 1 = |\mathcal{J}|. \end{aligned}$$
  Therefore, from the inductive hypothesis, there is an SDR $\{x_1, \ldots, x_{n-1}\}$ for $\{B_1, \ldots, B_{n-1}\}$, i.e., we have $x_i \neq x_j \quad \forall i \neq j \in [n-1]$ and $x_i \in B_i \subseteq A_i$ for all $i \in [n-1]$. Note that $x_n \in A_n$. Moreover for all $i \in [n-1]$, we have $x_i \in B_i = A_i \setminus \{x_n\}$ which implies that $x_n \neq x_i$. Therefore $\{x_1, \ldots, x_n\}$ is an SDR for $\mathcal{A}$.

- Case 2: $\exists\, \emptyset \neq \mathcal{I} \subsetneq [n]$ such that $|\cup_{i\in\mathcal{I}} A_i| = |\mathcal{I}|$. Note that $|\mathcal{I}| < n$ therefore from the inductive hypothesis there is an SDR $\{x_i\}_{i\in\mathcal{I}}$ for $\cup_{i\in\mathcal{I}} A_i$. Set $B_j = A_j \setminus \big(\cup_{i\in\mathcal{I}} A_i\big)$ for all $j \in \mathcal{I}^c = [n] \setminus \mathcal{I}$. Then $\{B_j\}_{j\in\mathcal{I}^c}$ satisfies Hall's condition because for all $\mathcal{J} \subseteq \mathcal{I}^c$ we have

$$\begin{aligned}|\cup_{j\in\mathcal{J}} B_j| &= |\cup_{j\in\mathcal{J}} (A_j \setminus \cup_{i\in\mathcal{I}} A_i)| = |\cup_{j\in\mathcal{J}} A_j \setminus \cup_{i\in\mathcal{I}} A_i| \\ &= |\cup_{j\in\mathcal{I}\cup\mathcal{J}} A_j| - |\cup_{i\in\mathcal{I}} A_i| \geq |\mathcal{I}\cup\mathcal{J}| - |\mathcal{I}| = (|\mathcal{I}| + |\mathcal{J}|) - |\mathcal{I}| = |\mathcal{J}|.\end{aligned}$$

Because $|\mathcal{I}^c| < n$, from the inductive hypothesis, there is an SDR $\{y_j\}_{j\in\mathcal{I}^c}$ for $\{B_j\}_{j\in\mathcal{I}^c}$. Then it is easy to check that $\{\{x_i\}_{i\in\mathcal{I}}, \{y_j\}_{j\in\mathcal{I}^c}\}$ is an SDR for $\mathcal{A}$.

□

5.2. Partial SDRs

Definition 5.2.1. Given a collection $\mathcal{A}$ of n sets $\mathcal{A} = \{A_1, \ldots, A_n\}$. A partial SDR for $\mathcal{A}$ is a set of distinct elements $\{x_i\}_{i\in\mathcal{I}}$ ($\emptyset \neq \mathcal{I} \subseteq [n]$) such that $x_i \in A_i, \quad \forall i \in \mathcal{I}$.
The maximum size of a partial SDR for the collection $\mathcal{A}$ is denoted by $\lambda(\mathcal{A}) = \max\{|\mathcal{I}| : \mathcal{I} \subseteq [n], \{A_i\}_{i\in\mathcal{I}}$ has an SDR$\}$.

Example 5.2.1. For the collection $\mathcal{A} = \{A_i\}_{i=1}^4$ with $A_1 = A_2 = A_3 = \{a, b\}, A_4 = \{b, c, d, e\}$. Then $\{a, b, c\}$ is a partial SDR which represents $\{A_1, A_2, A_4\}$ and $\lambda(\mathcal{A}) = 3$.

Theorem 5.2.1.

$$\lambda(\mathcal{A}) = \min_{\mathcal{I}\subseteq[n]} \{n - |\mathcal{I}| + |\cup_{i\in\mathcal{I}} A_i|\}$$

Proof. Set

$$\alpha(\mathcal{A}) = \min_{\mathcal{I}\subseteq[n]} \{n - |\mathcal{I}| + |\cup_{i\in\mathcal{I}} A_i|\}. \tag{5.2.1}$$

We prove that $\lambda(\mathcal{A}) = \alpha(\mathcal{A})$ by induction on n. In fact,

- Base case: for $n = 1$, we have $\mathcal{A} = \{A_1\}$ with $|A_1| \geq 1$. There are two subsets of $[1]$ which are $\emptyset$ and $\{1\}$. Therefore

  $$\alpha(\mathcal{A}) = \min\{1 - 0 + 0, 1 - 1 + |A_1|\} = 1.$$

  On the other hand, because $|A_1| \geq 1$, there is $x_1 \in A_1$ and $\{x_1\}$ is a SDR for $\{A_1\}$ therefore $\lambda(\mathcal{A}) = 1$. Thus, the statement is true for $n = 1$.

- Inductive step: Assume that $\lambda(\mathcal{A}) = \alpha(\mathcal{A})$ for all $|\mathcal{A}| < n$. We prove that it is also true for $|\mathcal{A}| = n$. In fact, consider $\mathcal{A} = \{A_1, \ldots, A_n\}$. Note that by choosing $\mathcal{I} = \emptyset$ we have $\alpha(\mathcal{A}) \leq n - 0 + 0 = n$. We consider two cases
  - Case 1: If $\alpha(\mathcal{A}) = n$ then $\mathcal{A}$ satisfies the Hall's condition because for all $\mathcal{I} \subseteq [n]$ we have $n - |\mathcal{I}| + |\cup_{i\in\mathcal{I}} A_i| \geq \alpha(\mathcal{A}) = n$ therefore $|\cup_{i\in\mathcal{I}} A_i| \geq |\mathcal{I}|$. Therefore, from the Hall theorem, there is an SDR for $\mathcal{A}$, i.e., $\lambda(\mathcal{A}) = n$.
  - Case 2: If $\alpha(\mathcal{A}) = m < n$ then there is a minimizer $\mathcal{I}_{\min} \subseteq [n]$ (with the smallest size in all minimizers of (5.2.1)) so that

    $$m = \min_{\mathcal{I}\subseteq[n]} \{n - |\mathcal{I}| + |\cup_{i\in\mathcal{I}} A_i|\} = n - |\mathcal{I}_{\min}| + |\cup_{i\in\mathcal{I}_{\min}} A_i|. \tag{5.2.2}$$

    It implies that

    $$-|\mathcal{I}| + |\cup_{i\in\mathcal{I}} A_i| \geq -|\mathcal{I}_{\min}| + |\cup_{i\in\mathcal{I}_{\min}} A_i|, \quad \forall \mathcal{I} \subseteq [n]. \tag{5.2.3}$$

    We consider two subcases:

* Case 2.1: If $|\mathcal{I}_{\min}| = k < n$ then, from the inductive hypothesis,

$$\lambda(\{A_i\}_{i\in\mathcal{I}_{\min}}) = \alpha(\{A_i\}_{i\in\mathcal{I}_{\min}}) = \min_{\mathcal{J}\subseteq\mathcal{I}_{\min}} \{k - |\mathcal{J}| + |\cup_{j\in\mathcal{J}} A_j|\}$$
$$\overset{(5.2.3)}{=} |\cup_{j\in\mathcal{I}_{\min}} A_j| \overset{(5.2.2)}{=} m - n + k.$$

It means that $\{A_i\}_{i\in\mathcal{I}_{\min}}$ has a partial SDR $\{x_1, \dots, x_{m-n+k}\}$. Set $C_j = A_j - \cup_{i\in\mathcal{I}_{\min}} A_i$ for all $j \in \mathcal{I}^c_{\min}$. Then $\{C_j\}_{j\in\mathcal{I}^c_{\min}}$ satisfies the Hall condition because for all $\mathcal{J} \subseteq \mathcal{I}^c_{\min}$ we have

$$|\cup_{i\in\mathcal{J}} C_j| = |\cup_{j\in\mathcal{J}\cup\mathcal{I}_{\min}} A_j| - |\cup_{i\in\mathcal{I}_{\min}} A_i|$$
$$\overset{(5.2.3)}{\geq} |\mathcal{J} \cup \mathcal{I}_{\min}| - |\mathcal{I}_{\min}| = |\mathcal{J}|.$$

Therefore, from the Hall theorem, there is a SDR $\{y_j\}_{j\in\mathcal{I}^c_{\min}}$ for $\{C_j\}_{j\in\mathcal{I}^c_{\min}}$. Thus, $\{\{x_1, \dots, x_{m-n+k}\}, \{y_j\}_{j\in\mathcal{I}^c_{\min}}\}$ is a partial SDR for $\mathcal{A}$ with size $m-n+k+(n-k) = m$. Therefore $\lambda(\mathcal{A}) = m = \alpha(\mathcal{A})$.

* Case 2.2: If $|\mathcal{I}_{\min}| = n$, i.e., there is unique minimizer $\mathcal{I}_{\min} = [n]$ then $\alpha(\mathcal{A}) = n - n + |\cup_{i\in[n]} A_i| = |\cup_{i=1}^n A_i|$ and for every $\mathcal{I} \subsetneq [n]$ we have

$$n - |\mathcal{I}| + |\cup_{i\in\mathcal{I}} A_i| > |\cup_{i=1}^n A_i|. \tag{5.2.4}$$

In particular, for $\mathcal{I} = [n-1]$ we have

$$n - (n-1) + |\cup_{i=1}^{n-1} A_i| > |\cup_{i=1}^n A_i|.$$

Because both sides are integers, it implies that $|\cup_{i=1}^{n-1} A_i| \geq |\cup_{i=1}^n A_i|$ and therefore we have

$$|\cup_{i=1}^{n-1} A_i| = |\cup_{i=1}^n A_i|. \tag{5.2.5}$$

Note that

$$\alpha(\{A_1, \dots, A_{n-1}\}) = \min_{\mathcal{I}\subseteq[n-1]} \{n - 1 - |\mathcal{I}| + |\cup_{i\in\mathcal{I}} A_i|\} \tag{5.2.6}$$

We prove that $[n-1]$ is a minimizer of (5.2.6)). In fact, if else, $[n-1]$ is not a minimizer of (5.2.6)) and $\mathcal{I}_0 \subsetneq [n-1]$ is a minimizer of (5.2.6)) (with $|\mathcal{I}_0| < n-1$) then

$$(n-1) - |\mathcal{I}_0| + |\cup_{i\in\mathcal{I}_0} A_i| < |\cup_{i=1}^{n-1} A_i|$$

which implies that

$$n - |\mathcal{I}_0| + |\cup_{i\in\mathcal{I}_0} A_i| < |\cup_{i=1}^{n-1} A_i| + 1.$$

On the other hand, for $\mathcal{I} = \mathcal{I}_0$ in (5.2.4) we have

$$|\cup_{i=1}^n A_i| < n - |\mathcal{I}_0| + |\cup_{i\in\mathcal{I}_0} A_i|.$$

Thus,

$$|\cup_{i=1}^{n-1} A_i| \overset{(5.2.5)}{=} |\cup_{i=1}^n A_i| < n - |\mathcal{I}_0| + |\cup_{i\in\mathcal{I}_0} A_i| < |\cup_{i=1}^{n-1} A_i| + 1$$

which is a contradiction to the integer property of $n - |\mathcal{I}_0| + |\cup_{i\in\mathcal{I}_0} A_i|$. Therefore $[n-1]$ is a minimizer of (5.2.6)), i.e., $\alpha(\{A_1, \dots, A_{n-1}\}) = |\cup_{i=1}^{n-1} A_i|$. From the inductive hypothesis, $\lambda(\{A_1, \dots, A_{n-1}\}) = \alpha(\{A_1, \dots, A_{n-1}\}) = |\cup_{i=1}^{n-1} A_i|$, i.e., there is a partial SDR for $\{A_1, \dots, A_{n-1}\}$ and then also for $\mathcal{A}$ of size $|\cup_{i=1}^{n-1} A_i| = |\cup_{i=1}^n A_i|$. Therefore $\lambda(\mathcal{A}) = |\cup_{i=1}^n A_i| = \alpha(\mathcal{A})$.

It completes the proof.

□

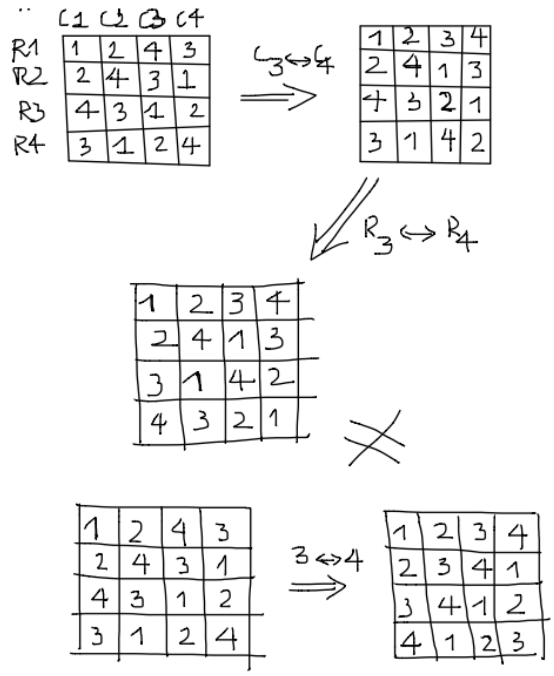
5.3. Latin squares

Definition 5.3.1. A Latin square of order n is an $n \times n$ grid filled with n symbols so that each symbol appears once in each row and column.

Example 5.3.1. A Latin square of order 4:

4	2	3	1
2	3	1	4
3	1	4	2
1	4	2	3

Definition 5.3.2. A reduced Latin square of a Latin square with n-symbols $[n]$ is one in which the first row and first column is $1, 2, \ldots, n$.



Definition 5.3.3. Two Latin squares are isotopic if each can be turned into the other by permuting the rows, columns, and symbols. This isotopy relation is an equivalence relation; the equivalence classes are the isotopy classes.

Below are the first few values for the number of all Latin squares, reduced Latin squares, and non-isotopic Latin squares (that is, the number of isotopy classes):

n	All	Reduced	Non-isotopic
1	1	1	1
2	2	1	1
3	12	1	1
4	576	4	2
5	161280	56	2

Example 5.3.2. The multiplication table of any finite group is a Latin square. The addition table for the integers modulo 6 is a Latin square.

0	1	2	3	4	5
1	2	3	4	5	0
2	3	4	5	0	1
3	4	5	0	1	2
4	5	0	1	2	3
5	0	1	2	3	4

Definition 5.3.4. Suppose $A = [a_{ij}]_{i,j=1}^n$ and $B = [b_{ij}]_{i,j=1}^n$ are two Latin squares of order n. Form the square $M = [m_{ij}]_{i,j=1}^n$ with entries $m_{ij} = (a_{ij}, b_{ij})$, we will denote by $M = A \cup B$. We say that A and B are orthogonal if M contains all n^2 ordered pairs.

Example 5.3.3. Example of orthogonal and not orthogonal:

$$\begin{array}{|c|c|c|}\hline 0 & 1 & 2 \\ \hline 1 & 2 & 0 \\ \hline 2 & 0 & 1 \\ \hline \end{array} \cup \begin{array}{|c|c|c|}\hline 0 & 1 & 2 \\ \hline 2 & 0 & 1 \\ \hline 1 & 2 & 0 \\ \hline \end{array} = \begin{array}{|c|c|c|}\hline (0,0) & (1,1) & (2,2) \\ \hline (1,2) & (2,0) & (0,1) \\ \hline (2,1) & (0,2) & (1,0) \\ \hline \end{array} \Rightarrow A \perp B$$

$$\begin{array}{|c|c|c|}\hline 0 & 1 & 2 \\ \hline 1 & 2 & 0 \\ \hline 2 & 0 & 1 \\ \hline \end{array} \cup \begin{array}{|c|c|c|}\hline 1 & 2 & 0 \\ \hline 2 & 0 & 1 \\ \hline 0 & 1 & 2 \\ \hline \end{array} = \begin{array}{|c|c|c|}\hline (0,1) & (1,2) & (2,0) \\ \hline (1,2) & (2,0) & (0,1) \\ \hline (2,0) & (0,1) & (1,2) \\ \hline \end{array} \Rightarrow A \not\perp B$$

Definition 5.3.5. Let A be a Latin square of order m with symbols $[m]$, and B one of order n with symbols $[n]$. Let $\{c_{i,j}\}_{i\in[m],j\in[n]}$ be mn new symbols. Form an $mn \times mn$ grid by replacing each entry of B with a copy of A. Then replace each entry i in this copy of A with $c_{i,j}$, where j is the entry of B that was replaced. We denote this new square $A \times B$.

More particular, if $A = [a_{ij}]_{i,j\in[m]}$ and $B = [b_{kl}]_{k,l\in[n]}$ then

$$A \times B = \begin{array}{|c|c|c|c|c|c|c|}\hline c_{a_{11},b_{11}} & \cdots & c_{a_{1m},b_{11}} & \cdots & c_{a_{11},b_{1n}} & \cdots & c_{a_{1m},b_{1n}} \\ \hline \vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots \\ \hline c_{a_{m1},b_{11}} & \cdots & c_{a_{mm},b_{11}} & \cdots & c_{a_{m1},b_{1n}} & \cdots & c_{a_{mm},b_{1n}} \\ \hline \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \hline c_{a_{11},b_{n1}} & \cdots & c_{a_{1m},b_{n1}} & \cdots & c_{a_{11},b_{nn}} & \cdots & c_{a_{1m},b_{nn}} \\ \hline \vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots \\ \hline c_{a_{m1},b_{n1}} & \cdots & c_{a_{mm},b_{n1}} & \cdots & c_{a_{m1},b_{nn}} & \cdots & c_{a_{mm},b_{nn}} \\ \hline \end{array}$$

Example 5.3.4.
$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 & 1 \\ \hline \end{array} \times \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 2 & 3 & 1 \\ \hline 3 & 1 & 2 \\ \hline \end{array} = \begin{array}{|c|c|c|c|c|c|}\hline c_{1,1} & c_{2,1} & c_{1,2} & c_{2,2} & c_{1,3} & c_{2,3} \\ \hline c_{2,1} & c_{1,1} & c_{2,2} & c_{1,2} & c_{2,3} & c_{1,3} \\ \hline c_{1,2} & c_{2,2} & c_{1,3} & c_{2,3} & c_{1,1} & c_{2,1} \\ \hline c_{2,2} & c_{1,2} & c_{2,3} & c_{1,3} & c_{2,1} & c_{1,1} \\ \hline c_{1,3} & c_{2,3} & c_{1,1} & c_{2,1} & c_{1,2} & c_{2,2} \\ \hline c_{2,3} & c_{1,3} & c_{2,1} & c_{1,1} & c_{2,2} & c_{1,2} \\ \hline \end{array}$$

Theorem 5.3.1. *If A and B are Latin squares then so is $A \times B$.*

Proof. Suppose that $A = [a_{ij}]_{i,j\in[m]}$ is a Latin square with symbols $[m]$ and $B = [b_{kl}]_{k,l\in[n]}$ is a Latin square with symbols $[n]$. Consider the r^{th} row of $A \times B$ with $1 \le r \le mn$. There is a unique representation $r = m(k-1) + i$ for $i \in [m]$ and $k \in [n]$. Then the r^{th} row of $A \times B$ is of the form

$$\begin{array}{|c|c|c|c|c|c|c|}\hline c_{a_{i1},b_{k1}} & \cdots & c_{a_{im},b_{k1}} & \cdots & c_{a_{i1},b_{kn}} & \cdots & c_{a_{im},b_{kn}} \\ \hline \end{array}$$

Note that each element in this row, a cell (r, s), is of form $c_{a_{ij},b_{kl}}$ where $s = m(j-1) + l$ is the unique representation of s for some $j \in [m]$ and $l \in [n]$. Assume that there exist different two cells (r, s) and (r, s') on this row with the same symbol, i.e., $c_{a_{ij},b_{kl}} = c_{a_{ij'},b_{kl'}}$ for the unique representations $s = m(j-1)+l$ and $s' = m(j'-1)+l'$. It implies that $a_{ij} = a_{ij'}$ and $b_{kl} = b_{kl'}$. Because A and B are Latin, $j = j'$ and $l = l'$ therefore $s = s'$ which is a contradiction. Therefore it is impossible to have two different cells in the same row with the same symbols. Similarly it is impossible to have two different cells in the same column with the same symbols. It means that $A \times B$ is Latin. It completes the proof. □

Theorem 5.3.2. *If A_1, A_2 are Latin squares of order m and B_1, B_2 are Latin squares of order n such that A_1 and A_2 are orthogonal, B_1 and B_2 are orthogonal, then $A_1 \times B_1$ and $A_2 \times B_2$ are orthogonal.*

Proof. Suppose that $A_u = [a^{(u)}_{ij}]_{i,j\in[m]}$ is a Latin square with symbols $[m]$ and $B_u = [b^{(u)}_{kl}]_{k,l\in[n]}$ is a Latin square with symbols $[n]$, where $u = 1, 2$. Because $A_1 \perp A_2$ and $B_1 \perp B_2$, $[(a^{(1)}_{ij}, a^{(2)}_{ij})]_{i,j\in[m]}$ contains m^2 ordered pairs and $[(b^{(1)}_{kl}, b^{(2)}_{kl})]_{k,l\in[n]}$ contains n^2 ordered pairs. From Theorem 5.3.1, $A_u \times B_u$ is Latin for $u = 1, 2$. Assume that $A_1 \times B_1 \not\perp A_2 \times B_2$, i.e., there are two different cells (r, s) and (r', s') of $M = A_1 \times B_1 \cup A_2 \times B_2$ with the same ordered pair of symbols, i.e.,

$$(c_{a^{(1)}_{ij}, b^{(1)}_{kl}}, c_{a^{(2)}_{ij}, b^{(2)}_{kl}}) = (c_{a^{(1)}_{i'j'}, b^{(1)}_{k'l'}}, c_{a^{(2)}_{i'j'}, b^{(2)}_{k'l'}})$$

where $r = m(i-1)+k, s = m(j-1)+l, r' = m(i'-1)+k', s' = m(j'-1)+l'$. It implies that $a^{(1)}_{ij} = a^{(1)}_{i'j'}$, $b^{(1)}_{kl} = b^{(1)}_{k'l'}$, $a^{(2)}_{ij} = a^{(2)}_{i'j'}$, and $b^{(2)}_{kl} = b^{(2)}_{k'l'}$. Therefore $(a^{(1)}_{ij}, a^{(2)}_{ij}) = (a^{(1)}_{i'j'}, a^{(2)}_{i'j'})$ and $(b^{(1)}_{kl}, b^{(2)}_{kl}) = (b^{(1)}_{k'l'}, b^{(2)}_{k'l'})$. Because $A_1 \perp A_2$ and $B_1 \perp B_2$ we must have $(i, j) = (i', j')$ and $(k, l) = (k', l')$ that implies $(r, s) = (r', s')$, a contradiction. Therefore $A_1 \times B_1 \perp A_2 \times B_2$. It completes the proof. □

Theorem 5.3.3. *1. There are pairs of orthogonal Latin squares of order n when n is odd.*

2. There are pairs of orthogonal Latin squares of order n when $n = 4k$.

3. There are pairs of orthogonal Latin squares of order n when $n = 4k + 2$ except 2 and 6.

Proof.
1. Suppose that n is odd. Construct a pair of orthogonal Latin squares of order n as follows: $A = [a_{ij}]_{i,j\in[n]}$ with $a_{ij} = i + j \mod n$ and $B = [b_{ij}]_{i,j\in[n]}$ with $b_{ij} = 2i + j \mod n$. We prove first that A and B are Latin. In fact, assume that A is not Latin and W.O.L.G. there are two different cells in the same row but with the same symbol, e.g., $a_{ij} = a_{ij'}$. It implies that $i + j = i + j' \mod n$ therefore $j' - j = 0 \mod n$ which is impossible because $j \neq j' \in [n]$. Therefore A is Latin. Similarly for B with a note that $2(j' - j) \neq 0 \mod n$ when n is odd. We then prove that $A \perp B$. In fact, assume that there are two cells $(i_1, j_1), (i_2, j_2)$ such that $(a_{i_1j_1}, b_{i_1j_1}) = (a_{i_2j_2}, b_{i_2j_2})$. It means that $(i_1+j_1, 2i_1+j_1) = (i_2+j_2, 2i_2+j_2) \mod n$ which implies that $i_1 = i_2$ and $j_1 = j_2$ therefore $(i_1, j_1) = (i_2, j_2)$.

2. We want to construct orthogonal Latin squares of order $n = 4k$. Write $4k = 2^r m$ for m odd and $r \geq 2$. Note that we can construct orthogonal Latin squares of order 4

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 4 & 1 & 2 \\ 4 & 3 & 2 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \\ 4 & 3 & 2 & 1 \\ 2 & 1 & 4 & 3 \end{bmatrix}$$

and order 8

$$A = \begin{bmatrix} 1 & 3 & 4 & 5 & 6 & 7 & 8 & 2 \\ 5 & 2 & 7 & 1 & 8 & 4 & 6 & 3 \\ 6 & 4 & 3 & 8 & 1 & 2 & 5 & 7 \\ 7 & 8 & 5 & 4 & 2 & 1 & 3 & 6 \\ 8 & 7 & 2 & 6 & 5 & 3 & 1 & 4 \\ 2 & 5 & 8 & 3 & 7 & 6 & 4 & 1 \\ 3 & 1 & 6 & 2 & 4 & 8 & 7 & 5 \\ 4 & 6 & 1 & 7 & 3 & 5 & 2 & 8 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 4 & 5 & 6 & 7 & 8 & 2 & 3 \\ 8 & 2 & 6 & 5 & 3 & 1 & 4 & 7 \\ 2 & 8 & 3 & 7 & 6 & 4 & 1 & 5 \\ 3 & 6 & 2 & 4 & 8 & 7 & 5 & 1 \\ 4 & 1 & 7 & 3 & 5 & 2 & 8 & 6 \\ 5 & 7 & 1 & 8 & 4 & 6 & 3 & 2 \\ 6 & 3 & 8 & 1 & 2 & 5 & 7 & 4 \\ 7 & 5 & 4 & 2 & 1 & 3 & 6 & 8 \end{bmatrix}$$

Therefore we can construct orthogonal Latin squares of order $2^r (r \geq 2)$ by using Theorem 5.3.2 and then we can construct orthogonal Latin squares of order $n = 2^r m$ for $(r \geq 2)$ and m odd by using Theorem 5.3.2 and the Theorem 5.3.3 1.

3. See, for example, [4].

□

Exercise 5.3.1 (E4.1: The number of SDRs)**.** *Given $n \geq 2$. How many different systems of distinct representatives are there for $\mathcal{A} = \{A_1, \ldots, A_n\}$ with*

(i) $A_1 = \{1, 2\}, A_2 = \{2, 3\}, \ldots, A_n = \{n, 1\}$?

(ii) $A_i = [n] \setminus \{i\}, \quad i \in [n]$?

Exercise 5.3.2 (E4.2: Partial SDR)**.** *Find the size of a maximum SDR for $\mathcal{A} = \{A_1, \ldots, A_6\}$ with*

$$A_1 = \{a, b, c\}, A_2 = \{a, b, c, d, e\}, A_3 = \{a, b\}, A_4 = \{b, c\}, A_5 = \{a\}, A_6 = \{a, c, e\}.$$

Exercise 5.3.3 (E4.3: Latin squares)**.** *A Latin square $A = [a_{ij}]_{i,j=1}^n$ is symmetric if $a_{ij} = a_{ji}$ for all $i, j \in [n]$. It is idempotent if every symbol appears on the main diagonal.*

(i) Show that if A is both symmetric and idempotent, then n is odd.

(ii) Find a 5×5 symmetric, idempotent Latin square.

6. Introduction to Graph theory

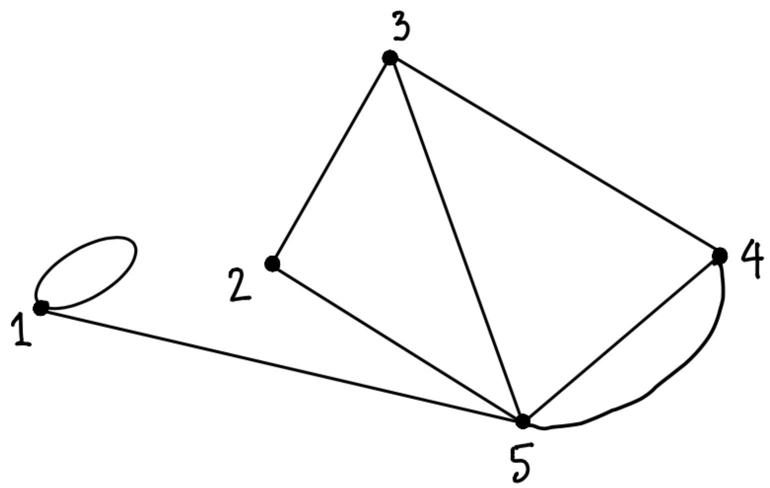
6.1. Basic concepts

Definition 6.1.1.
- A graph G consists of a pair $G = (V, E)$, where V is the set of vertices and E the set of edges. We write $V(G)$ for the vertices of G and $E(G)$ for the edges of G when necessary to avoid ambiguity.
- G has no multiple edges if no two edges have the same endpoints.
- G has no loops if no edge has a single vertex as both endpoints.

Example 6.1.1. Consider a graph $G = (V, E)$ with $V = \{1, 2, 3, 4, 5\}$ and

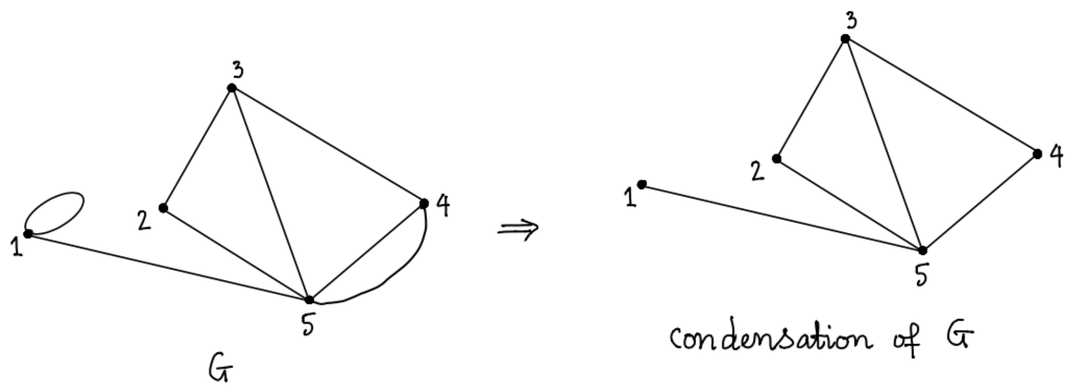
$$E = \{\{1, 1\}, \{1, 5\}, \{2, 3\}, \{2, 5\}, \{3, 4\}, \{3, 5\}, \{4, 5\}, \{4, 5\}\}$$

which has a loop $\{1, 1\}$ and a multiple edge $\{4, 5\}$.



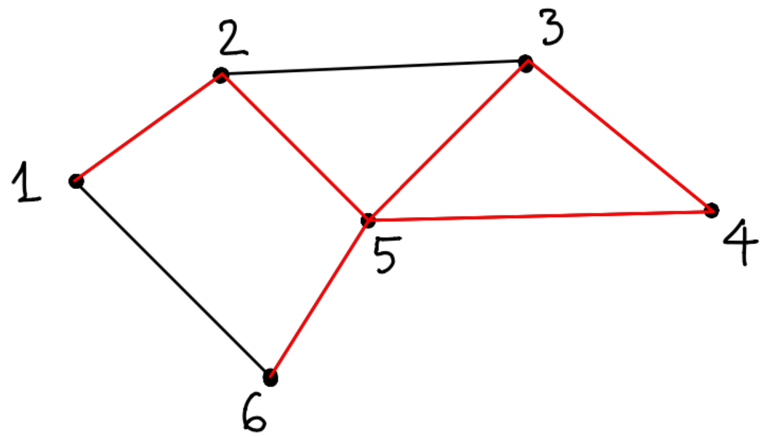
Definition 6.1.2.
- G is a simple graph if it has no loops and no multiple edges.
- G is a multigraph if it has no loops, but possibly has multiple edges.
- The condensation of a graph is the simple graph formed by eliminating multiple edges and loops.

Example 6.1.2. A condensation of G

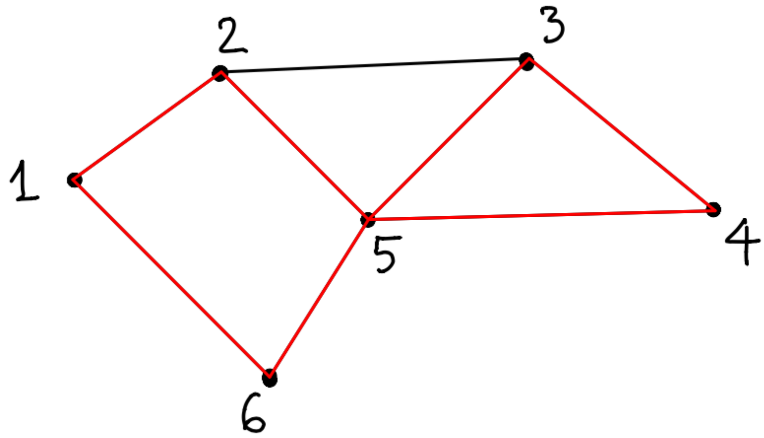


Definition 6.1.3.
- Given two vertices v, w in G. A walk in G from v to w is a sequence of vertices and edges,
$$v = v_1, e_1, v_2, e_2, \ldots, v_k, e_k, v_{k+1} = w$$
such that the endpoints of edge e_i are v_i and v_{i+1}, $\forall i = 1, \ldots, k$.
If $v = w$, the walk is a closed walk or a circuit.
- G is connected if each pair of vertices $v, w \in V$ is connected by a walk from v to w.

Example 6.1.3. Given a graph $G = (V, E)$ with $V = \{1, 2, 3, 4, 5, 6\}$ and
$$E = \{\{1, 2\}, \{1, 6\}, \{2, 3\}, \{2, 5\}, \{3, 4\}, \{3, 5\}, \{4, 5\}, \{5, 6\}\}.$$
A walk from 1 to 6:
$$W = \{1, \{1, 2\}, 2, \{2, 5\}, 5, \{5, 3\}, 3, \{3, 4\}, 4, \{4, 5\}, 5, \{5, 4\}, 4, \{4, 3\}, 3, \{3, 5\}, 5, \{5, 6\}, 6\}.$$

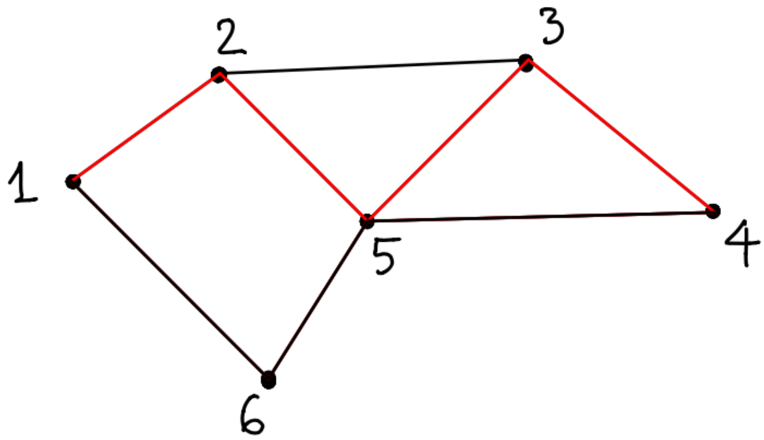


A closed walk (circuit):
$$W = \{1, \{1, 2\}, 2, \{2, 5\}, 5, \{5, 3\}, 3, \{3, 4\}, 4, \{4, 5\}, 5, \{5, 4\}, 4, \{4, 3\}, 3, \{3, 5\}, 5, \{5, 6\}, 6, \{6, 1\}, 1\}.$$

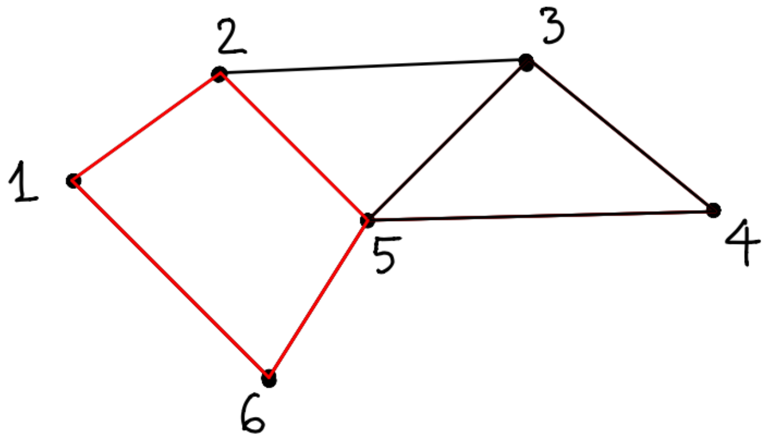


Definition 6.1.4.
- A path in G is a walk in which all edges are distinct and all vertices are distinct. Notation P_n
- A cycle in G is a path with an extra edge by joining the first and last vertices. Notation C_n

Example 6.1.4. For the graph above, the walk
$$W = \{1, \{1, 2\}, 2, \{2, 5\}, 5, \{5, 3\}, 3, \{3, 4\}, 4, \{4, 5\}, 5, \{5, 4\}, 4, \{4, 3\}, 3, \{3, 5\}, 5, \{5, 6\}, 6\}.$$
is not a path because vertices $3, 4, 5$ are repeated and edges $\{3, 4\}, \{3, 5\}, \{4, 5\}$ are repeated.
A path $P_4 = \{1, \{1, 2\}, 2, \{2, 5\}, 5, \{5, 3\}, 3, \{3, 4\}, 4\} = \{1, 2, 5, 3, 4\}$ is a path from 1 to 4 and its length is 4.



A cycle $C_4 = \{1, \{1,2\}, 2, \{2,5\}, 5, \{5,6\}, 6, \{6,1\}, 1\} = \{1,2,5,6,1\}$ has length 4.

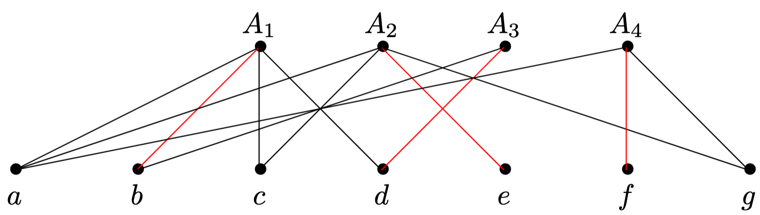


Definition 6.1.5.
- If two vertices in G are connected by an edge, we say they are adjacent.
- If a vertex v is an endpoint of edge e, we say they are incident.
- The set of vertices adjacent to v is called the neighbourhood of v, denoted $N(v)$.
- The degree of a vertex v is the number of edges incident with v, denoted $d(v)$.

Definition 6.1.6.
- A complete graph K_n is a graph with n vertices $v_1, \dots, v_n$ in which every two distinct vertices are joined by an edge.
- A graph G is bipartite if its vertices can be partitioned into two distinct parts $V = V_1 \sqcup V_2$ so that all edges has one endpoint in V_1 and the other one in V_2.

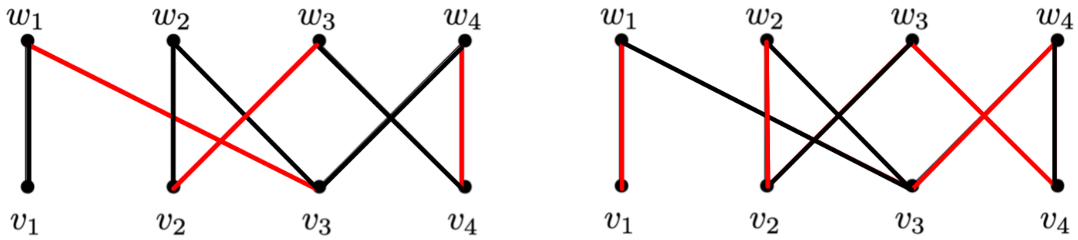
6.2. Matchings

Definition 6.2.1. A matching in a graph is a set of edges with no common endpoints.



Definition 6.2.2. A maximal matching cannot be enlarged by adding another edge. A maximum matching is one of maximum size.

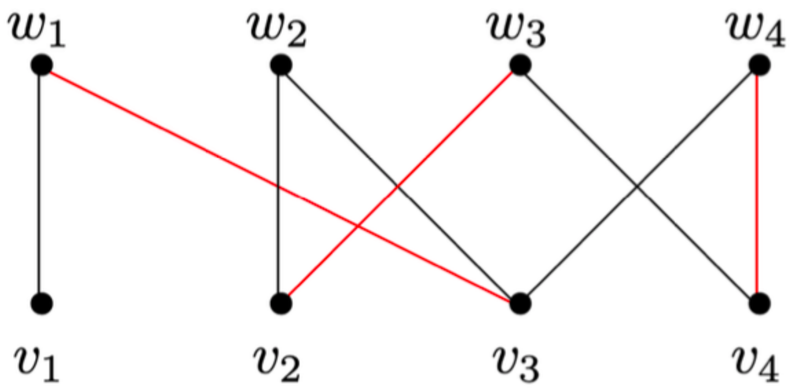
Example 6.2.1. The left panel is a maximal matching but not a maximum matching. The right panel is a maximum matching.



Definition 6.2.3. Suppose M is a matching in a graph G. Given $k \in \mathbb{N}$. An alternating chain is a sequence of vertices $v_1, w_1, v_2, w_2, \dots, v_k, w_k$ such that

- v_1, w_k are unsaturated by M, i.e., no edge in M is incident with v_1 or w_k;
- $\{v_i, w_i\} \in M^c := E - M$ for $i = \overline{1,k}$ and $\{w_i, v_{i+1}\} \in M$ for $i = \overline{1, k-1}$.

Example 6.2.2. An alternating chain $\{v_1, w_1, v_3, w_4, v_4, w_3, v_2, w_2\}$ with 4 edges outside M and 3 edges inside M.



Theorem 6.2.1. *Suppose that M is a matching in a bipartite graph G, and there is no alternating chain. Then M is a maximum matching.*

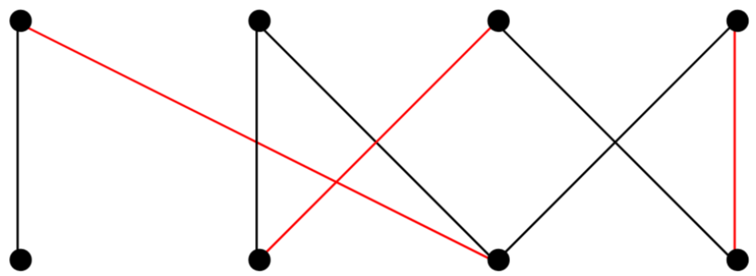
Proof. Suppose that M is a matching in a bipartite graph G, and there is no alternating chain. Assume that M is not maximum, i.e., there is a matching N with $|N| > |M|$. Construct a new graph G' with $V(G') = V(G)$ and $E(G') = (M \setminus N) \cup (N \setminus M) := E_M \cup E_N$. We first prove that $d_{G'}(v) \leq 2$ for all $v \in V$. In fact, if otherwise, there exist three edges $e_1, e_2, e_3 \in E(G')$ incident to some vertex v then from the pigeon-hole principle there are two edges in the same set, e.g., $e_1, e_2 \in E_M$ which implies that two edges $e_1, e_2 \in M$ has a common vertex, a contradiction to M is a matching. Therefore $d_{G'}(v) \leq 2$ for all $v \in V$ which implies that $E(G')$ consists of disjoint paths and cycles. Because $|N| > |M|$, $|E_N| > |E_M|$ which implies that there is a path P in $E(G')$ such that the number of edges in E_N is larger than the number of edges in E_M. We will prove that P is an alternating chain. In fact, because M and N are matchings, there is no path $\{u, v, w\}$ in $E(G')$ with two edges $\{u, v\}, \{v, w\}$ in the same set E_M or E_N. Therefore P changes alternatively between E_M and E_N and then must start and end in E_N. This implies that P is an alternating chain which is a contradiction to the fact that there is no alternating chain. Therefore M is maximum matching. It completes the proof. □

Algorithm. Suppose we have a bipartite graph $G = ((V; W), E)$ with vertex partition $V = \{v_1, v_2, \dots, v_n\}$ and $W = \{w_1, w_2, \dots, w_n\}$ and a matching M. An algorithm for finding the alternating chain:

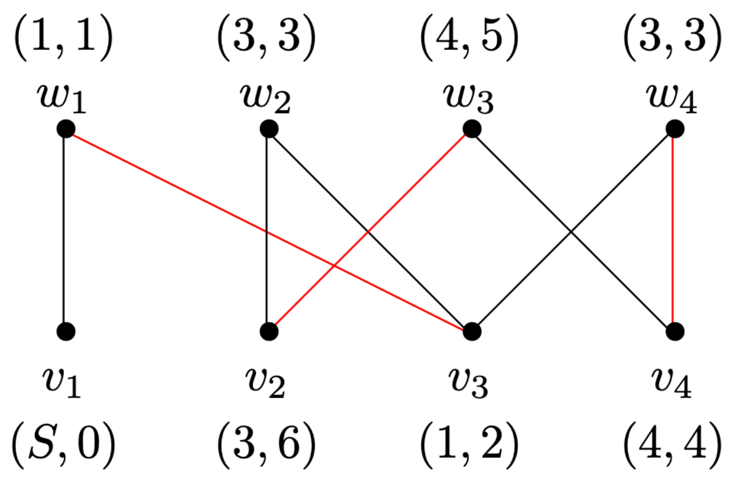
0. Step = 0. Label $(S, 0)$ all vertices in V that are unsaturated by M; Now repeat the next two steps until no vertex acquires a new label:
1. Step = Step + 1. If v_i is labelled, w_j is not labelled and $\{v_i, w_j\} \in M^c$ then we label w_j as $(i, step)$
2. Step = Step + 1. If w_i is labelled, v_j is not labelled and $\{w_i, v_j\} \in M$ then we label v_j as $(i, step)$

At the conclusion of the algorithm, if there is a labeled vertex w_i that is unsaturated by M, then there is an alternating chain, and we say the algorithm succeeds. If there is no such w_i, then there is no alternating chain, and we say the algorithm fails. $\square$

Example 6.2.3. By using the algorithm we end up with a labeled graph.

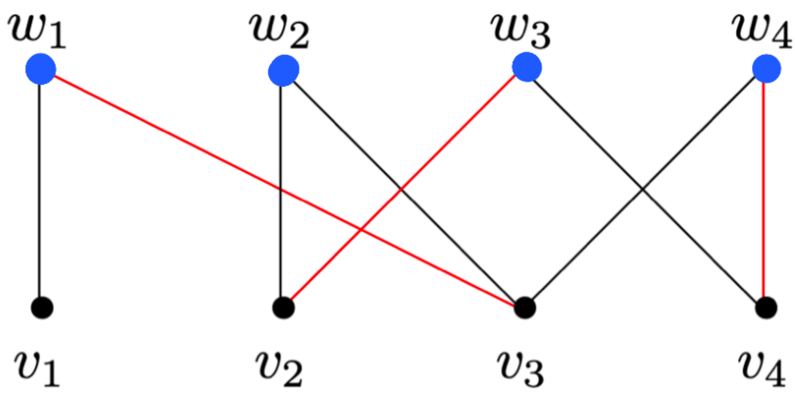


$\Downarrow$



Definition 6.2.4. A vertex cover in a graph is a set of vertices S such that every edge in the graph has at least one endpoint in S.

Example 6.2.4. A vertex cover in blue $\{w_1, w_2, w_3, w_4\}$ of the graph.



Theorem 6.2.2. *If M is a matching in a graph and S is a vertex cover, then $|M| \leq |S|$.*

Proof. Denote by $\overline{M}$ a maximum matching and $\underline{S}$ a minimum vertex cover. All we need to prove is that $|\overline{M}| \leq |\underline{S}|$. In fact, assume that $|\overline{M}| > |\underline{S}|$. From the definition of a vertex cover $\underline{S}$, for every $e \in \overline{M}$, there is a vertex $v \in \underline{S}$ so that $v \in e$. Because $|\overline{M}| > |\underline{S}|$, from the pigeon-hole principle, there are two edges $e_1, e_2 \in \overline{M}$ with the same endpoint in $\underline{S}$ which is a contradiction to $\overline{M}$ is a matching. Therefore $|\overline{M}| \leq |\underline{S}|$. It completes the proof. □

Corollary 6.2.1. *If M is a matching and S is a vertex cover in a graph G satisfying $|M| = |S|$ then M is the maximum matching and S is the minimum vertex cover.*

Theorem 6.2.3. *Suppose the algorithm fails on the bipartite graph G with matching M. Let U be the set of labeled w_i, L the set of unlabeled v_i, and $S = L \cup U$. Then S is a vertex cover and $|M| = |S|$.*

Proof. See [3], pp.86. □

Theorem 6.2.4. *In a bipartite graph G, the size of a maximum matching is the same as the size of a minimum vertex cover.*

Proof. Given a bipartite $G = (V = \{[n];[m]\}, E)$. Construct its associated collection $\mathcal{A} = \{A_1, \ldots, A_n\}$ with $A_i = \{y \in [m] : \{i, y\} \in E\}$ for each $i \in [n]$. It is easy to see that the size of a maximum matching in G is the same as the size of a maximum partial SDR for $\mathcal{A}$, i.e., $\lambda(\mathcal{A})$ and the size of a minimum vertex cover in G is nothing but

$$\alpha(\mathcal{A}) = \min\{n - |\mathcal{I}| + | \cup_{i \in \mathcal{I}} A_i|, \mathcal{I} \subseteq [n]\}.$$

Therefore the proof follows from the Theorem 5.2.1 where we have $\lambda(\mathcal{A}) = \alpha(\mathcal{A})$. □

Exercise 6.2.1 (E5.1: Maximum matching)**.** *A mathematician has five grandchildren (name Alice, Bob, Charles, Dot, Edward). He prepared six gifts to give them at Christmas (labeled 1,2,3,4,5,6). After asking their parents, he knows that:*

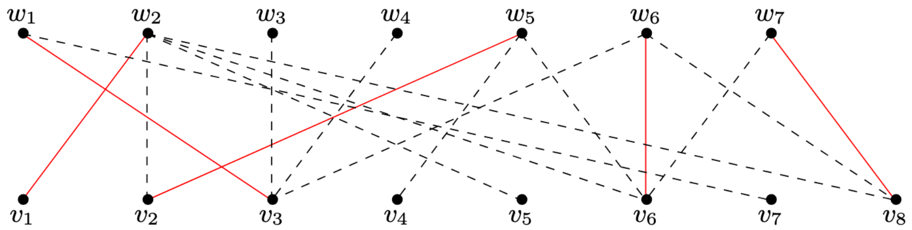
- *Alice wants: 1,3*
- *Bob wants: 2,4,5,6*
- *Charles wants: 2,3*
- *Dot wants: 1,2,3*
- *Edward wants: 2*

(i) Can he distribute one gift to each person so that everyone gets something they want?

(ii) If not, what is the maximum number of people those could get something they want?

Exercise 6.2.2 (E5.2: Maximal matching vs maximum matching)**.** *Prove that in any graph G, the size of any maximal matching is greater than or equal the half of the size of maximum matching.*

Exercise 6.2.3 (E5.3: Algorithm)**.** *Given a bipartite graph as below and a matching M shown in red.*



Use the algorithm of finding the alternative chain:

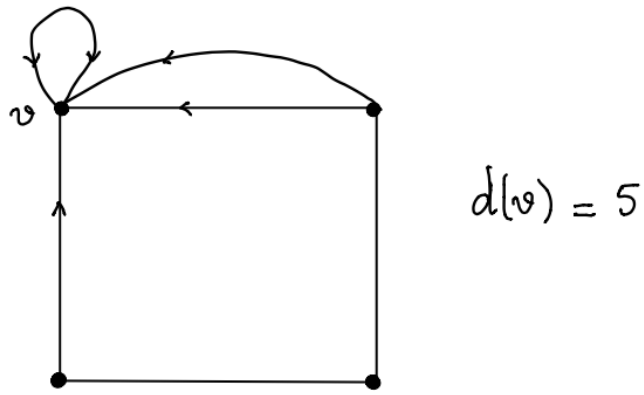
(i) What are the labels of vertices after the algorithm stops?

(ii) Find a maximum matching.

6.3. Degree sequence

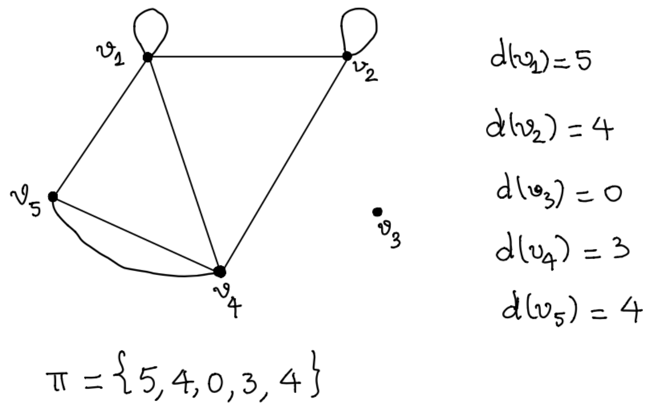
Definition 6.3.1. The degree of a vertex v is the number of edges incident with v, denoted $d(v)$ (where loops are counted twice).

Example 6.3.1. Loops are counted twice:



Definition 6.3.2. The degree sequence of a graph is a list of its degree; the order does not matter, but usually we list the degrees in increasing or decreasing order.

Example 6.3.2. Degree sequence of a graph:



Theorem 6.3.1. *In any graph $G = (V, E)$, the sum of the degree sequence is equal to twice the number of edges, that is,*

$$\sum_{v \in V} d(v) = 2|E|.$$

Proof. Note that by deleting an edge of a loop or multiple edges, the sum of degrees decreases by two. Therefore we only need to prove the theorem for a simple graph. By defining the set $S = \{(v, e) \in V \times E : v \in e\}$ and using the double counting principle we have $2|E| = \sum_{e \in E} 2 = |S| = \sum_{v \in V} d(v)$. ☐

Corollary 6.3.1. *The number of odd numbers in a degree sequence is even.*

Proof. Suppose that $\pi = \{d_1, \ldots, d_n\}$ is a degree sequence. Then $|\pi| = \sum_{i=1}^{n} d_i = 2|E|$ is even which implies the proof.

☐

Theorem 6.3.2. *A finite sequence of non-negative integers is a degree sequence of a graph (allow with loops and multiple edges) if and only if the sum of the sequence is even.*

Proof. ($\Rightarrow$:) Suppose that $\pi = \{d_1, \ldots, d_n\}$ is a degree sequence of a graph, then from the Theorem 6.3.1 $|\pi|$ is even.
($\Leftarrow$:) Suppose that $\pi = \{d_1, \ldots, d_n\}$ is a sequence of non-negative integers with $|\pi|$ is even, we construct a graph G as follows. First put $I = \{i \in [n] : d_i \text{ is odd}\}$. Because $|\pi|$ is even, $|I|$ is even, say $|I| = 2m$ and we can divide $I = I_1 \dot\cup I_2$ with $I_1 = \{i_1, \ldots, i_m\}$ and $I_2 = \{i_{m+1}, \ldots, i_{2m}\}$ where $i_1 < i_2 < \ldots < i_{2m}$. For each $j \in I^c = [n] - I$, d_j is even, we add a vertex v_j with $d_j/2$ loops. For each $i \in I$, d_i is odd, we add a vertex v_i with $\lfloor d_i/2 \rfloor$ loops. Finally, we add m edges $\{\{v_{i_1}, v_{i_{m+1}}\}, \ldots, \{v_{i_m}, v_{i_{2m}}\}\}$. ☐

Definition 6.3.3. A sequence that is the degree sequence of a simple graph is said to be graphical.

Theorem 6.3.3 (Erdös-Gallai's 1960)**.** *A finite sequence of non-negative integers $d_1 \geq d_2 \geq \cdots \geq d_n$ is graphical if and only if $\sum_{i=1}^{n} d_i$ is even and for all $k \in [n]$,*

$$\sum_{i=1}^{k} d_i \leq k(k-1) + \sum_{i=k+1}^{n} \min\{d_i, k\}. \tag{6.3.1}$$

Proof. ($\Rightarrow$:) Suppose that there is a simple graph $G = (V = \{v_1, \ldots, v_n\}, E)$ with $d(v_i) = d_i$ for all $i \in [n]$. From the Theorem 6.3.1 we have $\sum_i d_i$ is even. To this ends we prove (6.3.1) for every $k \in [n]$. In fact, fix $k \in [n]$ and set $V_1 = \{v_1, \ldots, v_k\}$ and $V_2 = \{v_{k+1}, \ldots, v_n\}$. Note that for every $i \in [k]$

$$d_i = |N(v_i)| = |N(v_i) \cap V_1| + |N(v_i) \cap V_2| \leq (k-1) + \#\{\text{edges connects } v_i \text{ and } V_2\}.$$

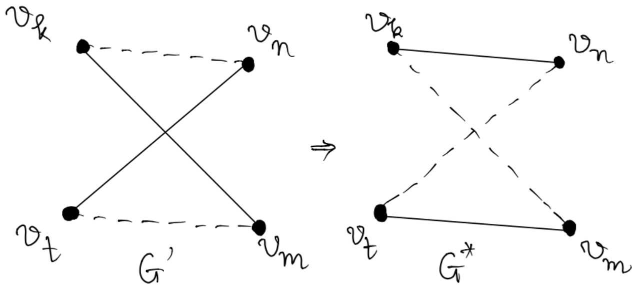
Therefore

$$\begin{aligned}
\sum_{i=1}^{k} d_i &\leq k(k-1) + \#\{\text{edges connects } V_1 \text{ and } V_2\} \\
&= k(k-1) + \sum_{i=k+1}^{n} \#\{\text{edges connects } V_1 \text{ and } v_i\} \\
&\leq k(k-1) + \sum_{i=k+1}^{n} \min\{k, d_i\}.
\end{aligned}$$

($\Leftarrow$:) Suppose that we have a finite sequence of non-negative integers $\pi = \{d_1 \geq d_2 \geq \cdots \geq d_n\}$ satisfying $|\pi| = \sum_{i=1}^{n} d_i$ is even and (6.3.1). We prove that π is graphical, i.e., there is a simple graph G with degree sequence π, by induction on $|\pi| = s$. In fact,

- Base cases: for $s = 0$, it implies that $d_i = 0$ for all $i \in [n]$. Then the graph $G = (V = \{v_1, \ldots, v_n\}, E = \emptyset)$ is what we need to construct.
  for $s = 2$, d_1 can not be 2 because if otherwise $d_i = 0$ for all $i = 2, \ldots, n$ which implies that (6.3.1) is not true for $k = 1$. Thus, $d_1 = d_2 = 1$ and $d_i = 0$ for all $i = 3, \ldots, n$. Then the graph $G = (V = \{v_1, \ldots, v_n\}, E = \{\{v_1, v_2\}\})$ is what we need to construct.

- Inductive step: Assume that it is true until $s - 2 \geq 0$, we prove that it also true for s. In fact, set
  $$t = \min\{k : d_k > d_{k+1}\} \cap (n-1).$$
  We construct another sequence $\pi' = \{d'_1, \ldots, d'_n\}$ with $d'_t = d_t - 1$, $d'_n = d_n - 1$, and $d'_i = d_i$ for all other i. Then we can check easily that π' satisfies $|\pi'| = s - 2$ is even and (6.3.1), therefore from the inductive hypothesis there is a graph $G' = (V(G') = \{v_1, \ldots, v_n\}, E(G'))$ so that $d_{G'}(v_i) = d'_i$. Now we construct G based on two cases as follows
  - Case 1: If $\{v_t, v_n\} \notin E(G')$ then we construct $G = G' \cup \{v_t, v_n\}$.
  - Case 2: $\{v_t, v_n\} \in E(G')$ then we construct G^* with the same degree sequence as G' but $\{v_t, v_n\} \notin E(G^*)$ and back to Case 1. In fact, because $d_{G'}(v_t) = d'_t = d_t - 1 \leq n-2$, it implies that there exists $v_m : \{v_m, v_t\} \notin E(G')$. Moreover $d_{G'}(v_n) = d'_n < d_n \leq d_m = d'_m = d_{G'}(v_m)$ there exists $v_k : \{v_m, v_k\} \in E(G')$ and $\{v_n, v_k\} \notin E(G')$. We then construct G^* by change two edges $\{v_k, v_m\}$ and $\{v_t, v_n\}$ into $\{v_k, v_n\}$ and $\{v_t, v_m\}$ (see figure below).

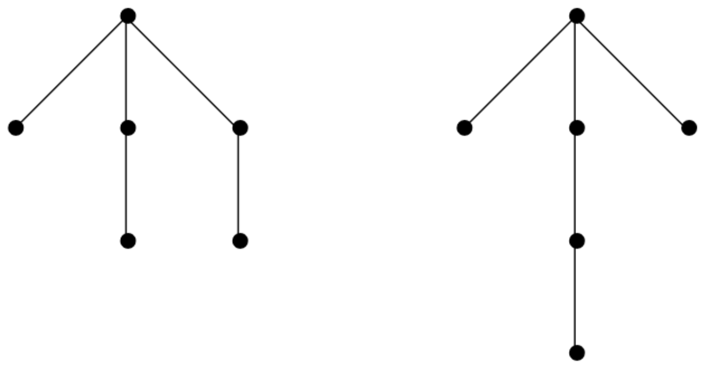


This completes the proof. □

6.4. Graph isomorphism

Definition 6.4.1. Given two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$. G_1 and G_2 are isomorphic, denoted by $G_1 \cong G_2$, if there is a bijection $f : V_1 \to V_2$ such that $\{v_1, v_2\} \in E_1$ if and only if $\{f(v_1), f(v_2)\} \in E_2$. In addition, the repetition numbers of $\{v_1, v_2\}$ and $\{f(v_1), f(v_2)\}$ are the same if multiple edges or loops are allowed. This bijection f is called a graph isomorphism.

Remark. Clearly, if two graphs are isomorphic, their degree sequences are the same. The converse is not true. For example, two following graphs have the same degree sequence $1, 1, 1, 2, 2, 3$ but not isomorphic:

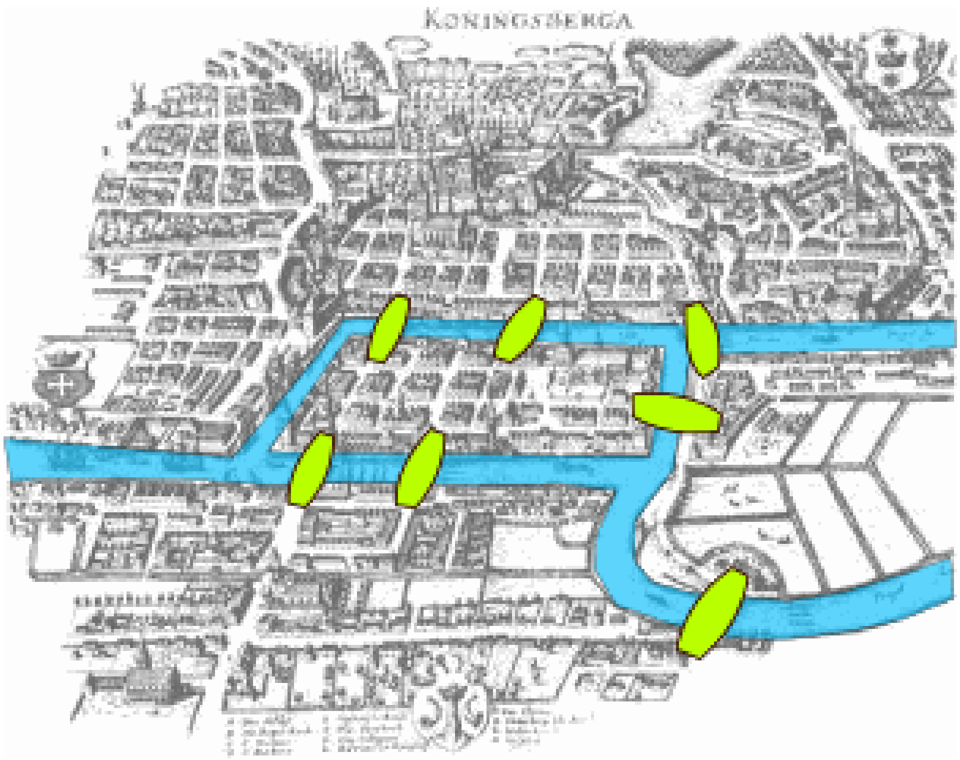


Definition 6.4.2. A graph $H = (W, F)$ is a subgraph of the graph $G = (V, E)$ if $W \subseteq V$ and $F \subseteq E$. H is an induced subgraph if F consists of all edges in E with endpoints in W. When $U \subseteq V$, we denote the induced subgraph of G on vertices U as $G[U]$.

Definition 6.4.3. If a graph G is not connected, define $v \sim_G w$ if and only if there is a walk connecting v and w. This is an equivalence relation. Each equivalence class corresponds to an induced subgraph of G, called the connected components of G.

6.5. Euler walks/circuits

Definition 6.5.1.
- Given two vertices v, w in G. A walk in G from v to w is a sequence of vertices and edges,
$$v = v_1, e_1, v_2, e_2, \ldots, v_k, e_k, v_{k+1} = w$$
such that the endpoints of edge e_i are v_i and v_{i+1}, $\forall i = 1, \ldots, k$.
If $v = w$, the walk is a closed walk or a circuit.
- A circuit in G in which every edge of G is used exactly once is called an Euler circuit.
- A walk in G in which every edge of G is used exactly once but is not an Euler circuit is called an Euler walk.



Theorem 6.5.1. *If G is a connected graph, then G contains an Euler circuit if and only if every vertex has even degree.*

Proof. ($\Rightarrow$:) Suppose that $G = (V = \{v_1, \ldots, v_n\}, E = \{e_1, \ldots, e_m\})$ is connected and contains an Euler circuit, then this Euler circuit is of the form

$$W = \{v_{i_1}, e_1, v_{i_2}, e_2, \ldots, v_{i_m}, e_m, v_{i_1}\}.$$

For each $k \in [n]$, denote by $I_k = \{j \in [m] : v_{i_j} = v_k\}$ the set of times the circuit visits v_k. Note that for each time $j \in I_k$ there are exactly two edges e_{j-1} and e_j which are incident to v_k and $\{e_{j-1}, e_j\} \neq \{e_{j'-1}, e_{j'}\}$ for all $j \neq j' \in I_k$, therefore for all $k \in [n]$, $d(v_k) = 2|I_k|$, i.e., even.
($\Leftarrow$:) Suppose that G is connected and every vertex has even degree. We prove that G contains an Euler circuit by induction on $m = |E(G)|$. In fact, if $m = 0$, the problem is trivial. Assume that it is true until $m - 1 \geq 0$, we prove that it is also true for m. In fact, we start from v and construct a longest walk W go through v and does not go through any edge twice. Clearly, W must end at v and contain all of edges incident to v. If $E(W) = E(G)$ then W is an Euler circuit. If otherwise, consider $G' = (V(G') = V, E(G') = E(G) - E(W)$ and assume that $\{G_i\}_{i\in[k]}$ are all connected components of G'. Note that G_i is connected, has even degree (when we delete $E(W)$, we reduce the degree of every vertex by an even number) and $|E(G_i)| < m$ therefore from the inductive hypothesis, there is an Euler circuit W_i for G_i. Then $\{W_1, \ldots, W_k, W\}$ will be an Euler circuit of G. ☐

Theorem 6.5.2. *A connected graph G has an Euler walk if and only if exactly two vertices have odd degree.*

Proof. ($\Rightarrow$:) Suppose that G is connected and has an Euler walk, say

$$W = \{v_{i_1}, e_1, v_{i_2}, e_2, \ldots, v_{i_m}, e_m, v_{i_{m+1}}\}$$

where $v_{i_1} \neq v_{i_{m+1}}$. Therefore the new graph $G' = G \cup \{v_{i_{m+1}}, v_{i_1}\}$ is connected and has an Euler circuit $W \cup \{v_{i_{m+1}}, v_{i_1}\}$ which implies from the Theorem 6.5.1 that all vertex of G' has even degree. This means that G has exactly two vertices v_{i_1} and $v_{i_{m+1}}$ of odd degree.
($\Leftarrow$:) Suppose that G is connected and has exactly two vertices v and w of odd degree. Then the new graph $G' = G \cup \{v, w\}$ is connected and has even degree therefore from the Theorem 6.5.1 there is an Euler circuit $W' = \{v = v_{i_1}, e_1, \ldots, v_{i_m}, e_m, w, \{w, v\}, v\}$ of G'. Thus, $W = \{v = v_{i_1}, e_1, \ldots, v_{i_m}, e_m, w\}$ is an Euler walk in G. ☐

Exercise 6.5.1 (E6.1: Degree sequence). *(i) Given non-negative integers $d_1 \geq d_2 \geq \cdots \geq d_n$ with $\sum_{i=1}^n d_i$ is even. Prove that there is a multigraph (no loops) with degree sequence $\{d_1, d_2, \ldots, d_n\}$ if and only if $d_1 \leq \sum_{i=2}^n d_i$.*

(ii) Show that the condition on the degrees in theorem Erdös-Gallai is equivalent to the condition: $\sum_{i=1}^n d_i$ is even and for all non-empty subset $I \subseteq [n]$,

$$\sum_{i\in I} d_i \leq |I|(|I| - 1) + \sum_{j\in I^c} \min\{d_j, |I|\} \qquad (\textit{where } I^c = [n] - I).$$

Exercise 6.5.2 (E6.2: Self-complementary). *The complement $\bar{G}$ of the simple graph G is a simple graph with the same vertices as G, and $\{v, w\}$ is an edge of $\bar{G}$ if and only if it is not an edge of G. A graph G is self-complementary if $G \cong \bar{G}$.*

(i) Show that if G is self-complementary then it has $4k$ or $4k + 1$ vertices for some k.

(ii) Find self-complementary graphs on 4 and 5 vertices.

Exercise 6.5.3 (E6.3: Euler circuits/walks). *(i) Which complete graphs $K_n, n \geq 2$, have Euler circuits?*

(ii) Which complete graphs $K_n, n \geq 2$, have Euler walks?

6.6. Hamilton cycles and paths

Problem:
Suppose a number of cities are connected by a network of roads. Is it possible to visit all the cities exactly once, without traveling any road twice? We assume that these roads do not intersect except at the cities. Again there are two versions of this problem, depending on whether we want to end at the same city in which we started.

In graph theory:
This problem can be represented by a graph: the vertices represent cities, the edges represent the roads. We want to know if this graph has a cycle, or path, that uses every vertex exactly once.

Remark: loops can never be used in a Hamilton cycle or path (except in the trivial case of a graph with a single vertex), and at most one of the edges between two vertices can be used. So we assume that all graphs are simple.

Definition 6.6.1. A cycle that uses every vertex in a graph exactly once is called a Hamilton cycle. A path that uses every vertex in a graph exactly once is called a Hamilton path.

Remark:
-There is no good characterization of graphs with Hamilton paths and cycles.
-If a graph has a Hamilton cycle then it also has a Hamilton path but the inverse way is not true.

Example 6.6.1.

Theorem 6.6.1 (Ore's theorem)**.** *Given $n \geq 3$. If G is a simple graph on n vertices satisfying the Ore property, i.e., $d(v)+d(w) \geq n$ whenever v and w are not adjacent, then G has a Hamilton cycle.*

Proof. Suppose that $G = (V, E)$ is a simple graph on $n \geq 3$ vertices satisfying the Ore property.
Claim 1: G is connected. In fact, if otherwise, there are two connected components $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ of G. It implies that for $v \in V_1, w \in V_2$ we have v and w are not adjacent but $d(v) + d(w) \leq |V_1| - 1 + |V_2| - 1 \leq n - 2$ which is a contradiction to the Ore property of G. Therefore G is connected.
Put $P_k = \{v_1, \ldots, v_k\}$ is a longest path in G, we prove that $k = n$. In fact, assume that $k < n$. Because G is connected, there is $w \in V - P_k$ such that w is adjacent to some vertex in P_k, say v_i for $1 \leq i \leq k$. Because P_k is longest, i can not be 1 or k, therefore $2 \leq i \leq k - 1$. We have two cases

- Case 1: if $v_1 \sim v_k$ then $P' = \{w, v_i, v_{i+1}, \ldots, v_k, v_1, v_2, \ldots, v_{i-1}\}$ is another path in G with $|P'| = k + 1 > |P_k|$ which is a contradiction to P_k of longest path in G.
- Case 2: if $v_1 \not\sim v_k$. Put $W = \{v_{j+1|v_j \in N(v_k)}\}$. Then $|W| = |N(v_k)|$. Note that $N(v_k), N(v_1) \subseteq \{v_2, \ldots, v_{k-1}\}$ therefore $W \subseteq \{v_3, \ldots, v_k\}$ and $N(v_1) \cup W \subseteq \{v_2, \ldots, v_k\}$. Thus $|N(v_1) \cup W| \leq k - 1 < n$. On the other hand, we have from the Ore property of G that $|N(v_1) \cup W| = d(v_1) + d(v_k) \geq n$. Therefore, there is $v_j \in N(v_1) \cap W$ for some $3 \leq j \leq k - 1$ which implies that $v_{j-1} \in N(v_k)$. Therefore we have a cycle $C = \{v_1, v_j, v_{j+1}, \ldots, v_k, v_{j-1}, v_{j-2}, \ldots, v_2, v_1\}$ with length k. By adding C with the edge $\{w, v_i\}$ and rearranging the vertices we can find a path with length $k + 1$ which is a contradiction to P_k is a longest path.

Therefore $k = n$ and our path $P_n = \{v_1, \ldots, v_n\}$ is a longest path. We consider the same two cases as above. If $v_1 \sim v_n$ then $C_n = P_n \cup \{v_n, v_1\}$ is a Hamilton cycle. If $v_1 \not\sim v_n$ then we can rearrange vertices so that we have a cycle with length n which is also a Hamilton cycle. $\square$

Theorem 6.6.2. *Given $n \geq 3$. If G is a simple graph on n vertices satisfying $d(v)+d(w) \geq n-1$ whenever v and w are not adjacent, then G has a Hamilton path.*

Proof. Follow the above proof until we have a path P_n which is nothing but a Hamilton path. □

Corollary 6.6.1. *Given $n \geq 3$. If G is a graph on n vertices so that its condensation graph satisfies $d(v) + d(w) \geq n - 1$ whenever v and w are not adjacent, then G has a Hamilton path.*

Proof. Note that if the condensation graph of G satisfies $d(v) + d(w) \geq n - 1$ whenever v and w are not adjacent then it has a Hamilton path which is also a Hamilton path in G. □

6.7. Bipartite graphs

Definition 6.7.1. The distance between vertices v and w is the length of a shortest walk between them, denoted by $d(v, w)$. If there is no walk between v and w, the distance is undefined.

Theorem 6.7.1. *A graph G is bipartite if and only if all closed walks in G are of even length.*

Proof. ($\Rightarrow$:) Suppose that $G = ((X; Y), E)$ is bipartite and $W = \{v_1, e_1, \ldots, v_m, e_m, v_1\}$ is a closed walk in G. W.L.O.G. assume that $v_1 \in X$ then $v_2 \in Y$ which implies $v_3 \in X$ and so on. If m is odd then $v_m \in X$ and is a contradiction to $\{v_m, v_1\} \in E$. Therefore $|W| = m$ is even.
($\Leftarrow$:) Note that if there is a closed walk of odd length then there is a cycle of odd length. It can be easily done by induction on the size of closed walk. □

Corollary 6.7.1. *A graph G is bipartite if and only if all cycles in G are of even length.*

Proof. ($\Rightarrow$:) It follows directly from Theorem 6.7.1.
($\Leftarrow$:) Suppose that all closed walks in $G = (V, E)$ are of even length, we prove that G is bipartite. In fact, W.L.O.G, G is connected. Pick $v \in V$ and define $X = \{w \in V : d(v, w) \text{ is even}\}$ and $Y = \{w \in V : d(v, w) \text{ is odd}\}$. Assume that there are $u, w \in X$ with $u \sim w$. Denote by $P_{v,u}$ the shortest path from v to u and $P_{w,v}$ the shortest path from w to v. Then $|P_{v,u}| = d(v, u)$ and $|P_{w,v}| = d(v, w)$ are even because $u, w \in X$. Therefore the closed walk $W = \{P_{v,u}, \{u, w\}, P_{w,v}\}$ of odd length which is a contradiction. Thus, there is no edge in X and similarly in Y which implies that G is biparttite.

□

Definition 6.7.2. A complete bipartite graph $G = (V, W, E) = K_{m,n}$ is a bipartite graph with $|V| = m, |W| = n$, and every vertex of V is adjacent to every vertex of W.

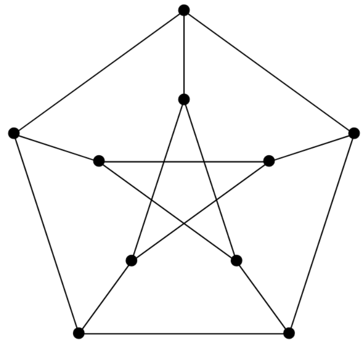
Theorem 6.7.2. *A complete bipartite graph $K_{m,n}$ has a Hamilton cycle if and only if $m = n$.*

Proof. ($\Rightarrow$:) Suppose that a complete bipartite graph $K_{m,n} = ((X; Y), E)$ has a Hamilton cycle. Because any cycle alternates between vertices of the two parts X and Y, $m = |X| = |Y| = n$.
($\Leftarrow$:) Consider a complete bipartite graph $K_{n/2,n/2} = ((X; Y), E)$. Note that for every $v, w \in K_{n/2,n/2}$ we have $d(v) + d(w) = n/2 + n/2 = n$ therefore from Ore's theorem, $K_{n/2,n/2}$ has a Hamilton cycle. □

Exercise 6.7.1 (E7.1: Hamilton cycles). *(i) Suppose G is a simple graph on $n \geq 2$ vertices with $|E(G)| \geq \frac{(n-1)(n-2)}{2} + 2$. Prove that G has a Hamilton cycle.*

(ii) For $n \geq 2$, show that there is a simple graph G with $|E(G)| = \frac{(n-1)(n-2)}{2} + 1$ that has no Hamilton cycle.

(iii) The Petersen graph is a graph with 10 vertices and 15 edges as follows:



Does it have a Hamilton cycle?

Exercise 6.7.2 (E7.2: Bipartite multigraphs)**.** *Prove that there is a bipartite multigraph with degree sequence $\{d_1, \ldots, d_n\}$ if and only if there is a partition $[n] = I \dot\cup J$ such that*

$$\sum_{i \in I} d_i = \sum_{i \in J} d_i.$$

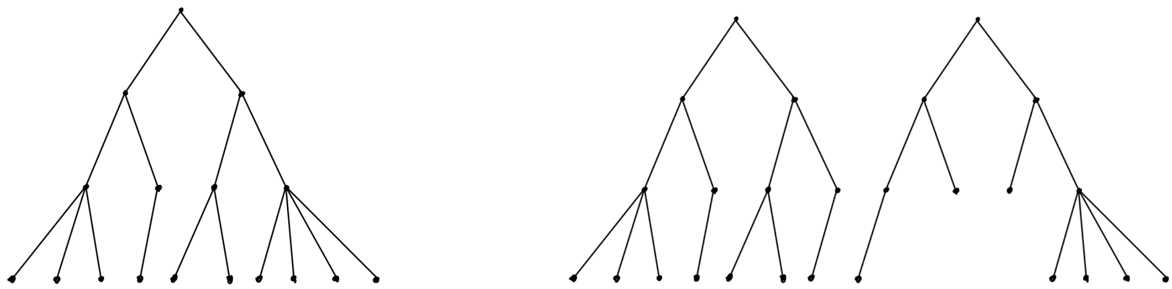
Exercise 6.7.3 (E7.3: Regular bipartite graph)**.** *(i) A regular graph is one in which the degree of every vertex is the same. Show that if G is a regular bipartite graph, and the common degree of the vertices is at least 1, then the two parts are the same size.*

(ii) A perfect matching is one in which all vertices of the graph are incident with exactly one edge in the matching. Show that a regular bipartite graph with common degree at least 1 has a perfect matching.

6.8. Trees

Definition 6.8.1. A connected graph G is a tree if it is acyclic, that is, it has no cycles. An acyclic graph is called a forest.

Example 6.8.1. The following left panel is a tree and the right panel is a forest:

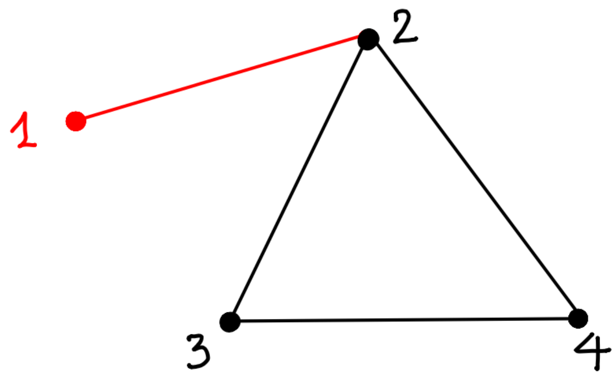


Theorem 6.8.1. *Every tree T is bipartite.*

Proof. We use the Corollary 6.7.1, a graph G is bipartite if and only if all cycle has even length. Because there is no cycle in a tree T, T is a bipartite. ☐

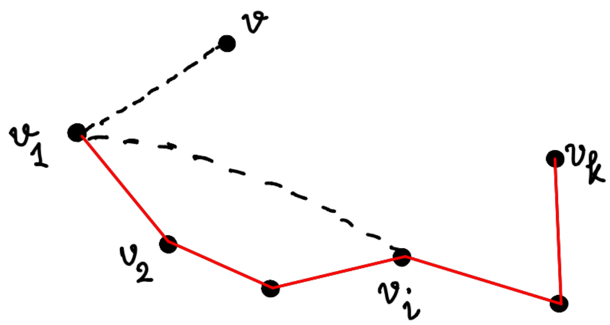
Definition 6.8.2. A vertex of degree one is called a pendant vertex, and the edge incident to it is a pendant edge.

Example 6.8.2. In this following graph, the vertex 1 is the unique pendant vertex and the edge $\{1, 2\}$ is the unique pendant edge.



Theorem 6.8.2. *Every tree on $n \geq 2$ vertices has at least one pendant vertex.*

Proof. Let $P_k = \{v_1, \dots, v_k\}$ be a longest path in T. For any vertex $v \in V(T) - P_k$, v should not be adjacent to v_1 because if otherwise we have $P_{k+1} = \{v, v_1, \dots, v_k\}$ is a path which is longer than P_k. On the other hand, there is no vertex $v_i, i \in [k]$ which is adjacent to v_1 because if otherwise there is a cycle in T. Therefore $d(v_1) = 1$ or v_1 is a pendant vertex of T.

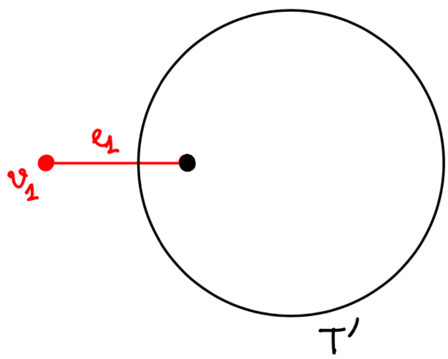


□

Theorem 6.8.3. *A tree on n vertices has exactly $n - 1$ edges.*

Proof. We proof this by induction on n.

- Base case: $n = 1$ we have $T_1 = (V = \{v_1\}, E = \emptyset)$ and it is true.
- Inductive step: Assume that this is true for $n-1$, i.e., T_{n-1} has exactly $n-2$ edge. Consider a tree on n vertices T_n. From the Theorem 6.8.2, there is a pendant vertex v_1 of T_n. Now we note that $T' = T - \{v_1\}$ is a tree on $n - 1$ vertices therefore by the inductive hypothesis T' has exactly $n - 2$ edges. Therefore $|E(T_n)| = |E(T')| + 1 = n - 1$.



It completes the proof. □

Theorem 6.8.4. *A tree with a vertex of degree $k \geq 1$ has at least k pendant vertices. In particular, every tree on at least two vertices has at least two pendant vertices.*

Proof. Suppose T is a tree on n vertices $\{v_1, \ldots, v_n\}$ and $d(v_j) = k$ for some $j \in [n]$. Set $I = \{i \in [n] : d(v_i) = 1\}$ is the set of all pendant vertices in T.

- Case 1: If $k = 1$ then there is at least one pendant vertex due to Theorem 6.8.2.
- Cases 2: If $k > 1$ then from Theorem 6.8.3

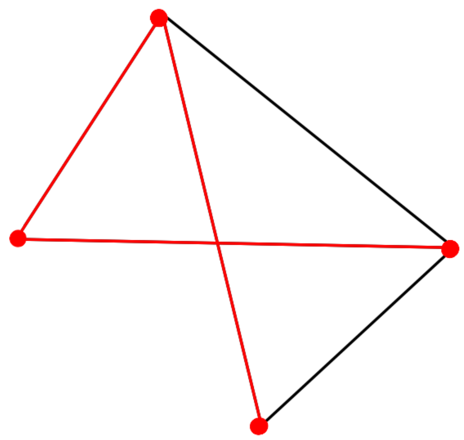
$$\begin{aligned} 2(n-1) = 2|E(T)| = \sum_{i=1}^{n} d(v_i) &= k + \sum_{i \in I} d(v_i) + \sum_{i \notin I \cup \{j\}} d(v_i) \\ &\geq k + |I| + 2(n - |I| - 1). \end{aligned}$$

  It implies that $|I| \geq k$, i.e., T has at least k pendant vertices.

For $n \geq 2$. We consider 2 cases: either $T_2 = (V = \{1, 2\}, E = \{\{1, 2\}\})$ or T_n for $n \geq 3$. The first case has two pendant vertices and the second case there is a vertex v with degree at least two. Therefore, in both cases, there are at least two pendant vertices. □

Definition 6.8.3. If G is a connected graph on n vertices, a spanning tree for G is a subgraph of G that is a tree on n vertices.

Example 6.8.3. The red subgraph is a spanning tree of G.



Theorem 6.8.5. *Every connected graph has a spanning tree.*

Proof. We prove this claim by induction on $m = |E(G)|$. In fact,

- Base case: if $m = 0$, because G is connected then $n = |V(E)| = 1$ and G is indeed a tree T_1.
- Assume that the claim is true until $m - 1 \geq 0$, we prove that it is also true for m. In fact, if G has no cycle then by definition G is a tree. If G has a cycle C, delete one edge e of C we have $G' = G - e$ is still connected with $|G'| = m - 1$. From the inductive hypothesis, there is a spanning tree T of G' which is also a spanning tree of G.

This completes the proof. □

Remark. In general, spanning trees are not unique, that is, a graph may have many spanning trees. Some edges could be in every spanning tree whenever there are multiple spanning trees. For example, any pendant edge must be in every spanning tree, as must any edge whose removal disconnects the graph (such an edge is called a bridge.)

Theorem 6.8.6. *If G is connected, it has at least $n-1$ edges; moreover, it has exactly $n-1$ edges if and only if it is a tree.*

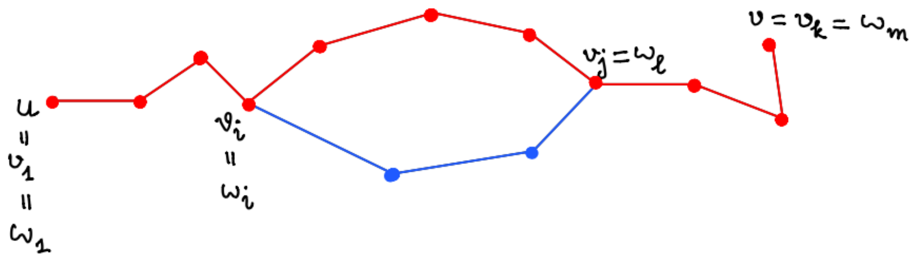
Proof. Suppose that G is a connected graph on n vertices. From Theorem 6.8.5 there is a spanning tree T and $|E(T)| = n - 1$ due to Theorem 6.8.3. Therefore $|E(G)| \geq |E(T)| = n - 1$. Moreover

- if G is a tree then $|E(G)| = n - 1$ due to Theorem 6.8.3 and
- if $|E(G)| = n - 1$ then $G = T$, i.e. G is a tree.

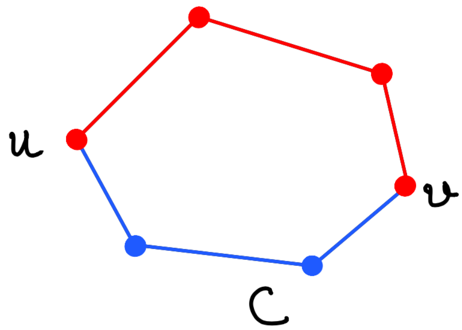
□

Theorem 6.8.7. *G is a tree if and only if there is a unique path between any two vertices.*

Proof. ($\Rightarrow$:) Suppose that G is a tree. Then G is connected, i.e. for every pair $u, v \in G$ there is a path $P = \{v_1, \dots, v_k\}$ from u to v in G. Assume that there is another path $P' = \{w_1, \dots, w_m\}$ from u to v in G. Consider the path P and set the index i is the first time two paths separate (it happens because P and P' are different paths and start at the same vertex $v_1 = w_1 = u$) and the index j is the first time they meet back (it happens because P and P' return at the same vertex $v_k = w_m = v$). Clearly, $P_{[v_i,v_j]} \cup P'_{[v_i,v_j]}$ is a cycle in G which is a contradiction to G is a tree. Therefore there is unique path from u to v.



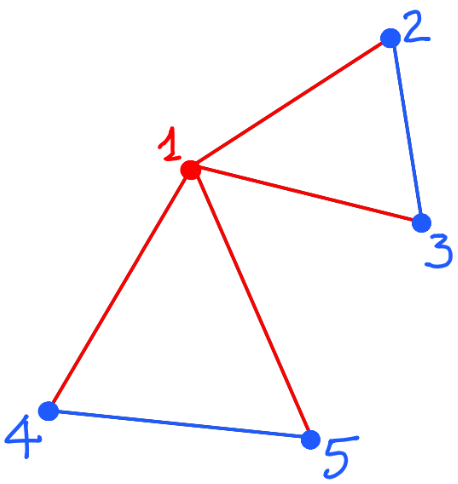
($\Leftarrow$:) Suppose that for every pair $u, v \in G$ there is a unique path from u to v. The existence of a path between every pair implies that G is connected. If G is not a tree, it means there is a cycle C in G. Choose two distinct points $u, v \in C$ we see that there are two paths from u to v in G which is a contradiction. Therefore G is a tree.



□

Definition 6.8.4. A cutpoint in a connected graph G is a vertex whose removal disconnects the graph.

Example 6.8.4. This graph has the unique cutpoint 1:



Theorem 6.8.8. *Every connected graph has a vertex that is not a cutpoint.*

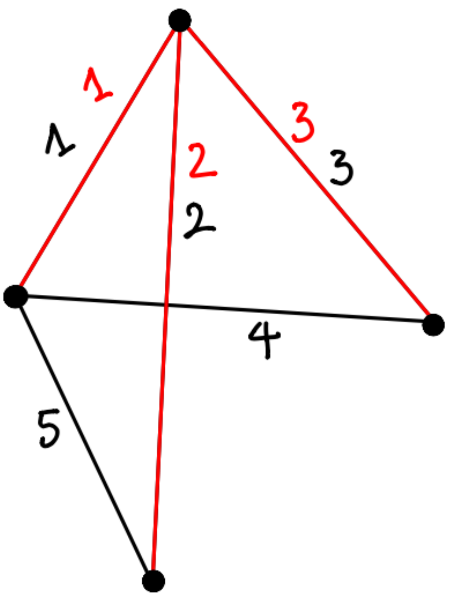
Proof. Suppose G is a connected graph. From Theorem 6.8.5, G has a spanning tree T and from Theorem 6.8.2 there is a pendant vertex v in T which is not a cutpoint of G. ☐

6.9. Optimal Spanning Trees

In some applications, a graph G is augmented by associating a weight or cost with each edge; such a graph is called a weighted graph. For example, if a graph represents a network of roads, the weight of an edge might be the length of the road between its two endpoints, or the amount of time required to travel from one endpoint to the other, or the cost to bury cable along the road from one end to the other. In such cases, instead of being interested in just any spanning tree, we may be interested in a least cost spanning tree, that is, a spanning tree such that the sum of the costs of the edges of the tree is as small as possible. For example, this would be the least expensive way to connect a set of towns by a communication network, burying the cable in such a way as to minimize the total cost of laying the cable.

Definition 6.9.1. A weighted graph is a graph G together with a cost function $c : E(G) \to \mathbb{R}_{>0}$. If H is a subgraph of G, the cost of H is $c(H) = \sum_{e \in E(H)} c(e)$.

Example 6.9.1. The red subgraph H with cost $c(H) = 1 + 2 + 3 = 6$.



The Jarník Algorithm:

Given a weighted connected graph G, we construct a minimum cost spanning tree T as follows:

- Choose any vertex $v_0 \in G$ and include it in T.
- If vertices $S = \{v_0, v_1, \ldots, v_k\}$ have been chosen, choose an edge with one endpoint in S and one endpoint not in S and with smallest weight among all such edges. Let v_{k+1} be the endpoint of this edge not in S, and add it and the associated edge to T.
- Continue until all vertices of G are in T.

Theorem 6.9.1. *The Jarník Algorithm produces a minimum cost spanning tree.*

Proof. Let $T = \{\{v_0, \ldots, v_{n-1}\}, \{e_1, \ldots, e_{n-1}\}\}$ be a spanning tree after the algorithm. Assume that $T_{\min}$ is an optimal spanning tree. We would like to prove that $c(T_{\min}) = c(T)$. In fact, we construct a sequence of spanning trees $T_0 = T_{\min}, T_1, \ldots, T_{n-1} = T$ as follows: if $e_1 \in T_{\min} = T_0$, set $T_1 = T_0$; if $e_1 \notin T_0$ then $T_0 \cup \{e_1\}$ has a cycle C which contains e_1 therefore there exists $f_1 \in C$ such that $f_1 \sim v_0$ and $f_1 \neq e_1$. The algorithm implies that $c(e_1) \leq c(f_1)$. We then construct $T_1 = T_0 \cup \{e_1\} - \{f_1\}$ which is a spanning tree because does not contain a cycle and connected. If $c(e_1) < c(f_1)$ then $c(T_1) < c(T_0)$, a contradiction to T_0 is an optimal spanning tree. Therefore $c(e_1) = c(f_1)$ which implies that $c(T_1) = c(T_0)$. Assume that we constructed $T_1, T_2, \ldots, T_i$ with $c(T_0) = c(T_1) = \ldots = c(T_i)$ for some $1 \leq i \leq n-2$, let construct T_{i+1} as follows: if $e_{i+1} \in T_i$ we construct $T_{i+1} = T_i$; if $e_{i+1} \notin T_i$, then $T_i \cup \{e_{i+1}\}$ has a cycle C which contains e_{i+1} therefore there exists $f_{i+1} \in C$ such that $f_{i+1} \sim v_i$ and $f_{i+1} \neq e_{i+1}$. The algorithm implies that $c(e_{i+1}) \leq c(f_{i+1})$. We then construct $T_{i+1} = T_i \cup \{e_{i+1}\} - \{f_{i+1}\}$ which is a spanning tree because does not contain a cycle and connected. If $c(e_{i+1}) < c(f_{i+1})$ then $c(T_{i+1}) < c(T_i)$, a contradiction to T_i is optimal spanning tree (due to $c(T_i) = c(T_0)$). Therefore $c(e_{i+1}) = c(f_{i+1})$ which implies that $c(T_{i+1}) = c(T_i)$. It implies that we can continue until $T_{n-1} = T$ with $c(T_{n-1}) = c(T_{\min})$. It implies the proof.

□

Exercise 6.9.1 (E8.1: Bridges and pendants). *(i) Suppose that G is a connected graph, and that every spanning tree contains edge e. Show that e is a bridge.*

(ii) Show that every edge in a tree is a bridge.

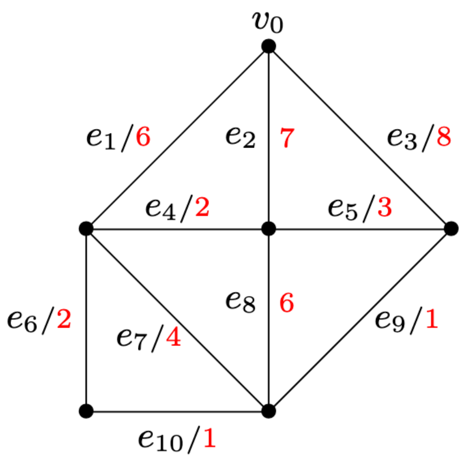
(iii) Suppose T is a tree on n vertices, k of which have degree larger than 1, e.g., $d_1, d_2, \ldots, d_k$. How many pendant vertices does T have?

Exercise 6.9.2 (E8.2: Degree sequence of trees). *(i) Let $n \geq 2$. Show that there is a tree with degree sequence $\{d_1, d_2, \ldots, d_n\}$ if and only if $d_i > 0$, $\forall i \in [n]$ and $\sum_{i=1}^{n} d_i = 2(n-1)$.*

(ii) A multitree is a multigraph whose condensation is a tree. Let $n \geq 2$. Let $\{d_1, d_2, \ldots, d_n\}$ be positive integers, and let g be the greatest common divisor of the d_i. Show that there is a multitree with degree sequence $\{d_1, d_2, \ldots, d_n\}$ if and only if $\sum_{i=1}^{n} d_i \geq 2g(n-1)$ and for every partition $[n] = I \dot\cup J$, $\sum_{i \in I} d_i = \sum_{i \in J} d_i$.

Exercise 6.9.3 (E8.3: Greedy algorithm). *(i) Kruskal's Algorithm is also a greedy algorithm that produces a minimum cost spanning tree for a connected graph G. Begin by choosing an edge in G of smallest cost. Assuming that edges $e_1, e_2, \ldots, e_i$ have been chosen, pick an edgee $i+1$ that does not form a cycle together with $e_1, e_2, \ldots, e_i$ and that has smallest cost among all such edges. The edges $e_1, e_2, \ldots, e_{n-1}$ form a spanning tree for G. Prove that this spanning tree has minimum cost.*

(ii) *In both the Jarník (in the lecture) and Kruskal algorithms, it may be that two or more edges can be added at any particular step, and some method is required to choose one over the other. For the graph below, use both algorithms to find a minimum cost spanning tree. Using the labels e_i on the graph, at each stage pick the edge e_i that the algorithm specifies and that has the lowest possible i among all edges available. For the Jarník algorithm, use the designated v_0 as the starting vertex. For each algorithm, list the edges in the order in which they are added. The edge weights $e_1, e_2, \ldots, e_{10}$ are 6,7,8,2,3,2,4,6,1,1, shown in red.*



6.10. Connectivity

Definition 6.10.1. Given a connected graph G, a vertex $v \in G$ is called a cutpoint if its removal disconnects G. If G has a cutpoint then we say G has connectivity 1.

Definition 6.10.2. Given a connected graph G, any set of vertices whose removal disconnects the graph is called a cutset. G has connectivity k if there is a cutset of size k but no smaller cutset.
The connectivity of G is denoted $\kappa(G)$. G is k-connected if $\kappa(G) \geq k$.

Remark. If G has at least two vertices and has no cutset, we say $\kappa(G) = n - 1$; If G has one vertex, its connectivity is undefined; If G is not connected, we say $\kappa(G) = 0$.

Definition 6.10.3. If a graph G is connected, any set of edges whose removal disconnects the graph is called a cut. G has edge connectivity k if there is a cut of size k but no smaller cut; the edge connectivity of a one-vertex graph is undefined. G is k-edge-connected if the edge connectivity of G is at least k. The edge connectivity is denoted $\lambda(G)$.

Theorem 6.10.1. $\kappa(G) \leq \lambda(G)$.

Proof. We prove this claim by induction on $\lambda(G) = k$. In fact,

- Base case: for $k = 0$: G is disconnected and therefore $\kappa(G) = 0$.
  for $k = 1$: if $G = K_2$ then $\kappa(G) = 1$. If $|V(G)| \geq 3$ then there exists $e = \{v, w\} \in E(G)$ so that $G - \{e\}$ is disconnected, i.e., there exist $v \in U \subseteq V(G)$ and $w \in W \subseteq V(G)$ with $G[U] \cap G[W] = \emptyset$. Because $|V(G)| \geq 3$ there is, e.g., $u \neq v \in U$ therefore $G - \{v\}$ is disconnected which implies that $\kappa(G) = 1$.

- Inductive step: Assume that the claim is true until $k - 1 \geq 1$, we prove that it is also true for k. In fact, there are $\{e_1, \ldots, e_k\} \subseteq E(G)$ so that $G - \{e_1, \ldots, e_k\}$ is disconnected. Therefore $G_1 = G - \{e_k\}$ has $\lambda(G_1) = k - 1$. From the inductive hypothesis, there are $\{v_1, \ldots, v_l\}(l \leq k - 1) \subseteq V(G_1) = V(G)$ such that $G_2 = G_1 - \{v_1, \ldots, v_l\}$ is disconnected. There are three cases:

- Case 1: if there are $v, w \in G_2$ with $G_3 = G_2 \cup \{v, w\}$ is connected then $G_4 = G - \{v_1, \ldots, v_l, v\}$ is disconnected which implies that $\kappa(G) \leq k$.
- Case 2: if there are $v, w \in G_2$ with $G_3 = G_2 \cup \{v, w\}$ is disconnected then $G - \{v_1, \ldots, v_l\}$ is disconnected which implies that $\kappa(G) < k$.
- Case 3: if there is $v \notin G_2$ so that $G - \{v_1, \ldots, v_l\}$ is disconnected then $\kappa(G) < k$.

It implies that $\kappa(G) \leq k$.

This completes the proof. □

Theorem 6.10.2. *If G has at least three vertices, the following are equivalent:*

1. *G is 2-connected*
2. *G is connected and has no cutpoint*
3. *For all distinct vertices u, v, w in G there is a path from u to v that does not contain w.*

Proof. (1. $\Rightarrow$ 3.) : Given three distinct vertices u, v, w. Because of 1., $G - \{w\}$ is connected. Therefore there is a path from u to v in $G - \{w\}$. This path is a path from u to v and does not contains w.
(3. $\Rightarrow$ 2.) : from 3., for each two distinct vertices u, v there is a path from u to v therefore G is connected. Assume that there is a cutpoint $w \in G$. Then $G' = G - \{w\}$ is disconnected, i.e., there are two components U, W with $u \in U$, $v \in W$. Therefore there is no path from u to v in G' which means that every path in G from u to v must go through w which is a contradiction to 3.
(2. $\Rightarrow$ 1.) : from 2., G is connected and has no cutpoint therefore $\kappa(G) \geq 2$ and it is 2-connected. □

Theorem 6.10.3. *If G has at least three vertices, then G is 2-connected if and only if every two vertices u and v are contained in a cycle.*

Proof. ($\Rightarrow$:) Suppose that G has at least three vertices and is 2-connected, i.e., $\kappa(G) \geq 2$. Consider two arbitrary vertices $u, v \in G$, we need to prove that there is a cycle C contains u and v. In fact, set $U = \{w \neq u : \text{ there exists a cycle } C \text{ contains } u \text{ and } v\}$, we prove that $v \in U$. Assume that $v \notin U$.
Claim 1: $U \neq \emptyset$.
Because G is connected there exists $w \sim u$. From Theorem 6.10.1 $\lambda(G) \geq \kappa(G) \geq 2$ therefore $G - \{u, w\}$ is connected. It implies that there is a path P from u to w in $G - \{u, w\}$. Therefore $P \cup \{u, w\}$ is a cycle which contains u and w. It implies that $w \in U$.

Let $w_0 \in U$ such that $d(v, w_0) = \min\{d(v, w), w \in U\}$ and let P_{v,w_0} be a shortest path from v to w_0 and C_0 be a cycle contains u and w_0. Note that $C_0 \cap P_{v,w_0} = \emptyset$ because if otherwise, $C_0 \cap P_{v,w_0} = \{w\}$ then $w \in U$ and $d(v, w) < d(v, w_0)$ which is a contradiction to the way to choose w_0. On the other hand, from the Theorem 6.10.2 3., there is a path Q from u to v does not contain w_0. If $Q \cap P = \emptyset$ then v is in a cycle (some part of C_0, some part of Q and P_{v,w_0}) which contains u and therefore $v \in U$, contradiction to the assumption $v \notin U$. If $Q \cap P = \{w\}$ then $w \in U$ because there is a cycle contains w and u (some part of C_0, some part of Q and some part of P_{v,w_0}). Moreover $d(v, w) < d(v, w_0)$ which is a contradiction to the way to choose w_0. Thus, $v \in U$.
($\Leftarrow$:) Suppose that G has at least three vertices and every two vertices u and v are contained in a cycle. For every three distinct vertices u, v, w in G there is a cycle contains u and v which is divided into two paths P_1 and P_2 from u to v. Therefore there is at least one path does not contains w. This implies that G is 2-connected from the Theorem 6.10.2. □

Corollary 6.10.1. *If G has at least three vertices, then G is 2-connected if and only if between every two vertices u and v there are two internally disjoint paths, that is, paths that share only the vertices u and v.*

Proof. It is because the existence of a cycle contains u and v is equivalent to the existence of two internally disjoint paths from u to v. $\square$

Definition 6.10.4. If v and w are non-adjacent vertices in G, $\kappa_G(v, w)$ is the smallest number of vertices whose removal separates v from w, that is, disconnects G leaving v and w in different components. A cutset that separates v and w is called a separating set for v and w. $p_G(v, w)$ is the maximum number of internally disjoint paths between v and w.

Theorem 6.10.4. *If v and w are non-adjacent vertices in G then $\kappa_G(v, w) = p_G(v, w)$.*

Proof. See, for example, [3] pp. 111-112. $\square$

Exercise 6.10.1 (E9.1: Connected graph)**.** *(i) Suppose a simple graph G on $n \geq 2$ vertices has at least $(n-1)(n-2)+1$ edges. Prove that G is connected.*

(ii) Show that the complement of a disconnected graph is connected. Is the complement of a connected graph always disconnected?

Exercise 6.10.2 (E9.2: 2-connected graph)**.** *Suppose G has at least one edge. Show that G is 2-connected if and only if for all vertices v and edges e there is a cycle containing v and e.*

Exercise 6.10.3 (E9.3: The edge connectivity $\lambda(G)$)**.** *(i) Denote by $\kappa(G), \lambda(G)$ the connectivity and the edge connectivity of G and $\delta(G)$ the minimum degree of any vertex in G. Find a simple graph with $\kappa(G) < \lambda(G) < \delta(G)$.*

(ii) Suppose $\lambda(G) = k > 0$. Show that there are sets of vertices U and V that partition the vertices of G, and such that there are exactly k edges with one endpoint in U and one endpoint in V.

(iii) Given $m, n \in \mathbb{N}$. Find $\lambda(K_{m,n})$, where $K_{m,n}$ is a complete bipartite graph.

Theorem 6.10.5 (Menger's Theorem)**.** *If G has at least $k+1$ vertices, then G is k-connected if and only if between every two vertices u and v there are k pairwise internally disjoint paths.*

Proof. ($\Rightarrow$:) Suppose that G is k-connected, i.e., $\kappa(G) \geq k$. For every two vertices $u \neq v$ we have two cases:

- Case 1: if $\{u, v\} \notin E(G)$ then $P_G(u, v) = \kappa_G(u, v) \geq k$.
- Case 2: if $\{u, v\} = e \in E(G)$. Consider $G - e$. If there is a cutset S of $G - e$ with $|S| < k - 1$ then $S' = S \cup \{u\}$ or $S' = S \cup \{v\}$ is a cutset of G with $|S'| < k$ which is a contradiction to $\kappa(G) \geq k$. Therefore $\kappa_{G-e}(u, v) \geq k - 1$. Note that $\{u, v\}$ is not an edge in $G - e$ therefore $p_{G-e}(u, v) = \kappa_{G-e}(u, v) \geq k - 1$, i.e., there are at least $k - 1$ pairwise internally disjoint paths in $G - e$ from u to v. Therefore by adding $\{u, v\}$ we have k pairwise internally disjoint paths in G.

($\Leftarrow$:) Suppose that between every two vertices $u \neq v \in G$ there are k pairwise internally disjoint paths in G. Assume that G is not k-connected, i.e., $\kappa(G) \leq k - 1$. Therefore there is a cutset S of G with $|S| \leq k - 1$, i.e., $G - S$ is disconnected. It means there are $u \neq v$ on two different connected components of $G - S$. Therefore k pairwise internally disjoint paths in G from u to v must go through S which is of size only $k - 1$, a contradiction. Therefore G is k-connected. $\square$

Theorem 6.10.6 (The Handle Theorem)**.** *Suppose G is 2-connected and K is a 2-connected proper subgraph of G. Then there are subgraphs L and H (the handle) of G such that L is 2-connected, L contains K, H is a simple path, L and H share exactly the endpoints of H, and G is the union of L and H.*

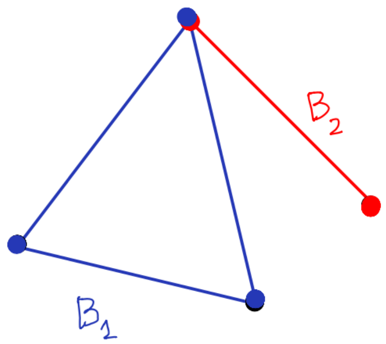
Proof. Let L be a maximal proper 2-connnected subgraph of G containing K. There are two cases:

- Case 1: If $V(L) = V(G)$, $e \in E(G) - E(L)$ then $L \cup \{e\}$ is 2-connected. Therefore $L \cup \{e\} = G$ due to the way to construct L. Then $H = \{e, endpoints(e)\}$.
- Case 2: If there is $v \in V(G) - V(L)$. Put $u \in L$. Because G is 2-connected, there cycle C contains u and v which cut L at x, w. It is easy to see that H is the part of C outside of L.

□

Definition 6.10.5. A block in a graph G is a maximal induced subgraph on at least two vertices without a cutpoint.

Example 6.10.1. A graph with two blocks:



Theorem 6.10.7. *The blocks of G partition the edges.*

Proof. We need to prove that for every edge $e \in E(G)$ there is a unique block contains e. In fact, for each $e \in E(G)$, if e is not inside any 2-connected induced subgraph then $e, endpoints(e)$ is a block. It means there always a block contains e. Assume that there are two blocks B_1, B_2 contains e. We prove that $B = G[V(B_1) \cup V(B_2)]$ has no cutpoint. In fact, if otherwise, there is a vertex $w \in V(B) = V(B_1) \cup V(B_2)$ such that $B - w$ is disconnected. It implies that either $B_1 - w$ is disconnected or $B_2 - w$ is disconnected which is a contradiction to B_1 and B_2 have no cutpoint. Therefore there is a unique block contains e. It completes the proof. □

Theorem 6.10.8. *If G is connected but not 2-connected, then every vertex that is in two blocks is a cutpoint of G.*

Proof. Suppose that B_1, B_2 are two blocks in a connected but not 2-connected graph G and $w \in B_1 \cap B_2$, we prove that w is a cutpoint. In fact, if otherwise, $G - w$ is connected. Because B_1, B_2 are blocks, there are $u \neq w \in B_1$ and $v \neq w \in B_2$. It means $u \neq v \in G - w$. Because $G - w$ is connected, there is a path P in $G - w$ from u to v. It is easy to see that $B = G[V(B_1) \cup V(B_2) \cup V(P)]$ is 2-connected with $|B| > |B_1|$ which is a contradiction to the definition of a block of B_1. It implies the proof. □

Exercise 6.10.4 (E10.1: Connected graph)**.** *(i) Suppose G is simple with degree sequence $d_1 \leq d_2 \leq \cdots \leq d_n$, and for $k \leq n - d_n - 1, d_k \geq k$. Show G is connected.*

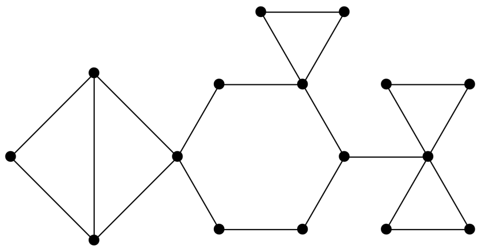
(ii) Suppose a general graph G has exactly two odd-degree vertices, v and w. Let G' be the graph created by adding an edge joining v to w. Prove that G' is connected if and only if G is connected.

Exercise 6.10.5 (E10.2: k-regular graph)**.** *Recall that a graph is k-regular if all the vertices have degree k. What is the smallest integer k (depends on n) that makes this true: "If G is simple, has n vertices, $m \geq k$, and G is m-regular, then G is connected."*

Exercise 6.10.6 (E10.3: Block-cutpoint graph)**.** *(i) Suppose G is a connected graph. The block-cutpoint graph of G, $BC(G)$ is formed as follows: Let vertices $c_1, \dots, c_k$ be the cutpoints of G, and let the blocks of G be $B_1, \dots, B_l$. The vertices of $BC(G)$ are $c_1, \dots, c_k, B_1, \dots, B_l$. Add an edge $\{B_i, c_j\}$ if and only if $c_j \in B_i$. Show that the block-cutpoint graph is a tree.*

Note that a cutpoint is contained in at least two blocks, so that all pendant vertices of the block-cutpoint graph are blocks. These blocks are called endblocks.

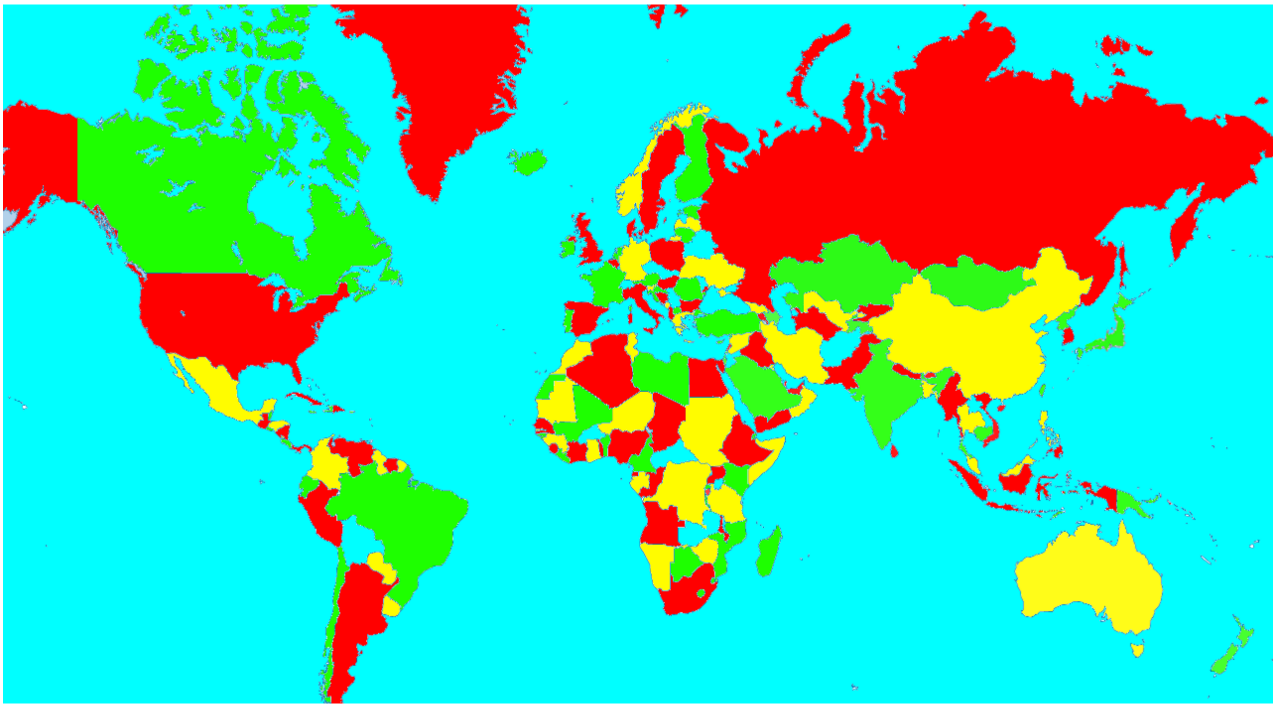
(ii) Draw the block-cutpoint graph of the graph below.



6.11. Graph coloring

6.11.1. Proper coloring

The map-colour problem:



Question: Can the world map be coloured with just four colours so that neighbouring countries receive different colours?

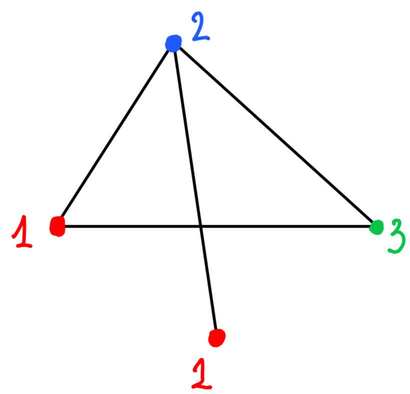
Application 6.11.1. 1. If the vertices of a graph represent academic classes, and two vertices are adjacent if the corresponding classes have people in common, then a coloring of the vertices can be used to schedule class meetings.

2. If the vertices of a graph represent radio stations, and two vertices are adjacent if the stations are close enough to interfere with each other, a coloring can be used to assign non-interfering frequencies to the stations.

3. If the vertices of a graph represent traffic signals at an intersection, and two vertices are adjacent if the corresponding signals cannot be green at the same time, a coloring can be used to designate sets of signals than can be green at the same time.

Definition 6.11.1. A proper coloring of a graph is an assignment of colors to the vertices of the graph so that no two adjacent vertices have the same color. The chromatic number of a graph G is the minimum number of colors required in a proper coloring; it is denoted by $\chi(G)$.

Example 6.11.1. The chromatic number of the below graph is $\chi(G) = 3$.



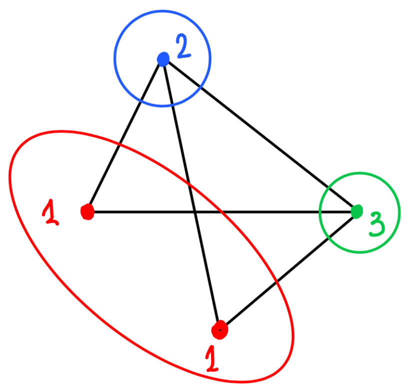
Theorem 6.11.1. *If H is a subgraph of G, $\chi(H) \leq \chi(G)$.*

Proof. Assume that $\{1, \dots, \chi(G)\}$ are colors of a proper coloring of G. This proper coloring of G provides a proper coloring of H, simply by assigning the same colors to vertices of H that they have in G. This means that H can be colored with $\chi(G)$ colors. Thus, $\chi(H) \leq \chi(G)$. $\square$

Remark. If G has a subgraph H that is a complete graph K_m, then $\chi(G) \geq \chi(K_m) = m$.

Definition 6.11.2. A set S of vertices in a graph is independent if no two vertices of S are adjacent. The independence number of G is the maximum size of an independent set; it is denoted by $\alpha(G)$.

Example 6.11.2. The independence number of the below graph is $\alpha(G) = 2$:

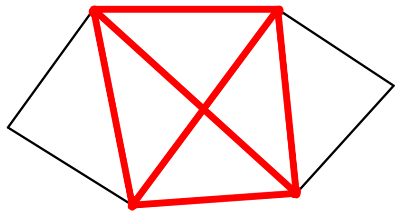


Theorem 6.11.2. *In any graph G on n vertices, $\frac{n}{\alpha(G)} \leq \chi(G)$.*

Proof. Denote by $\{1, \dots, \chi(G)\}$ is the set of the colors used in a proper coloring of G and for each $i \in \{1, \dots, \chi(G)\}$ denote by V_i the set of vertices of G of color i. Then V_i is independent therefore $|V_i| \leq \alpha(G)$. Then $n = \sum_{i=1}^{\chi(G)} |V_i| \leq \alpha(G)\chi(G)$. It implies the proof. $\square$

Definition 6.11.3. A subgraph of G that is a complete graph is called a clique. The clique number of a graph G is the largest m such that K_m is a subgraph of G.

Example 6.11.3. The clique number of the below graph is 4:

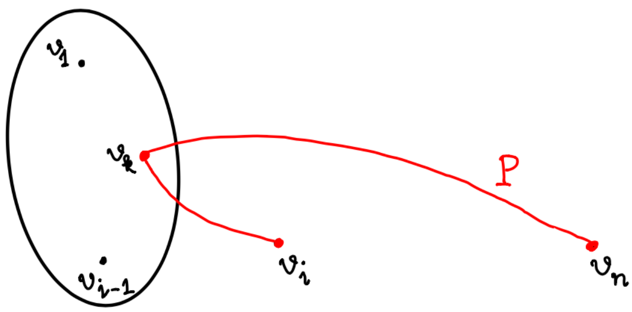


Theorem 6.11.3. *In any graph G, $\chi(G) \leq \Delta(G) + 1$, where $\Delta(G) = \max\{\deg(v), v \in G\}$.*

Proof. By using a greedy algorithm as follows:
An algorithm:

- Step 1: Choose some vertex $v \in G$;
- Step 2: For each other vertex $w \neq v$ calculate the distance between v and w, i.e., the length of the shortest path between v and w. Then we order vertices of G as $\{v_1, v_2, \ldots, v_n\}$ such that $v_n = v$ and $d(v_1, v_n) \geq d(v_2, v_n) \geq \ldots \geq d(v_{n-1}, v_n)$;
- Step 3: Color v_1 as 1;
- Step 4: Assume that we colored $\{v_1, \ldots, v_{i-1}\}$ with $2 \leq i \leq n$ we will color v_i as the smallest natural number which has not been colored to the neighbors $N(v_i)$ of v_i.

We will prove that this algorithm ends up with a proper coloring of at most $\Delta + 1$ colors. First, we prove the following claim:
Claim: By the algorithm, v_i (i<n) can be colored by one in $\mathcal{C} = \{1, \ldots, \Delta\}$.
In fact, this claim is true for $i = 1$ because v_1 is colored as 1 and $1 \in \mathcal{C}$. For $1 < i < n$, consider a shortest path P from v_i to v_n. Then $P \cap \{v_1, \ldots, v_{i-1}\} = \emptyset$ because if else there is $1 \leq k \leq i-1$ so that $v_k \in P \cap \{v_1, \ldots, v_{i-1}\}$, it implies that $d(v_k, v_n) < d(v_i, v_n)$ which is a contradiction to the ordering in Step 2.

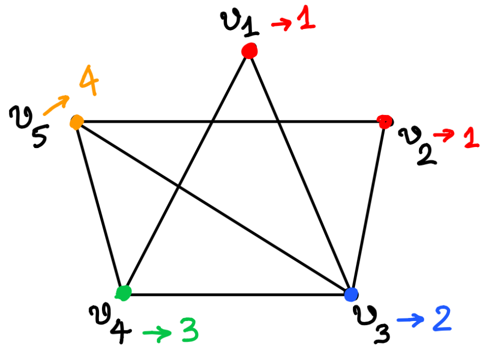


It implies that v_i has at least one neighbor that is not yet colored because if else we have $N(v_i) \subseteq \{v_1, \ldots, v_{i-1}\}$ and then the path P must go through one of neighbors of v_i which is in $\{v_1, \ldots, v_{i-1}\}$. On the other hand, because $|N(v_i)| = \deg(v_i) \leq \Delta$, $N(v_i)$ used at most $\Delta - 1$ colors from $\mathcal{C}$, leaving at least one color from $\mathcal{C}$ available for v_i. This completes the Claim. At time $i = n$, from the Claim, $\{v_1, \ldots, v_{n-1}\}$ have been colored from $\mathcal{C}$. Moreover, $N(v_n) \subseteq \{v_1, \ldots, v_{n-1}\}$, i.e., all neighbors of v_n have been colored using color from $\mathcal{C}$, so color $\Delta + 1$ may be used to color v_n. This completes the proof. □

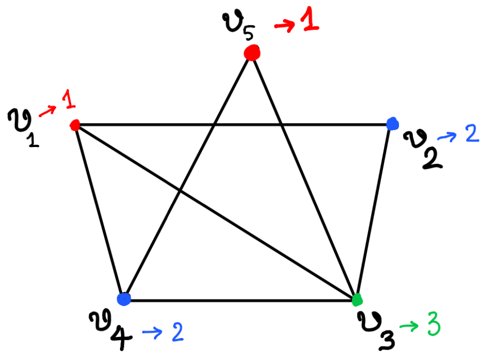
Corollary 6.11.1. *If G is not regular, then $\chi(G) \leq \Delta(G)$.*

Proof. If G is not regular, there is a vertex, say v, with $d(v) < \Delta(G)$. We choose this vertex as v_n for the Step 1. Then after we color $\{v_1, \ldots, v_{n-1}\}$ from $\mathcal{C}$, because $|N(v_n)| = d(v) < \Delta$, v_n can also be colored by one color in $\mathcal{C}$. □

Remark. This algorithm is just give us a proper coloring, but not the one with the smallest size, i.e., not give us a $\chi(G)$. For example, if we choose the left-up vertex as v, the algorithm will end up with the ordering of vertices as below



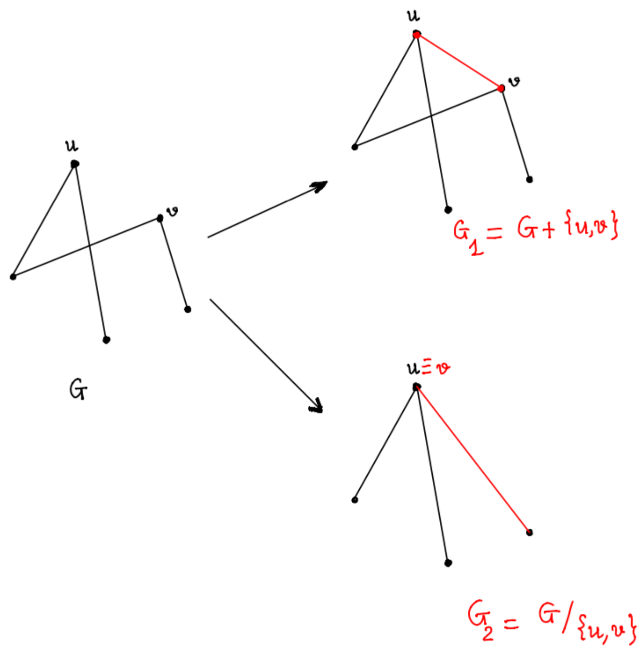
and use four colors for a proper coloring satisfying $4 < 5 = \Delta + 1$. But $\chi(G) < 4$ because for example, if we choose the top vertex as v, the algorithm will end up with the ordering of vertices as below



and use only three colors for a proper coloring. Indeed, $\chi(G) = 3$ because K_3 is a subgraph of G.

An algorithm to find $\chi(G)$:

- Step 1: If G is complete then stop the algorithm.
- Step 2: Else we choose some pair of non-adjacent vertices $\{u, v\}$ of G and generate two new graphs $G_1 = G + \{u, v\}$ and $G_2 = G/\{u, v\}$ where G_1 is generated by just adding one edge $\{u, v\}$ to G and G_2 is generated by moving v into u and add edges from u to neighbors of v which is not neighbor of u, i.e., add an edge $\{u, w\}$ if $w \in N(v) \setminus N(u)$;



- Step 3: If G_1 is complete we do not go further for G_1, else repeat Step 2 for G_1 to have two new graphs G_{11}, G_{12}. Similarly, if G_2 is complete we do not go further for G_2, else repeat Step 2 for G_2 to have two new graphs G_{21}, G_{22}. This will be repeated until all new graphs are complete.

- Step 4: Calculate the chromatic numbers for all new graphs and choose the smallest one for $\chi(G)$.

Remark. This algorithm is based on the following idea:
$\chi(G + \{u, v\})$ is the smallest number of colors needed to color G so that u and v have different colors.
$\chi(G/\{u, v\})$ is the smallest number of colors needed to color G so that u and v have the same colors.
Therefore $\chi(G) = \min\{\chi(G + \{u, v\}), \chi(G/\{u, v\})\}$.

Theorem 6.11.4. *The algorithm above correctly computes the chromatic number in a finite amount of time.*

Proof. Suppose that G has n vertices and m edges. We will prove the Algorithm will stop after a finite amount of time by induction on the number of pairs of non-adjacent vertices $na(G) := \binom{n}{2} - m$. In fact,

- Base case: $na(G) = 0$, i.e., G is complete then the algorithm stop by Step 1.

- Inductive step: if $na(G) > 0$, i.e., there are non-adjacent vertices u and v. We need only to prove that $na(G + \{u, v\}) < na(G)$ and $na(G/\{u, v\}) < na(G)$. In fact, we have

$$na(G + \{u, v\}) = \binom{n}{2} - (m + 1) = na(G) - 1.$$

Moreover, by denoting by $0 \leq c \leq n - 2$ is the number of neighbors that u and v have in common, we have

$$\begin{aligned} na(G/\{u, v\}) &= \binom{n-1}{2} - (m - c) \\ &\leq \binom{n-1}{2} - m + (n - 2) = na(G) - 1. \end{aligned}$$

□

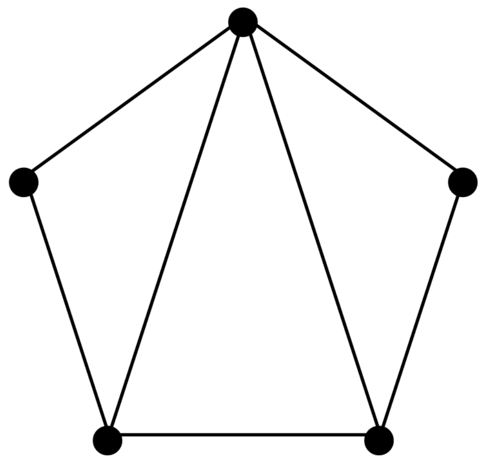
Theorem 6.11.5 (Brooks's Theorem)**.** *If G is a graph other than K_n or C_{2n+1}, then $\chi(G) \leq \Delta(G)$.*

Proof. Exercise E11.3. □

Exercise 6.11.1 (E11.1: Chromatic number $\chi(G)$)**.** *(i) Suppose $G = (V, E)$ is a simple connected graph on $|V| = n$ vertices and chromatic number $\chi(G) = k$. Prove that $E(G) \geq \binom{k}{2}$.*

(ii) Show that $\chi(G - v)$ is either $\chi(G)$ or $\chi(G) - 1$.

Exercise 6.11.2 (E11.2: Greedy algorithm to find $\chi(G)$)**.** *Find the chromatic number of the graph below by using the algorithm in the lecture. Draw all of the graphs $G + e$ and G/e generated by the algorithm in a "tree structure" with the complete graphs at the bottom, label each complete graph with its chromatic number, then propagate the values up to the original graph.*



Exercise 6.11.3 (E11.3: Brooks's theorem)**.** *Prove that if G is a simple connected graph other than K_n or C_{2n+1} then*

$$\chi(G) \leq \Delta(G),$$

where $\Delta(G)$ is the maximum degree of any vertex in G.

6.11.2. The chromatic polynomial

Given a graph G. Denote by $P_G(k)$ the number of ways to color G with k colors.

Example 6.11.4. (i) If $G = K_n$ then $P_G(k) = P_{n,k} = k(k-1)\dots(k-n+1)$.

(ii) If $G = (V = \{v_1, \dots, v_n\}, E = \emptyset)$ then $P_G(k) = k^n$.

Theorem 6.11.6. *Given a graph G on n vertices. Then P_G is a polynomial of degree n, called the chromatic polynomial of G.*

Proof. We first prove that for every $\{u, v\} \in E(G)$:

$$P_G(k) = P_{G-\{u,v\}}(k) - P_{G/\{u,v\}}(k). \tag{6.11.2}$$

In fact, put $G' = G - \{u, v\}$. It is easy to see that

1. $\#\{\text{ways to color } c : G' \to \mathbb{N} \text{ with } c(u) = c(v)\} = \#\{\text{ways to color } G/\{u,v\}\}$
2. $\#\{\text{ways to color } c : G' \to \mathbb{N} \text{ with } c(u) \neq c(v)\} = \#\{\text{ways to color } G\}$.

Therefore

$$\begin{aligned}
P_{G'}(k) &= \#\{\text{ways to color } G'\} \\
&= \#\{\text{ways to color } c \text{ of } G' \text{ with } c(u) = c(v)\} + \#\{\text{ways to color } c \text{ of } G' \text{ with } c(u) \neq c(v)\} \\
&= \#\{\text{ways to color } G/\{u,v\}\} + \#\{\text{ways to color } G\} \\
&= P_{G/\{u,v\}}(k) + P_G(k).
\end{aligned}$$

We then prove the theorem by induction on $m = |E(G)|$:

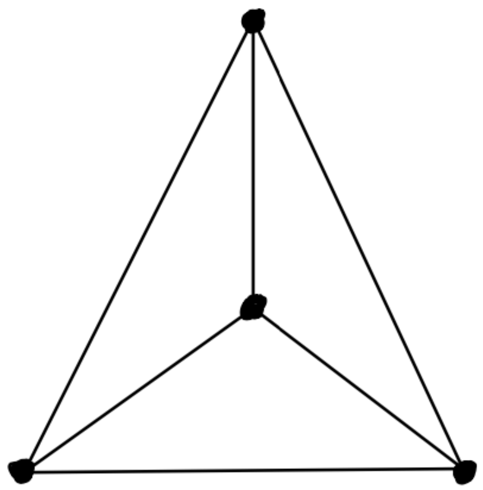
- Base case: for $m = 0$, we know from the Example 6.11.4 (ii) that $P_G(k) = k^n$ is a polynomial of degree $|V(G)| = n$.
- Inductive step: assume that the statement is true until $m - 1 \geq 0$, i.e., $P_{G'}(k)$ is a polynomial of degree $|V(G')|$ for all G' with $|E(G')| \leq m - 1$. Consider an arbitrary graph G with $|E(G)| = m > 0$. Then there is an edge $e \in E(G)$ and from Eq. (6.11.2) we have $P_G(k) = P_{G-e}(k) - P_{G/e}(k)$. Note that $|E(G - e)| = m - 1$ and $|E(G/e)| \leq m - 1$ therefore by the inductive hypothesis, $P_{G-e}(k)$ is a polynomial of degree $|V(G - e)| = n$ and $P_{G/e}(k)$ is a polynomial of degree $|V(G/e)| = n - 1$. Thus, $P_G(k)$ is a polynomial of degree $n = |V(G)|$. It completes the proof.

□

6.12. Coloring planar graphs

Definition 6.12.1. A graph G is planar if it can be represented by a drawing in the plane so that no edges cross.

Example 6.12.1. K_4 is planar because it can be drawn in the plane as follows



Theorem 6.12.1 (Euler's Formula)**.** *Suppose G is a connected planar graph, drawn so that no edges cross, with n vertices and m edges, and that the graph divides the plane into r regions. Then*

$$r = m - n + 2.$$

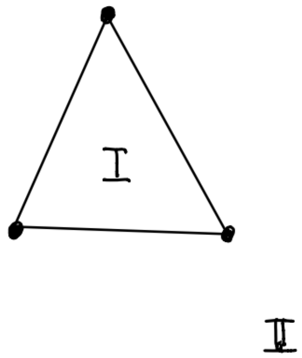
Proof. We prove by induction on m. Because G is connected, we have $m \geq n - 1$.

- Base case: for $m = n - 1$, it happens if and only if G is a tree which has only one region $r = 1$. Therefore the equality is hold.

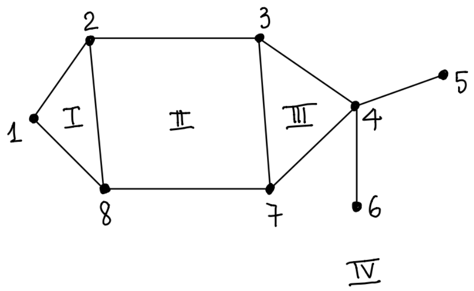
- Inductive step: assume that the equality is true until $m - 1 \geq n - 1$, we prove that it is also true for m. In fact, because $m \geq n$, G is not a tree, therefore there is a cycle C in G. Choose $e \in C$ then $G - e$ is connected planar graph with $|E(G - e)| = m - 1$ and $r(G - e) = r - 1$, therefore from the inductive hypothesis, $r - 1 = (m - 1) - n + 2$ which implies that $r = m - n + 2$. It completes the proof.

□

Example 6.12.2. Case $m = n = 3, r = 2$



Case $m = 10, n = 8, r = 4$

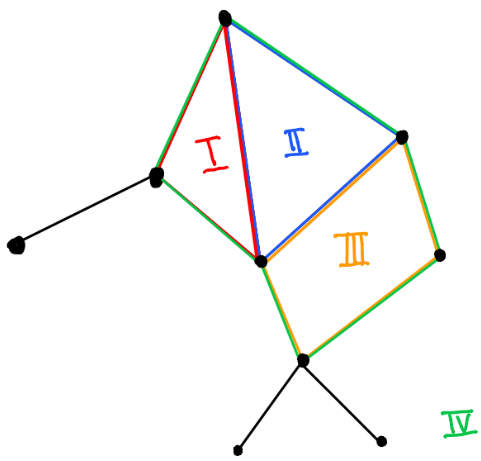


Lemma 6.12.1. *Suppose G is a simple connected planar graph, drawn so that no edges cross, with $n \geq 3$ vertices and m edges, and that the graph divides the plane into r regions. Then $m \leq 3n - 6$.*

Proof. Set f_i is the number of edges on the boundary of region i for all $i \in \{1, \ldots, r\}$. Because G is simple, $f_i \geq 3$ for all $i \in \{1, \ldots, r\}$. Therefore

$$3r \leq \sum_{i=1}^{r} f_i \leq 2m.$$

From Euler's formula we have $3r = 3m - 3n + 6 \leq 2m$ which implies that $m \leq 3n - 6$.



□

Theorem 6.12.2. *K_5 is not planar.*

Proof. Assume that K_5 is planar. Moreover it is simple, connected therefore from the lemma 6.12.1, we must have $10 = m \leq 3n - 6 = 3 \times 5 - 6 = 9$ which is a contradiction. Therefore K_5 is not planar.

□

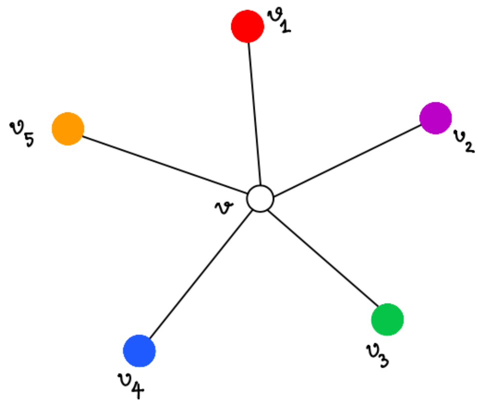
Lemma 6.12.2. *If G is simple planar then G has a vertex of degree at most 5.*

Proof. Suppose that G is connected, otherwise we consider its connected component. Assume that $d_i \geq 6$ for all i. Then $2m = \sum_{i=1}^{n} d_i \geq 6n$. Therefore, from the lemma 6.12.1, we have $6n - 12 \geq 2m \geq 6n$ which is a contradiction. Therefore G has a vertex of degree at most 5. □

Theorem 6.12.3 (Five Color Theorem)**.** *Every planar graph can be colored with 5 colors.*

Proof. We prove by induction on n.

- Base case: for $n \leq 5$, it is obviously, just color each vertex by a new color.
- Inductive step: Assume that it is true for $n - 1 \geq 5$, we will prove it is also true for n. In fact, from Lemma 6.12.2 there is a vertex v of degree at most 5. Consider $G - v$. From the inductive hypothesis, $G - v$ is 5-colorable and we color $G - v$ with 5 colors $\mathcal{C} = \{red, purple, green, blue, brown\}$. If $d(v) \leq 4$ then we can color v by one color in $\mathcal{C}$ which is not used by its neighbors so that we have a proper coloring of G. If $d(v) = 5$ and its neighbors are colored with at most 4 colors then again we can color v to have a proper coloring of G. If $d(v) = 5$ and all five neighbors of v have a different color, for example as in the below figure

  then we recolor $G - v$ as follows:

  We denote by $RedGreen(v_1)$ the set of vertices $w \in G - v$ so that there is an alternative red-green path from v_1 to w, i.e., there is a path $P = \{v_1 = x_1, x_2, \ldots, x_k = w\}$ such that x_i is colored by *red* or *green*. There are two cases:

  – Case 1: If $v_3 \notin RedGreen(v_1)$ then we recolor $G - v$ by change *red* into *green* and *green* into *red*. In this new coloring, look at neighbors of v there are 4 colors because v_1 is now colored by *green* therefore we can color v by *red* to have a proper coloring of G.

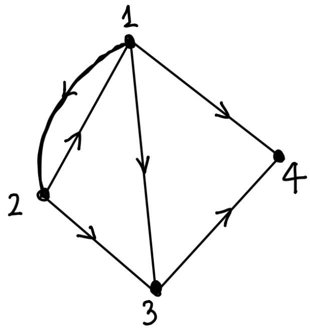
  – Case 2: If $v_3 \in RedGreen(v_1)$, i.e. there is an alternative red-green path P from v_1 to v_3. $P \cup \{v_1, v, v_3\}$ become a cycle which divide the plane into two regions, one contains v_2, one contains v_4, v_5. Now we denote by $PurpleBlue(v_2)$ the set of vertices $w \in G - v$ so that there is an alternative purple-blue path from v_2 to w. Because v_2 and v_4 lie into two different regions, $v_4 \notin PurpleBlue(v_2)$ and we go to Case 1.

It implies the proof. □

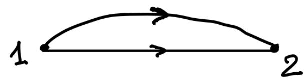
6.13. Directed graphs

Definition 6.13.1. A directed graph, also called a digraph, is a graph in which the edges have a direction, indicated with an arrow on the edge. If v and w are vertices, an edge is an unordered pair $\{v, w\}$, while a directed edge, called an arc or arrow, is an ordered pair (v, w) or (w, v). If a graph contains both arcs (v, w) and (w, v), this is not a "multiple edge", as the arcs are distinct. It is possible to have multiple arcs, namely, an arc (v, w) may be included multiple times in the multiset of arcs. As before, a digraph is called simple if there are no loops or multiple arcs.

Example 6.13.1. $\overrightarrow{G} = (V, \overrightarrow{E})$ with $V = \{1, 2, 3, 4\}$, $\overrightarrow{E} = \{(1, 2), (1, 3), (1, 4), (2, 1), (2, 3), (3, 4)\}$ is a simple digraph.



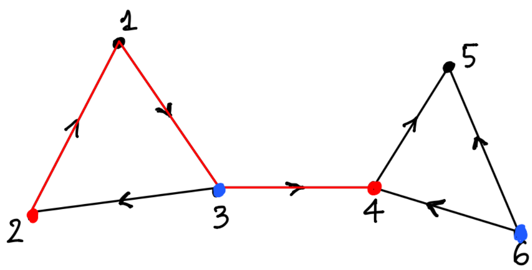
$\overrightarrow{G} = (V = \{1\}, \overrightarrow{E} = \{(1, 1)\})$ is a digraph with loop and $\overrightarrow{G} = (V = \{1, 2\}, \overrightarrow{E} = \{(1, 2), (1, 2)\})$ is a digraph with mutiple arcs.



Definition 6.13.2. We denote by E_v^- the set of all arcs of the form (w, v), and by E_v^+ the set of arcs of the form (v, w).

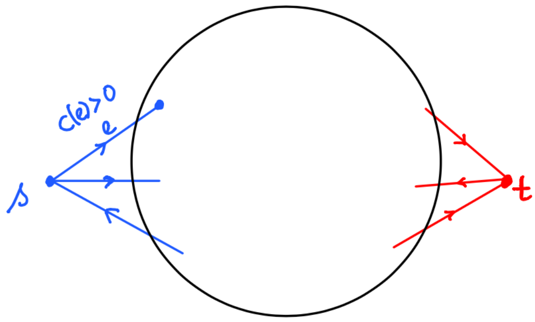
- The indegree of v, denoted by $d^-(v)$, is the number of arcs in E_v^-; The outdegree, $d^+(v)$, is the number of arcs in E_v^+. If the vertices are $v_1, v_2, \ldots, v_n$, the degrees are usually denoted $d_1^-, d_2^-, \ldots, d_n^-$ and $d_1^+, d_2^+, \ldots, d_n^+$. Note that both $\sum_{i=0}^n d_i^-$ and $\sum_{i=0}^n d_i^+$ count the number of arcs exactly once, and of course $\sum_{i=0}^n d_i^- = \sum_{i=0}^n d_i^+$.
- A walk in a digraph is a sequence $v_1, e_1, v_2, e_2, \ldots, v_{k-1}, e_{k-1}, v_k$ such that $e_k = (v_i, v_{i+1})$; if $v_1 = v_k$, it is a closed walk or a circuit.
- A path in a digraph is a walk in which all vertices are distinct. It is not hard to show that, as for graphs, if there is a walk from v to w then there is a path from v to w.

Example 6.13.2. Given $\overrightarrow{G} = (V, \overrightarrow{E})$ with $V = \{1, 2, 3, 4, 5, 6\}$, $\overrightarrow{E} = \{(1, 3), (2, 1), (3, 2), (3, 4), (4, 5), (6, 4), (6, 5)\}$.



Then $E_1^- = \{(2,1)\}, E_1^+ = \{(1,3)\}, E_2^- = \{(3,2)\}, E_2^+ = \{(2,1)\}, E_3^- = \{(1,3)\}, E_3^+ = \{(3,2),(3,4)\}, E_4^- = \{(3,4),(6,4)\}, E_4^+ = \{(4,5)\}, E_5^- = \{(4,5),(6,5)\}, E_5^+ = \emptyset, E_6^- = \emptyset, E_6^+ = \{(6,4),(6,5)\}$. $d_1^- = d_1^+ = d_2^- = d_2^+ = d_3^- = d_4^+ = 1$, $d_3^+ = d_4^- = d_5^- = d_6^+ = 2$, $d_5^+ = d_6^- = 0$. We see that $\sum_{i=1}^6 d_i^- = \sum_{i=1}^6 d_i^+ = |\overrightarrow{E}| = 7$.
A walk $\{2,(2,1),1,(1,3),3,(3,2),2,(2,1),1,(1,3),3,(3,4),4\}$ from 2 to 4.
A path $\{2,(2,1),1,(1,3),3,(3,4),4\}$ from 2 to 4. There is no path from 3 to 6.

Definition 6.13.3. A network is a digraph with a designated source s and target $t \neq s$. In addition, each arc e has a positive capacity, $c(e)$.

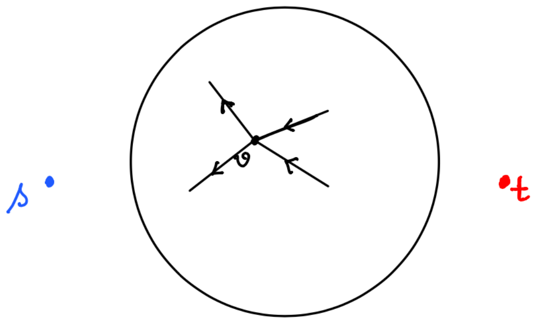


Example 6.13.3. Networks can be used to model transport through a physical network, of a physical quantity like oil or electricity, or of something more abstract, like information.

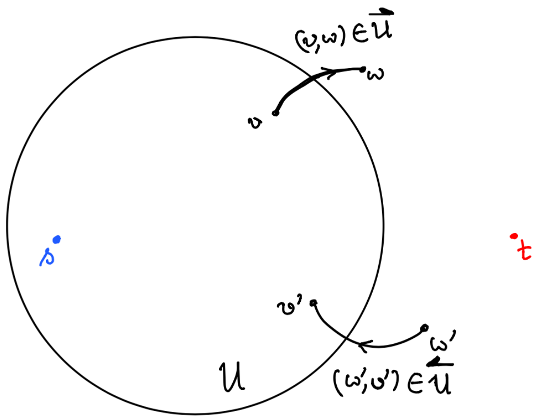
Definition 6.13.4. A flow in a network is a function f from the arcs of the digraph to $\mathbb{R}$, with $0 \le f(e) \le c(e)$ for all e, and such that

$$f(E_v^+) := \sum_{e \in E_v^+} f(e) = \sum_{e \in E_v^-} f(e) =: f(E_v^-),$$

for all v other than s and t.



Suppose that U is a set of vertices in a network with $s \in U$ and $t \notin U$. Let $\overrightarrow{U}$ be the set of arcs (v,w) with $v \in U, w \notin U$, and $\overleftarrow{U}$ be the set of arcs (v,w) with $v \notin U, w \in U$.



Theorem 6.13.1. *For any flow f in a network, the net flow out of the source is equal to the net flow into the target, namely,*

$$\sum_{e\in E_s^+} f(e) - \sum_{e\in E_s^-} f(e) = \sum_{e\in E_t^-} f(e) - \sum_{e\in E_t^+} f(e),$$

Proof. Let U be an arbitrary set of vertices which contains s but not t. We note that

$$f(E_s^+) - f(E_s^-) = \sum_{v\in U} f(E_v^+) - f(E_v^-)$$

because the sum under $v \in U$ has non-zero value only on $v = s$ due to the definition of a flow f. On the other hand, we have

$$\begin{aligned}
\sum_{v\in U} f(E_v^+) - f(E_v^-) = & \sum_{(v,w)\in \overrightarrow{E}: v\in U} f((v,w)) - \sum_{(w,v)\in \overrightarrow{E}: w\in U} f((w,v)) \\
= & \sum_{(v,w)\in \overrightarrow{E}: v\in U, w\in U} f((v,w)) + \sum_{(v,w)\in \overrightarrow{E}: v\in U, w\notin U} f((v,w)) \\
& - \sum_{(w,v)\in \overrightarrow{E}: w\in U, v\in U} f((w,v)) - \sum_{(w,v)\in \overrightarrow{E}: w\in U, v\notin U} f((w,v)) \\
= & \sum_{(v,w)\in \overrightarrow{E}: v\in U, w\notin U} f((v,w)) - \sum_{(w,v)\in \overrightarrow{E}: w\in U, v\notin U} f((w,v)) \\
= & f(\overrightarrow{U}) - f(\overleftarrow{U}).
\end{aligned}$$

Therefore for any U which contains s but not t

$$f(E_s^+) - f(E_s^-) = f(\overrightarrow{U}) - f(\overleftarrow{U}). \tag{6.13.3}$$

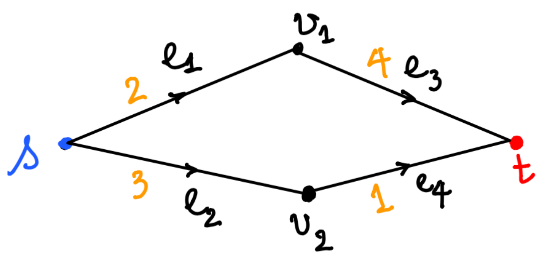
Now we choose $U = V - \{t\}$ and note that $\overrightarrow{U} = E_t^-$ and $\overleftarrow{U} = E_t^+$. Thus we have

$$f(E_s^+) - f(E_s^-) = f(E_t^-) - f(E_t^+).$$

It completes the proof. $\square$

Definition 6.13.5. The value of a flow, denoted $val(f)$, is $f(E_s^+) - f(E_s^-)$. A maximum flow in a network is any flow f whose value is the maximum among all flows.

Example 6.13.4. Consider a network with $\overrightarrow{G} = (V = \{s, v_1, v_2, t\}, \overrightarrow{E} = \{e_1 = (s, v_1), e_2 = (s, v_2), e_3 = (v_1, t), e_4 = (v_2, t)\})$ with capacities $c(e_1) = 2, c(e_2) = 3, c(e_3 = 4), c(e_4) = 1$ as in the figure:



Then the maximum flow is $f_{\max} : \overrightarrow{E} \to \mathbb{R}$, $f_{\max}(e_1) = 2$, $f_{\max}(e_2) = 1$, $f_{\max}(e_3) = 2$, $f_{\max}(e_4) = 1$ and the maximum value of this network is $val(f_{\max}) = f_{\max}(e_1) + f_{\max}(e_2) = 3$.

Definition 6.13.6. A cut in a network is a set C of arcs with the property that every path from s to t uses an arc in C, i.e., $\forall$ path P from s to t: $P \cap C \neq \emptyset$. In other words, if the arcs in C are removed from the network there is no path from s to t.

Definition 6.13.7. The capacity of a cut, denoted $c(C)$, is $\sum_{e\in C} c(e)$. A minimum cut is one with minimum capacity. A cut C is minimal if no cut is properly contained in C.

Remark.
- A minimum cut is a minimal cut.
- If U is a set of vertices containing s but not t, then $\overrightarrow{U}$ is a cut.

Lemma 6.13.1. *Suppose C is a minimal cut. Then there is a set U containing s but not t such that $C = \overrightarrow{U}$.*

Proof. Construct a set of vertices $U = \{v \in V : \text{ there is a path } P \text{ from } s \text{ to } v : P \cap C = \emptyset\}$. Then $s \in U$ by convention $P_1 \cap C = \emptyset$ and $t \notin U$ due to the definition of a cut C. Therefore we only need to prove that $C = \overrightarrow{U}$. In fact,

- $C \subseteq \overrightarrow{U}$: For every $e = (v, w) \in C$ we need to prove that $e \in \overrightarrow{U}$, i.e., $v \in U$ and $w \notin U$. In fact, assume that every path from s to t using e and another arc in C, then $C' = C - \{e\}$ will be a cut which is a contradiction to C is minimal. Therefore there is a path P from s to t using e but not other arc in C. Its restriction on $[s, v]$, i.e., $P_{[s,v]}$ is a path from s to v with $P_{[s,v]} \cap C = \emptyset$. Therefore $v \in U$.
  Moreover, assume that $w \in U$ then there is a path P' from s to w with $P' \cap C = \emptyset$. But then the path $P' \cup P_{[w,t]}$ from s to t does not cut C which is a contradiction to the definition of a cut C. Therefore $w \notin U$.
- $\overrightarrow{U} \subseteq C$: For every $e = (v, w) \in \overrightarrow{U}$, i.e., $v \in U$ and $w \notin U$, we need to prove that $e \in C$. In fact, because $v \in U$ there is a path P from s to v with $P \cap C = \emptyset$. Because $w \notin U$, every path from s to w must cut C. Therefore the path $P \cup \{e\}$ from s to w must cut C which implies that $e \in C$.

$\square$

Theorem 6.13.2 (Max-flow, min-cut theorem)**.** *Suppose in a network all arc capacities are integers. Then the value of a maximum flow is equal to the capacity of a minimum cut. Moreover, there is a maximum flow f for which all $f(e)$ are integers.*

Proof. We first prove that for any flow f and a minimum cut $C_{\min}$, $val(f) \leq c(C_{\min})$. In fact, by Lemma 6.13.1 there is some U such that $C_{\min} = \overrightarrow{U}$. From Eq. (6.13.3) we have

$$val(f) = f(E_s^+) - f(E_s^-) = f(\overrightarrow{U}) - f(\overleftarrow{U}) \leq f(\overrightarrow{U}) \leq c(\overrightarrow{U}) = c(C_{\min}).$$

Therefore if we can find a flow f and a cut C such that $val(f) = c(C)$ then f is a maximum flow and C is a minimum cut.

An algorithm to find a maximum flow f and a minimum cut C:

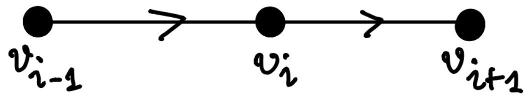
- Step 1: Start with a zero flow: $f \equiv 0$.
- Step 2: Let $U = \{s\}$. Repeat the next two steps until no new vertices are added to U.
  - Step 2.1: If there is $e = (v, w)$ with $v \in U$, $w \notin U$ and $f(e) < c(e)$ then add w to U
  - Step 2.2: If there is $e = (v, w)$ with $v \notin U$, $w \in U$ and $f(e) > 0$ then add v to U
- Step 3: Check if $t \in U$ or $t \notin U$. If $t \notin U$ then stop the algorithm. Otherwise if $t \in U$ then we choose a sequence of vertices (not need to be all vertices of G) $\{s = v_1, v_2, \ldots, v_k = t\}$ such that for each $1 \leq i < k$

- either case 1: $(v_i, v_{i+1}) \in \overrightarrow{E}$ with $f(v_i, v_{i+1}) < c(v_i, v_{i+1})$
- or case 2: $(v_{i+1}, v_i) \in \overrightarrow{E}$ with $f(v_{i+1}, v_i) > 0$.

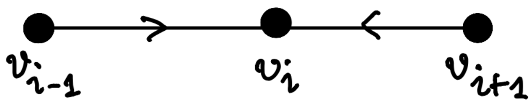
We then update the flow f to f' by $f'(v_i, v_{i+1}) = f(v_i, v_{i+1}) + 1$ for case 1 and $f'(v_{i+1}, v_i) = f(v_{i+1}, v_i) - 1$ for case 2. Now rename f' to f and return to Step 2.

To prove that the algorithm work well we only need to prove that the new function f' in Step 3 is also a flow with its value is greater than the value of f. In fact, in case 1, $f(e) < c(e)$ implies $f'(e) \leq c(e)$ and in case 2, $f(e) > 0$ implies $f'(e) \geq 0$. Moreover, f' is different from f at exactly $k-1$ arcs with endpoints at $\{v_i\}_{i=1}^k$. Therefore we only need to check $f(E_v^+) - f(E_v^-) = 0$ at $v = v_i$ for $1 < i < k$. There are four cases:

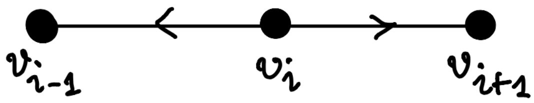
- Case a. $(v_{i-1}, v_i) \in \overrightarrow{E}$ with $f(v_{i-1}, v_i) < c(v_{i-1}, v_i)$ and $(v_i, v_{i+1}) \in \overrightarrow{E}$ with $f(v_i, v_{i+1}) < c(v_i, v_{i+1})$: $f'(E_{v_i}^+) - f'(E_{v_i}^-) = (f(E_{v_i}^+) + 1) - (f(E_{v_i}^-) + 1) = f(E_{v_i}^+) - f(E_{v_i}^-) = 0$.



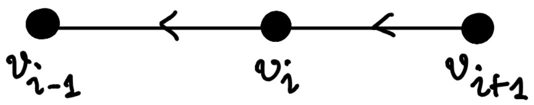
- Case b. $(v_{i-1}, v_i) \in \overrightarrow{E}$ with $f(v_{i-1}, v_i) < c(v_{i-1}, v_i)$ and $(v_{i+1}, v_i) \in \overrightarrow{E}$ with $f(v_{i+1}, v_i) > 0$: $f'(E_{v_i}^+) - f'(E_{v_i}^-) = f(E_{v_i}^+) - (f(E_{v_i}^-) + 1 - 1) = f(E_{v_i}^+) - f(E_{v_i}^-) = 0$.



- Case c. $(v_i, v_{i-1}) \in \overrightarrow{E}$ with $f(v_i, v_{i-1}) > 0$ and $(v_i, v_{i+1}) \in \overrightarrow{E}$ with $f(v_i, v_{i+1}) < c(v_i, v_{i+1})$: $f'(E_{v_i}^+) - f'(E_{v_i}^-) = (f(E_{v_i}^+) - 1 + 1) - f(E_{v_i}^-) = f(E_{v_i}^+) - f(E_{v_i}^-) = 0$.



- Case d. $(v_i, v_{i-1}) \in \overrightarrow{E}$ with $f(v_i, v_{i-1}) > 0$ and $(v_{i+1}, v_i) \in \overrightarrow{E}$ with $f(v_{i+1}, v_i) > 0$: $f'(E_{v_i}^+) - f'(E_{v_i}^-) = (f(E_{v_i}^+) - 1) - (f(E_{v_i}^-) - 1) = f(E_{v_i}^+) - f(E_{v_i}^-) = 0$.



Moreover, similarly we can see that at $v = s$ there are two cases $(s, v_1) \in \overrightarrow{E}$ with $f(s, v_1) < c(s, v_1)$ or $(v_1, s) \in \overrightarrow{E}$ with $f(v_1, s) > 0$. Both cases give us $val(f') = f'(E_s^+) - f'(E_s^-) = val(f) + 1$.

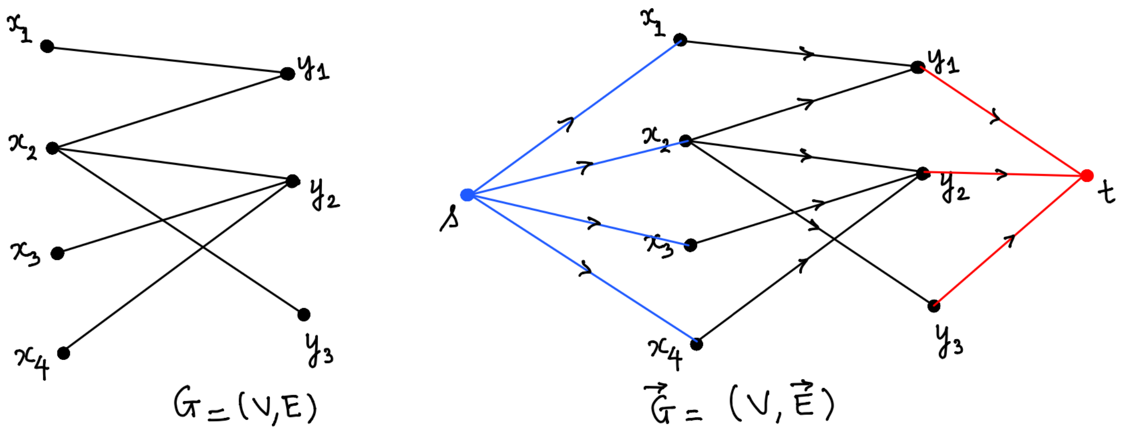
Now, the algorithm stops when $t \notin U$ in Step 3. At this time we have a flow f and U for which no new vertices can be added in the sense of Step 2. We prove that this flow f is a maximum flow and $\overrightarrow{U}$ is a minimum cut. In fact, we have now for every $e = (v, w) \in \overrightarrow{U}$, $f(e) = c(e)$ because if otherwise w can be added into U and similarly for every $e = (w, v) \in \overleftarrow{U}$, $f(e) = 0$ because if otherwise v can be added into U. Therefore

$$val(f) = f(\overrightarrow{U}) - f(\overleftarrow{U}) = \sum_{e \in \overrightarrow{U}} c(e) - \sum_{e \in \overleftarrow{U}} 0 = c(\overrightarrow{U}).$$

Thus, f is a maximum flow and $C = \overrightarrow{U}$ is a minimum cut. $\square$

Corollary 6.13.1. *In a bipartite graph G, the size of a maximum matching is the same as the size of a minimum vertex cover.*

Proof. Suppose that the bipartite graph G with parts $X = \{x_1, \ldots, x_k\}$ and $Y = \{y_1, \ldots, y_l\}$. Now we construct a network as follows: $\overrightarrow{G} = (V, \overrightarrow{E})$ with $V = \{s; x_1, \ldots, x_k, y_1, \ldots, y_l; t\}$ and $\overrightarrow{E} = \{(s, x_i)\}_{i=1}^{k} \cup \{(y_j, t)\}_{j=1}^{l} \cup \{(x_i, y_j)\}_{i\in[k], j\in[l]:\{x_i,y_j\}\in E(G)}$ and capacities $c(e) = 1$ for all $e \in \overrightarrow{E}$.



Let C be a minimum cut. Note that if $(x_i, y_j) \in C$ then $(s, x_i) \notin C$ because if otherwise, $C' = C - \{(s, x_i)\}$ is also a cut and strictly subset of C which is a contradiction to C is a minimum cut. We construct C' from C by replacing every arc $(x_i, y_j) \in C$ into (s, x_i) then $|C'| = |C|$ and C' is also a cut because every path from s to t containing (x_i, y_j) must contain (s, x_i). Thus, we suppose that C contains only arcs of the form (s, x_i) and (y_j, t). Consider

$$K = \{x_i : (s, x_i) \in C\} \cup \{y_j : (y_j, t) \in C\}$$

which has the same size as C. K is a vertex cover of G because if otherwise there is an edge in G, say $\{x_i, y_j\}$ such that $x_i, y_j \notin K$. Then the path $\{s, (s, x_i), x_i, (x_i, y_j), y_j, (y_j, t), t\}$ from s to t but does not cut C which is a contradiction to the definition of a cut C.

Now, from the Theorem 6.13.2, there is a maximum flow f with $f(e) \in \mathbb{N}_0$ and therefore $f(e) \in \{0, 1\}$ for all $e \in \overrightarrow{E}$ and $val(f) = c(C)$. Consider the set of edges

$$M = \{\{x_i, y_j\} : f(x_i, y_j) = 1\}.$$

Assume that there is $\{x_i, y_j\}, \{x_i, y_r\} \in M$ then $f(E^+_{x_i}) \geq 2 > 1 = f(E^-_{x_i})$ which is a contradiction to the definition of a flow f. Likewise, if $\{x_i, y_j\}, \{x_s, y_j\} \in M$ then $f(E^+_{y_j}) = 1 < 2 \leq f(E^-_{y_j})$ which is also a contradiction. Therefore M is a matching. Moreover, choose $U = \{s, x_1, \ldots, x_k\}$ we have

$$f(\overrightarrow{U}) - f(\overleftarrow{U}) = f(\overrightarrow{U}) = |M|.$$

Thus, $|M| = val(f) = c(C) = |K|$. It means that we find a matching M and a vertex cover K with the same size. It implies that M is a maximum matching and K is a minimum vertex cover. It completes the proof. □

Exercise 6.13.1 (E13.1: Connectivity in digraphs)**.** *A digraph is connected if the underlying graph is connected. (The underlying graph of a digraph is produced by removing the orientation of the arcs to produce edges, that is, replacing each arc (v, w) by an edge $\{v, w\}$.). A digraph is strongly connected if for every vertices v and w there is a walk from v to w. Give an example of a digraph that is connected but not strongly connected.*

Exercise 6.13.2 (E13.2: Euler circuit and Hamilton path in digraphs)**.** *(i) A digraph has an Euler circuit if there is a closed walk that uses every arc exactly once. Show that a digraph with no vertices of degree 0 has an Euler circuit if and only if it is connected and $d^+(v) = d^-(v)$ for all vertices v.*

(ii) *A tournament is an oriented complete graph. That is, it is a digraph on n vertices, containing exactly one of the arcs (v, w) and (w, v) for every pair of vertices. A Hamilton path is a walk that uses every vertex exactly once. Show that every tournament has a Hamilton path.*

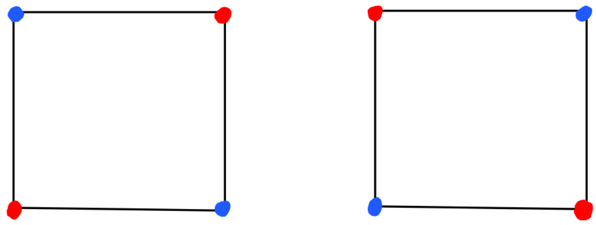
Exercise 6.13.3 (E13.3: Champion)**.** *Interpret a tournament as follows: the vertices are players. If (v, w) is an arc, player v beat w. Say that v is a champion if for every other player w, either v beat w or v beat a player who beat w.*

(i) *Show that a player with the maximum number of wins is a champion.*

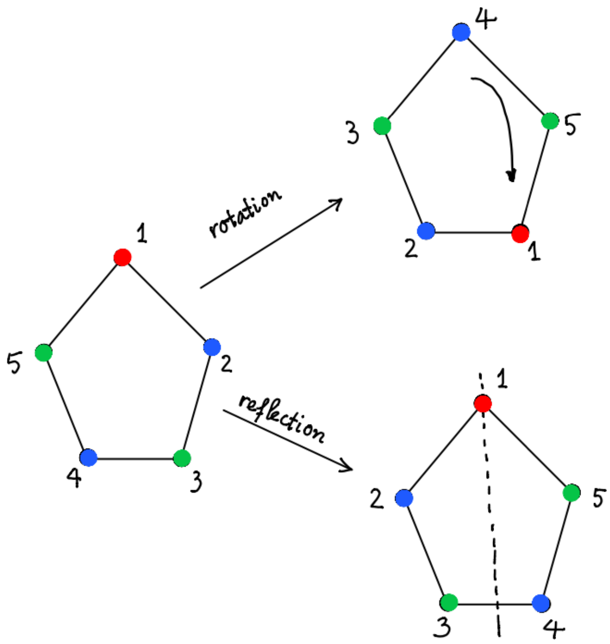
(ii) *Find a 5-vertex tournament in which every player is a champion.*

6.14. Pólya-Redfield Counting

Motivation: We have known about the number of ways to properly color a graph with k colors, based on the chromatic polynomial. For example, the chromatic polynomial for the graph C_4 is $P_{C_4}(k) = k^4 - 4k^3 + 6k^2 - 3k$, and then $P_{C_4}(2) = 2$. The two colorings are shown in the figure below, but they are the same coloring in the sense that one can be turned into the other by simply rotating the graph. We will then consider a slightly different sort of coloring problem, in which we count the "truly different" colorings of objects.



To simplify the problem, in this section we will not require that adjacent vertices must have different colors (counting the number of different proper colorings of graphs can also be done, but it is more complicated) therefore the number of colorings of a graph of n vertices with k colors is $|\mathcal{C}| = k^n$. We first define in which sense two colorings are the same and then count how many distinct colorings. For example, consider a regular pentagon with a coloring c. We can think of a rotation or a reflection through a vertical line to have another coloring which is considered as the same c.



Both two above motions (rotation and reflection) can be thought of as permutations $\sigma_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 1 & 2 \end{pmatrix}$ for the rotation and $\mu_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 5 & 4 & 3 & 2 \end{pmatrix}$ for the reflection through a vertical line which contains 1. Note that if G is a set of permutations that we wish to use to define the "same coloring" relation then G must satisfy the group property.

Definition 6.14.1. A set of permutations G is called a group of permutations if G has the group property, i.e., satisfies:

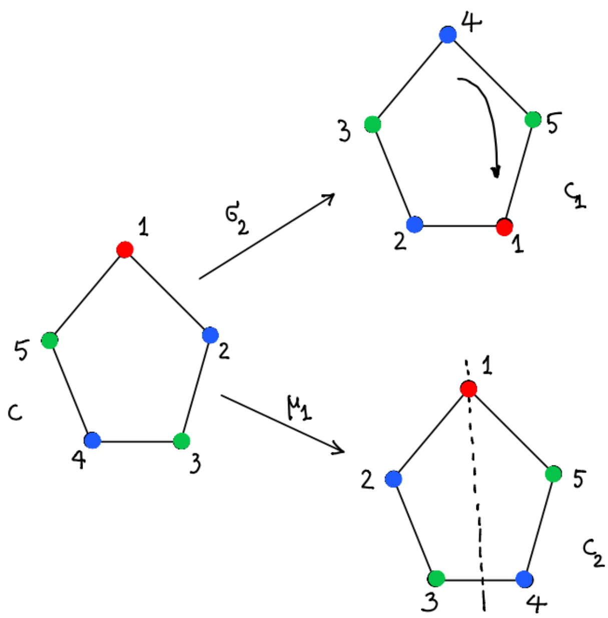
1. If σ_1 and σ_2 are in G then so is $\sigma_1 \circ \sigma_2$;
2. The identity permutation id is in G;
3. If $\sigma \in G$ then $\sigma^{-1} \in G$.

Example 6.14.1. (i) The group of all permutations of $\{1, 2, \dots, n\}$ is denoted S_n, the symmetric group on n items. It satisfies the three required conditions by simple properties of bijections.

(ii) The dihedral group of "rigid motions", i.e., combinations of rotations and reflections $D_5 = \{id, \sigma_1, \sigma_2, \sigma_3, \sigma_4, \mu_1, \mu_2, \mu_3, \mu_4, \mu_5\}$ where
$id = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix}$ is the identity element or the trivial rotation,
$\sigma_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \end{pmatrix}$ is the rotation to the right one,
$\sigma_2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 1 & 2 \end{pmatrix}$ is the rotation to the right two,
$\sigma_3 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 5 & 1 & 2 & 3 \end{pmatrix}$ is the rotation to the right three,
$\sigma_4 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 1 & 2 & 3 & 4 \end{pmatrix}$ is the rotation to the right four,
$\mu_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 5 & 4 & 3 & 2 \end{pmatrix}$ is the reflection through a line contains 1,
$\mu_2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 1 & 5 & 4 \end{pmatrix}$ is the reflection through a line contains 2,
$\mu_3 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 4 & 3 & 2 & 1 \end{pmatrix}$ is the reflection through a line contains 3,
$\mu_4 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 5 & 4 & 3 \end{pmatrix}$ is the reflection through a line contains 4,
$\mu_5 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 3 & 2 & 1 & 5 \end{pmatrix}$ is the reflection through a line contains 5.

Definition 6.14.2. Given two colorings $c_1, c_2 \in \mathcal{C}$. We say c_1 and c_2 are the same in modulo G, denoted by $c_1 \sim_G c_2$, if there is $\sigma \in G$ such that $\sigma(c_1) = c_2$.

Example 6.14.2. Three colorings c, c_1, c_2 are the same in modulo D_5 because $c_1 = \sigma_2(c)$ and $c_2 = \mu_1(c)$ with $\sigma_2, \mu_1 \in D_5$ as in Example 6.14.1(ii):



Definition 6.14.3. If $c \in \mathcal{C}$ then $[c] := \{d \in \mathcal{C} : d \sim_G c\}$ is called the orbit of c, also called as the equivalence class of c, which groups colorings that are the same as c. The number of truly different colorings that we want to count is then the number of orbits. Denote by $G(c) = \{\sigma \in G : \sigma(c) = c\}$ the set of permutations in G that fix c.

Lemma 6.14.1. *$G(c)$ is a group of permutations.*

Proof. We check the group property of $G(c)$:

- For any $\sigma_1, \sigma_2 \in G(c)$ we have $\sigma_1 \circ \sigma_2(c) = \sigma_1(c) = c$ therefore $\sigma_1 \circ \sigma_2 \in G(c)$;
- We always have $id(c) = c$ therefore $id \in G(c)$;
- For any $\sigma \in G(c)$ we have $\sigma^{-1}(c) = \sigma^{-1}(\sigma(c)) = id(c) = c$ therefore $\sigma^{-1} \in G(c)$.

□

Lemma 6.14.2. $|G| = |[c]||G(c)|$.

Proof. Because $G(c)$ is a group (Lemma 6.14.1), it is well known from abstract algebra that the relation $\varphi \sim \sigma$ if $\sigma^{-1} \circ \varphi \in G(c)$ is an equivalence relation in G and for any $\sigma \in G$, the equivalence class $[\sigma] = \{\sigma \circ \varphi, \varphi \in G(c)\}$ which has the same size to $G(c)$, i.e., $|[\sigma]| = |G(c)|, \forall \sigma \in G$. Therefore we only need to show that the set of equivalence classes $E := \{[\sigma], \sigma \in G\}$ has the same size to the equivalence of c, i.e. $|E| = |[c]|$. In fact, define a map $g : [c] \to E$ as follows: if $d \in [c]$ then there is $\sigma \in G$ such that $\sigma(c) = d$, then we define $g(d) := [\sigma]$. We will prove that g is well-defined and bijective. In fact,

- if $d = \sigma_1(c) = \sigma_2(c)$ then $\sigma_2^{-1} \circ \sigma_1(c) = c$, so $\sigma_1 \sim \sigma_2$ and $[\sigma_1] = [\sigma_2]$. Therefore g is well-defined.
- if $g(d_1) = g(d_2)$ then $d_1 = \sigma_1(c)$ and $d_2 = \sigma_2(c)$ for some $\sigma_1, \sigma_2 \in G$ and $[\sigma_1] = [\sigma_2]$. Hence $\sigma_2^{-1} \circ \sigma_1(c) = c$, so $d_1 = \sigma_1(c) = \sigma_2(c) = d_2$. Therefore g is 1-1.
- for any $[\sigma] \in E$ we have $\sigma(c) \in [c]$ such that $g(\sigma(c)) = [\sigma]$ therefore g is onto.

Thus, $|[c]| = |E| = \frac{|G|}{|G(c)|}$ which implies the proof. □

Corollary 6.14.1. *If $c \sim d$ then $|G(c)| = |G(d)|$.*

Proof. Because $c \sim_G d$, $[c] = [d]$ and therefore $|G(c)| = \frac{|G|}{|[c]|} = \frac{|G|}{|[d]|} = |G(d)|$. □

Definition 6.14.4. If group G acts on the colorings of an object and $\sigma \in G$, $fix(\sigma)$ is the set of colorings that are fixed by σ.

Theorem 6.14.1 (Burnside's theorem)**.** *If group G acts on the colorings of an object, the number of distinct colorings modulo G is*

$$K_G = \frac{1}{|G|} \sum_{\sigma \in G} |fix(\sigma)|.$$

Proof. Denote by K the number of distinct colorings modulo G, i.e., $\mathcal{C}$ is divided into K distinct orbits $[c_1], \ldots, [c_K]$. Then we have

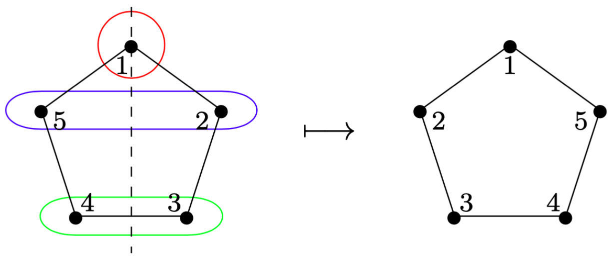
$$\sum_{c \in \mathcal{C}} |G(c)| = \sum_{i=1}^{K} \sum_{c \in [c_i]} |G(c)| \overset{Lemma\ 6.14.2}{=} \sum_{i=1}^{K} \sum_{c \in [c_i]} \frac{|G|}{|[c]|} = \sum_{i=1}^{K} \sum_{c \in [c_i]} \frac{|G|}{|[c_i]|} = \sum_{i=1}^{K} |G| = K|G|.$$

Therefore $K = \frac{1}{|G|} \sum_{c \in \mathcal{C}} |G(c)|$. On the other hand, we have

$$\sum_{c \in \mathcal{C}} |G(c)| = \sum_{c \in \mathcal{C}} \sum_{\sigma \in G(c)} 1 = \sum_{c \in \mathcal{C}, \sigma \in G : \sigma(c) = c} 1 = \sum_{\sigma \in G} \sum_{c \in fix(\sigma)} 1 = \sum_{\sigma \in G} |fix(\sigma)|.$$

Thus, $K = \frac{1}{|G|} \sum_{c \in \mathcal{C}} |G(c)| = \frac{1}{|G|} \sum_{\sigma \in G} |fix(\sigma)|$. This completes the proof. □

Definition 6.14.5. Every permutation can be written in its cycle form: for example, the permutation $\sigma = \mu_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 5 & 4 & 3 & 2 \end{pmatrix}$



can be written in its cycle form $(1)(2,5)(3,4)$, where a cycle is a sequence $(x_1, x_2, \ldots, x_k)$, meaning that $\sigma(x_1) = x_2, \sigma(x_2) = x_3$, and so on until $\sigma(x_k) = x_1$.

Corollary 6.14.2. *If group G acts on the colorings of an object, the number of distinct colorings modulo G with k colors is*

$$K_G(k) = \frac{1}{|G|} \sum_{\sigma \in G} k^{\#\sigma},$$

where $\#\sigma$ is the number of cycles in cycle form of σ.

Proof. Note that if $c \in fix(\sigma)$ then c has the same value on the vertices in each cycle, and the total number of colors fixed by σ is $|fix(\sigma)| = k^{\#\sigma}$. □

Example 6.14.3. We try to find the number of distinct colorings modulo D_5 with k colors. Recall from Example 6.14.1 that $D_5 = \{id, \sigma_1, \sigma_2, \sigma_3, \sigma_4, \mu_1, \mu_2, \mu_3, \mu_4, \mu_5\}$. Note that $id = (1)(2)(3)(4)(5)$ therefore $\#id = 5$. $\sigma_1 = (1,2,3,4,5)$ therefore $\#\sigma_1 = 1$. Similarly we have $\#\sigma_2 = \#\sigma_3 = \#\sigma_4 = 1$. $\mu_1 = (1)(2,5)(3,4)$ therefore $\#\mu_1 = 3$. Similarly we have $\#\mu_2 = \#\mu_3 = \#\mu_4 = \#\mu_5 = 3$. Thus, the number of distinct colorings modulo D_5 with k colors is

$$K_{D_5}(k) = \frac{1}{|D_5|} \sum_{\sigma \in D_5} k^{\#\sigma} = \frac{1}{10}(k^5 + 4k + 5k^3).$$

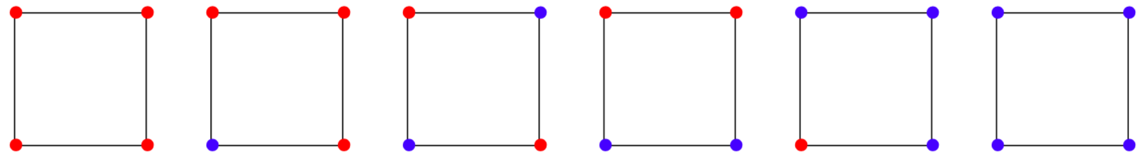
For $k = 3$ the number of different 3-colorings is $K_{D_5}(3) = 39$.

Example 6.14.4. We find the number of distinct colorings of the vertices of a square with k colors, modulo D_4 where D_4 consist of four rotations $r_0, r_{90}, r_{180}, r_{270}$ where r_i is the counterclockwise rotation by i degrees and the four reflections f_H, f_V, f_D, f_A as below

In cycle form, we have $r_0 = (1)(2)(3)(4), r_{90} = (1,4,3,2), r_{180} = (1,3)(2,4), r_{270} = (1,2,3,4), f_H = (1,4)(2,3), f_V = (1,2)(3,4), f_D = (1)(2,4)(3), f_A = (1,3)(2)(4)$. Therefore we have

$$K_{D_4}(k) = \frac{1}{8}(k^4 + 2k^3 + 3k^2 + 2k).$$

For $k = 2$ we have $K_{D_4}(2) = 6$ as below:



Example 6.14.5. How many non-isomorphic graphs are there on four vertices?

Solve. We translate this problem into the coloring problem: Color the edges of the complete graph K_4 with two colors, say black and white. The black edges form a graph, the white edges are the remaining ones. The group G we need to consider is all permutations of the six edges of K_4 induced by permutations of the vertices in the sense that if $\sigma \in S_4$ is a permutation of the vertices then $\tilde{\sigma} \in S_6$ is constructed by $\tilde{\sigma}(ij) = \sigma(i)\sigma(j)$. For example, if $\sigma = id = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}$ then $\tilde{\sigma} = \begin{pmatrix} 12 & 13 & 14 & 23 & 24 & 34 \\ 12 & 13 & 14 & 23 & 24 & 34 \end{pmatrix}$; if $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$ then $\tilde{\sigma} = \begin{pmatrix} 12 & 13 & 14 & 23 & 24 & 34 \\ 23 & 24 & 12 & 34 & 13 & 14 \end{pmatrix}$. The number of non-isomorphic graphs on four vertices is nothing but the number of distinct colorings on six edges with two colors, modulo $G = \{\tilde{\sigma}, \sigma \in S_4\}$. Note that $|G| = 4! = 24$. All we need to know is the number of cycles in each permutation; we consider a number of cases.

Case 1. The identity permutation on the vertices induces the identity permutation on the 6 edges, with 6 cycles, so the contribution to the sum is 2^6.

Case 2. A 4-cycle on the vertices induces a permutation of the edges consisting of one 4-cycle and one 2-cycle, that is, two cycles. There are $3! = 6$ 4-cycles on the vertices, so the contribution of all of these is $6 \cdot 2^2$.

Case 3. A permutation of the vertices consisting of a 3-cycle and a 1-cycle induces a permutation of the edges consisting of two 3-cycles. There are $4 \cdot 2 = 8$ such permutations of the vertices, so the contribution of all is $8 \cdot 2^2$.

Case 4. A permutation of the vertices consisting of two 2-cycles induces a permutation of the edges consisting of two 1-cycles and two 2-cycles. There are $\frac{1}{2}\binom{4}{2} = 3$ such permutations, so the contribution is $3 \cdot 2^4$.

Case 5. A permutation of the vertices consisting of a 2-cycle and two 1-cycles induces a permutation of the edges consisting of two 1-cycles and two 2-cycles. There are $\binom{4}{2} = 6$ such permutations, so the contribution is $6 \cdot 2^4$.

Therefore the number of distinct colorings, i.e., the number of non-isomorphic graphs on four vertices is

$$\frac{1}{24}(2^6 + 6\cdot 2^2 + 8\cdot 2^2 + 3\cdot 2^4 + 6\cdot 2^4) = 11.$$

□

Suppose we are interested in a more detailed inventory of the colorings of an object, namely, instead of the total number of colorings we seek the number of colorings with a given number of each color.

Example 6.14.6. How many distinct ways are there to color the vertices of a regular pentagon modulo D_5 so that one vertex is red, two are blue, and two are green?

Solve. We can approach this as before, that is, the answer is

$$\frac{1}{|D_5|}\sum_{\sigma\in D_5}|fix(\sigma)|,$$

where $fix(\sigma)$ now means the colorings with one red, two blues, and two greens that are fixed by σ. More precisely, we have $|fix(id)| = \binom{5}{2}\binom{3}{2}$, $|fix(\sigma_i)| = 0$ for $i = 1,\ldots,4$, and $|fix(\mu_i)| = 2$ for $i = 1,\ldots,5$. Thus, the number of distinct colorings is

$$\frac{1}{10}(30+0+0+0+0+2+2+2+2+2) = 4.$$

□

Definition 6.14.6. The type of a permutation $\sigma\in S_n$ is $\tau(\sigma) = (\tau_1(\sigma),\tau_2(\sigma),\ldots,\tau_n(\sigma))$, where $\tau_i(\sigma)$ is the number of i-cycles in the cycle form of σ.

Example 6.14.7. For $\sigma = id\in D_5$, $\tau(id) = (5,0,0,0,0)$. For $\sigma=\sigma_1 = (1,2,3,4,5)\in D_5$, $\tau(\sigma_1) = (0,0,0,0,1)$. For $\sigma=\mu_1 = (1)(2,5)(3,4)\in D_5$, $\tau(\mu_1) = (1,2,0,0,0)$.

Definition 6.14.7. The cycle index of G is

$$\mathcal{P}_G = \frac{1}{|G|}\sum_{\sigma\in G}\prod_{i=1}^{n} x_i^{\tau_i(\sigma)}.$$

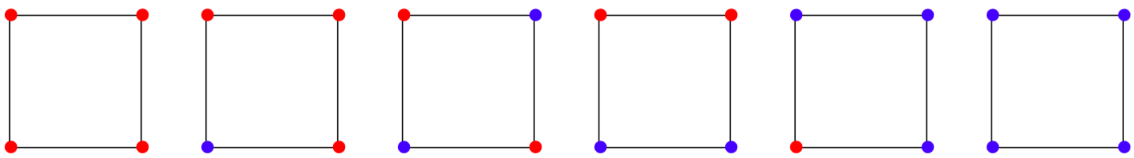
Example 6.14.8. The cycle index of D_4 is

$$\frac{1}{8}(x_1^4 + x_4^1 + x_2^2 + x_4^1 + x_2^2 + x_2^2 + x_1^2x_2 + x_1^2x_2) = \frac{1}{8}x_1^4 + \frac{1}{4}x_1^2x_2 + \frac{3}{8}x_2^2 + \frac{1}{4}x_4.$$

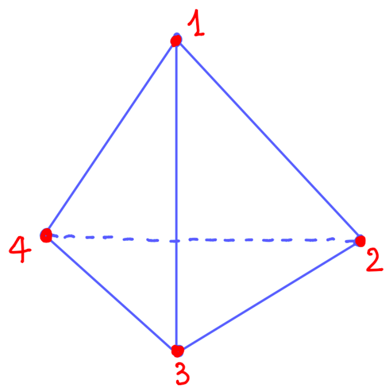
Substituting as above gives

$$\frac{1}{8}(r+b)^4 + \frac{1}{4}(r+b)^2(r^2+b^2) + \frac{3}{8}(r^2+b^2)^2 + \frac{1}{4}(r^4+b^4) = r^4 + r^3b + 2r^2b^2 + rb^3 + b^4.$$

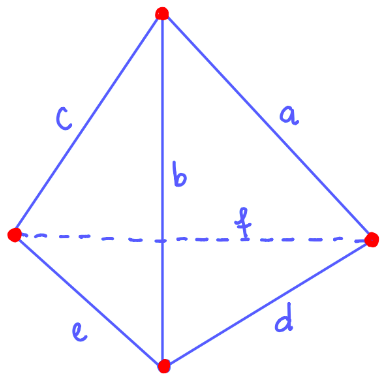
Thus there is one all red coloring, one with three reds and one blue, and so on, as shown below:



Exercise 6.14.1 (E14.1: Permutations in the regular tetrahedron)**.** *(i) Find the* 12 *permutations of the vertices of the regular tetrahedron corresponding to the* 12 *rigid motions of the regular tetrahedron and write them in cycle form. Use the labeling below.*



(ii) Find the 12 *permutations of the edges of the regular tetrahedron corresponding to the* 12 *rigid motions of the regular tetrahedron and write them in cycle form. Use the labeling below.*



Exercise 6.14.2 (E14.2: Number of different colorings)**.** *(i) Find the number of different colorings of the vertices of a regular tetrahedron with k colors, modulo the rigid motions.*

(ii) Find the number of different colorings of the edges of a regular tetrahedron with k colors, modulo the rigid motions.

Exercise 6.14.3 (E14.3: Cycle index)**.** *(i) Find the cycle index $\mathcal{P}_G$ for the group of permutations of the vertices of a regular tetrahedron induced by the rigid motions.*

(ii) Find the cycle index $\mathcal{P}_G$ for the group of permutations of the edges of K_5.

A. Solutions

E1.1: Dance partners

Denote by M the set of 20 men and W the set of 12 women. A pair of dance partner is of the form $(m, w) \in M \times W$. Therefore the number of pairs are

$$|M| \times |W| = 20 \times 12 = 240.$$

E1.2: Poker hand

Denote by V the set of 13 distinct values and S the set of 4 suits. Each card will then be represented by AB with $A \in V$ and $B \in S$.

(i) Full house: A full house hand is of the form $(A_1B_1, A_1B_2, A_1B_3, A_2B_4, A_2B_5)$ where $A_1 \neq A_2 \in V$, $B_1 \neq B_2 \neq B_3 \in S$, and $B_4 \neq B_5 \in S$. Therefore the number of full house hands is

$$\begin{aligned}&\#\{(A_1 \neq A_2) \in V^2 \text{ with order}\} \times \#\{(B_1 \neq B_2 \neq B_3) \in S^3 \text{ without order}\} \times \\ &\#\{(B_4 \neq B_5) \in S^2 \text{ without order}\} \\ &= \binom{13}{1}\binom{12}{1} \times \binom{4}{3} \times \binom{4}{2} = 3744.\end{aligned}$$

(ii) Flush: A flush hand is of the form $(A_1B, A_2B, A_3B, A_4B, A_5B)$ where $A_1 \neq A_2 \neq A_3 \neq A_4 \neq A_5 \in V$, $B \in S$. Therefore the number of flush hands is

$$\begin{aligned}&\#\{(A_1 \neq A_2 \neq A_3 \neq A_4 \neq A_5) \in V^5 \text{ without order}\} \times \#\{B \in S\} \\ &= \binom{13}{5} \times \binom{4}{1} = 5148.\end{aligned}$$

E1.3: Boarding

Consider n passengers, for each $k = 1, \ldots, n$ denote by P_k is the k^{th} passenger in line, $Seat_k$ his designated seat, and $P_{n,k}$ the probability that P_k sits in $Seat_k$.

Claim 1: for $3 \leq k \leq n$, when P_k turns to sit, $Seat_2, Seat_3, \ldots, Seat_{k-1}$ are all taken.
In fact, we prove this claim by induction on k:

- For $k = 3$, when P_3 turns to sit, $Seat_2$ is taken because either P_1 sat on it or otherwise P_2 sat on it.

- Assume the claim is true until $k = m$ with $3 \leq m \leq n - 1$, we will prove the claim is also true for $k = m + 1$. From the assumption, when P_m turns to sit, $Seat_2, \ldots, Seat_{m-1}$ are taken. If at this time, $Seat_m$ was not taken, P_m must sit on it. Therefore when P_{m+1} turns to sit, $Seat_2, \ldots, Seat_m$ are taken. It proves the Claim 1.

Claim 2: We prove a stronger result as follows: for $3 \le k \le n$ then

$$\alpha_{k,i}^{(n)} := \mathbb{P}(P_i \text{ sits on } Seat_k) = \begin{cases} \frac{1}{n}, & \text{if } i = 1, \\ \frac{1}{n-i+2}\frac{1}{n-i+1}, & \text{if } 2 \le i \le k-1, \\ P_{n,k}, & \text{if } i = k, \\ 0, & \text{if } k+1 \le i \le n. \end{cases}$$

In fact, for $i = 1$, we have $\alpha_{k,1}^{(n)} = \mathbb{P}(P_1 \text{ sits on } Seat_k) = \frac{1}{n}$ because P_1 chosen uniformly random from $\{Seat_1, \dots, Seat_n\}$. For $i \ge k+1$, we have $\alpha_{k,i}^{(n)} = 0$ due to the Claim 1.
We now prove the case $2 \le i \le k-1$ by induction on k. In fact,

- For $k = 3$, we have $i = 2$:

  $$\mathbb{P}(P_2 \text{ sits on } Seat_3) = \mathbb{P}(P_1 \text{ sits on } Seat_2)\mathbb{P}(P_2 \text{ sits on } Seat_3 | P_1 \text{ sat on } Seat_2) = \frac{1}{n}\frac{1}{n-1}.$$

  The first term is $\frac{1}{n}$ because P_1 chosen uniformly random from $\{Seat_1, \dots, Seat_n\}$ and the second term is $\frac{1}{n-1}$ because P_2 chosen uniformly random from $\{Seat_1, Seat_3, \dots, Seat_n\}$ (P_1 sat on $Seat_2$).

- Assume that the claim is true until k, i.e.,

  $$\alpha_{i,j}^{(n)} = \frac{1}{n-j+2}\frac{1}{n-j+1}, \quad \forall 2 \le i \le k, 2 \le j \le i-1. \tag{1.0.1}$$

  We will prove that the claim is true for $k+1$. In fact, for every $2 \le i \le k$ we have

  $$\begin{aligned} \alpha_{k+1,i}^{(n)} &= \mathbb{P}(P_i \text{ sits on } Seat_{k+1}) = \mathbb{P}(Seat_i \text{ is taken by someone from } \{P_1, \dots, P_{i-1}\}) \\ &\quad \times \mathbb{P}(P_i \text{ sits on } Seat_{k+1} | Seat_i \text{ is taken by someone from } \{P_1, \dots, P_{i-1}\}) \\ &= \sum_{j=1}^{i-1} \alpha_{i,j}^{(n)} \frac{1}{n-i+1}. \end{aligned}$$

  The first term is $\sum_{j=1}^{i-1} \alpha_{i,j}^{(n)}$ because it is a sum of the probability of P_j sits on $Seat_i$ for $j = 1, \dots, i-1$ and the second term is $\frac{1}{n-i+1}$ because P_i chosen uniformly random from the set of $n-i+1$ remain seats. From the assumption (1.0.5) we obtain for all $2 \le i \le k$

  $$\alpha_{k+1,i}^{(n)} = \Big(\frac{1}{n} + \sum_{j=2}^{i-1} \frac{1}{n-j+2}\frac{1}{n-j+1}\Big)\frac{1}{n-i+1} = \frac{1}{n-i+2}\frac{1}{n-i+1}.$$

  This proves the Claim 2.

Note that $\sum_{i=1}^{n} \alpha_{k,i}^{(n)} = \mathbb{P}(Seat_k \text{ is taken from someone from } \{P_1, \dots, P_n\}) = 1$, we have

$$P_{n,k} = 1 - \sum_{i=1}^{k-1} \alpha_{k,i}^{(n)} = \frac{n-k+1}{n-k+2}.$$

When $k = n$ we have $P_{n,n} = \frac{1}{2}$.

E2.1: Binomial coefficients

1. Given a natural number $n \in \mathbb{N}_0$. We have

$$\begin{aligned}\sum_{k=0}^{n} \frac{(-1)^k}{k+1}\binom{n}{k} &= \sum_{k=0}^{n} \frac{(-1)^k}{k+1}\frac{n!}{k!(n-k)!} = \sum_{k=0}^{n} \frac{(-1)^k}{n+1}\frac{(n+1)!}{(k+1)!(n-k)!} \\ &= -\frac{1}{n+1}\sum_{k=0}^{n}(-1)^{k+1}\binom{n+1}{k+1} = -\frac{1}{n+1}\sum_{k=1}^{n+1}(-1)^k\binom{n+1}{k} \\ &= -\frac{1}{n+1}\left((1-1)^{n+1} - (-1)^0\binom{n+1}{0}\right) = \frac{1}{n+1}.\end{aligned}$$

Another way to prove:

$$\frac{1}{n+1} = \int_0^1 t^n dt = \int_{-1}^0 (1+x)^n dx = \int_{-1}^0 \sum_{k=0}^{n}\binom{n}{k}x^k dx = \sum_{k=0}^{n}\binom{n}{k}\int_{-1}^0 x^k dx = \sum_{k=0}^{n}\frac{(-1)^k}{k+1}\binom{n}{k}.$$

2. Given three natural numbers $m, n, k \in \mathbb{N}_0$ with $k \leq \min\{m, n\}$. We first apply the binomial theorem to have

$$(1+x)^{m+n} = \sum_{l=0}^{m+n}\binom{n+m}{l}x^l$$

and

$$\begin{aligned}(1+x)^m(1+x)^n &= \left(\sum_{i=0}^{m}\binom{m}{i}x^i\right)\left(\sum_{j=0}^{n}\binom{n}{j}x^j\right) \\ &= \sum_{l=0}^{m+n} a_l x^l\end{aligned}$$

where $a_l = \sum_{i=0}^{l}\binom{m}{i}\binom{n}{l-i}$ with the convention that $\binom{m}{i} = 0$ when $i > m$ and $\binom{n}{l-i} = 0$ when $l - i > n$. Because that $(1+x)^m(1+x)^n = (1+x)^{m+n}$, all of the corresponding coefficients of these two polynomials are the same. Therefore at the k^{th} coefficient we have $\sum_{i=0}^{k}\binom{m}{i}\binom{n}{k-i} = a_k = \binom{m+n}{k}$. It completes the proof.

E2.2: Number of integer solutions

1. Find the number of integer solutions to

$$x_1 + x_2 + x_3 + x_4 + x_5 = 2020, \quad x_1 \geq -3, x_2 \geq -2, x_3 \geq 4, x_4 \geq 3, x_5 \geq 10:$$

First, we substitute $y_1 = x_1 + 3; y_2 = x_2 + 2; y_3 = x_3 - 4; y_4 = x_4 - 3; y_5 = x_5 - 10$. Then, our problem becomes finding the number of non-negative integer solutions for

$$y_1 + y_2 + y_3 + y_4 + y_5 = 2008.$$

This number is also the number of submultisets of size 2008 in the multiset $\{\infty \cdot y_1, \infty \cdot y_2, \infty \cdot y_3, \infty \cdot y_4, \infty \cdot y_5\}$ and is given by $\binom{2008+5-1}{5-1} = \binom{2012}{4} = 680776881995$.

2. Find the number of integer solutions to

$$x_1 + x_2 + x_3 + x_4 = 30, \quad 1 \le x_1 \le 6, 2 \le x_2 \le 8, 0 \le x_3 \le 10, 3 \le x_4 \le 12:$$

First, we substitute $y_1 = x_1 - 1; y_2 = x_2 - 2; y_3 = x_3; y_4 = x_4 - 3$. Then, our problem becomes finding the number of non-negative integer solutions for

$$y_1 + y_2 + y_3 + y_4 = 24, \quad y_1 \le 5, y_2 \le 6, y_3 \le 10, y_4 \le 9.$$

Denote by

$$\begin{aligned} S &= \{(y_1, y_2, y_3, y_4) \in \mathbb{N}_0^4 : y_1 + y_2 + y_3 + y_4 = 24\} \\ A_1 &= \{(y_1, y_2, y_3, y_4) \in S : y_1 \ge 6\} \\ A_2 &= \{(y_1, y_2, y_3, y_4) \in S : y_2 \ge 7\} \\ A_3 &= \{(y_1, y_2, y_3, y_4) \in S : y_3 \ge 11\} \\ A_4 &= \{(y_1, y_2, y_3, y_4) \in S : y_4 \ge 10\}. \end{aligned}$$

We also denote by $AB := A \cap B$ and $\bar{A} = S - A$ for $A, B \subseteq S$. Then the number we need to find is $|\bar{A}_1\bar{A}_2\bar{A}_3\bar{A}_4|$ which can be calculated by using the Inclusion-Exclusion principle as follows

$$\begin{aligned} |\bar{A}_1\bar{A}_2\bar{A}_3\bar{A}_4| = |S| &- (|A_1| + |A_2| + |A_3| + |A_4|) + (|A_1A_2| + |A_1A_3| + |A_1A_4| + |A_2A_3| \\ &+ |A_2A_4| + |A_3A_4|) - (|A_1A_2A_3| + |A_1A_2A_4| + |A_1A_3A_4| + |A_2A_3A_4|) + |A_1A_2A_3A_4| \end{aligned}$$

We know that $|S|$ is the number of submultisets of size 24 in the multiset $\{\infty \cdot y_1, \infty \cdot y_2, \infty \cdot y_3, \infty \cdot y_4\}$ and is given by $\binom{24+4-1}{4-1} = \binom{27}{3}$.
Note that $A_1A_3A_4 = \emptyset$ because if $\exists y = (y_1, y_2, y_3, y_4) \in A_1A_3A_4$ then $y_1 + y_3 + y_4 \ge 6 + 11 + 10 > 24$ therefore $y \notin S$ which is a contradiction. Thus, $|A_1A_3A_4| = 0$. Similarly we have $|A_2A_3A_4| = |A_1A_2A_3A_4| = 0$.
To calculate other terms we use again corresponding substitutions so that the four new variables are non-negative integers and apply the formula of the number of non-negative integer solutions. Then we have

$$|A_1| = |\{(y_1 - 6, y_2, y_3, y_4) \in \mathbb{N}_0^4 : (y_1 - 6) + y_2 + y_3 + y_4 = 24 - 6\}| = \binom{24 - 6 + 4 - 1}{4 - 1} = \binom{21}{3};$$

$$|A_2| = |\{(y_1, y_2 - 7, y_3, y_4) \in \mathbb{N}_0^4 : y_1 + (y_2 - 7) + y_3 + y_4 = 24 - 7\}| = \binom{24 - 7 + 4 - 1}{4 - 1} = \binom{20}{3};$$

$$|A_3| = |\{(y_1, y_2, y_3 - 11, y_4) \in \mathbb{N}_0^4 : y_1 + y_2 + (y_3 - 11) + y_4 = 24 - 11\}| = \binom{24 - 11 + 4 - 1}{4 - 1} = \binom{16}{3};$$

$$|A_4| = |\{(y_1, y_2, y_3, y_4 - 10) \in \mathbb{N}_0^4 : y_1 + y_2 + y_3 + (y_4 - 10) = 24 - 10\}| = \binom{24 - 10 + 4 - 1}{4 - 1} = \binom{17}{3};$$

$$|A_1A_2| = |\{(y_1-6, y_2-7, y_3, y_4) \in \mathbb{N}_0^4 : (y_1-6)+(y_2-7)+y_3+y_4 = 11\}| = \binom{11+4-1}{4-1} = \binom{14}{3};$$
$$|A_1A_3| = |\{(y_1-6, y_2, y_3-11, y_4) \in \mathbb{N}_0^4 : (y_1-6)+y_2+(y_3-11)+y_4 = 7\}| = \binom{10}{3};$$
$$|A_1A_4| = |\{(y_1-6, y_2, y_3, y_4-10) \in \mathbb{N}_0^4 : (y_1-6)+y_2+y_3+(y_4-10) = 8\}| = \binom{11}{3};$$
$$|A_2A_3| = |\{(y_1, y_2-7, y_3-11, y_4) \in \mathbb{N}_0^4 : y_1+(y_2-7)+(y_3-11)+y_4 = 6\}| = \binom{9}{3};$$
$$|A_2A_4| = |\{(y_1, y_2-7, y_3, y_4-10) \in \mathbb{N}_0^4 : y_1+(y_2-7)+y_3+(y_4-10) = 7\}| = \binom{10}{3};$$
$$|A_3A_4| = |\{(y_1, y_2, y_3-11, y_4-10) \in \mathbb{N}_0^4 : y_1+y_2+(y_3-11)+(y_4-10) = 3\}| = \binom{6}{3};$$
$$|A_1A_2A_3| = |\{(y_1-6, y_2-7, y_3-11, y_4) \in \mathbb{N}_0^4 : (y_1-6)+(y_2-7)+(y_3-11)+y_4 = 0\}| = \binom{3}{3};$$
$$|A_1A_2A_4| = |\{(y_1-6, y_2-7, y_3, y_4-10) \in \mathbb{N}_0^4 : (y_1-6)+(y_2-7)+y_3+(y_4-10) = 1\}| = \binom{4}{3}.$$

In total, we have

$$\begin{aligned}|\bar{A}_1\bar{A}_2\bar{A}_3\bar{A}_4| &= \binom{27}{3} - \left(\binom{21}{3}+\binom{20}{3}+\binom{16}{3}+\binom{17}{3}\right) + \left(\binom{14}{3}+\binom{10}{3}+\binom{11}{3}+\binom{9}{3}\right.\\ &\quad \left.+\binom{10}{3}+\binom{6}{3}\right) - \left(\binom{3}{3}+\binom{4}{3}+0+0\right)+0\\ &= 83.\end{aligned}$$

E2.3: Derangements

Let D_n be the number of derangements of $[n]$. We denote by S_n is the set of all permutations of $[n]$. Then $|S_n| = n!$. On the other hand, we divide S_n into $n+1$ disjoint subsets $A_k (0 \le k \le n)$ which consists of permutations with precisely k elements placed on their right positions. For each k disjoint numbers $i_1, \ldots, i_k \in [n]$, denote by $B_{i_1,\ldots,i_k}$ the subset of A_k which consists of permutations with numbers $i_1, \ldots, i_k$ placed on their right positions. Note that each element of $B_{i_1,\ldots,i_k}$ is 1-1 corresponding with a permutation of $[n] - \{i_1, \ldots, i_k\}$ which there is no number in its right position. Therefore $|B_{i_1,\ldots,i_k}| = D_{n-k}$. Because there are $\binom{n}{k}$ ways to have k disjoint numbers $i_1, \ldots, i_k \in [n]$, we have $|A_k| = \binom{n}{k} D_{n-k}$ and thus

$$n! = |S_n| = \sum_{k=0}^{n} |A_k| = \sum_{k=0}^{n} \binom{n}{k} D_{n-k}.$$

It completes the proof.

Another way to prove:

We use the result from the lecture: $D_n = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}$ for all $n \ge 0$. Then we need to prove that

$$n! = \sum_{k=0}^{n} \binom{n}{k} (n-k)! \sum_{i=0}^{n-k} \frac{(-1)^i}{i!}$$

which is equivalent to

$$A_n := \sum_{k=0}^{n} \frac{1}{k!} \sum_{i=0}^{n-k} \frac{(-1)^i}{i!} = 1. \qquad (1.0.2)$$

We prove the Claim (1.0.5) by induction over n. In fact, $A_0 = \frac{1}{0!}\frac{(-1)^0}{0!} = 1$. Assume that the claim is true until $n = m$. We will prove it is also true for $n = m + 1$. Indeed, we have

$$
\begin{aligned}
A_{m+1} &= \sum_{k=0}^{m+1} \frac{1}{k!} \sum_{i=0}^{m+1-k} \frac{(-1)^i}{i!} \\
&= \sum_{k=0}^{m} \frac{1}{k!} \sum_{i=0}^{m+1-k} \frac{(-1)^i}{i!} + \frac{1}{(m+1)!}\frac{(-1)^0}{0!} \\
&= \sum_{k=0}^{m} \frac{1}{k!} \left(\sum_{i=0}^{m-k} \frac{(-1)^i}{i!} + \frac{(-1)^{m+1-k}}{(m+1-k)!} \right) + \frac{1}{(m+1)!} \\
&= A_m + \sum_{k=0}^{m} \frac{(-1)^{m+1-k}}{k!(m+1-k)!} + \frac{1}{(m+1)!} \\
&= A_m + \sum_{k=0}^{m+1} \frac{(-1)^{m+1-k}}{k!(m+1-k)!} \\
&= A_m + (1-1)^{m+1} = A_m = 1.
\end{aligned}
$$

It completes the proof.

E3.1: Exponential generating functions

(i) The exponential generating function for the number of permutations with repetition of length n of the set $\{a, b, c\}$, in which there are an odd number of $a's$, an even number of $b's$, and an even number of $c's$ is

$$
\begin{aligned}
f(x) &= \sum_{a\in\mathbb{N}_0\colon a \text{ is odd}} \frac{x^a}{a!} \sum_{b\in\mathbb{N}_0\colon b \text{ is even}} \frac{x^b}{b!} \sum_{c\in\mathbb{N}_0\colon c \text{ is even}} \frac{x^c}{c!} \\
&= \Big(\frac{e^x - e^{-x}}{2}\Big)\Big(\frac{e^x + e^{-x}}{2}\Big)^2 = \frac{1}{8}\Big(e^{3x} + e^x - e^{-x} - e^{-3x}\Big).
\end{aligned}
$$

(ii) The exponential generating function for the number of permutations with repetition of length n of the set $\{a, b, c\}$, in which the number of $a's$ is even and at least 2, the number of $b's$ is even and at most 6, and the number of $c's$ is at least 3 is

$$
\begin{aligned}
f(x) &= \sum_{a\in\mathbb{N}_0\colon a\geq 2,\, a \text{ is even}} \frac{x^a}{a!} \sum_{b\in\mathbb{N}_0\colon b\leq 6,\, b \text{ is even}} \frac{x^b}{b!} \sum_{c\in\mathbb{N}_0\colon c\geq 3} \frac{x^c}{c!} \\
&= \Big(\frac{e^x + e^{-x}}{2} - 1\Big)\Big(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!}\Big)\Big(e^x - 1 - x - \frac{x^2}{2!}\Big).
\end{aligned}
$$

E3.2: Partition of integers

Given two subsets $\mathcal{I}, \mathcal{N} \subseteq \mathbb{N}_0$ and denote by $\mathcal{I}^* = \mathcal{I} - \{0\}$. We have

$$
\prod_{i\in\mathcal{I}^*} \Big(\sum_{n_i\in\mathcal{N}} x^{in_i} \Big) = \sum_{n=0}^{\infty} \Bigg(\sum_{\substack{i_1 n_{i_1} + \cdots + i_k n_{i_k} = n \\ i_1,\ldots,i_k \in \mathcal{I}^* \\ n_{i_1},\ldots,n_{i_k} \in \mathcal{N}}} 1 \Bigg) x^n.
$$

This means that $\prod_{i\in\mathcal{I}^*}\left(\sum_{n_i\in\mathcal{N}} x^{in_i}\right)$ is the generating function for $\{a_n\}_{n\in\mathbb{N}_0}$ with a_n is the number of partitions of n into parts in $\mathcal{I}^*$ with repetitions in $\mathcal{N}$. For distinct parts, it means every part i_j appear at most one time $n_{i_j} \le 1$. In other words, $\mathcal{N} = \{0,1\}$. For even parts, it means $\mathcal{I} = 2\mathbb{N}_0$ and $\mathcal{I}^* = \{2,4,6,\ldots\}$.

(i) The generating function for the number of partitions of an integer into distinct even parts is

$$f(x) = \prod_{i\in\{2,4,6,\ldots\}}\left(\sum_{n_i\in\{0,1\}} x^{in_i}\right) = (1+x^2)(1+x^4)(1+x^6)(1+x^8)\cdots = \prod_{k=1}^{\infty}(1+x^{2k}).$$

(ii) By directing computation, we have

$$\begin{aligned}(1+x^2)(1+x^4)(1+x^6)\cdots(1+x^{30}) &= 1+x^2+x^4+2x^6+2x^8+3x^{10}+4x^{12}+5x^{14}+6x^{16}+\\ &\quad +8x^{18}+10x^{20}+12x^{22}+15x^{24}+18x^{26}+22x^{28}+27x^{30}+\cdots\end{aligned}$$

therefore the number of distinct even partitions of $n = 30$ is 27.

E3.3: Recurrence relations

(i) Denote by $f(x)$ the generating function for $\{h_n\}_{n\ge 0}$ where $h_0 = 2, h_1 = 5$ and $h_n = 4h_{n-1} - 3h_{n-2}, n \ge 2$. Note that

$$\begin{aligned}4xf(x) &= \sum_{n=0}^{\infty} 4h_n x^{n+1} = \sum_{n=1}^{\infty} 4h_{n-1}x^n;\\ 3x^2 f(x) &= \sum_{n=0}^{\infty} 3h_n x^{n+2} = \sum_{n=2}^{\infty} 3h_{n-2}x^n.\end{aligned}$$

Therefore

$$\begin{aligned}f(x) - 4xf(x) + 3x^2 f(x) &= \left(h_0 + h_1 x + \sum_{n=2}^{\infty} h_n x^n\right) - \left(4h_0 x + \sum_{n=2}^{\infty} 4h_{n-1}x^n\right) + \sum_{n=2}^{\infty} 3h_{n-2}x^n\\ &= h_0 + (h_1 - 4h_0)x + \sum_{n=2}^{\infty}\Big(h_n - 4h_{n-1} + 3h_{n-2}\Big)x^n\\ &= 2 - 3x.\end{aligned}$$

Thus, $f(x) = \frac{2-3x}{1-4x+3x^2}$.
Moreover, note that $1 - 4x + 3x^2 = (1-3x)(1-x)$, we can rewrite

$$f(x) = \frac{2-3x}{1-4x+3x^2} = \frac{c_1}{1-3x} + \frac{c_2}{1-x}$$

where c_1, c_2 solve $2 - 3x = c_1(1-x) + c_2(1-3x) = (c_1+c_2) - (c_1+3c_2)x$ which implies that $c_1 + c_2 = 2$ and $c_1 + 3c_2 = 3$. Therefore $c_1 = \frac{3}{2}$ and $c_2 = \frac{1}{2}$ which implies that

$$f(x) = \frac{3}{2(1-3x)} + \frac{1}{2(1-x)} = \sum_{n=0}^{\infty}\left(\frac{3}{2}3^n + \frac{1}{2}\right)x^n.$$

Therefore, $h_n = \frac{3^{n+1}+1}{2}$.

(ii) Denote by $f(x)$ the generating function for $\{h_n\}_{n\geq 0}$ where $h_0 = 0, h_1 = 1$ and $h_n = 3h_{n-1} + 4h_{n-2}, n \geq 2$. Note that

$$3xf(x) = \sum_{n=0}^{\infty} 3h_n x^{n+1} = \sum_{n=1}^{\infty} 3h_{n-1}x^n;$$
$$4x^2 f(x) = \sum_{n=0}^{\infty} 4h_n x^{n+2} = \sum_{n=2}^{\infty} 4h_{n-2}x^n.$$

Therefore

$$\begin{aligned} f(x) - 3xf(x) - 4x^2 f(x) &= \left(h_0 + h_1 x + \sum_{n=2}^{\infty} h_n x^n\right) - \left(3h_0 x + \sum_{n=2}^{\infty} 3h_{n-1}x^n\right) - \sum_{n=2}^{\infty} 4h_{n-2}x^n \\ &= h_0 + (h_1 - 3h_0)x + \sum_{n=2}^{\infty}\Big(h_n - 3h_{n-1} - 4h_{n-2}\Big)x^n \\ &= x. \end{aligned}$$

Thus, $f(x) = \frac{x}{1-3x-4x^2}$.
Moreover, note that $1 - 3x - 4x^2 = (1 - 4x)(1 + x)$, we can rewrite

$$f(x) = \frac{x}{1 - 3x - 4x^2} = \frac{c_1}{1 - 4x} + \frac{c_2}{1 + x}$$

where c_1, c_2 solve $x = c_1(1 + x) + c_2(1 - 4x) = (c_1 + c_2) + (c_1 - 4c_2)x$ which implies that $c_1 + c_2 = 0$ and $c_1 - 4c_2 = 1$. Therefore $c_1 = \frac{1}{5}$ and $c_2 = -\frac{1}{5}$ which implies that

$$f(x) = \frac{1}{5(1 - 4x)} - \frac{1}{5(1 + x)} = \sum_{n=0}^{\infty}\left(\frac{1}{5}4^n - \frac{1}{5}(-1)^n\right)x^n.$$

Therefore, $h_n = \frac{(-1)^{n+1}+4^n}{5}$.

E4.1: The number of SDRs

(i) Assume that $\{x_1, \dots, x_n\}$ is an SDR for $\mathcal{A} = \{A_1, \dots, A_n\}$ with $A_1 = \{1, 2\}, A_2 = \{2, 3\}, \dots, A_n = \{n, 1\}$. There are two cases of x_1:

- If $x_1 = 1$ then the SDR is followed by
$$x_1 = 1 \xRightarrow{A_n=\{n,1\}} x_n = n \xRightarrow{A_{n-1}=\{n-1,n\}} x_{n-1} = n{-}1 \xRightarrow{A_{n-2}=\{n-2,n-1\}} \cdots \xRightarrow{A_2=\{2,3\}} x_2 = 2.$$
It means $SDR = \{1, 2, \dots, n - 1, n\}$.
- If $x_1 = 2$ then the SDR is followed by
$$x_1 = 2 \xRightarrow{A_2=\{2,3\}} x_2 = 3 \xRightarrow{A_3=\{3,4\}} x_3 = 4 \xRightarrow{A_4=\{4,5\}} \cdots \xRightarrow{A_n=\{n,1\}} x_n = 1.$$
It means $SDR = \{2, 3, \dots, n, 1\}$.

Therefore there are two SDRs for $\mathcal{A}$.

(ii) Assume that $\{x_1, \dots, x_n\}$ is an SDR for $\mathcal{A} = \{A_1, \dots, A_n\}$ with $A_i = [n] \setminus \{i\}$, $i \in [n]$. Then $\{x_1, \dots, x_n\}$ is a permutation of $[n]$ with $x_i \neq i$, $\forall i \in [n]$, i.e., is a derangement of $[n]$. Therefore the number of SDRs for $\mathcal{A}$ is $D_n = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}$.

E4.2: Partial SDR

Consider $\mathcal{A} = \{A_1, \ldots, A_6\}$ with $A_1 = \{a, b, c\}$, $A_2 = \{a, b, c, d, e\}$, $A_3 = \{a, b\}$, $A_4 = \{b, c\}$, $A_5 = \{a\}$, $A_6 = \{a, c, e\}$. First, $\lambda(\mathcal{A}) < 6$ because otherwise assume $\{x_1, x_2, x_3, x_4, x_5, x_6\}$ is an SDR for $\mathcal{A}$ then

$$\xRightarrow{A_5=\{a\}} x_5 = a \xRightarrow{A_3=\{a,b\}} x_3 = b \xRightarrow{A_4=\{b,c\}} x_4 = c.$$

However $A_1 = \{a, b, c\}$ therefore we can not pick $x_1 \in A_1$ such that $x_1 \neq x_3, x_4, x_5$. This contradiction means that $\lambda(\mathcal{A}) < 6$. On the other hand, we can find a partial SDR of size 5, for example, $\{d, b, c, a, e\}$ is an SDR for $\{A_2, A_3, A_4, A_5, A_6\}$. Therefore $\lambda(\mathcal{A}) = 5$.

E4.3: Latin squares

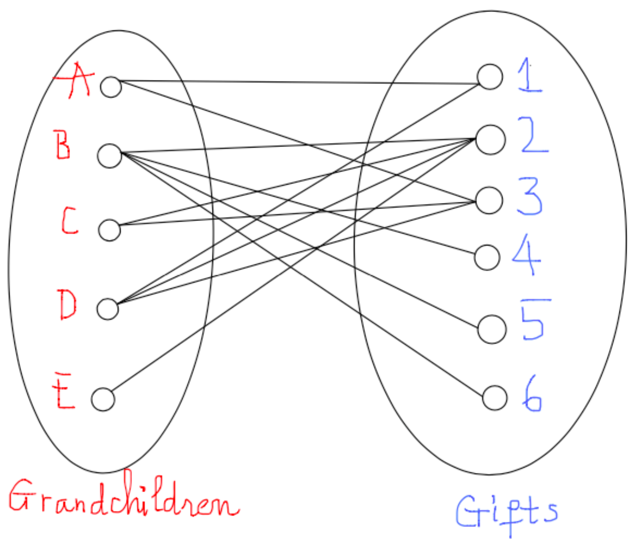
(i) Assume that A is an $(n \times n)$-Latin square with symbols in $[n]$ which is symmetric and idempotent. Because A is Latin, 1 appears exactly n times in A. Because A is idempotent, 1 appears exactly one time in the diagonal of A. Therefore 1 appears exactly $(n-1)$ times in non-diagonal of A. Because A is symmetric, $n-1$ must be even which implies that n is odd.

(ii) We give out an 5×5 symmetric, idempotent Latin square as follows:

1	3	4	5	2
3	2	5	1	4
4	5	3	2	1
5	1	2	4	3
2	4	1	3	5

E5.1: Maximum matching

Denote Alice, Bob, Charles, Dot and Edward by A, B, C, D, E. We construct a bipartite graph based on what children wants:



(i) Assume that the mathematician can distribute one gift to each person so that everyone gets something they want, i.e., there is a matching $M = \{\{A, x_A\}, \{B, x_B\}, \{C, x_C\}, \{D, x_D\}, \{E, x_E\}\}$ where x_A, x_B, x_C, x_D, x_E are distinct presents labeled in $\{1, 2, 3, 4, 5, 6\}$. Since E wants only 2 therefore $x_E = 2$. Since C wants only $2, 3$ therefore $x_C = 3$ (because 2 has been taken). Since D wants only $1, 2, 3$ therefore $x_D = 1$ (because $2, 3$ have been taken). Since A wants only $1, 3$ therefore A can not have a gift she wants (because $1, 3$ have been taken). It means that the mathematician can not distribute the gifts as required.

Another way: based on $\{A, C, D, E\}$ *wants* $\{1, 2, 3\}$.

(ii) Because of (i), the maximum matching has size at most 4, i.e., the maximum number of people those could get something they want is at most 4. This maximum would be 4, for example by a maximum matching $M = \{\{A, 1\}, \{B, 4\}, \{C, 2\}, \{D, 3\}\}$.

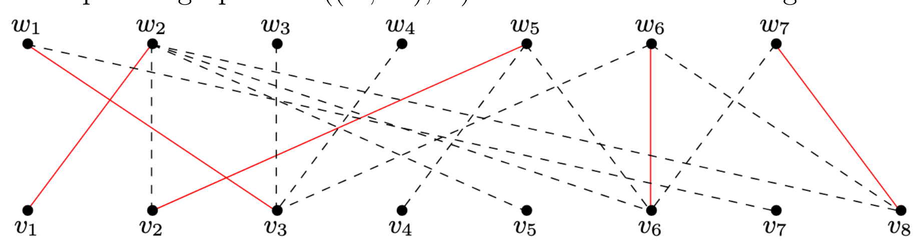
E5.2: Maximal matching vs maximum matching

Let M be any maximal matching and M^* be any maximum matching. Assume that $|M| < \frac{1}{2}|M^*|$. Consider an arbitrary edge $\{a, b\} \in M$, there is at most one edge in M^* that is incident to a (because M^* is a matching) and there is at most one edge in M^* that is incident to b. Therefore there are at most two edges in M^* those share with e at least one common vertex. It implies that there are at most $2|M|$ edges in M^* those share with M at least one common vertex. By assumption, we have $2|M| < |M^*|$, therefore there is $e^* \in M^*$ that does not have any common vertex with every edge in M. Thus, $M \cup \{e^*\}$ is also a matching with whose size is strictly larger than the size of M, which is a contradiction with the definition of maximal matching of M. Therefore $|M| \geq \frac{1}{2}|M^*|$.

Remark: In fact, we proved $|M| \geq \frac{1}{2}|N|$ *for any maximal matching* M *and any matching* N.

E5.3: Algorithm

Consider the bipartite graph $G = ((V; W), E)$ as below and a matching M shown in red.



(i) The algorithm of finding the alternative chain goes as follows:

Step 0: Label v_4, v_5, v_7 as $(S, 0)$ (they are unsaturated unlabelled vertices on vertex set V);

Step 1: Label w_1 as $(7, 1)$, w_2 as $(5, 1)$ and w_5 as $(4, 1)$ (because they are unlabelled vertices on vertex set W and $\{v_4, w_5\}, \{v_5, w_2\}, \{v_7, w_1\} \in M^c$);

Step 2: Label v_1 as $(2, 2)$, v_2 as $(5, 2)$ and v_3 as $(1, 2)$ (because they are unlabelled vertices on vertex set V and $\{w_2, v_1\}, \{w_5, v_2\}, \{w_1, v_3\} \in M$);

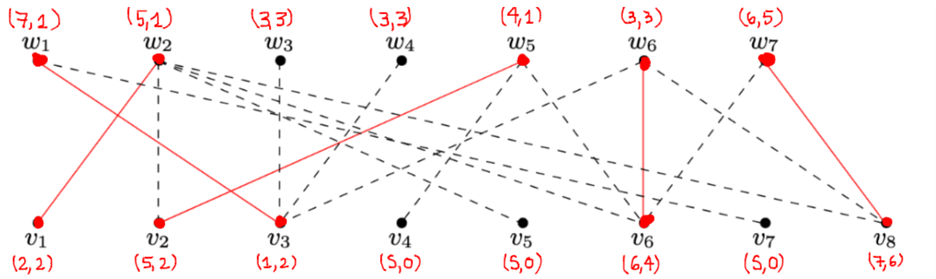
Step 3: Label w_3 as $(3, 3)$, w_4 as $(3, 3)$ and w_6 as $(3, 3)$ (because they are unlabelled vertices on vertex set W and $\{v_3, w_3\}, \{v_3, w_4\}, \{v_3, w_6\} \in M^c$);

Step 4: Label v_6 as $(6, 4)$ (because it is an unlabelled vertex on vertex set V and $\{w_6, v_6\} \in M$);

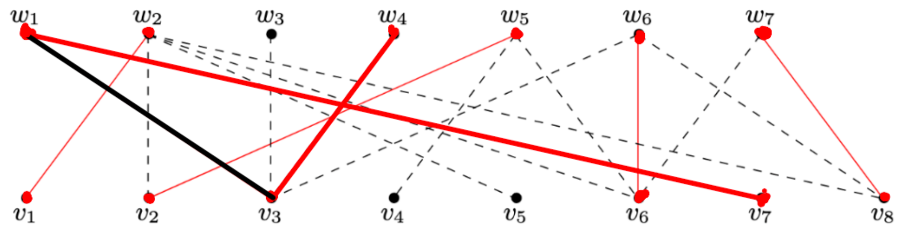
Step 5: Label w_7 as $(6, 5)$ (because it is an unlabelled vertex on vertex set W and $\{v_6, w_7\} \in M^c$);

Step 6: Label v_8 as $(7, 6)$ (because it is an unlabelled vertex on vertex set V and $\{w_7, v_8\} \in M$); Stop because all vertices have been labelled.

Therefore, when the algorithm stops, we end up with labels as follows



(ii) Maximum matching: We see that both w_3 and w_4 are unsaturated by M and labelled. Therefore M is not a maximum matching and we can construct an alternative chain, for example, $\{w_4, v_3, w_1, v_7\}$. Now, by deleting the edge $\{v_3, w_1\}$ and adding two edges $\{w_4, v_3\}$ and $\{w_1, v_7\}$ to M we have a matching M' of size 6 shown in red as follows



Moreover we can check that $S = \{w_2, w_5, v_3, v_6, v_7, v_8\}$ is a vertex cover of size 6 (each of total 17 edges of the graph G is incident to at least one vertex in S), i.e. we have a matching M' and a vertex cover S such that $|M'| = |S| = 6$. Therefore, from a theorem in the lecture, M' is a maximum matching.

Another way: Instead of using a vertex cover S, we use the algorithm again and see that when the algorithm stops, there is no w which is labelled and unsaturated by M'. Therefore M' is a maximum matching.

E6.1: Degree sequence

(i) Given non-negative integers $d_1 \geq d_2 \geq \cdots \geq d_n$ with $\sum_{i=1}^n d_i$ is even. We want to prove that there is a multigraph (no loops) with degree sequence $\{d_1, d_2, \dots, d_n\}$ if and only if $d_1 \leq \sum_{i=2}^n d_i$:

($\Rightarrow$) Assume $G = (V = \{v_1, \dots, v_n\}, E)$ is a multigraph with the degree sequence $\{d_1, \dots, d_n\}$, i.e., $d(v_i) = d_i$ for all $i \in [n]$. If otherwise, $d_1 > \sum_{i=2}^n d_i$ then there has to have an edge which is incident to v_1 but not incident to any v_i for $i \in \{2, \dots, n\}$. In other words, there has to have a loop at v_1, which is a contradiction to G is a multigraph. Therefore $d_1 \leq \sum_{i=2}^n d_i$.

Another proof: Using the result $\sum_{i=1}^n d_i = 2|E|$ and in a multigraph $d_1 \leq |E|$. We have $d_1 \leq |E| = \frac{1}{2} \sum_{i=1}^n d_i$ which implies $d_1 \leq \sum_{i=2}^n d_i$.

($\Leftarrow$) We will prove this statement by induction on $s = \sum_{i=1}^n d_i \in 2\mathbb{N}_0$:

- Base step:

- $s = 0$: it happens only when $d_i = 0$, $\forall i \in [n]$. The empty graph $G = (V = \{v_1, \dots, v_n\}, E = \emptyset)$ have the degree sequence $\{0,\dots,0\}$.
- $s = 2$: Since $d_1 \leq \sum_{i=2}^{n} d_i$ and $d_1 \geq d_2 \geq \dots \geq d_n$, it happens only when $d_1 = d_2 = 1$ and $d_i = 0$, $\forall i \in \{3, \dots, n\}$. The multigraph $G = (V = \{v_1, \dots, v_n\}, E = \{\{v_1, v_2\}\})$ have the degree sequence $\{1, 1, 0, \dots, 0\}$.

- Inductive step:
  Assume that the statement holds true with $s - 2 \in 2\mathbb{N}$, i.e., for given non-negative integers $d_1 \geq d_2 \geq \cdots \geq d_n$ with $\sum_{i=1}^{n} d_i = s - 2$ and $d_1 \leq \sum_{i=2}^{n} d_i$, we can construct a multigraph with degree sequence $\{d_1, d_2, \dots, d_n\}$. We prove that the statement is also true for s. In fact, for given non-negative integers $d_1 \geq d_2 \geq \cdots \geq d_n$ with $\sum_{i=1}^{n} d_i = s$ and $d_1 \leq \sum_{i=2}^{n} d_i$, there are two cases:
  - $d_1 = \sum_{i=2}^{n} d_i$: The multigraph $G = (V = \{v_1, \dots, v_n\}$, $E = \{\underbrace{\{v_1, v_2\}, \dots, \{v_1, v_2\}}_{d_2 \text{ times}}, \dots, \underbrace{\{v_1, v_n\}, \dots, \{v_1, v_n\}}_{d_n \text{ times}}\})$ has the degree sequence $\{d_1, \dots, d_n\}$.
  - $d_1 < \sum_{i=2}^{n} d_i$: Because $\sum_{i=2}^{n} d_i - d_1 = 2|E| - 2d_1$ is even, we have $\sum_{i=2}^{n} d_i - d_1 \geq 2$. Without loss of generality, we assume that $d_n > 0$. Let $d'_{n-1} = d_{n-1} - 1, d'_n = d_n - 1$ and $d'_i = d_i$, $\forall i \in \{1, \dots, n-2\}$. We have $d'_1 \geq \dots \geq d'_n$, $\sum_{i=1}^{n} d'_i = s - 2$ and $\sum_{i=2}^{n} d'_i - d'_1 = \sum_{i=2}^{n} d_i - d_1 - 2 \geq 0$. Therefore, by induction hypothesis, there is a multigraph $G' = (V, E')$ with degree sequence $\{d'_1, \dots, d'_n\}$. Then, the multigraph $G = (V, E = E' \cup \{v_{n-1}, v_n\})$ has the degree sequence $\{d_1, \dots, d_n\}$.

(ii) We only need to show that: given a sequence of non-negative integers $d_1 \geq d_2 \geq \cdots \geq d_n$ with $\sum_{i=1}^{n} d_i$ is even, the condition

$$\sum_{i=1}^{k} d_i \leq k(k-1) + \sum_{j=k+1}^{n} \min\{d_j, k\}, \qquad \forall k \in [n] \tag{1.0.3}$$

is equivalent to the condition

$$\sum_{i \in I} d_i \leq |I|(|I| - 1) + \sum_{j \in [n] - I} \min\{d_j, |I|\} \qquad (\forall \emptyset \neq I \subseteq [n]). \tag{1.0.4}$$

($\Leftarrow$): For any $k \in [n]$, then by applying (1.0.4) for the non-empty subset $I = \{1, \dots, k\} \subseteq [n]$ we obtain (1.0.5).

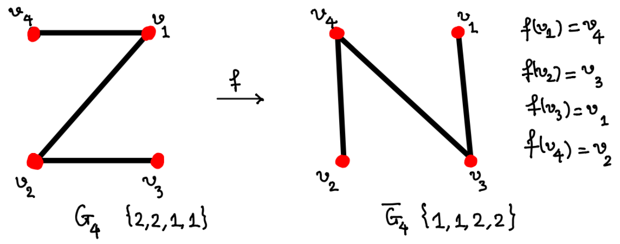
($\Rightarrow$): Because $\{d_i\}_{i=1}^{n}$ is non-increasing we have for every non-empty subset $I \subseteq [n]$:

$$\begin{aligned}\sum_{i \in I} d_i \leq \sum_{i=1}^{|I|} d_i \underbrace{\leq}_{\text{because of (1.0.5) for } k=|I|} |I|(|I| - 1) + \sum_{i=|I|+1}^{n} \min\{d_i, |I|\} \\ \leq |I|(|I| - 1) + \sum_{j \in [n] - I} \min\{d_j, |I|\}.\end{aligned}$$

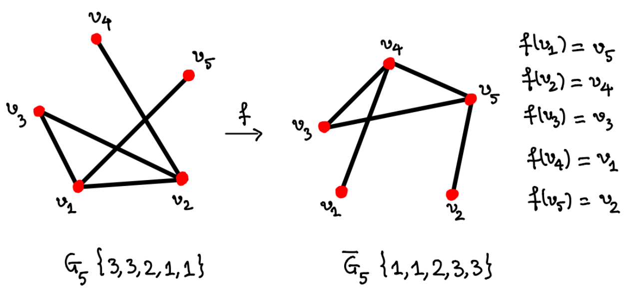
E6.2: Self-complementary

(i) Let G be a self-complementary graph on n vertices, i.e., $G \cong \bar{G}$ which implies that $|E(G)| = |E(\bar{G})|$. Moreover $G \dot{\bigcup} \bar{G} = K_n$, the complete graph. Therefore $|E(G)| = \frac{1}{2}|K_n| = \frac{n(n-1)}{4}$. Because n and $n - 1$ can not both even, it happens only when $n = 4k$ or $n = 4k + 1$ for some $k \in \mathbb{N}$.

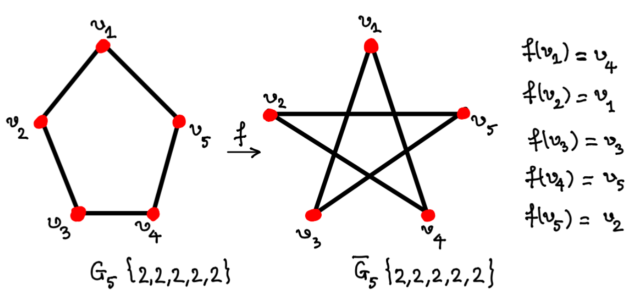
(ii) For $n = 4$: Note that if a simple graph G is self-complementary then its degree sequence $\{d_1, d_2, d_3, d_4\}$ is equal to the degree sequence $\{3 - d_1, 3 - d_2, 3 - d_3, 3 - d_4\}$ of $\bar{G}$. If $d_1 \geq d_2 \geq d_3 \geq d_4$ then $d_1 = 3 - d_4, d_2 = 3 - d_3$ which implies that $d_1 \leq 3$ and $d_1 \geq d_4 = 3 - d_1$ therefore $d_1 \in \{2, 3\}$. If $d_1 = 3$ then in one hand G is connected but in another hand $d_4 = 3 - d_1 = 0$ therefore G is not connected. This contradiction implies $d_1 = 2$. Therefore $d_1 = d_2 = 2$ and $d_3 = d_4 = 1$. For example: $G = (V = \{v_1, v_2, v_3, v_4\}, E = \{\{v_1, v_2\}, \{v_2, v_3\}, \{v_3, v_4\}\})$ is self-complementary with the graph isomorphism $f : G \to \bar{G}$, $f(v_1) = v_3, f(v_2) = v_1, f(v_3) = v_4, f(v_4) = v_2$.



For $n = 5$: Similarly, if $\{d_1, d_2, d_3, d_4, d_5\}$ is the degree sequence of a simple self-complementary graph G with $d_1 \geq \ldots \geq d_5$ then $d_1 = 4 - d_5, d_2 = 4 - d_4, d_3 = 2$. It implies that $d_1, d_2 \in \{2, 3, 4\}$. Because $d_1 = 4$ is impossible with the same argument as above, we have 3 cases $\{3, 3, 2, 1, 1\}; \{3, 2, 2, 2, 1\}; \{2, 2, 2, 2, 2\}$. Note that there is no graph isomorphism for the second case $\{3, 2, 2, 2, 1\}$ because assume $f : G \to \bar{G}$ is such a graph isomorphism, f must map the vertex of degree 3 to the vertex of degree 3 and map the vertex of degree 1 to vertex of degree 1, it means $f(v_1) = v_5$ and $f(v_5) = v_1$, i.e., $\{v_1, v_5\} \in E(G) \cap E(\bar{G}) = \emptyset$, which is a contradiction. Thus, there are only two cases $\{d_1, d_2, d_3, d_4, d_5\} = \{3, 3, 2, 1, 1\}$ and $\{d_1, d_2, d_3, d_4, d_5\} = \{2, 2, 2, 2, 2\}$.
In case $\{d_1, d_2, d_3, d_4, d_5\} = \{3, 3, 2, 1, 1\}$:



In case $\{d_1, d_2, d_3, d_4, d_5\} = \{2, 2, 2, 2, 2\}$:



E6.3: Euler circuits/walks

Note that all $K_n, n \geq 2$ are connected graphs and every vertex in K_n has degree $n-1$.

(i) By a theorem from the lecture, we know that K_n has an Euler circuit if and only if every vertex has even degree, which happens if and only if n is odd.

(ii) By a theorem from the lecture, we know that K_n has an Euler walk if and only if there are exactly two vertices with odd degree, which happens if and only if $n = 2$.

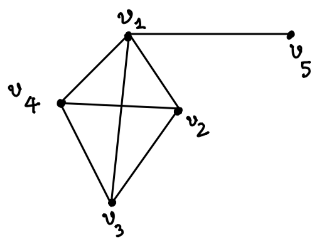
E7.1: Hamilton cycles

(i) Suppose G is a simple graph on $n \geq 2$ vertices with $|E(G)| \geq \frac{(n-1)(n-2)}{2} + 2$. First, we note that n can not be 2 because if else $|E(G)| \geq 2$ implies that G is not simple, which is a contradiction. It implies $n \geq 3$. On the other hand, for any pair of vertices those are not adjacent, say $u \not\sim v$, we have

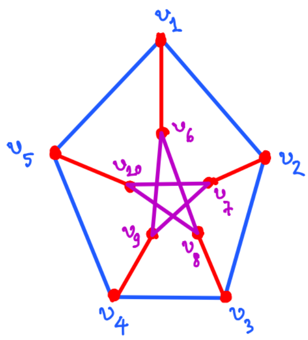
$$\begin{aligned} d(u) + d(v) &= |E(G)| - \#\{e \in E(G) : endpoints(e) \in V(G) \setminus \{u, v\}\} \\ &\geq \frac{(n-1)(n-2)}{2} + 2 - |K_{n-2}| = \frac{(n-1)(n-2)}{2} + 2 - \frac{(n-2)(n-3)}{2} = n. \end{aligned}$$

Therefore G satisfies the Ore property which implies that G has a Hamilton cycle due to the Ore theorem.

(ii) For $n \geq 2$, construct a graph G by connecting a complete graph K_{n-1} on $\{v_1, \dots, v_{n-1}\}$ with a vertex v_n by an edge, e.g., $E(G) = E(K_{n-1}) \cup \{v_n, v_1\}$. Then G is a simple graph with $|E(G)| = \frac{(n-1)(n-2)}{2} + 1$ and G has no Hamilton cycle because otherwise there is a Hamilton cycle which must go through v_n which implies $d(v_n) \geq 2$. This is a contradiction to $d(v_n) = 1$ by the construction of G. (See figure for $n = 5$).



(iii) We prove that the Petersen graph G has no Hamilton cycle. In fact, assume that there is a Hamilton cycle C_{10} in G. Note that the blue C_5 (the outer pentagon) and the purple C_5 (the inner pentagram) are connected by five red crossing edges (see Figure).



Because C_{10} must return to its starting point, it must use an even number of crossing edges, i.e., there are two or four crossing edges in C_{10}.

Case 1: Two crossing edges, say, R_1, R_2: Then $C_{10} = R_1 P_4 R_2 P_4$. WLOG $R_1 = \{v_1, v_6\}$ then $P_4 = \{v_6, v_8, v_{10}, v_7, v_9\}$ or $\{v_6, v_9, v_7, v_{10}, v_8\}$ while $P_4 = \{v_1, v_2, v_3, v_4, v_5\}$ or $\{v_1, v_5, v_4, v_3, v_2\}$. Note that ends of P_4 (i.e. v_9 or v_8) and ends of P_4 (i.e. v_5 or v_2) are not adjacent. Therefore this case is impossible.

Case 2: Four crossing edges, WLOG, say, $\{v_2, v_7\}, \{v_3, v_8\}, \{v_4, v_9\}, \{v_5, v_10\}$: If $\{v_4, v_5\} \in C_{10}$ then $\{v_5, v_1\}, \{v_4, v_3\} \notin C_{10}$ (if else then $\{v_5\}$ or $\{v_4\}$ has degree 3 in C_{10}). It implies that $\{v_1, v_2\}, \{v_2, v_3\} \in C_{10}$ (because C_{10} must use all vertices). But this implies that $\{v_2\}$ has degree 3 in C_{10}. This contradiction implies that $\{v_4, v_5\} \notin C_{10}$. Similarly we have $\{v_2, v_3\}, \{v_7, v_9\}, \{v_8, v_{10}\} \notin C_{10}$. In other words, C_{10} does not contain P_3 with two crossing edges. Thus, there is a unique remaining configuration for C_{10} which however consists of two C_5: $\{v_1, v_2, v_7, v_{10}, v_5\}$ and $\{v_6, v_8, v_3, v_4, v_9\}$. Therefore this case is also impossible.

E7.2: Bipartite multigraphs

Given a sequence of positive integer numbers $D_n = \{d_1, \dots, d_n\}$. We prove that there is a bipartite multigraph with degree sequence D_n if and only if D_n satisfies condition A, i.e., there is a partition $[n] = I \,\dot\cup\, J$ such that $\sum_{i \in I} d_i = \sum_{i \in J} d_i$:

($\Rightarrow$): Assume that $G = (V = \{v_1, \dots, v_n\}, E)$ is a bipartite multigraph with degree sequence $\{d_1, \dots, d_n\}$, i.e., $d(v_i) = d_i$, $\forall i \in [n]$. Because G is bipartite, there are a partition $V = V_1 \,\dot\cup\, V_2$ so that every $e \in E(G)$ connects one vertex in V_1 with another vertex in V_2. Put $I = \{i \in [n] : v_i \in V_1\}$ and $J = \{i \in [n] : v_i \in V_2\}$ we have $[n] = I \,\dot\cup\, J$ and

$$\sum_{i \in I} d_i = \sum_{v \in V_1} d(v) = |E(G)| = \sum_{v \in V_2} d(v) = \sum_{i \in J} d_i.$$

($\Leftarrow$): We will prove by induction on n. In fact,

- for $n = 2$: Condition A means that the sequence $D_2 = \{d_1, d_2\}$ satisfies $d_1 = d_2$. We construct G consists of two vertices $V = \{v_1, v_2\}$ with $E = \{d_1$ edges connecting v_1 and $v_2\}$. Then $G = (\{\{v_1\}; \{v_2\}\}, E)$ is a bipartite multigraph with degree sequence D_2.

- for $n = 3$: Condition A means that the sequence $D_3 = \{d_1, d_2, d_3\}$ satisfies $d_i = d_j + d_k$ for some permutation $\{i, j, k\}$ of $\{1, 2, 3\}$. We construct G consists of three vertices $V = \{v_1, v_2, v_3\}$ with $E = \{d_j$ edges connecting v_1 and v_2, and d_k edges connecting v_1 and $v_3\}$. Then $G = (\{\{v_1\}; \{v_2, v_3\}\}, E)$ is a bipartite multigraph with degree sequence D_3.

- Assume that the claim is true until $n - 1$, we prove it is also true for n: In fact, given a sequence of positive integer numbers $\{d_1, \dots, d_n\}$. Assume that there is a partition $[n] = I \,\dot\cup\, J$ such that $\sum_{i \in I} d_i = \sum_{i \in J} d_i$.

Case 1: If $|I| = 1$, w.l.o.g. we assume that $I = \{1\}$ then $J = \{2, \dots, n\}$ and Condition A means that $d_1 = d_2 + \cdots + d_n$. We construct G consists of n vertices $V = \{v_1, \dots, v_n\}$ with $E = \cup_{i=2}^{n} \{d_i$ edges connecting v_1 and $v_i\}$. Then $G = (\{\{v_1\}; \{v_2, \dots, v_n\}\}, E)$ is a bipartite multigraph with degree sequence D_n.

Case 2: If $|I| = n - 1$, we have $|J| = 1$ and do the same as Case 1 to have a bipartite multigraph with degree sequence D_n.

Case 3: If $2 \le |I| = k \le n - 2$, w.l.o.g. we assume that $I = \{1, \dots, k\}$ and $J = \{k + 1, \dots, n\}$ and consider three cases:

Case 3a: If $d_1 > d_n$, then the sequence of positive integer numbers $D'_{n-1} = \{d'_1 = d_1 - d_n, d'_2 = d_2, \ldots, d'_{n-1} = d_{n-1}\}$ satisfies $\sum_{i=1}^{k} d'_i = (d_1 - d_n) + d_2 + \cdots + d_k = d_{k+1} + \cdots + d_{n-1} = \sum_{i=k+1}^{n-1} d'_i$, i.e., the condition A with $[n-1] = \{1, \ldots, k\} \dot{\cup} \{k+1, \ldots, n-1\}$. From the induction hypothesis we can construct a bipartite multigraph

$$G' = (\{\{v_1, \ldots, v_k\}; \{v_{k+1}, \ldots, v_{n-1}\}\}, E(G'))$$

with degree sequence D'_{n-1}, i.e., $d(v_i) = d'_i$, $\forall i \in [n-1]$. Now we construct G by adding one vertex $V(G) = V(G') \cup \{v_n\}$ and $E(G) = E(G') \cup \{d_n$ edges connecting v_n and $v_1\}$. Then $G = (\{\{v_1, \ldots, v_k\}; \{v_{k+1}, \ldots, v_n\}\}, E(G))$ is a bipartite multigraph with degree sequence D_n.

Case 3b: If $d_1 = d_n$, then the sequence of positive integer numbers $D'_{n-2} = \{d'_1 = d_2, \ldots, d'_{n-2} = d_{n-1}\}$ satisfies $\sum_{i=1}^{k-1} d'_i = \sum_{i=2}^{k} d_i = \sum_{i=k+1}^{n-1} d_i = \sum_{i=k}^{n-2} d'_i$, i.e., the condition A with $[n-2] = \{1, \ldots, k-1\} \dot{\cup} \{k, \ldots, n-2\}$. From the induction hypothesis we can construct a bipartite multigraph

$$G' = (\{\{v_2, \ldots, v_k\}; \{v_{k+1}, \ldots, v_{n-1}\}\}, E(G'))$$

with degree sequence D'_{n-2}, i.e., $d(v_i) = d'_{i-1} = d_i$, $\forall i \in \{2, \ldots, n-1\}$. Now we construct G by $V(G) = V(G') \cup \{v_1, v_n\}$ and $E(G) = E(G') \cup \{d_n$ edges connecting v_n and $v_1\}$. Then $G = (\{\{v_1, \ldots, v_k\}; \{v_{k+1}, \ldots, v_n\}\}, E(G))$ is a bipartite multigraph with degree sequence D_n.

Case 3c: If $d_1 < d_n$, then the sequence of positive integer numbers $D'_{n-1} = \{d'_1 = d_2, \ldots, d'_{n-2} = d_{n-1}, d'_{n-1} = d_n - d_1\}$ satisfies $\sum_{i=1}^{k-1} d'_i = d_2 + \cdots + d_k = d_{k+1} + \cdots + d_{n-1} + d_n - d_1 = \sum_{i=k+1}^{n-1} d'_i$, i.e., the condition A with $[n-1] = \{1, \ldots, k-1\} \dot{\cup} \{k, \ldots, n-1\}$. From the induction hypothesis we can construct a bipartite multigraph

$$G' = (\{\{v_2, \ldots, v_k\}; \{v_{k+1}, \ldots, v_n\}\}, E(G'))$$

with degree sequence D'_{n-1}, i.e., $d(v_i) = d'_{i-1}$, $\forall i \in \{2, \ldots, n\}$. Now we construct G by adding one vertex $V(G) = V(G') \cup \{v_1\}$ and $E(G) = E(G') \cup \{d_1$ edges connecting v_1 and $v_n\}$. Then $G = (\{\{v_1, \ldots, v_k\}; \{v_{k+1}, \ldots, v_n\}\}, E(G))$ is a bipartite multigraph with degree sequence D_n.

E7.3: Regular bipartite graph

(i) Assume that $G = (\{V; W\}, E)$ is a regular bipartite graph with the common degree $d \geq 1$. Then we have

$$d|V| \overbrace{=}^{\textit{regular}} \sum_{v \in V} d(v) \overbrace{=}^{\textit{bipartite}} |E| \overbrace{=}^{\textit{bipartite}} \sum_{w \in W} d(w) \overbrace{=}^{\textit{regular}} d|W|.$$

Since $d \geq 1$, it implies that $|V| = |W|$.

(ii) Assume that $G = (\{V; W\}, E)$ is a regular bipartite graph with the common degree $d \geq 1$. From (i) we have $|V| = |W|$ and $|E| = d|V|$. Let C be an arbitrary vertex cover of G. Because each vertex in C covers exactly d edges, C covers at most $d|C|$ edges. It implies that $d|C| \geq |E|$, i.e., $|C| \geq |V|$. We know from the lecture that in a bipartite graph, the size of a maximum matching is the same as the size of a minimum vertex cover. Therefore if M is a maximum matching, $|M| \geq |V|$. Moreover, if $|M| > |V|$ then by the pigeon-hole principle, there are at least two edges in M have the same vertex which is a contradiction to M is a matching. Therefore $|M| = |V| = |W|$. Now for each $v \in V$, because M is a matching there is at most one edge of M incident to v. On the other hand, if there is no edge in M incident to v then by the pigeon-hole principle, there is another vertex in V incident to two edges in M which is a contradiction. Therefore there is exactly one edge in M incident to v. Thus, M is a perfect matching.

Another proof: Assume that $G = (\{V; W\}, E)$ is a regular bipartite graph with the common degree $d \geq 1$. From (i) we have $|V| = |W|$ therefore we can assume that $V = \{v_1, \ldots, v_k\}$ and $W = \{w_1, \ldots, w_k\}$. Denote by $A_i = \mathcal{N}(v_i) := \{w_j \in W : w_j \sim v_i\} \subseteq W$ the neighborhood of v_i for each $i \in [k]$. We will prove that the collection $\mathcal{A} = \{A_1, \ldots, A_k\}$ has a SDR. In fact, for every $I \subseteq [k]$, denote by $v_I = \{v_i, i \in I\}$, $\mathcal{B} = \{e \in E : e \sim v_I\}$, $\mathcal{C} = \{e \in E : e \sim \cup_{i\in I} A_i\}$. Note that $|\mathcal{B}| = d|I|$ because each vertex in v_I has degree d; $|\mathcal{C}| = d|\cup_{i\in I} A_i|$ because each vertex in $\cup_{i\in I} A_i$ has degree d. Moreover, if $e \in \mathcal{B}$, i.e., $e \sim v_I$ then $e \sim \mathcal{N}(v_I) = \cup_{i\in I} A_i$, i.e., $e \in \mathcal{C}$. Therefore $|\mathcal{B}| \leq |\mathcal{C}|$ which implies that $|I| \leq |\cup_{i\in I} A_i|$. By applying the Hall's theorem, $\mathcal{A}$ has a SDR, i.e., there is a set of k distinct elements $\{w_1^*, \ldots, w_k^*\}$ with $w_i^* \in A_i$. The matching $M = \{\{v_1, w_1^*\}, \ldots, \{v_k, w_k^*\}\}$ is then a perfect matching of G.

E8.1: Bridges and pendants

(i) Suppose that G is a connected graph, and that every spanning tree in G contains edge e. Assume that e is not a bridge, i.e., $G - \{e\}$ is a connected graph. By a theorem from the lecture, there exists a spanning tree T for $G - \{e\}$. Since $V(G) = V(G - \{e\})$, T is also a spanning tree for G. Therefore T is a spanning tree for G and does not contain e. This contradiction implies that e must be a bridge.

(ii) Let T be a tree on n vertices and consider an arbitrary edge e of T. Assume that e is not a bridge, i.e., $T - \{e\}$ is a connected graph. Therefore there is a spanning tree T' for $T - \{e\}$. Thus $V(T') = V(T - \{E\}) = n$ and $E(T') \leq E(T - \{e\}) = n - 2 \neq n - 1$ which is a contradiction to T' is a tree. Therefore e must be a bridge.

(iii) Suppose T is a tree on n vertices. It implies that T is connected and then all vertices of T have degree at least 1. Since we know that there are exactly k vertices have degree larger than 1, there are exactly $n - k$ vertices have degree 1. This means T have $n - k$ pendant vertices.

E8.2: Degree sequence of trees

(i) ($\Rightarrow$): Assume that T is a tree on $\{v_1, \ldots, v_n\}$ with degree sequence $\{d_1, d_2, \ldots, d_n\}$ $(n \geq 2)$. Therefore T is connected and thus $d_i > 0$, $\forall i \in [n]$. Moreover, T is a tree on n vertices, therefore $E(T) = n - 1$ and

$$\sum_{i=1}^{n} d_i = \sum_{i=1}^{n} d(v_i) = 2|E(T)| = 2(n-1).$$

($\Leftarrow$): We prove by induction on n:

- Base step: Case $n = 2$: We have $d_1, d_2 > 0$ and $d_1 + d_2 = 2$. It implies $d_1 = d_2 = 1$. We construct $T = P_2$ which is a tree with degree sequence $\{d_1, d_2\}$.
- Inductive step: Case $n > 2$:
  Assume that the statement holds true for $n - 1$, we prove that the statement is also true for n. In fact, for every sequence of positive integers $d_1, \ldots, d_n > 0$ with $d_1 + \cdots + d_n = 2(n-1)$. WLOG, we assume that $d_1 \geq \ldots \geq d_n > 0$. Then $d_n = 1$ because if otherwise, $\sum_{i=1}^{n} d_i \geq nd_n \geq 2n > 2(n-1)$ and $d_1 > 1$ because if otherwise $\sum_{i=1}^{n} d_i = n < 2(n-1)$. Denote by $k \in [n] : d_k > 1$ and $d_i = 1, \forall i > k$. Consider a new sequence $d'_1 = d_1, \ldots, d'_{k-1} = d_{k-1}, d'_k = d_k - 1, d'_{k+1} = d_{k+1}, \ldots, d'_{n-1} = d_{n-1}$. Because $d_1 \geq \ldots \geq d_n$ and $d_k > 1 = d_{k+1}$, we have $d'_1 \geq \cdots \geq d'_{n-1}$ with $\sum_{i=1}^{n-1} d'_i = \sum_{i=1}^{n-1} d_i - 1 = 2(n-1) - d_n - 1 = 2(n-2)$. By the induction hypothesis, there is a tree T_{n-1} on $\{v_1, \ldots, v_{n-1}\}$ with the degree sequence $\{d'_1, \ldots, d'_{n-1}\}$. By adding a vertex v_n and one edge $\{v_n, v_k\}$ we obtain a new graph T with $|V(T)| = |V(T_{n-1})| + 1 = n$ vertices and $|E(T)| = |E(T_{n-1})| + 1 = n - 1$ edges. Therefore T is a tree and have $\{d_1, \ldots, d_n\}$ as its degree sequence.

(ii) For a proof, see, for example Fact 15 in the paper: Link

E8.3: Greedy algorithm

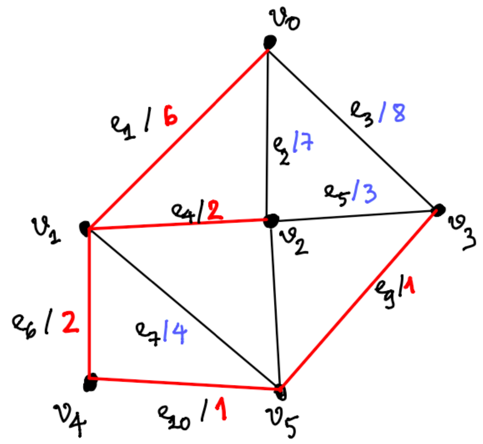
(i) Let T be the spanning tree obtained by Kruskal's algorithm and $T_{\min}$ an optimal spanning tree ($\exists T_{\min}$ because G has a finite number of spanning trees). Let $e_1, \ldots, e_{n-1}$ be the edges in the order they are added to T. If $E(T) = E(T_{\min})$ then T has minimum cost by the definition of $T_{\min}$. If $E(T) \neq E(T_{\min})$, let e_{i_1} be the first edge in which $e_{i_1} \notin T_{\min}$, $i \in [n]$. Since $T_{\min}$ is a spanning tree, $T_{\min} + \{e_{i_1}\}$ has n edges and then contains a cycle C_1. Since T is also a tree, T can not contain C_1 therefore there is an edge $f_1 \in C_1 - T$. The subgraph $T_1 = T_{\min} + \{e_{i_1}\} - \{f_1\}$ has $n - 1$ edges therefore is also a spanning tree. By definition of $T_{\min}$ we have $c(T_1) \geq c(T_{\min})$. On the other hand, if $c(T_1) > c(T_{\min})$ then $c(e_{i_1}) > c(f_1)$ therefore f_1 must be considered before e_{i_1} by the algorithm. But because f_1 in not added into T, $C'_1 = \{f_1, e_1, \ldots, e_{i_1 - 1}\}$ must be a cycle. However, we see that $C'_1 \subseteq T_{\min}$ which is a contradiction to $T_{\min}$ is a tree. Therefore $c(T_1) = c(T_{\min})$. By applying $k = |E(T) - E(T_{\min})| > 0$ times we obtain a sequence spanning trees $T_1, \ldots, T_k = T$ with the same cost. Therefore $c(T) = c(T_{\min})$.

(ii) Jarník algorithm:

- Step 1: $T = \{V = \{v_0\}, E = \emptyset\}$; $S = \{v_0\}$
- Step 2: $T = \{V = \{v_0, v_1\}, E = \{e_1\}\}$; $S = \{v_0, v_1\}$
- Step 3: $T = \{V = \{v_0, v_1, v_2, v_4\}, E = \{e_1, e_4, e_6\}\}$; $S = \{v_0, v_1, v_2, v_4\}$
- Step 4: $T = \{V = \{v_0, v_1, v_2, v_4, v_5\}, E = \{e_1, e_4, e_6, e_{10}\}\}$; $S = \{v_0, v_1, v_2, v_4, v_5\}$
- Step 5: $T = \{V = \{v_0, v_1, v_2, v_4, v_5, v_3\}, E = \{e_1, e_4, e_6, e_{10}, e_9\}\}$; $S = \{v_0, v_1, v_2, v_4, v_5, v_3\}$

Kruskal algorithms:

- Step 1: $T = \{V = \{v_3, v_4, v_5\}, E = \{e_9, e_{10}\}\}$
- Step 2: $T = \{V = \{v_3, v_4, v_5, v_1, v_2\}, E = \{e_9, e_{10}, e_4, e_6\}\}$
- Step 3: $T = \{V = \{v_3, v_4, v_5, v_1, v_2, v_0\}, E = \{e_9, e_{10}, e_4, e_6, e_1\}\}$



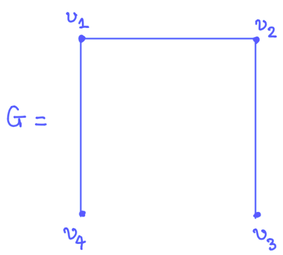
E9.1: Connected graph

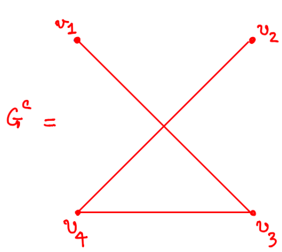
(i) Suppose a simple graph G on $n \geq 2$ vertices has at least $(n-1)(n-2)+1$ edges. Assume that G is not connected, then G can be divided into at least two connected components. Set G_1 is one connected component and G_2 is the remaining part, i.e., $G_2 = G - G_1$. Then $|V(G_1)| = k \in \{1, \ldots, n-1\}, |V(G_2)| = n-k$. Therefore $|E(G)| = |E(G_1)| + |E(G_2)| \leq \binom{k}{2} + \binom{n-k}{2} = f(k)$. Note that $f(x) := \frac{x(x-1)}{2} + \frac{(n-x)(n-x-1)}{2}$ is a convex function on $[1, n-1]$ and attains it maximum at some endpoint: $\max f = \max\{f(1), f(n-1)\} = \frac{(n-1)(n-2)}{2} < (n-1)(n-2)+1$ which is a contradiction. This implies that G must be connected.

Remark: - We can prove that there is no simple graph on $n > 3$ vertices which has at least $(n-1)(n-2)+1$ edges therefore we need only to check the connectedness for $n \leq 3$.
- If we change the problem to the statement: "Suppose a simple graph G on $n \geq 2$ vertices has at least $\frac{(n-1)(n-2)}{2} + 1$ edges. Prove that G is connected." then the proof could be done in the same way.

(ii)
- Suppose that G is a disconnected graph and G^c is its complement. For any pair of distinct vertices $u \neq v$, we have two cases.
  - Case 1: if $\{u, v\} \in E(G)$ then u and v are in the same connected component of G. Because G is disconnected, there is a vertex w in another connected component of G. It implies that $\{w, u\}, \{w, v\} \notin E(G)$ and therefore $\{w, u\}, \{w, v\} \in E(G^c)$, i.e., there is a path $\{u, w, v\}$ in G^c connect u and v.
  - Case 2: if $\{u, v\} \notin E(G)$ then $\{u, v\} \in E(G^c)$, i.e., u and v are connected in G^c.

  Therefore G^c is connected.
- The complement of a connected graph is not always disconnected. For example, graphs G and G^c are both connected.

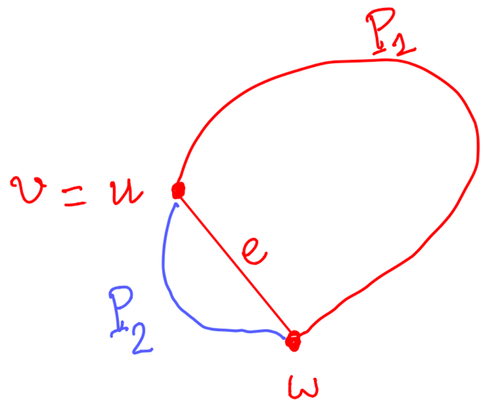




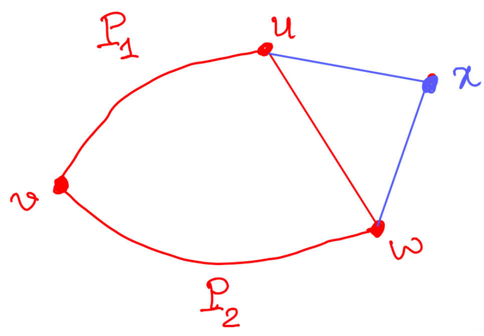
E9.2: 2-connected graph

($\Rightarrow$): Suppose that G is 2-connected. Consider an arbitrary pair of $\{v, e\}$ with v is a vertex and $e = \{u, w\}$ is an edge in G. There are two cases:

- Case 1: $v \in e$, for example, $v = u$. Because G is 2-connected, from a theorem in the lecture, there are two internally disjoint paths from u to w, one of them must be different from $\{u, w\}$, say P_1. Then $P_1 \cup \{u, w\}$ is a cycle contains v and e.



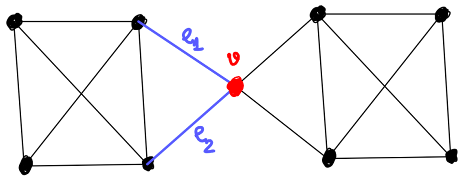
- Case 2: $v \notin e$. We construct a new graph $G' = (V(G') = V(G) \cup \{x\}, E(G') = E(G) - \{u, w\} \cup \{\{u, x\}, \{w, x\}\})$. Because G is 2-connected, $G - \{u, w\}$ is connected, and therefore G' is connected. Moreover, note that G is 2-connected implies that $G' - x = G - e$, $G' - u = G - u + \{x, w\}$, $G' - w = G - w + \{x, u\}$, $G' - t = G - t + \{x, u\} + \{x, w\}$ for all $t \in V(G') - \{u, w, x\}$ are connected. It means that G' has no cut point. Therefore G' is 2-connected. It implies from a theorem in the lecture that there is a cycle in G' contains v and x. Because $\deg_{G'}(x) = 2$ this cycle consists of one path P_1 from v to u, $\{u, x\}, \{x, w\}$, one path P_2 from w to v ($P_1 \cap P_2 = \{v\}$). Then the cycle $P_1, \{u, w\}, P_2$ will be a cycle in G contains v and e.



($\Leftarrow$): Let u and v be two vertices of G. If, for example, $\deg(u) = 0$ then because G has at least one edge, say e, there is no cycle contains u and e. Therefore there is an edge incident to u, say e. Thus, there is a cycle contains v and e which then contains u and v. This implies that G is 2-connected from a theorem in the lecture.

E9.3: The edge connectivity $\lambda(G)$

(i) Consider the following graph:



We see that

- $\kappa(G) = 1$ because G is connected and v is a cut point of the graph.
- $\lambda(G) = 2$ because $\{e_1, e_2\}$ is a minimal cut.
- $\delta(G) = 3$ because all vertices have degree at least 3.

(ii) Suppose $\lambda(G) = k > 0$. It means that there is a cut $T = \{e_1, \ldots, e_k\}$ of G with smallest size so that $G - T$ is disconnected. $G - T$ has only two connected components because if otherwise, $\{e_1, \ldots, e_{k-1}\}$ will also be a cut (e_k connects at most two connected components) which is a contradiction to $\lambda(G) = k$. In other words, $G - T = U \cup V$ with U, V are two connected components. We will prove that every e_i for $i \in [k]$ has one endpoint in U and one endpoint in V and there are no other edge has this property. In fact, if there is e_i, $i \in [k]$ with two endpoints in the same connected component, say U. Then $T' = T - \{e_i\}$ is also a cut of G with size $k - 1$ which is a contradiction. Moreover, if there is $e \in E(G) - T$ such that e has one endpoint in U and one endpoint in V. Then $G - T$ contains e and U is then connected to V by e which is a contradiction. Thus, there are exactly k edges with one endpoint in U and one endpoint in V.

(iii) WLOG we assume that $m \leq n$. Note that $G = K_{m,n} = (V = \{\{u_1, \ldots, u_m\}; \{v_1, \ldots, v_n\}\}, E = \{\{u_i, v_j\}\}_{1 \leq i \leq m, 1 \leq j \leq n})$ is a complete bipartite graph with m vertices of degree n and n vertices of degree m. Therefore $\delta(K_{m,n}) = \min\{\deg(v), v \in K_{m,n}\} = m$. First, by removing m edges incident to a vertex of degree m we can isolate this vertex. Therefore $\lambda(G) \leq m$. We will prove that $\lambda(G) = m$. In fact, assume that T is a cut in G with the smallest size and $K_{m,n} - T$ is disconnected. Then there is a connected component U of $K_{m,n} - T$ with vertex set $V(U) = \{u_1, \ldots, u_p, v_1, \ldots, v_q\}$ with $0 \leq p \leq m, 0 \leq q \leq n$ and $1 \leq p + q \leq m + n - 1$. Denote by G_1 is the induced subgraph on $V(U)$ and G_2 is the induced subgraph on $V(G) - V(U)$. Then

$$|T| = |E(G)| - |E(G_1)| - |E(G_2)| \geq mn - pq - (m - p)(n - q).$$

It is easy to see that the function $f(x, y) := mn - xy - (m - x)(n - y)$ defined on $0 \leq x \leq m, 0 \leq y \leq n, 1 \leq x + y \leq m + n - 1$ attains its minimum at boundary of the domain which is m. It implies that $|T| = m$.

E10.1: Connected graph

(i) Suppose $G = (V = \{v_1, \ldots, v_n\}, E)$ with $d(v_i) = d_i$ is simple and satisfies

(1) $d_1 \leq d_2 \leq \cdots \leq d_n$

(2) $d_k \geq k$ if $k \leq n - d_n - 1$.

- For $n = 1$, because G is simple, $G = (V = \{v_1\}, E = \emptyset)$ is connected.
- For $n \geq 2$, assume that G is disconnected. Then there is a connected component K which does not contain v_n. It implies that $V(K) \subseteq V - \{v_n\} - N(v_n)$. Therefore $1 \leq k := |V(K)| \leq n - 1 - d_n$ which implies from (2) that $k \leq d_k$. However, since

there are k vertices in K, there is an index $m \geq k$ such that $v_m \in K$ which implies that $d_m \leq k-1$. This is a contradiction to $d_m \overset{(1)}{\geq} d_k \geq k$. This contradiction implies that G is connected.

(ii) Suppose a general graph G has exactly two odd-degree vertices, v and w. Let G' be the graph created by adding an edge joining v to w.

($\Rightarrow$): If G' is connected then G' is a connected graph whose vertices are of even degree. Therefore, from a theorem in lecture, G' contains an Euler circuit. Removing the edge $\{v, w\}$ give us an Euler walk which connects every pair of vertices. Therefore G is connected.

($\Leftarrow$): If G is connected, it is easy to see that the graph is still connected after adding an edge. Therefore G' is connected.

E10.2: k-regular graph

Denote by $k_{\min}(n) = argmin\{k : \mathcal{A}_k \text{ is true}\}$ where $\mathcal{A}_k =$ "If G is simple, has n vertices, $m \geq k$, and G is m-regular, then G is connected." We will prove that

$$k_{\min}(n) = \begin{cases} p, & \text{if } n = 2p, \text{ for some } p \in \mathbb{N} \\ (2p-1) \vee 0, & \text{if } n = 4p+1, \text{ for some } p \in \mathbb{N}_0 \\ 2p+1, & \text{if } n = 4p+3, \text{ for some } p \in \mathbb{N}_0. \end{cases} \tag{1.0.5}$$

Claim 1:

(i) We can construct a simple $2k$-regular graph G with $2k+2$ vertices ($k \in \mathbb{N}_0$).

(ii) We can construct a simple k-regular graph G with $k+3$ vertices ($k \in \mathbb{N}_0$).

This claim could be proven by induction on k. In fact, we prove (i), the proof for (ii) is similar.

- Base cases: For $k = 0$ we have $G = (V = \{v_1, v_2\}, E = \emptyset)$.
  For $k = 1$ we have $G = (V = \{v_1, v_2, v_3, v_4\}, E = \{\{v_1, v_2\}, \{v_3, v_4\}\})$.
- Assume that we could construct a simple $2k$-regular graph G with $2k+2$ vertices ($k > 1$). We construct G' by adding G with two extra vertices each of which connects to all vertices in G. Then G' is a simple $(2k+2)$-regular graph on $2k+4$ vertices. It completes the proof.

We now prove (1.0.5):

- For $n = 1$: the only simple graph is $G = (V = \{v_1\}, E = \emptyset)$ therefore if G is simple m-regular, then $m = 0$. It means $k_{\min}(1) = 0$.
- For $n = 2$: the simple 0-regular graph $G = (V = \{v_1, v_2\}, E = \emptyset)$ is not connected therefore $k_{\min}(2) > 0$. On the other hand, $\mathcal{A}_1$ is true because if G is simple m-regular with $m \geq 1$ then $m = 1$ and $G = K_2$ which is connected. Therefore $k_{\min}(2) = 1$.
- For $n > 2, n = 2p$: $\mathcal{A}_{p-1}$ is not true because there is a simple $(p-1)$-regular graph $G = K_p \cup K_p$ which is not connected. On the other hand $\mathcal{A}_p$ is true because for all simple m-regular graph G with $m \geq p$, every two vertices u and v in G satisfy the Ore's condition, i.e., $d(u) + d(v) = 2m \geq n$. Therefore G has a Hamilton cycle which implies that G is connected.

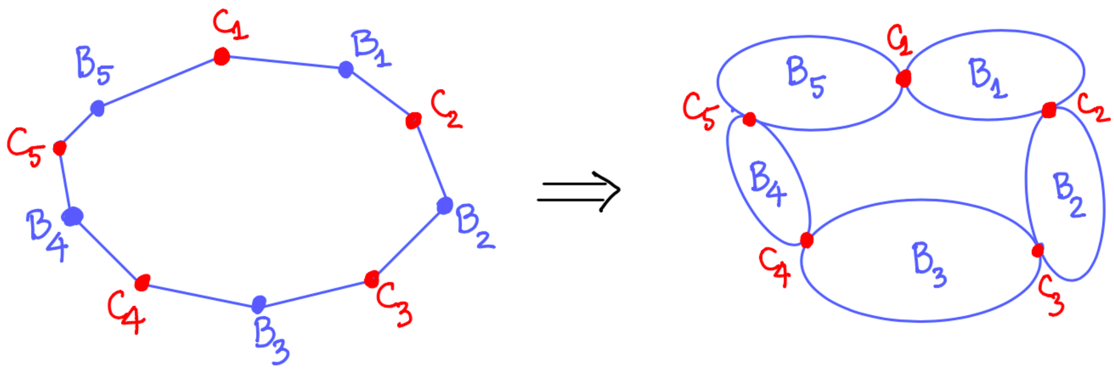
- For $n > 2, n = 4p+1$: $\mathcal{A}_{2p-2}$ is not true because from the Claim 1, we can construct a simple $(2p-2)$-regular graph G_1 with $2p+1$ vertices and a simple $(2p-2)$-regular graph G_2 with $2p$ vertices. Therefore we have a simple $(2p-2)$-regular graph $G = G_1 \cup G_2$ which is not connected. On the other hand $\mathcal{A}_{2p-1}$ is true because for all simple m-regular graph G with $m \geq 2p-1$, because n is odd, m must be even ($mn = 2|E|$), therefore $m \geq 2p$ which implies that for every two vertices u and v in G, $d(u) + d(v) = 2m \geq n-1$. Therefore G has a Hamilton path which implies that G is connected.

- For $n > 2, n = 4p+3$: $\mathcal{A}_{2p}$ is not true because from the Claim 1, we can construct a simple $2p$-regular graph G_3 with $2p+2$ vertices. Therefore we have a simple $2p$-regular graph $G = G_3 \cup K_{2p+1}$ which is not connected. On the other hand $\mathcal{A}_{2p+1}$ is true because for all simple m-regular graph G with $m \geq 2p+1$, because n is odd, m must be even, therefore $m \geq 2p+2$ which implies that for every two vertices u and v in G, $d(u)+d(v) = 2m > n$. Therefore G has a Hamilton cycle which implies that G is connected.

E10.3: Block-cutpoint graph

(i) Suppose G is a connected graph. To prove that $BC(G) = (V = \{c_1, \ldots, c_k, B_1, \ldots, B_l\}, E = \{\{B_i, c_j\}, 1 \leq i \leq k, 1 \leq j \leq l$ such that $c_j \in B_i\})$ is a tree we prove that it is connected and acyclic.

  - Acyclic: Assume that there exists a cycle C_r in $BC(G)$. Because each edge in $BC(G)$ connect one cutpoint with one block (with at least two vertices), it implies that $r \geq 3$ and the cycle of the form $C_r = \{c_1, B_1, c_2, B_2, \ldots, c_r, B_r, c_1\}$. But then $G' = G[\{V(B_1) \cup \ldots \cup V(B_r)\}]$ is an induced subgraph on at least two vertices without cutpoints and strictly contains B_1 which is a contradiction to B_1 is a block. It implies that $BC(G)$ is acyclic.



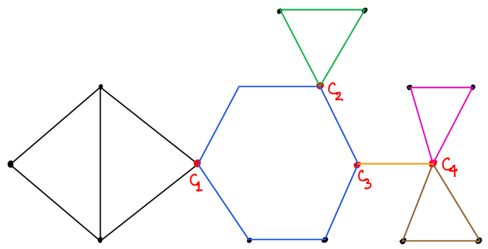
  - Connected: For any two vertices in $BC(G)$ we consider three cases:
    - $v, w \in \{c_1, \ldots, c_k\}$ are cutpoints in G, say $v = c_i, w = c_j$ with $i < j$. Because G is connected there is a path P in G from c_i to c_j. Divide P into subpaths $\{P_k\}_{i \leq k \leq j-1}$ which connects two consecutive cutpoints, i.e., $P_k = \{c_k, u_1, \ldots, u_{i_k}, c_{k+1}\}$. Note that G has at least one cutpoint ($k \geq 1$) therefore it is a connected but not 2-connceted which implies from a theorem in the lecture that every vertex that is in two blocks is a cutpoint of G. Therefore P_k must be into exactly one block, say B_k (we can renumber indexes if necessary). Therefore we have a path $= \{c_i, B_i, c_{i+1}, B_{i+1}, \ldots, B_{j-1}, c_j\}$ in $BC(G)$ connect c_i to c_j.
    - $v \in \{c_1, \ldots, c_k\}$ is a cutpoint and $w \in \{B_1, \ldots, B_l\}$ is a block in G. Because the block w contains least one cutpoint, say v'. If $v = v'$ then we have an edge $\{v, w\}$

in $BC(G)$. If else we know from Case 1 that there is a path P in $BC(G)$ connect v to v'. Then $P \cup \{v', w\}$ is a path in $BC(G)$ connect v to w.

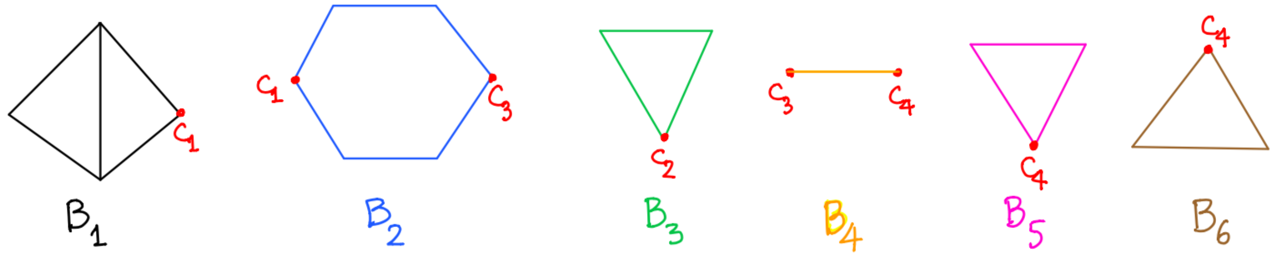
– $v, w \in \{B_1, \ldots, B_l\}$ is two blocks in G. Because each block v, w contains least one cutpoint, say v', w'. If $v' = w'$ then we have a path $\{v, v' = w', w\}$ in $BC(G)$. If else we know from Case 1 that there is a path P in $BC(G)$ connect v' to w'. Then $\{v, v'\} \cup P \cup \{w', w\}$ is a path in $BC(G)$ connect v to w.

It implies that $BC(G)$ is connected.

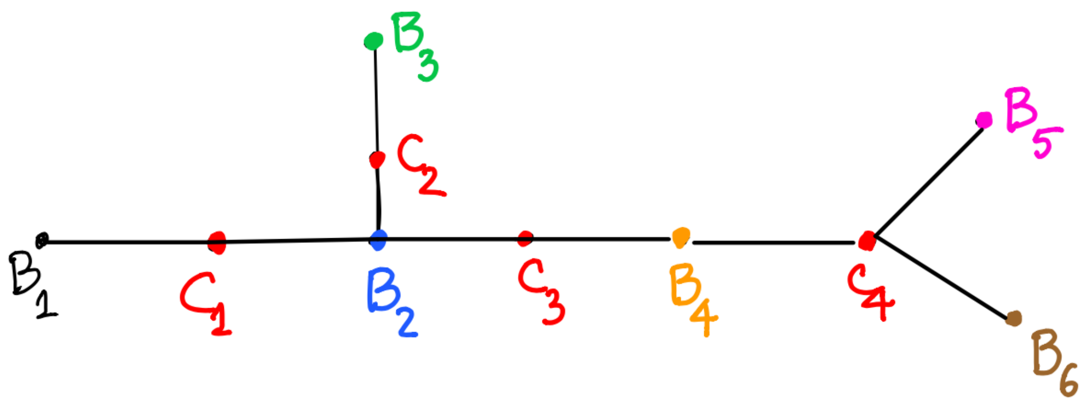
(ii) Cutpoints:



Blocks:



The block-cutpoint graph:

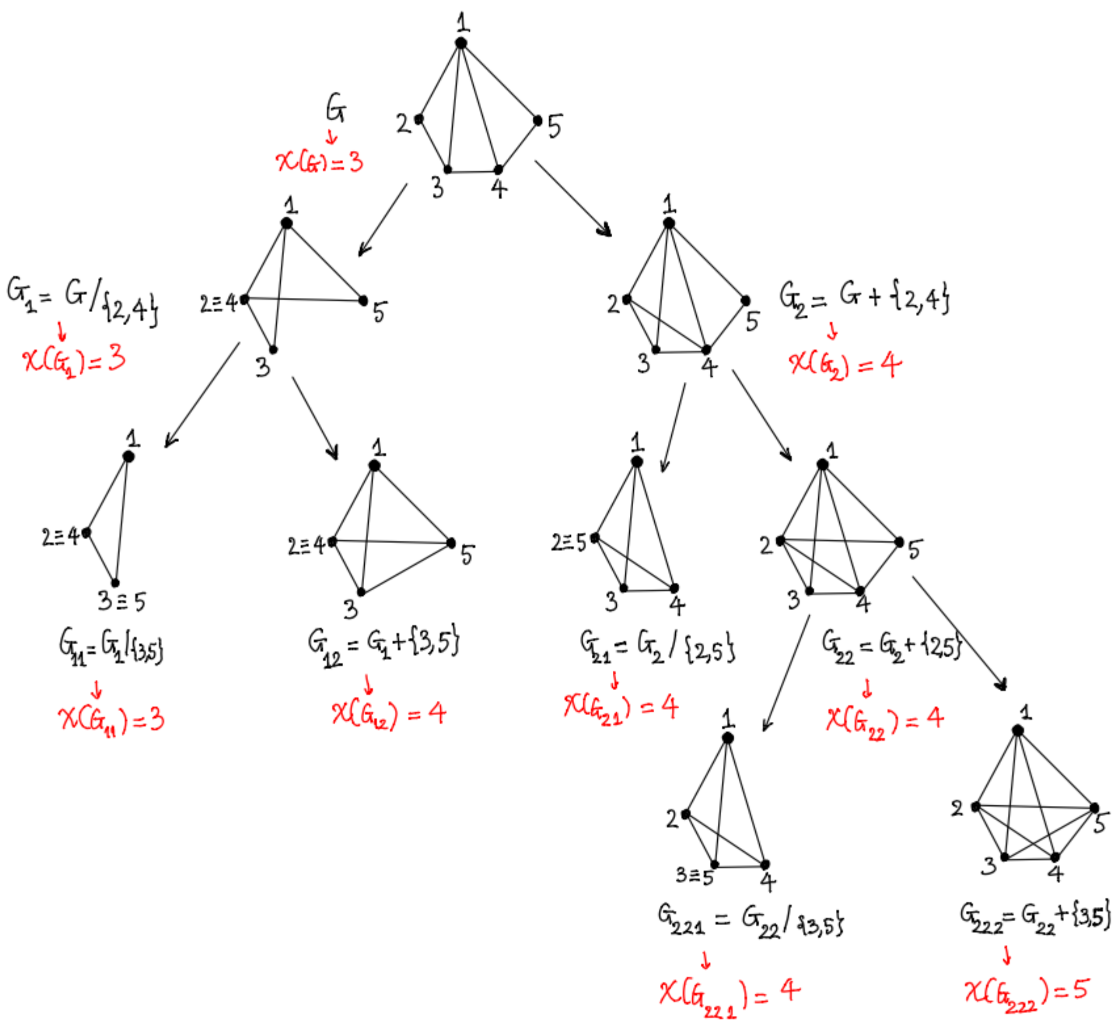


E11.1: Chromatic number $\chi(G)$

(i) Suppose $G = (V, E)$ is a simple connected graph on $|V| = n$ vertices with its chromatic number $\chi(G) = k$ and we have a proper coloring of G with k colors $\{1, \ldots, k\}$. Denote by A_i the set of vertices with color i for $i \in \{1, \dots, k\}$. Then A_i is an independent set. Moreover, for every pair $i \neq j \in \{1, \dots, k\}$, if there is no edge connection A_i and A_j then $A_i \cup A_j$ is also an independent set and we can recolor by changing j into i to have a proper coloring with $k-1$ colors $\{1, \dots, j-1, j+1, \dots, k\}$ which is a contradiction to $\chi(G) = k$. Therefore for each $i \neq j \in \{1, \dots, k\}$ there is at least an edge of G which connects A_i to A_j. It implies that $|E(G)| \geq \binom{k}{2}$.

(ii) First, note that $G - v$ is a subgraph of G therefore from a result in the lecture, $\chi(G-v) \leq \chi(G)$. Furthermore, assume that $\chi(G-v) \leq \chi(G) - 2$, i.e., we can find a proper coloring of $G - v$ with only $\chi(G) - 2$ colors.By adding an extra color for v we have then a proper coloring of G with only $\chi(G) - 1$ colors which is a contradiction to the definition of $\chi(G)$. Therefore $\chi(G-v) \geq \chi(G) - 1$. This implies that $\chi(G-v)$ is either $\chi(G)$ or $\chi(G) - 1$.

E11.2: Greedy algorithm to find $\chi(G)$



- Step 1: Label vertices of G with $\{1, 2, 3, 4, 5\}$ then we have the set of pairs of two non-adjacent vertices $E(\bar{G}) = \{\{2, 4\}, \{2, 5\}, \{3, 5\}\}$.
- Step 2: Because G is not complete, we find a pair of non-adjacent vertices in order, e.g., $\{2, 4\}$ and construct $G_1 = G/\{2, 4\}$ and $G_2 = G + \{2, 4\}$.

- Step 3: Because G_1 is not complete with $E(\bar{G}_1) = \{\{3,5\}\}$ we construct $G_{11} = G_1/\{3,5\}$ and $G_{12} = G_1 + \{3,5\}$. Because G_2 is not complete with $E(\bar{G}_2) = \{\{2,5\},\{3,5\}\}$ we construct $G_{21} = G_2/\{2,5\}$ and $G_{22} = G_2 + \{2,5\}$.

- Step 4: Because G_{11}, G_{12}, G_{21} are complete we do not continue with these branches. Because G_{22} is not complete with $E(\bar{G}_{22}) = \{\{3,5\}\}$ we construct $G_{221} = G_{22}/\{3,5\}$ and $G_{222} = G_{22} + \{3,5\}$.

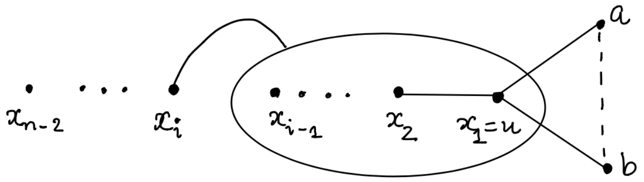
- Step 5: Because G_{221} and G_{222} are complete we stop our algorithm by counting all the chromatic numbers on leave graphs and take the minimum. It means we count $\chi(G_{11}) = 3, \chi(G_{12}) = 4, \chi(G_{21}) = 4, \chi(G_{221}) = 4, \chi(G_{222}) = 5$ and therefore $\chi(G) = \min\{3,4,4,4,5\} = 3$.

Moreover, by using the formula $\chi(G) = \min\{\chi(G/e), \chi(G+e)\}$, we can find the chromatic numbers for all graphs in this tree as follows $\chi(G_1) = \min\{\chi(G_{11}), \chi(G_{12})\} = \min\{3,4\} = 3$, $\chi(G_{22}) = \min\{\chi(G_{221}), \chi(G_{222})\} = \min\{4,5\} = 4$, $\chi(G_2) = \min\{\chi(G_{21}), \chi(G_{22})\} = \min\{4,4\} = 4$ and $\chi(G) = \min\{\chi(G_1), \chi(G_2)\} = \min\{3,4\} = 3$.

E11.3: Brooks's theorem

Suppose that G is a simple, connected graph other than K_n or C_{2n+1}. We know from a result in the lecture that if G is not regular then $\chi(G) \leq \Delta(G)$. Therefore we only need to prove the claim for a regular graph G other than K_n or C_{2n+1}. In fact, because G is regular, $d = \Delta(G)$ is the common degree of G with $2 \leq d \leq n-1$. Because G is not K_n it implies that $d \leq n-2$. For $d = 2$, because G is not C_{2n+1}, we only need to check for C_{2n}. This can be done easily by coloring 2-colors alternatively. Now we check for $3 \leq d \leq n-2$. We first prove the following claim:

Claim: If a simple, connected, regular graph G contains two vertices a, b with the distance $d(a,b) = 2$ and $G - a - b$ is connected then $\chi(G) \leq d$.
Proof of Claim: In fact, because $d(a,b) = 2$, it implies that $a \not\sim b$ and there is a vertex u with $u \sim a$, $u \sim b$. We label vertices of $G - a - b$ as follows: $x_1 = u, x_2, \ldots, x_{n-2}$ such that each label x_i $(i \geq 2)$ is labeled for an unlabelled vertex which is adjacent to one of labeled vertices (i.e. $\{x_1, \ldots, x_{i-1}\}$). In other words, $N(x_i) \cap \{x_1, \ldots, x_{i-1}\} \neq \emptyset$, $\quad \forall 2 \leq i \leq n-2$.



Now we color G as follows: First, color vertices a and b with color 1. We then color $x_{n-2}, x_{n-3}, \ldots, x_2$ with one of colors $\mathcal{C} = \{1, \ldots, d\}$. This is always possible because when we would like to color x_i, because $N(x_i) \cap \{x_1, \ldots, x_{i-1}\} \neq \emptyset$ it means x_i has fewer than d neighbours previously colored. When we would like to color x_1, because x_1 has $d-2$ neighbours other a, b therefore we have used at most $d - 2 + 1 = d - 1$ colors for coloring neighbours of x_1. It means that we could color x_1 with colors in $\mathcal{C}$. This completes the Claim.

Back to the theorem: We consider two following cases:

1. G is 2-connected: we will prove that we always can pick two vertices a, b with $d(a,b) = 2$ and $G - a - b$ is connected so that we have $\chi(G) \leq d$ from the Claim. In fact, pick an arbitrary vertex x and consider $G - x$:

- If $G - x$ is also 2-connected, then choose $a = x$ and b is a vertex with $d(x, b) = 2$ (this can be done because $d(x) = d \leq n - 2$ then there is some vertex $y \not\sim x$ and because $d(y) = d \geq 3$ we can pick b from neighbors of y). Moreover $G - a - b = (G - x) - b$ is connected.
- If $G - x$ is not 2-connected. Consider two different endblocks B_1, B_2 (an endblock B is a 2-connected component containing a point z such that for any other 2-connected component B' either $B \cap B' = \emptyset$ or $B \cap B' = \{z\}$). Since G is 2-connected, $G - z_1$ is connected, therefore there is $a \in B_1 - z_1$ which is adjacent to x and similarly there is $b \in B_2 - z_2$ which is adjacent to x. Then $a \not\sim b$ because they are in two different blocks and not the cutpoint. Moreover the path $\{a, x, b\}$ has lenght 2. It implies that $d(a, b) = 2$. Furthermore, because a, b are not cutpoints of $G - x$ and $d(x) = d \geq 3$, it implies that $G - a - b$ is connected.

2. G is not 2-connected: from a result in the lecture, we know that the blocks partition $E(G)$. Denote by $\{B_1, \ldots, B_r\}$ $(r \geq 2)$ the set of all blocks of G which connects to each other by cutpoints. Note that $\Delta(B_i) \leq d$ and B_i either 2-connected or C_2 therefore each B_i can be colored by $d \geq 3$ colors. Then the coloring may be altered slightly so that they combine to give a proper coloring of G with at most d colors.

Remark: For other proofs, see for example, http://faculty.wwu.edu/sarkara/brooks.pdf or https://www.sciencedirect.com/science/article/pii/S0972860018300100

E12.1: Chromatic polynomial P_G

Suppose that $G = (V, E)$ is a graph on $|V| = n$ vertices and $|E| = m$ edges with $P_G(k) = \sum_{i=0}^{n} a_{n,i} k^i$. We prove

(i) $a_{n,n} = 1$.

(ii) $a_{n,0} = 0$.

(iii) $a_{n,n-1} = -|E|$.

(iv) $(-1)^{n-i} a_{n,i} \geq 0$ for all $i \in [n]$.

by induction on m. In fact

- Base case: if $m = 0$ then $P_G(k) = k^n$ therefore $a_{n,n} = 1$, $a_{n,n-1} = 0 = -|E|$, $a_{n,0} = 0$ and $(-1)^{n-i} a_{n,i} \geq 0$ for all $i \in [n]$.
- Inductive step: Assume that (i)-(iv) are true until $m - 1 \geq 0$, we prove that they are also true for m. In fact, because $m \geq 1$, there is $e \in E$. We know from the lecture that

  $$P_G(k) = P_{G-e}(k) - P_{G/e}(k).$$

  Because $G - e$ is a graph on n vertices and $m - 1$ edges, we have from the inductive hypothesis

  $$P_{G-e}(k) = k^n - (m-1)k^{n-1} + \cdots + b_{n,1}k$$

  with $(-1)^{n-i} b_{n,i} \geq 0$ for all $i \in [n]$. Because G/e is a graph on $n - 1$ vertices and $m - 1 - c$ edges ($c \geq 0$ is the number of vertices which are adjacent to both endpoints of e), we have from the inductive hypothesis

  $$P_{G/e}(k) = k^{n-1} - (m-1-c)k^{n-2} + \cdots + c_{n-1,1}k$$

with $(-1)^{n-1-i}c_{n-1,i} \geq 0$ for all $i \in [n-1]$. Therefore

$$P_G(k) = k^n - mk^{n-1} + \cdots + a_{n,1}k$$

with $(-1)^{n-i}a_{n,i} = (-1)^{n-i}(b_{n,i} - c_{n-1,i}) = (-1)^{n-i}b_{n,i} + (-1)^{n-1-i}c_{n-1,i} \geq 0$. Thus, $a_{n,n} = 1$, $a_{n,n-1} = -m = -|E|$, $a_{n,0} = 0$, and $(-1)^{n-i}a_{n,i} \geq 0$ for all $i \in [n]$. It implies the proof.

E12.2: Characterize of P_G

(i) ($\Rightarrow$:) Suppose that G is a tree on n vertices, we prove that $P_G(k) = k(k-1)^{n-1}$ by induction on n. In fact

- Base case: if $n = 1$ then there are k ways to color G with k colors. Therefore $P_G(k) = k = k(k-1)^{n-1}$.
- Inductive step: assume that this is true until $n - 1 \geq 0$. We know from the lecture that there is at least one pendant vertex v in a tree and denote by e the edge incident to v. Note that $G - e$ is a disjoint union of a tree T' on $n-1$ vertices and v while G/e is nothing but T'. Therefore we have from E12.2(iii) and the inductive hypothesis that

$$P_G(k) = P_{G-e}(k) - P_{G/e}(k) = P_{T'}(k)P_{\{v\}}(k) - P_{T'}(k) = k(k-1)^{n-2}k - k(k-1)^{n-2} = k(k-1)^{n-1}$$

which implies that the statement is also true for n. Therefore it is true for all $n \geq 1$.
($\Leftarrow$:) Suppose that $P_G(k) = k(k-1)^{n-1} = k^n - (n-1)k^{n-1} + \cdots + (-1)^{n-1}k$ for all $n \geq 1$, we prove that G is a tree. In fact, from E12.1(ii) and E12.2(ii) we know that $|E(G)| = n - 1$ and G is connected. Therefore G is a tree from a result in the lecture.

(ii) ($\Rightarrow$:) Suppose that G is connected whose chromatic polynomial is $P_G(k) = \sum_{i=0}^{n} a_{n,i}k^i$. We prove that $a_{n,1} \neq 0$ by induction on m. In fact

- Base case: if $m = 0$ then $n = 1$ because G is connected, therefore $a_{1,1} = 1 \neq 0$.
- Inductive step: Assume that the claim is true until $m - 1 \geq 0$, we prove that it is also true for m. In fact, similar to E12.1(iv) because $m \geq 1$ there is $e \in E$ and we have

$$\begin{aligned} P_G(k) &= P_{G-e}(k) - P_{G/e}(k) \\ &= \left(k^n - (m-1)k^{n-1} + \cdots + b_{n,1}k\right) - \left(k^{n-1} - (m-1-c)k^{n-2} + \cdots + c_{n-1,1}k\right) \end{aligned}$$

where $(-1)^{n-1}b_{n,1} \geq 0$ and $(-1)^{n-2}c_{n-1,1} \geq 0$. On the other hand, G/e is connected with the number of edges is less than m, therefore from the inductive hypothesis, $c_{n-1,1} \neq 0$ which implies $(-1)^{n-2}c_{n-1,1} > 0$. Therefore $(-1)^{n-1}a_{n,1} = (-1)^{n-1}b_{n,1} + (-1)^{n-2}c_{n-1,1} > 0$ or $a_{n,1} \neq 0$. It implies the proof.

($\Leftarrow$:) Suppose that a graph G whose chromatic polynomial is $P_G(k) = \sum_{i=0}^{n} a_{n,i}k^i$ with $a_{n,1} \neq 0$ we prove that G is connected. In fact, assume that G is not connected, assume that G has $L \geq 2$ connected components $C_1, \ldots, C_L$. Then from E12.2(iii) and E12.1(ii)

$$P_G(k) = \prod_{i=1}^{L} P_{C_i} = \prod_{i=1}^{L}(k^{|V(G_1)|} + \cdots + c_i k) = k^n + \cdots + (c_1 \ldots c_L)k^L$$

therefore $a_{n,1} = 0$ which is a contradiction. Therefore G is connected.

(iii) Suppose that G is not connected and has components $C_1, \dots, C_L$. We prove that $P_G = \prod_{i=1}^{L} P_{C_i}$. In fact, because there is no connection between connected components, a proper coloring c of G is an one-to-one corresponding to L proper colorings $\{c_{|C_1}, \dots, c_{|C_L}\}$ on connected components. Therefore the number of proper colorings of G is nothing but the product of the number of proper colorings of its connected components. In other words, $P_G = \prod_{i=1}^{L} P_{C_i}$.

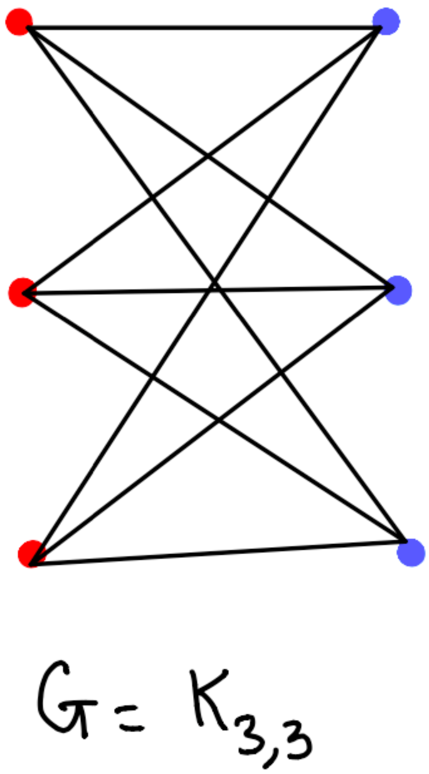
E12.3: $K_{3,3}$

(i) Assume that $K_{3,3}$ is planar. From Euler's formula, there are $r = m - n + 2 = 9 - 6 + 2 = 5$ distinct regions. Set f_i is the number of edges on the boundary of region i for all $i \in \{1, \dots, 5\}$. Because G is a simple bipartite graph, $f_i \geq 4$ for all $i \in \{1, \dots, 5\}$. Therefore

$$4r \leq \sum_{i=1}^{5} f_i \leq 2m$$

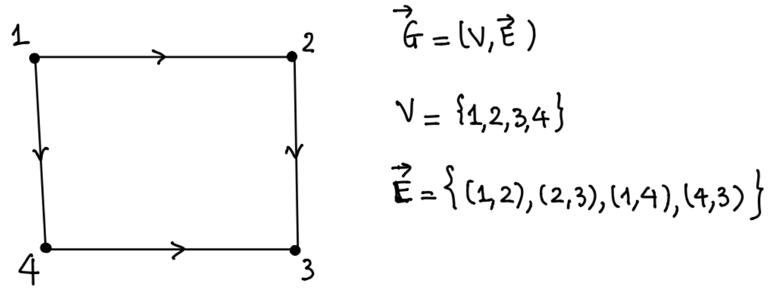
which implies that $20 = 4 \times r \leq 2 \times 9 = 18$. This contradiction means that $K_{3,3}$ is not planar.

(ii) The chromatic number of $K_{3,3} = 2$ because we can not color $K_{3,3}$ with only one color (two endpoints of an edge should have different colors) and we can color with two colors as follows



E13.1: Connectivity in digraphs

An example of a digraph that is connected but not strongly connected:



The underline graph $G = (V, E)$ is connected but $\vec{G}$ is not connected because there is no path from 2 to 4.

E13.2: Euler circuit and Hamilton path in digraphs

(i) ($\Rightarrow$:) Suppose $\vec{G}$ is a digraph with no vertices of degree 0 has an Euler circuit $\vec{C}$. Because every edge of the underlying graph G is inside the Euler circuit C and there are no vertices of degree 0, $\vec{G}$ is connected. Moreover, for any vertex $v \in V$, the number of outgoing arcs of v is equal to the number of ingoing arcs of v (to see it more carefully, we just write the Euler circuit as $\vec{C} = \{v_{i_1}, \vec{e}_1, v_{i_2}, \vec{e}_2, \dots, v_{i_m}, \vec{e}_m, v_{i_1}\}$ where $m = |\vec{E}|$). Thus $d^+(v) = d^-(v)$.

($\Leftarrow$:) Suppose that $\vec{G}$ is a connected digraph with $d^+(v) = d^-(v)$ for all vertices $v \in V$. We prove that there is an Euler circuit by induction on m.

- Base case: we can check easily for $m = 2$ where n must be 2 and there is an Euler circuit $\vec{C} = \{1, (1,2), 2, (2,1), 1\}$.
- Inductive step: assume that it is true until $m \geq 2$, we prove that it is also true for $m+1$. In fact, because $d^+(v) = d^-(v)$ for all vertices $v \in V$, starting from $v \in V$ we can construct a cycle $\vec{C}_0$ in $\vec{G}$ which starts and ends at v. By deleting all arcs in $\vec{C}_0$ we have a new digraph with $r \geq 1$ connected components $\vec{H}_1, \dots, \vec{H}_r$ and $d^+(v) = d^-(v)$ for all vertices $v \in \vec{H}_i$, $\forall i \in [r]$. Clearly, $|E(\vec{H}_i)| \leq m$ therefore from the inductive hypothesis, there is an Euler circuit $\vec{C}_i$ for each $\vec{H}_i$. Because $\vec{G}$ is connected, $\exists v_i \in \vec{C}_0 \cap \vec{H}_i$. Therefore $v_1, \dots, v_r$ breaks $\vec{C}_0$ into r paths, say $v_1 P_1 v_2, v_2 P_2 v_3, \dots, v_r P_r v_1$. Arrange $\vec{C}_i$ so that the starting and ending vertex is v_i. Then we end up with an Euler circuit $\vec{C} = \{\underbrace{v_1, \dots, v_1}_{\vec{C}_1}, P_1, \underbrace{v_2, \dots, v_2}_{\vec{C}_2}, P_2, \dots, \underbrace{v_r, \dots, v_r}_{\vec{C}_r}, P_r, v_1\}$.

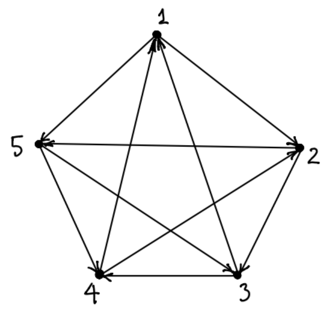
(ii) We prove that every tournament has a Hamilton path by induction on n:

- Base case: $n \leq 2$ can be easily checked.
- Inductive step: Assume that the statement is true until $n-1 \geq 2$, we prove that it is also true for n. In fact, pick an arbitrary player, say v_n. From the inductive hypothesis for $n-1$ remaining players, we can arrange them in a path $P_{n-1} = \{v_1, \dots, v_{n-1}\}$, i.e., $(v_i, v_{i+1}) \in \vec{E}$ for all $i = 1, \dots, n-2$. In case there is no other person who beats v_n then $(v_n, v_1) \in \vec{E}$ and we have a Hamilton path $P = \{v_n, v_1, \dots, v_{n-1}\}$. In case there is someone who beats v_n, we consider the last one who beats v_n, say v_k $(1 \leq k \leq n-1)$. Then we have $(v_n, v_{k+1}) \in \vec{E}$ and a Hamilton path $P = \{v_1, \dots, v_k, v_n, v_{k+1}, \dots, v_{n-1}\}$ for $k \leq n-2$ or $P = \{v_1, \dots, v_n\}$ for $k = n-1$.

E13.3: Champion

(i) Suppose that v is the player with the maximum number of wins and denote by $N_v^+ := \{u \in V : (v,u) \in \vec{E}\}$ all players who are beaten by v. Assume that v is not a champion, then there is a player w who is not beaten by v and every $u \in N_v^+$ must be beaten by w. Because a tournament is a complete digraph, then w beats v and therefore has more wins than v which is a contradiction. Thus v is a champion.

(ii) A 5-vertex tournament in which every player is a champion:



$\vec{G} = (V, \vec{E})$

$V = \{1,2,3,4,5\}$

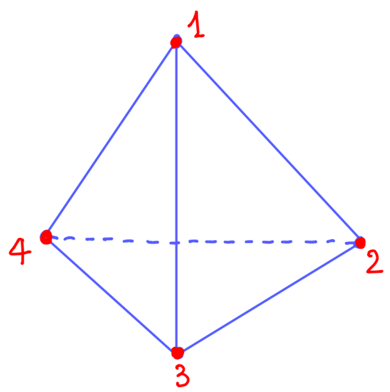
$\vec{E} = \{(1,2),(3,1),(4,1),(1,5),(2,3),(4,2),(2,5),(3,4),(5,3),(5,4)\}$

E14.1: Permutations in the regular tetrahedron

(i) The 12 permutations (and their cycle form) of the vertices of the regular tetrahedron corresponding to the 12 rigid motions of the regular tetrahedron are:

- No motion: $\pi_0 = id = (1)(2)(3)(4)$.
- Rotations 120^o and -120^o around the line which goes through 1 and the center of the triangle (234):
  $\pi_1^+ = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 2 \end{pmatrix} = (1)(2,3,4).$
  $\pi_1^- = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 2 & 3 \end{pmatrix} = (1)(2,4,3).$
- Rotations 120^o and -120^o around the line which goes through 2 and the center of the triangle (134):
  $\pi_2^+ = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 4 & 1 \end{pmatrix} = (1,3,4)(2).$
  $\pi_2^- = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 1 & 3 \end{pmatrix} = (1,4,3)(2).$
- Rotations 120^o and -120^o around the line which goes through 3 and the center of the triangle (124):
  $\pi_3^+ = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 1 \end{pmatrix} = (1,2,4)(3).$
  $\pi_3^- = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 3 & 2 \end{pmatrix} = (1,4,2)(3).$
- Rotations 120^o and -120^o around the line which goes through 4 and the center of the triangle (123):
  $\pi_4^+ = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix} = (1,2,3)(4).$
  $\pi_4^- = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 4 \end{pmatrix} = (1,3,2)(4).$

- Rotation 180^o around the line which goes through the midpoint of (12) and the midpoint of (34):
$\pi_{12,34} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} = (1,2)(3,4).$
- Rotation 180^o around the line which goes through the midpoint of (13) and the midpoint of (24):
$\pi_{13,24} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix} = (1,3)(2,4).$
- Rotation 180^o around the line which goes through the midpoint of (14) and the midpoint of (23):
$\pi_{14,23} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix} = (1,4)(2,3).$

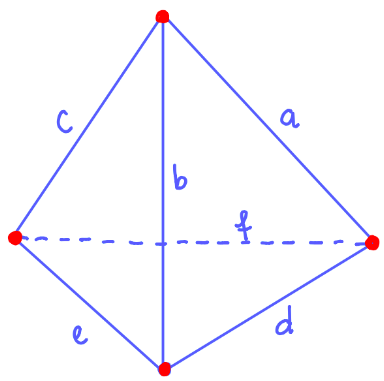


(ii) The 12 permutations (and their cycle form) of the edges of the regular tetrahedron corresponding to the 12 rigid motions of the regular tetrahedron are:

- No motion: $\tilde{\pi}_0 = id = (a)(b)(c)(d)(f)(e)$.
- Rotations 120^o and -120^o around the line which goes through 1 and the center of the triangle (234):
$\tilde{\pi}_1^+ = \gamma \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 2 \end{pmatrix} = \begin{pmatrix} 12 & 13 & 14 & 23 & 24 & 34 \\ 13 & 14 & 12 & 34 & 32 & 42 \end{pmatrix} = \begin{pmatrix} a & b & c & d & f & e \\ b & c & a & e & d & f \end{pmatrix} = (a,b,c)(d,e,f).$
$\tilde{\pi}_1^- = \gamma \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 12 & 13 & 14 & 23 & 24 & 34 \\ 14 & 12 & 13 & 42 & 43 & 23 \end{pmatrix} = \begin{pmatrix} a & b & c & d & f & e \\ c & a & b & f & e & d \end{pmatrix} = (a,c,b)(d,f,e).$
- Rotations 120^o and -120^o around the line which goes through 2 and the center of the triangle (134):
$\tilde{\pi}_2^+ = \gamma \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 4 & 1 \end{pmatrix} = \begin{pmatrix} 12 & 13 & 14 & 23 & 24 & 34 \\ 32 & 34 & 31 & 24 & 21 & 41 \end{pmatrix} = \begin{pmatrix} a & b & c & d & f & e \\ d & e & b & f & a & c \end{pmatrix} = (a,d,f)(b,e,c).$
$\tilde{\pi}_2^- = \gamma \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 12 & 13 & 14 & 23 & 24 & 34 \\ 42 & 41 & 43 & 21 & 23 & 13 \end{pmatrix} = \begin{pmatrix} a & b & c & d & f & e \\ f & c & e & a & d & b \end{pmatrix} = (a,f,d)(b,c,e).$
- Rotations 120^o and -120^o around the line which goes through 3 and the center of the triangle (124):
$\tilde{\pi}_3^+ = \gamma \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 12 & 13 & 14 & 23 & 24 & 34 \\ 24 & 23 & 21 & 43 & 41 & 31 \end{pmatrix} = \begin{pmatrix} a & b & c & d & f & e \\ f & d & a & e & c & b \end{pmatrix} = (a,f,c)(b,d,e).$

$\tilde{\pi}_3^- = \gamma\begin{pmatrix}1 & 2 & 3 & 4\\ 4 & 1 & 3 & 2\end{pmatrix} = \begin{pmatrix}12 & 13 & 14 & 23 & 24 & 34\\ 41 & 43 & 42 & 13 & 12 & 32\end{pmatrix} = \begin{pmatrix}a & b & c & d & f & e\\ c & e & f & b & a & d\end{pmatrix} = (a,c,f)(b,e,d)$.

- Rotations 120^o and -120^o around the line which goes through 4 and the center of the triangle (123):
  $\tilde{\pi}_4^+ = \gamma\begin{pmatrix}1 & 2 & 3 & 4\\ 2 & 3 & 1 & 4\end{pmatrix} = \begin{pmatrix}12 & 13 & 14 & 23 & 24 & 34\\ 23 & 21 & 24 & 31 & 34 & 14\end{pmatrix} = \begin{pmatrix}a & b & c & d & f & e\\ d & a & f & b & e & c\end{pmatrix} = (a,d,b)(c,f,e)$.
  $\tilde{\pi}_4^- = \gamma\begin{pmatrix}1 & 2 & 3 & 4\\ 3 & 1 & 2 & 4\end{pmatrix} = \begin{pmatrix}12 & 13 & 14 & 23 & 24 & 34\\ 31 & 32 & 34 & 12 & 14 & 24\end{pmatrix} = \begin{pmatrix}a & b & c & d & f & e\\ b & d & e & a & c & f\end{pmatrix} = (a,b,d)(c,e,f)$.
- Rotation 180^o around the line which goes through the midpoint of (12) and the midpoint of (34):
  $\tilde{\pi}_{12,34} = \gamma\begin{pmatrix}1 & 2 & 3 & 4\\ 2 & 1 & 4 & 3\end{pmatrix} = \begin{pmatrix}12 & 13 & 14 & 23 & 24 & 34\\ 21 & 24 & 23 & 14 & 13 & 43\end{pmatrix} = \begin{pmatrix}a & b & c & d & f & e\\ a & f & d & c & b & e\end{pmatrix} = (a)(b,f)(c,d)(e)$.
- Rotation 180^o around the line which goes through the midpoint of (13) and the midpoint of (24):
  $\tilde{\pi}_{13,24} = \gamma\begin{pmatrix}1 & 2 & 3 & 4\\ 3 & 4 & 1 & 2\end{pmatrix} = \begin{pmatrix}12 & 13 & 14 & 23 & 24 & 34\\ 34 & 31 & 32 & 41 & 42 & 12\end{pmatrix} = \begin{pmatrix}a & b & c & d & f & e\\ e & b & d & c & f & a\end{pmatrix} = (a,e)(b)(c,d)(f)$.
- Rotation 180^o around the line which goes through the midpoint of (14) and the midpoint of (23):
  $\tilde{\pi}_{14,23} = \gamma\begin{pmatrix}1 & 2 & 3 & 4\\ 4 & 3 & 2 & 1\end{pmatrix} = \begin{pmatrix}12 & 13 & 14 & 23 & 24 & 34\\ 43 & 42 & 41 & 32 & 31 & 21\end{pmatrix} = \begin{pmatrix}a & b & c & d & f & e\\ e & f & c & d & b & a\end{pmatrix} = (a,e)(b,f)(c)(d)$.



E14.2: Number of different colorings

(i) The number of different colorings of the vertices of a regular tetrahedron with k colors, modulo the rigid motions $G = \{id, \pi_1^+, \pi_1^-, \pi_2^+, \pi_2^-, \pi_3^+, \pi_3^-, \pi_4^+, \pi_4^-, \pi_{12,34}, \pi_{13,24}, \pi_{14,23}\}$ is

$$\frac{1}{|G|}\sum_{\sigma\in G} k^{\#\sigma} = \frac{1}{12}\Big(k^4 + 11\cdot k^2\Big).$$

(ii) The number of different colorings of the edges of a regular tetrahedron with k colors, modulo the rigid motions $\tilde{G} = \{id, \tilde{\pi}_1^+, \tilde{\pi}_1^-, \tilde{\pi}_2^+, \tilde{\pi}_2^-, \tilde{\pi}_3^+, \tilde{\pi}_3^-, \tilde{\pi}_4^+, \tilde{\pi}_4^-, \tilde{\pi}_{12,34}, \tilde{\pi}_{13,24}, \tilde{\pi}_{14,23}\}$ is

$$\frac{1}{|\tilde{G}|} \sum_{\sigma \in \tilde{G}} k^{\#\sigma} = \frac{1}{12}\Big(k^6 + 8 \cdot k^2 + 3 \cdot k^4\Big).$$

E14.3: Cycle index

(i) The cycle index $\mathcal{P}_G$ for the group of permutations of the vertices of a regular tetrahedron induced by the rigid motions G is

$$\mathcal{P}_G = \frac{1}{|G|} \sum_{\sigma \in G} \prod_{i=1}^{n} x_i^{\tau_i(\sigma)} = \frac{1}{12}\Big(x_1^4 + 8 \cdot x_1 x_3 + 3 \cdot x_2^2\Big).$$

(ii) Note that the group G of permutations of the vertices of K_5 consists of $5! = 120$ elements in which

- there is one identity permutation of the form: five 1-cycles $(1)(2)(3)(4)(5)$;
- $\binom{5}{2} = 10$ permutations of the form: three 1-cycles and one 2-cycle, e.g., $(1,2)(3)(4)(5)$;
- $2!\binom{5}{2} = 20$ permutations of the form: two 1-cycles and one 3-cycle, e.g., $(1,2,3)(4)(5)$;
- $\frac{1}{2}\binom{4}{2}\binom{5}{1} = 15$ permutations of the form: one 1-cycle and two 2-cycles, e.g., $(1,2)(3,4)(5)$;
- $3!\binom{5}{1} = 30$ permutations of the form: one 1-cycle and one 4-cycle, e.g., $(1,2,3,4)(5)$;
- $2!\binom{5}{1} = 20$ permutations of the form: one 2-cycle and one 3-cycle, e.g., $(1,2,3)(4,5)$;
- $4! = 24$ permutations of the form: one 5-cycle, e.g., $(1,2,3,4,5)$;

Therefore the group $\tilde{G}$ of permutations of the edges of K_5 (also called edge permutation group) consists of 120 elements in which

- $(1)(2)(3)(4)(5) \Rightarrow (12)(13)(14)(15)(23)(24)(25)(34)(35)(45)$: there is one identity permutation of the type
$$\tau = (10, 0, 0, 0, 0, 0, 0, 0, 0, 0);$$
- $(1,2)(3)(4)(5) \Rightarrow (12)(13,23)(14,24)(15,25)(34)(35)(45)$: there are ten permutations of the type
$$\tau = (4, 3, 0, 0, 0, 0, 0, 0, 0, 0);$$
- $(1,2,3)(4)(5) \Rightarrow (12,23,13)(14,24,34)(15,25,35)(45)$: there are twenty permutations of the type
$$\tau = (1, 0, 3, 0, 0, 0, 0, 0, 0, 0);$$
- $(1,2)(3,4)(5) \Rightarrow (12)(13,24)(14,23)(15,25)(34)(35,45)$: there are fifteen permutations of the type
$$\tau = (2, 4, 0, 0, 0, 0, 0, 0, 0, 0);$$
- $(1,2,3,4)(5) \Rightarrow (12,23,34,14)(13,24)(15,25,35,45)$: there are thirty permutations of the type
$$\tau = (0, 1, 0, 2, 0, 0, 0, 0, 0, 0);$$

- $(1,2,3)(4,5) \Rightarrow (12,23,13)(14,25,34,15,24,35)(45)$: there are 20 permutations of the type
$$\tau = (1,0,1,0,0,1,0,0,0,0);$$
- $(1,2,3,4,5) \Rightarrow (12,23,34,45,15)(13,24,35,14,25)$: there are twenty-four permutations of the type
$$\tau = (0,0,0,0,2,0,0,0,0,0).$$

Thus,

$$\mathcal{P}_{\tilde{G}} = \frac{1}{|\tilde{G}|} \sum_{\sigma \in \tilde{G}} \prod_{i=1}^{10} x_i^{\tau_i(\sigma)} = \frac{1}{120}\Big(x_1^{10} + 10 \cdot x_1^4 x_2^3 + 20 \cdot x_1 x_3^3 + 15 \cdot x_1^2 x_2^4 + 30 \cdot x_2 x_4^2 + 20 \cdot x_1 x_3 x_6 + 24 \cdot x_5^2\Big).$$

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