UNIVERSITÄT
LEIPZIG

Introduction to Combinatorics

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Contents

Definition

- A graph G consists of a pair $G = (V, E)$, where V is the set of vertices and E the set of edges. We write $V(G)$ for the vertices of G and $E(G)$ for the edges of G when necessary to avoid ambiguity.
- G has no multiple edges if no two edges have the same endpoints.
- G has no loops if no edge has a single vertex as both endpoints.

Example

Definition

- G is a simple graph if it has no loops and no multiple edges.
- G is a multigraph if it has no loops, but possibly has multiple edges.
- The condensation of a graph is the simple graph formed by eliminating multiple edges and loops.

Example

Definition

- Given two vertices v, w in G. A walk in G from v to w is a sequence of vertices and edges,

$$v = v_1, e_1, v_2, e_2, \ldots, v_k, e_k, v_{k+1} = w$$

  such that the endpoints of edge e_i are v_i and v_{i+1}, $\forall i = 1, \ldots, k$.
  If $v = w$, the walk is a closed walk or a circuit.
- G is connected if each pair of vertices $v, w \in V$ is connected by a walk from v to w.

Example

Definition

- A path in G is a walk in which all edges are distinct and all vertices are distinct. Notation P_n
- A cycle in G is a path with an extra edge by joining the first and last vertices. Notation C_n

Example

Definition

- If two vertices in G are connected by an edge, we say they are adjacent.
- If a vertex v is an endpoint of edge e, we say they are incident.
- The set of vertices adjacent to v is called the neighbourhood of v, denoted $N(v)$.
- The degree of a vertex v is the number of edges incident with v, denoted $d(v)$.

Example

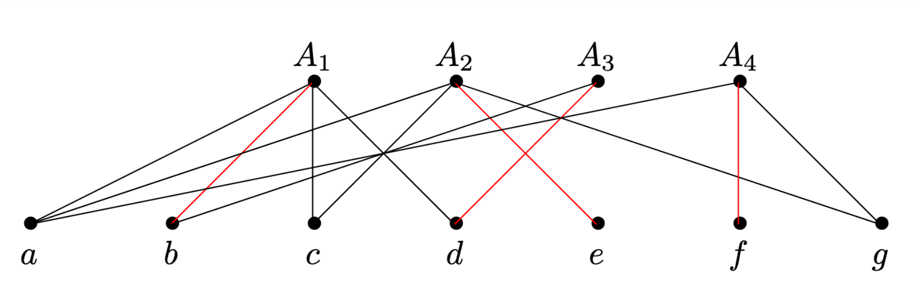
Definition

- A complete graph K_n is a graph with n vertices $v_1, \dots, v_n$ in which every two distinct vertices are joined by an edge.
- A graph G is bipartite if its vertices can be partitioned into two distinct parts $V = V_1 \sqcup V_2$ so that all edges has one endpoint in V_1 and the other one in V_2.

Example

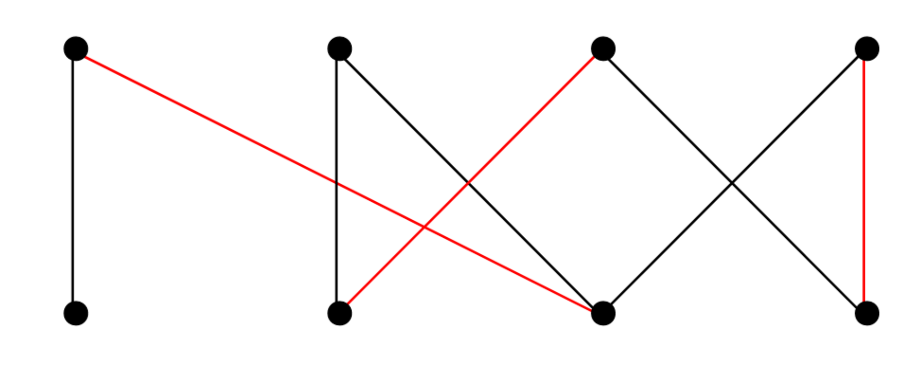
Definition

A matching in a graph is a set of edges with no common endpoints.



Definition

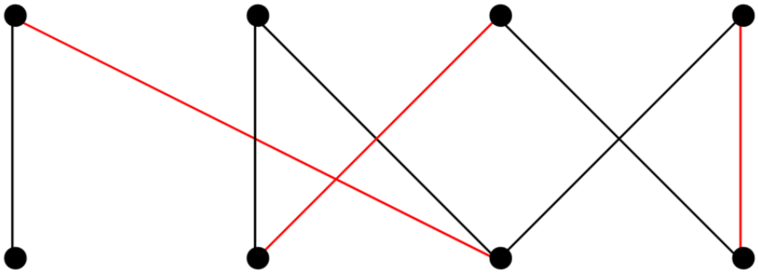
A maximal matching cannot be enlarged by adding another edge. A maximum matching is one of maximum size.



Definition

Suppose M is a matching in a graph G. Given $k \in \mathbb{N}$. An alternating chain is a sequence of vertices $v_1, w_1, v_2, w_2, \dots, v_k, w_k$ such that

- v_1, w_k are unsaturated by M, i.e., no edge in M is incident with v_1 or w_k;
- $\{v_i, w_i\} \in M^c := E - M$ for $i = \overline{1, k}$ and $\{w_i, v_{i+1}\} \in M$ for $i = \overline{1, k-1}$.



Theorem

Suppose that M is a matching in a bipartite graph G, and there is no alternating chain. Then M is a maximum matching.

Proof.

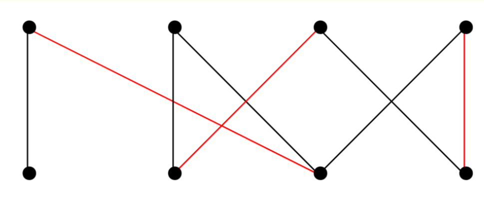
□

Algorithm.

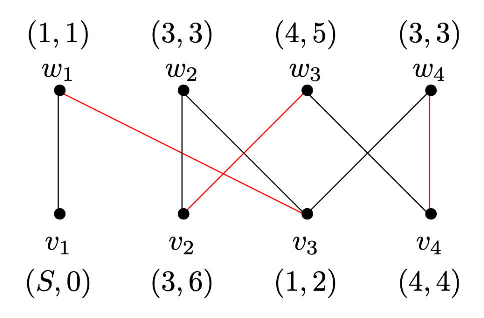
Suppose we have a bipartite graph with vertex partition $\{v_1, v_2, \ldots, v_n\}$ and $\{w_1, w_2, \ldots, w_n\}$ and a matching M. An algorithm for finding the alternating chain:

0. step = 0. Label $(S, 0)$ all vertices that are unsaturated by M; Now repeat the next two steps until no vertex acquires a new label:
1. step = step + 1. If v_i is labelled, w_j is not labelled and $\{v_i, w_j\} \in M^c$ then we label w_j as $(i, step)$
2. step = step + 1. If w_j is labelled, v_i is not labelled and $\{w_j, v_i\} \in M$ then we label v_i as $(i, step)$

At the conclusion of the algorithm, if there is a labeled vertex w_i that is unsaturated by M, then there is an alternating chain, and we say the algorithm succeeds. If there is no such w_i, then there is no alternating chain, and we say the algorithm fails. □



$$\Downarrow$$



Definition

A vertex cover in a graph is a set of vertices S such that every edge in the graph has at least one endpoint in S.

Theorem

If M is a matching in a graph and S is a vertex cover, then $|M| \leq |S|$.

Proof.

$\square$

Theorem

Suppose the algorithm fails on the bipartite graph G with matching M. Let U be the set of labeled w_i, L the set of unlabeled v_i, and $S = L \cup U$. Then S is a vertex cover and $|M| = |S|$.

Proof.

□

Theorem

In a bipartite graph G, the size of a maximum matching is the same as the size of a minimum vertex cover.

Proof.

□