Introduction to Combinatorics

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Contents

Definition

Given a collection $\mathcal{A}$ of n sets $\mathcal{A} = \{A_1, \ldots, A_n\}$. A system of distinct representatives (abbreviated as SDR) for $\mathcal{A}$ is a set of n distinct elements $\{x_1, \ldots, x_n\}$ with $x_i \in A_i, \forall i \in [n]$.

Example

Given $\mathcal{A} = \{A_1, A_2\}$ with $A_1 = \{a, b\}$, $A_2 = \{a, c\}$. Then there are totally three SDRs for $\mathcal{A}$ which are $\{a, c\}$, $\{b, a\}$, and $\{b, c\}$.

There are collections $\mathcal{A}$ which has no SDR. For example,

- $\mathcal{A} = \{A_1, A_2, A_3\}$ with
$$A_1 = A_2 = A_3 = \{a, b\}$$
- $\mathcal{A} = \{A_1, A_2, A_3, A_4\}$ with
$$A_1 = A_2 = A_3 = \{a, b\}; A_4 = \{b, c, d, e\}.$$

Theorem (Hall's theorem)

A collection $\mathcal{A} = \{A_1, \ldots, A_n\}$ has an SDR if and only if $\mathcal{A}$ satisfies the Hall's condition, i.e., for every subset $\mathcal{I} \subseteq [n]$,

$$|\cup_{i \in \mathcal{I}} A_i| \geq |\mathcal{I}|$$

(from now on we use the convention that $|\emptyset| = 0$ and $\cup_{i \in \emptyset} A_i = \emptyset$).

Proof.

$\square$

Definition

Given a collection $\mathcal{A}$ of n sets $\mathcal{A} = \{A_1, \dots, A_n\}$. A partial SDR for $\mathcal{A}$ is a set of distinct elements $\{x_i\}_{i\in\mathcal{I}}$ ($\emptyset \neq \mathcal{I} \subseteq [n]$) such that $x_i \in A_i, \quad \forall i \in \mathcal{I}$.
The maximum size of a partial SDR for the collection $\mathcal{A}$ is denoted by $\lambda(\mathcal{A}) = \max\{|\mathcal{I}| : \mathcal{I} \subseteq [n], \{A_i\}_{i\in\mathcal{I}} \text{ has an SDR}\}$.

Example

For the collection $\mathcal{A} = \{A_i\}_{i=1}^4$ with $A_1 = A_2 = A_3 = \{a, b\}, A_4 = \{b, c, d, e\}$.
Then $\{a, b, c\}$ is a partial SDR which represents $\{A_1, A_2, A_4\}$ and $\lambda(\mathcal{A}) = 3$.

Theorem

$$\lambda(\mathcal{A}) = \min\{n - |\mathcal{I}| + |\cup_{i\in\mathcal{I}} A_i|, \ \textit{for all subsets } \mathcal{I} \subseteq [n]\}$$

Proof.

$\square$

Definition

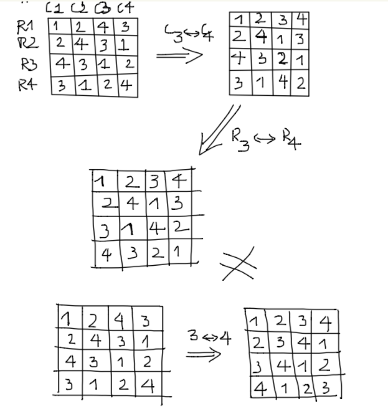
A Latin square of order n is an $n \times n$ grid filled with n symbols so that each symbol appears once in each row and column.

Example

4	2	3	1
2	3	1	4
3	1	4	2
1	4	2	3

Definition

A reduced Latin square of a Latin square with n-symbols $[n]$ is one in which the first row and first column is $1, 2, \ldots, n$.



Definition

Two Latin squares are isotopic if each can be turned into the other by permuting the rows, columns, and symbols. This isotopy relation is an equivalence relation; the equivalence classes are the isotopy classes.

Below are the first few values for the number of all Latin squares, reduced Latin squares, and non-isotopic Latin squares (that is, the number of isotopy classes):

n	All	Reduced	Non-isotopic
1	1	1	1
2	2	1	1
3	12	1	1
4	576	4	2
5	161280	56	2

Example

The multiplication table of any finite group is a Latin square. The addition table for the integers modulo 6 is a Latin square.

0	1	2	3	4	5
1	2	3	4	5	0
2	3	4	5	0	1
3	4	5	0	1	2
4	5	0	1	2	3
5	0	1	2	3	4

Definition

Suppose $A = [a_{ij}]_{i,j=1}^n$ and $B = [b_{ij}]_{i,j=1}^n$ are two Latin squares of order n. Form the square $M = [m_{ij}]_{i,j=1}^n$ with entries $m_{ij} = (a_{ij}, b_{ij})$, we will denote by $M = A \cup B$. We say that A and B are orthogonal if M contains all n^2 ordered pairs.

Example

$$\begin{array}{|c|c|c|} \hline 0 & 1 & 2 \\ \hline 1 & 2 & 0 \\ \hline 2 & 0 & 1 \\ \hline \end{array} \cup \begin{array}{|c|c|c|} \hline 0 & 1 & 2 \\ \hline 2 & 0 & 1 \\ \hline 1 & 2 & 0 \\ \hline \end{array} = \begin{array}{|c|c|c|} \hline (0,0) & (1,1) & (2,2) \\ \hline (1,2) & (2,0) & (0,1) \\ \hline (2,1) & (0,2) & (1,0) \\ \hline \end{array} \Rightarrow A \perp B$$

$$\begin{array}{|c|c|c|} \hline 0 & 1 & 2 \\ \hline 1 & 2 & 0 \\ \hline 2 & 0 & 1 \\ \hline \end{array} \cup \begin{array}{|c|c|c|} \hline 1 & 2 & 0 \\ \hline 2 & 0 & 1 \\ \hline 0 & 1 & 2 \\ \hline \end{array} = \begin{array}{|c|c|c|} \hline (0,1) & (1,2) & (2,0) \\ \hline (1,2) & (2,0) & (0,1) \\ \hline (2,0) & (0,1) & (1,2) \\ \hline \end{array} \Rightarrow A \not\perp B$$

Definition

Let A be a Latin square of order m with symbols $[m]$, and B one of order n with symbols $[n]$. Let $\{c_{i,j}\}_{i\in[m],j\in[n]}$ be mn new symbols. Form an $mn \times mn$ grid by replacing each entry of B with a copy of A. Then replace each entry i in this copy of A with $c_{i,j}$, where j is the entry of B that was replaced. We denote this new square $A \times B$.

Example

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 & 1 \\ \hline \end{array} \times \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 2 & 3 & 1 \\ \hline 3 & 1 & 2 \\ \hline \end{array} = \begin{array}{|c|c|c|c|c|c|}\hline c_{1,1} & c_{2,1} & c_{1,2} & c_{2,2} & c_{1,3} & c_{2,3} \\ \hline c_{2,1} & c_{1,1} & c_{2,2} & c_{1,2} & c_{2,3} & c_{1,3} \\ \hline c_{1,2} & c_{2,2} & c_{1,3} & c_{2,3} & c_{1,1} & c_{2,1} \\ \hline c_{2,2} & c_{1,2} & c_{2,3} & c_{1,3} & c_{2,1} & c_{1,1} \\ \hline c_{1,3} & c_{2,3} & c_{1,1} & c_{2,1} & c_{1,2} & c_{2,2} \\ \hline c_{2,3} & c_{1,3} & c_{2,1} & c_{1,1} & c_{2,2} & c_{1,2} \\ \hline \end{array}$$

Theorem

If A and B are Latin squares then so is $A \times B$.

Proof.

$\square$

Theorem

If A_1, A_2 are Latin squares of order m and B_1, B_2 are Latin squares of order n such that A_1 and A_2 are orthogonal, B_1 and B_2 are orthogonal, then $A_1 \times B_1$ and $A_2 \times B_2$ are orthogonal.

Proof.

$\square$

Theorem

1. *There are pairs of orthogonal Latin squares of order n when n is odd.*
2. *There are pairs of orthogonal Latin squares of order n when $n = 4k$.*
3. *There are pairs of orthogonal Latin squares of order n when $n = 4k + 2$ except 2 and 6.*

Proof.

$\square$