UNIVERSITÄT LEIPZIG

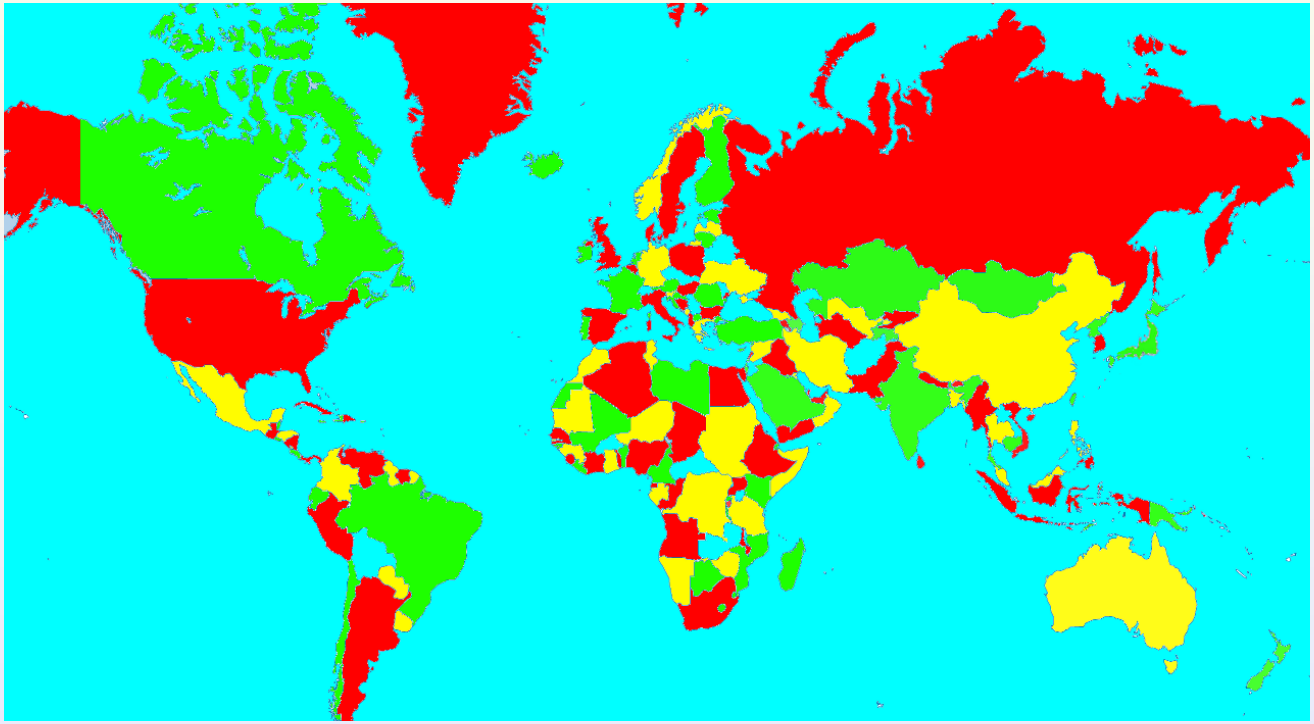
Introduction to Combinatorics

Dat Tran (FMI, Leipzig University)

SoSe 2020 (Day 11 - 16/06/2020)

Contents

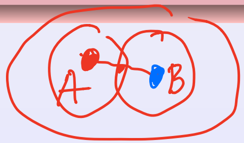
The map-colour problem



Question: Can the world map be coloured with just four colours so that neighbouring countries receive different colours?

Application

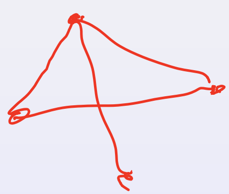
1. If the vertices of a graph represent academic classes, and two vertices are adjacent if the corresponding classes have people in common, then a coloring of the vertices can be used to schedule class meetings.
2. If the vertices of a graph represent radio stations, and two vertices are adjacent if the stations are close enough to interfere with each other, a coloring can be used to assign non-interfering frequencies to the stations.
3. If the vertices of a graph represent traffic signals at an intersection, and two vertices are adjacent if the corresponding signals cannot be green at the same time, a coloring can be used to designate sets of signals than can be green at the same time.



Definition

A proper coloring of a graph is an assignment of colors to the vertices of the graph so that no two adjacent vertices have the same color. The chromatic number of a graph G is the minimum number of colors required in a proper coloring; it is denoted $\chi(G)$.

Example



Theorem

If H is a subgraph of G, $\chi(H) \leq \chi(G)$.

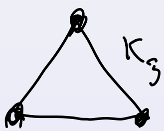
Proof.

$\square$

A proper colouring of G can used for H.

$\chi(H) \leq \underline{\underline{\chi(G)}}$.

$K_m \leq G \Rightarrow \chi(G) \geq m = \chi(K_m)$



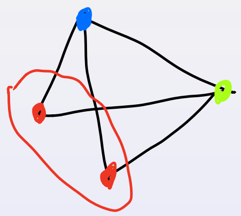
$\chi(K_3) = 3$

$\chi(K_m) = m$

Definition

A set S of vertices in a graph is independent if no two vertices of S are adjacent. The independence number of G is the maximum size of an independent set; it is denoted $\alpha(G)$.

Example



Theorem

In any graph G on n vertices, $\frac{n}{\alpha(G)} \leq \chi(G)$

Proof.

□

Denote by $V_1, \dots, V_\chi$ are sets of the colour i in graph G.

V_i : independent set $\forall i \in [\chi]$

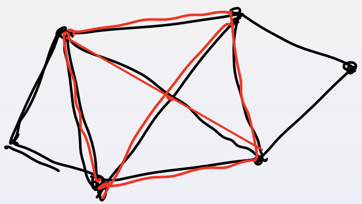
$$n = \sum_{i=1}^{\chi} |V_i| \leq \alpha \chi .$$

$$\Rightarrow \frac{n}{\alpha} \leq \chi \quad \square$$

Definition

A subgraph of G that is a complete graph is called a clique. The clique number of a graph G is the largest m such that K_m is a subgraph of G.

Example



clique number = 4.

Theorem

In any graph G, $\chi(G) \leq \underbrace{\Delta(G)}_{= \max\{\deg(v),\, v \in G\}} + 1$.

Proof.

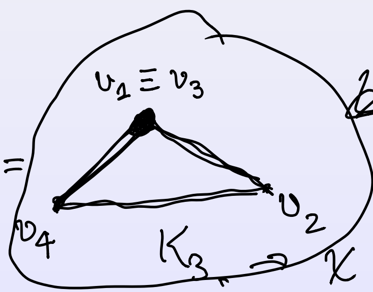
By using a greedy algorithm: □

* An algorithm to find $\chi(G)$:

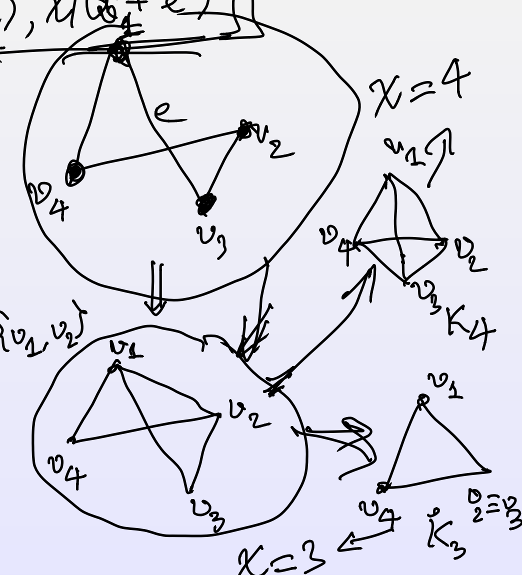
$$\boxed{\chi(G) = \min\{\chi(G/e), \chi(G+e)\}}$$

- $e = \{u, v\} \in E(G) \Rightarrow \chi(G) = \chi(G/e)$
- $\{u, v\} \notin E(G) \Rightarrow \chi(G) = \chi(G+e)$

$\{v_1, v_3\} \in E(G)$

$G/e =$ 

$e = \{v_1, v_2\}$



Theorem

The algorithm above correctly computes the chromatic number in a finite amount of time.

Proof.

□

$\underline{na}(G)$ = number of pairs of non-adjacent vertices in $G = (V, E)$, $|V| = n$, $|E| = m$.

$$= \binom{n}{2} - m$$

$$\chi(G) = \min\{\chi(G/e), \chi(G+e)\}$$

Prove by induction on $\underline{na(G)}$.

$$\begin{array}{ccc} & G & \\ \swarrow & & \searrow \\ G+e & & G/e \end{array}$$

- $na(G) = 0 \Rightarrow$ complete graph. ✓.
- $na(G)$

$$\boxed{na(G/e) < na(G) \, ; \; na(G+e) < na(G)}$$

- $na(G+e) = \binom{n}{2} - (m+1) < na(G)$
- $na(G/e) = \binom{n-1}{2} - (m - c)$

$c = |N(v) \cap N(w)|$ $(e = \{v, w\})$

$\leq na(G) - 1 < na(G)$.

Corollary

If G is not regular, then $\chi(G) \leq \Delta(G)$*.*

Proof.

$\square$

Theorem (Brooks's Theorem)

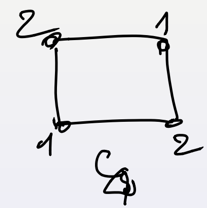
If G is a graph other than K_n or C_{2n+1}, then $\boxed{\chi(G) \le \Delta(G)}$.

Proof.

Exercise E11.3. □

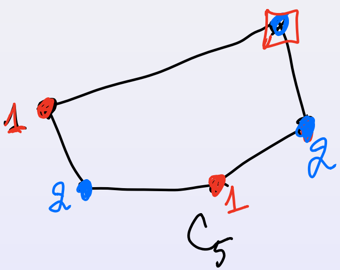
(*) - $\chi(K_n) = n$ $\rightarrow$ $\boxed{\chi = \Delta + 1}$.

$\Delta(K_n) = n - 1$



$\Delta = 2$
$\chi = 2$

(*) -

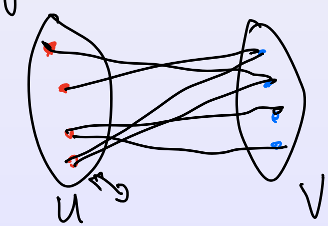


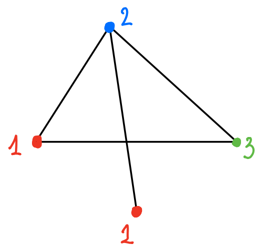
$\chi(C_5) = 3$

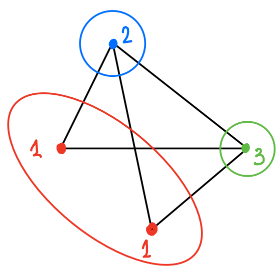
$\Delta(C_5) = 2$

$\boxed{\chi = \Delta + 1}$

bipartite graph : 2-colour.




2
1
3
1


2
1
3
1