Some well known probability mass, density and distribution functions

1 Poisson Distribution

Poisson distribution is a special case of binomial distribution when N is very large, p is very small and Np is finite. Let X be a random variable denoting number of positive outcomes after N Bernoulli trials (experiments with only two possible outcomes, i.e bit errors). The PMF of X would be

$$p(X = i) = {}^N C_i p^i (1-p)^{N-i} = \frac{N!}{i!(N-i)!} p^i (1-p)^{N-i} \tag{1}$$

Let Np be denoted as λ, so $p = \frac{\lambda}{N}$. If p is replaced accordingly in equation 1, then for $N \to \infty$

$$\lim_{N\to\infty} \frac{N!}{i!(N-i)!} (\frac{\lambda}{N})^i (1 - \frac{\lambda}{N})^{N-i} = \frac{\lambda^i}{i!} \lim_{N\to\infty} \frac{N!}{(N-i)!} \frac{1}{N^i} (1-\frac{\lambda}{N})^N (1-\frac{\lambda}{N})^{-i} \tag{2}$$

Now,

$$\begin{aligned} \lim_{N\to\infty} \frac{N!}{(N-i)!} \frac{1}{N^i} = \lim_{N\to\infty} \frac{N(N-1)(N-2)\cdots(N-i+1)}{N^i} &= \\ \lim_{N\to\infty} \frac{N}{N} . \frac{N-1}{N} . \frac{N-2}{N} \cdots \frac{N-i+1}{N} &= \\ 1^i = 1. \end{aligned} \tag{3}$$

$$\lim_{N\to\infty} (1 - \frac{\lambda}{N})^N = e^{-\lambda} \tag{4}$$

as it is a well known result that

$$\lim_{x\to\infty} (1 + \frac{1}{x})^x = e.$$

$$\lim_{N\to\infty} (1 - \frac{\lambda}{N})^{-i} = 1 \tag{5}$$

From 2, 3, 4 and 5,

$$\lim_{N\to\infty} \frac{N!}{i!(N-i)!} (\frac{\lambda}{N})^i (1 - \frac{\lambda}{N})^{N-i} = \frac{\lambda^i}{i!} e^{-\lambda} \tag{6}$$

So, the binomial PMF reduces to Poisson PMF for large N and small p. A notable point is that, $\lambda = np$ is the average number of positive outcomes among N Bernoulli trials. So, arrival of entities from a very large set in a queue can be modelled based on a Poisson random variable, where λ is the average number of arrivals in the queue within a certain time duration.

2 Exponential distribution and density functions

Let the arrivals in a queue be Poisson distributed with mean arrival rate of λ per unit time, and let a random variable T denote the inter-arrival time of successive arrivals in the queue. Hence T is a continuous random variable. Let us try to find the distribution and density function of T. $Prob(T > t) = Prob(No\, arrivals\, within\, t) = e^{-\lambda t}$ So, $Prob(T \leq t) = 1 - Prob(No\, arrivals\, within\, t) = 1 - e^{-\lambda t}$ So, the distribution function of T $= F(t) = 1 - e^{-\lambda t}$ As per definition, the density function of t, $f(t) = \frac{d}{dt}(1 - e^{-\lambda t}) = \lambda e^{-} \lambda t$

3 Gamma distribution and Density

Let the arrivals in a queue be Poisson distributed with mean arrival rate of λ per unit time, and let a random variable T denote the time of $k_t h$ successive arrivals in the queue. Hence T is a continuous random variable. Let us try to find the distribution and density function of T.

$$Prob\ (T > t) = Prob(0/1/2/ \cdots k-1\ arrivals\ within\ t) = e^{-\lambda t} \sum_{i=0}^{k-1} \frac{(\lambda t)^i}{i!}$$

So,

$$F_T(t) = Prob(T \leq t) = 1 - e^{-\lambda t} \sum_{i=0}^{k-1} \frac{(\lambda t)^i}{i!} \tag{7}$$

As T is a continuous random variable, the density function is

$$\begin{aligned} f_T(t) = \frac{dF_T(t)}{dt} &= \frac{d}{dt}(1 - e^{-\lambda t}\sum_{i=0}^{k-1}\frac{(\lambda t)^i}{i!}) \\ &= \frac{d}{dt}(1 - e^{\lambda t} - \sum_{i=1}^{k-1}(\frac{(\lambda t)^i}{i!}e^{-\lambda t}) \\ &= \lambda e^{-\lambda t} - \frac{d}{dt}(\sum_{i=1}^{k-1}(\frac{(\lambda t)^i}{i!}e^{-\lambda t}) \\ &= \lambda e^{-\lambda t} - \sum_{i=1}^{k-1}(\frac{(\lambda t)^i}{i!} - \frac{(\lambda t)^{i-1}}{(i-1)!}) \\ &= \lambda e^{-\lambda t} + \lambda e^{-\lambda t}(\frac{(\lambda t)^{k-1}}{(k-1)!} - 1) \\ &= \frac{\lambda e^{-\lambda t}(\lambda t)^{k-1}}{(k-1)!} \\ &= \frac{\lambda^k t^{k-1} e^{-\lambda t}}{(k-1)!} \end{aligned} \tag{8}$$

The density function denoted in 8 is called a gamma density function, which is also denoted as $\Gamma(k, \lambda)$. So, the probability density function of arrival time of the k-th arrival in the queue is $\Gamma(k, \lambda)$.

4 Queuing system as an application of Poisson arrival process

A common application of Poisson random variables is in modelling a queuing system. If arrival of each entity in a queue is independent of each other, and can be viewed as an outcome of Bernoulli trial from a very large sample set, the number of arrivals within any interval is a Poisson distributed random variable. Such arrival process model is also called Markov model. Individual departure from the queue is most commonly Poisson distributed also, as inter-departure time of entities are independent of each other as well as the arrival process. But there can be other statistical distribution of the departure process. But irrespective of the arrival and departure statistics, it is intuitive that, the number of entities in a queuing system would reach a steady state only if the arrival rate is less than or equal to the departure rate. When the steady state is achieved, there is an important relationship (called as Little's Theorem) between the average number of entities in the system, average time spent by an entity in the system and average rate of arrival in the system. Let N be the average number of entities in the system at an instant, λ be the average rate of arrival in the system and T be the average time spent in the system by an entity. According to Little's theorem,

$$N = \lambda T \tag{9}$$

4.1 Markov Chain model

Let us consider a queuing system where arrival is modelled as a Markov model with average arrival rate = λ, and there is a single service point with average service rate as μ and service time for entities are independent and exponentially distributed; and $\mu \geq \lambda$. In such a system, number of entities $N(t)$ in the system at time t can be analyzed as follows : Let the system be viewed in time instances which are multiples of a very small time interval δ. So, $N(t) = N(k\delta), k = 0, 1, 2 \cdots \infty$. Let $N(k\delta)$ be referred as N_k; which implies N_k is the number of entities in the system at $t = k\delta$. N_k's are viewed as state-variables of the system, where there can be transitions from N_k to N_i. Let P_{ik} be the probability if transition from state i to k. We can see that, due to small δ,

$$\begin{aligned} P_{00} &= 1 - \lambda\delta \\ P_{ii} &= 1 - \lambda\delta - \mu\delta \ for \ i \neq 0 \\ P_{i,i-1} &= \mu\delta \\ P_{i,i+1} &= \lambda\delta \end{aligned} \tag{10}$$

Stationary state probabilities can be obtained in steady state as follows :

$$p_n = \lim_{k \to \infty} Prob(N_k = n)$$

Now from 10, $p_n\lambda\delta = p_{n+1}\mu\delta$, hence

$$p_n\lambda = p_{n+!}\mu \ for \ n = 0, 1, \cdots p_{n+1} = \rho p_n \ where \ \rho = \frac{\lambda}{\mu} \tag{11}$$

As $\rho < 1$, all state probabilities add up to 1, so

$$\begin{aligned} 1 = \sum_{n=0}^{\infty} \rho^n p_0 &= \frac{p_0}{1-\rho} \\ p_0 &= 1 - \rho \\ p_n &= \frac{\rho^n}{1-\rho} \end{aligned} \tag{12}$$

Equation 12 defines the steady-state probability of the system being in each state. Now, let us try to find out the average number of entities in the system and the average time spent by an entity in the system. Average number of entities in the system N can be evaluated by

$$N = \sum_{n=0}^{\infty} n p_n \tag{13}$$

From equations 12 and 13,

$$N = \sum_{n=0}^{\infty} n\rho^n(1-\rho) = \rho(1-\rho)\sum_{n=1}^{\infty} n\rho^{n-1} = \rho(1-\rho)\frac{\partial}{\partial\rho}\sum_{n=1}^{\infty}\rho^n = \rho(1-\rho) = \frac{\partial}{\partial\rho}\frac{1}{1-\rho}$$

$$= \rho(1-\rho)\frac{1}{(1-\rho)^2} = \frac{\rho}{1-\rho}$$

So, replacing $\rho = \frac{\lambda}{\mu}$, we get

$$N = \frac{\rho}{1-\rho} = \frac{\frac{\lambda}{\mu}}{1-\frac{\lambda}{\mu}} = \frac{\lambda}{\mu-\lambda} \tag{14}$$

From 9 and 14, average time spent by an entity is given by

$$T = \frac{N}{\lambda} = \frac{1}{\mu-\lambda} \tag{15}$$

Interestingly, average time spent by an entity in the system is the inverse of the difference in service rate and arrival rate.

Let W be the average time spent in the queue by an entity, which is the average time spent in the system - average time of being serviced. So

$$W = T - \frac{1}{\mu} = \frac{1}{\mu-\lambda} - \frac{1}{\mu} = \frac{\rho}{\mu-\lambda} \tag{16}$$

Hence, if N_Q denotes the average number of entities waiting in a queue, from Little's theorem

$$N_Q = \lambda W = \frac{\rho^2}{1-\rho} \tag{17}$$