# Introduction to Combinatorics

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# Contents

## Definition

Given a connected graph $G$, a vertex $v \in G$ is called a cutpoint if its removal disconnects $G$. If $G$ has a cutpoint then we say $G$ has connectivity 1.

## Example

## Definition

Given a connected graph $G$, any set of vertices whose removal disconnects the graph is called a cutset. $G$ has connectivity $k$ if there is a cutset of size $k$ but no smaller cutset.
The connectivity of $G$ is denoted $\kappa(G)$. $G$ is $k$-connected if $\kappa(G) \geq k$.

## Remark

If $G$ has at least two vertices and has no cutset, we say $\kappa(G) = n - 1$; If $G$ has one vertex, its connectivity is undefined; If $G$ is not connected, we say $\kappa(G) = 0$.

## Example

## Definition

If a graph $G$ is connected, any set of edges whose removal disconnects the graph is called a cut. $G$ has edge connectivity $k$ if there is a cut of size $k$ but no smaller cut; the edge connectivity of a one-vertex graph is undefined. $G$ is $k$-edge-connected if the edge connectivity of $G$ is at least $k$. The edge connectivity is denoted $\lambda(G)$.

## Example

### Theorem

$\kappa(G) \leq \lambda(G)$.

### Proof.

$\square$

## Theorem

*If $G$ has at least three vertices, the following are equivalent:*

1. *$G$ is 2-connected*
2. *$G$ is connected and has no cutpoint*
3. *For all distinct vertices $u, v, w$ in $G$ there is a path from $u$ to $v$ that does not contain $w$.*

## Proof.

$\square$

## Theorem

*If $G$ has at least three vertices, then $G$ is 2-connected if and only if every two vertices $u$ and $v$ are contained in a cycle.*

## Proof.

□

## Corollary

*If $G$ has at least three vertices, then $G$ is 2-connected if and only if between every two vertices $u$ and $v$ there are two internally disjoint paths, that is, paths that share only the vertices $u$ and $v$.*

## Proof.

□

## Definition

If $v$ and $w$ are non-adjacent vertices in $G$, $\kappa_G(v, w)$ is the smallest number of vertices whose removal separates $v$ from $w$, that is, disconnects $G$ leaving $v$ and $w$ in different components. A cutset that separates $v$ and $w$ is called a separating set for $v$ and $w$. $p_G(v, w)$ is the maximum number of internally disjoint paths between $v$ and $w$.

## Example

**Theorem**

*If $v$ and $w$ are non-adjacent vertices in $G$ then $\kappa_G(v, w) = p_G(v, w)$.*

**Proof.**

$\square$

**Theorem (Menger's Theorem)**

*If $G$ has at least $k+1$ vertices, then $G$ is $k$-connected if and only if between every two vertices $u$ and $v$ there are $k$ pairwise internally disjoint paths.*

**Proof.**

$\square$