

# CoinPress Extension: Linear Regression

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## 1 Introduction

With all the excitement around machine learning, there have been many concerns about privacy, and in particular, leaking information about the training data used for a particular model. We have seen attacks successfully reveal information about a training dataset from just the model. One of the most notable of these attacks being Membership Inference Attacks [SSSS17].

Linear Regression is a form of supervised machine learning used to generate a model that maps input data to a continuous set of outputs. The native method for solving linear regression finds parameters that minimize the distance between the predicted output and actual outputs. As it is, this native method has no privacy guarantees, i.e. information can be leveraged about the individuals used in the training dataset. This could be especially problematic if the model contains sensitive information about these individuals. The goal of this paper is to present a differentially-private algorithm to solve linear regression with a strong guarantee of protecting individuals in the training dataset.

Differential Privacy (DP) [DMNS06] is currently the most widely accepted solution to privacy in machine learning and statistics. CoinPress [BDKU20] is a private estimation confidence-interval based strategy that includes algorithms for finding the mean and covariance of a given dataset using DP. We use both of these estimation strategies to privately solve linear regression in the setting where the covariates and error terms are normally distributed.

## 2 Linear Regression

Linear regression is a form of supervised machine learning whose goal is to model the relationship between features and their labels, where the labels come from a continuous space.

Assume we have the distribution  $(\vec{x}_i, y_i)$  with  $\vec{x}_i \sim \mathcal{N}(\mu, \Sigma)$  a  $d$ -dimensional vector and  $y_i | \vec{x}_i \sim \mathcal{N}(\langle \vec{x}, \beta \rangle, \sigma)$  a scalar. Let  $X$ , composed of  $\vec{x}_1, \dots, \vec{x}_n$ , be the training data, and  $y$ , composed of  $y_1, \dots, y_n$ , be our training labels. Then to find the parameter vector,  $\hat{\beta}$  which minimizes the square of the error between the predicted value  $\beta X$  and the actual output  $y$ , i.e.  $\min_{\beta} \sum_{i=1}^d [\beta_i \cdot x_i - y_i]^2$ , the

closed form solution is

$$\hat{\beta} = (X^T X)^{-1} X^T y = \left( \frac{1}{n} X^T X \right)^{-1} \left( \frac{1}{n} X^T y \right)$$

Where  $\frac{1}{n} X^T X$  converges to the inverse of the covariance matrix of  $X$ , and  $\frac{1}{n} X^T y$  converges to the expected value  $\mathbb{E}(x \cdot \langle x, \beta \rangle)$ .

### 3 Our Algorithm

**Theorem 3.1** (MVMRec [BDKU20]). *As proven in the cited paper, if we pass  $n$  samples  $X_{1,\dots,n}$  from  $\mathcal{N}(\mu, I_{d \times d})$ , and  $B_2(c, r)$  containing  $\mu, \rho_s, \beta_s \geq 0$ , to MVMRec, it returns a  $(\Sigma_{i=1}^t \rho_i)$ -zCDP estimate of  $\mu$ .*

**Theorem 3.2** (MVCRec [BDKU20]). *As proven in the cited paper, if we pass  $n$  samples  $X_{1,\dots,n}$  from  $\mathcal{N}(0, \Sigma)$ , and  $L, u$  such that  $L \preceq \Sigma \preceq uI$ ,  $t \in \mathbb{N}^+$ ,  $\rho_1, \dots, t, \beta > 0$  to MVCRec, it returns a  $(\Sigma_{i=1}^t \rho_i)$ -zCDP estimate of  $\Sigma$ .*

From Theorem 2.1, we see that solving linear regression can be turned into a problem of estimating mean and covariance. Using MVMRec (Theorem 3.1) and MVCRec (Theorem 3.2), we can combine differentially private estimates for the mean and covariance to find a private estimate for  $\hat{\beta}$ .

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#### Algorithm 1 LinReg Mean Estimation

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**Input:**  $n$  samples of  $z_{1,\dots,n} \in \mathbb{R}^d$ ,  $c \in \mathbb{R}^d$ ,  $r, d$  and  $\rho \geq 0$ ,  $\|\beta\|_2^2$   
**Output:** estimate of  $\mu$  of  $z_i$ 's

- 1: **procedure** LINREGMEANEST( $z_{1,\dots,n}, c, r, d, \|\beta\|_2^2, \rho$ )
- 2:   Let  $\vec{\rho} = [\frac{1}{4}\rho, \frac{3}{4}\rho]$
- 3:   Let  $z_{1,\dots,n} = \frac{1}{\sqrt{2*\|\beta\|_2^2+1}} \cdot z_{1,\dots,n}$  ▷ Normalize  $z_i$ 's by  $\|\beta\|_2^2$
- 4:   **return** MVMRec( $z_{1,\dots,n}, c, r, 2, \vec{\rho}$ )  $\cdot \sqrt{2 \cdot \|\beta\|_2^2 + 1}$
- 5: **end procedure**

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#### Algorithm 2 Beta L2-norm Estimation

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**Input:**  $n$  samples from  $y_{1,\dots,n}$  from  $\mathcal{N}(\langle \beta, \vec{x}_i \rangle)$ , and  $d, t, \rho > 0$   
**Output:** estimate of  $\Sigma$  of  $y_i$ 's ▷ approximation for  $\|\beta\|_2^2$

- 1: **procedure** BETAL2NORMEST( $y_{1,\dots,n}, d, t, \rho$ )
- 2:   Let  $\vec{\rho} = [\frac{1}{4}\rho, \frac{3}{4}\rho]$
- 3:   Let  $u = 100 \cdot d$
- 4:   **return** MVCRec( $y_{1,\dots,n}, n, d, u, \vec{\rho}, t$ ) - 1
- 5: **end procedure**

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**Algorithm 3** Linear Regression using CoinPress

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**Input:**  $n$  samples  $X_{1,\dots,n}$  from  $\mathcal{N}(0, \mathbb{I}_{d \times d})$ ,  $n$  samples from  $y_{1,\dots,n}$  from  $\mathcal{N}(\langle \beta, \vec{x}_i \rangle)$ ,  $c \in \mathbb{R}^d$ ,  $\rho > 0$ , and  $r \geq \|\beta - c\|_2$

**Output:**  $\hat{\beta}$

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1: procedure COINPRESSLINREG( $X_{1,\dots,n}, y_{1,\dots,n}, c, r, \rho$ )
2:   Let  $\rho\text{Division} = [0.1 \cdot \rho, 0.45 \cdot \rho, 0.45 \cdot \rho]$ 
3:   Let  $z_i = X_i \cdot y_i$ 
4:   Let  $\|\beta\|_2^2 = \text{BetaL2NormEst}(y_{1,\dots,n}, d, t, \rho_s[0])$ 
5:   Let  $\text{meanEst} = \text{LinRegMeanEst}(z_{1,\dots,n}, c, r, d, \|\beta\|_2^2, \rho_s[1])$ 
6:   return meanEst
7: end procedure

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Our algorithm considers only the simplified case where  $\vec{x} \sim N(0, I_{d \times d})$ , and  $y|\vec{x} \sim N(\langle \vec{x}, \beta \rangle, \sigma^2)$ . Therefore,  $\frac{1}{n}X^T X \approx I_{d \times d}$ , and

$$\hat{\beta} = \mathbb{I} \frac{1}{n} X^T y = \frac{1}{n} X^T y$$

We can use Algorithm 1, **LinRegMeanEst**, to estimate  $(\frac{1}{n}X^T y)$ , which converges to the expected value  $\mathbb{E}(\vec{x} \cdot \langle \vec{x}, \beta \rangle)$ . Therefore, if we define a variable  $z_i = \vec{x}_i y_i \approx \vec{x} \cdot \langle \vec{x}, \beta \rangle$ , we can pass all the  $z_i$ 's to MVMRec. However, this first requires some modifications. First of all, we must transform the  $(\vec{x}_i, y_i)$  data into  $z_i$  before passing it to MVMRec. Furthermore, MVMRec is stated and implemented as an algorithm for a Gaussian with identity variance, but the same argument works for an arbitrary known covariance  $\Sigma$  if we rescale the data. The expected covariance of the  $z_i$ 's is in fact not the identity so a rescale is necessary. In particular, from Lemma 3.1, we can see that a good approximate to  $\Sigma$  is  $\sqrt{2\|\beta\|_2^2 + 1} \cdot \mathbb{I}_{d \times d}$ .

**Lemma 3.1.**  $\sqrt{2\|\beta\|_2^2 + 1} \cdot \mathbb{I}_{d \times d}$  dominates the covariance matrix,  $\Sigma$ , of  $z_1, \dots, z_n$ .

*Proof.* The off-diagonal entries  $Cov(z_j, z_k)$  take on the value  $\beta_j \beta_k$  (Lemma 3.4), and the diagonal entries of  $Cov(z_j, z_j)$  take on the value  $\beta_j^2 + \|\beta\|_2^2 + 1$  (Lemma 3.3). Combining these two results, we get that  $\Sigma \in \mathbb{R}^{d \times d}$ , with  $\Sigma = \beta \beta^T + (\|\beta\|_2^2 + 1) \mathbb{I}_{d \times d}$ . Furthermore, if we refer to Figure 1 below, we can see a pictorial representation of this proof. The inner red circle represents  $\|\beta\|_2^2$ , and the blue ellipse represents adding the term  $\beta \beta^T$ . Then we can see that the red circle which is approximately equal to  $\sqrt{2\|\beta\|_2^2 + 1} \cdot \mathbb{I}_{d \times d}$ , gives an overestimate for  $\Sigma$ . In using CoinPress, we would rather have an estimate of  $\|\beta\|_2^2$  that is too large rather than too small. ■

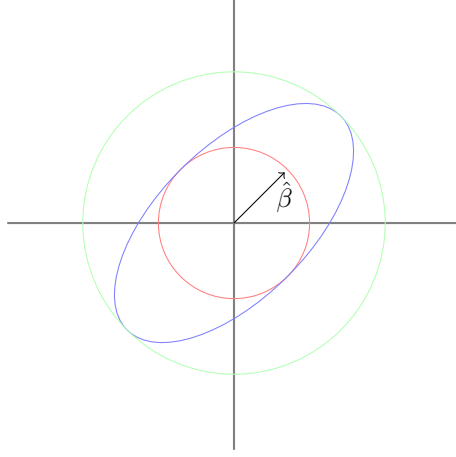


Figure 1: Using  $\sqrt{2\|\beta\|_2^2 + 1} \cdot \mathbb{I}_{d \times d}$  to approximate  $\Sigma$  of  $z_1, \dots, z_n$ .

**Lemma 3.2.** *The diagonal entries of  $\text{Cov}(z_j, z_j)$  take on the value  $\beta_j^2 + \|\beta\|_2^2 + 1$ .*

*Proof.* For every d-dimensional vector,  $\vec{z} = \vec{x} \cdot \mathbf{y}$ , if we assume  $\mathbf{y} = \langle x, \beta \rangle + v$ , where  $v$  is random noise with zero mean:

$$\mathbb{E}(z_j^2) - \mathbb{E}(z_j)^2 = \mathbb{E}(z_j^2) - \beta_j^2 \quad (1)$$

$$= \mathbb{E}\left((x_j y)^2\right) - \beta_j^2 \quad (2)$$

$$= \mathbb{E}\left(\left(x_j \cdot (\langle \vec{x}, \vec{\beta} \rangle + v)\right)^2\right) - \beta_j^2 \quad (3)$$

$$= \mathbb{E}\left(x_j^2 \cdot (\langle \vec{x}, \vec{\beta} \rangle + v^2 + 2v\langle \vec{x}, \vec{\beta} \rangle)\right) - \beta_j^2 \quad (4)$$

$$= \mathbb{E}\left(x_j^2 \langle \vec{x}, \vec{\beta} \rangle + x_j^2 v^2 + 2x_j^2 v \langle \vec{x}, \vec{\beta} \rangle\right) - \beta_j^2 \quad (5)$$

$$= \mathbb{E}\left(x_j^2 \langle \vec{x}, \vec{\beta} \rangle\right) + 1 - \beta_j^2 \quad (6)$$

$$= \mathbb{E}\left(x_j^2 \left(\sum_{k=1}^d x_k \beta_k\right)^2\right) + 1 - \beta_j^2 \quad (7)$$

$$= \mathbb{E}\left(x_j^2 \sum_{k=1}^d \sum_{l=1}^d x_k x_l \beta_k \beta_l\right) + 1 - \beta_j^2 \quad (8)$$

$$= \mathbb{E}(x_j^4 \beta_j^2) + \sum_{k \neq j} \mathbb{E}(x_k^2 \beta_k^2) + 1 - \beta_j^2 \quad (9)$$

$$= \beta_j^2 \mathbb{E}(\mathcal{N}(0, 1))^2 + \sum_{k \neq j} \mathbb{E}(x_k^2 \beta_k^2) + 1 - \beta_j^2 \quad (10)$$

$$= 3\beta_j^2 + \sum_{k \neq j} \beta_k^2 + 1 - \beta_j^2 \quad (11)$$

$$= 2\beta_j^2 + \sum_{k \neq j} \beta_k^2 + 1 \quad (12)$$

$$= \beta_j^2 + \sum_{k=1}^d \beta_k^2 + 1 \quad (13)$$

$$= \beta_j^2 + \|\beta\|_2^2 + 1 \quad (14)$$

■

**Lemma 3.3.** *The off-diagonal entries  $\text{Cov}(z_j, z_k)$  take on the value  $\beta_j \beta_k$ .*

*Proof.* For every  $d$ -dimensional vector,  $\vec{z} = \vec{x} \cdot y$ , if we assume  $y = \langle x, \beta \rangle + v$ , where  $v$  is random noise with zero mean, then for entry  $C_{jk}$ , we want  $\text{Cov}(z_j, z_k)$ :

$$\text{Cov}(z_j, z_k) = \mathbb{E}((z_j - \mathbb{E}(z_j))(z_k - \mathbb{E}(z_k))) \quad (15)$$

$$= \mathbb{E}(z_j z_k - z_j \mathbb{E}(z_k) - z_k \mathbb{E}(z_j) + \mathbb{E}(z_j) \mathbb{E}(z_k)) \quad (16)$$

$$= \mathbb{E}(z_j z_k) - \mathbb{E}(z_j \mathbb{E}(z_k)) - \mathbb{E}(z_k \mathbb{E}(z_j)) + \mathbb{E}(\mathbb{E}(z_j) \mathbb{E}(z_k)) \quad (17)$$

$$= \mathbb{E}(z_j z_k) - \mathbb{E}(z_j \mathbb{E}(z_k)) \quad (18)$$

$$= \mathbb{E}(z_j z_k) - \beta_j \beta_k \quad (19)$$

$$= \mathbb{E}((x_j y)(x_k y)) - \beta_j \beta_k \quad (20)$$

$$= \mathbb{E}\left(\left(x_j \cdot (\langle \vec{x}, \vec{\beta} \rangle + v)\right)\left(x_k \cdot (\langle \vec{x}, \vec{\beta} \rangle + v)\right)\right) - \beta_j \beta_k \quad (21)$$

$$= \mathbb{E}\left(\left(x_j \langle \vec{x}, \vec{\beta} \rangle + x_j v\right)\left(x_k \langle \vec{x}, \vec{\beta} \rangle + x_k v\right)\right) - \beta_j \beta_k \quad (22)$$

$$= \mathbb{E}\left(x_j x_k \langle \vec{x}, \vec{\beta} \rangle^2 + 2x_j x_k \langle \vec{x}, \vec{\beta} \rangle v + x_j x_k v^2\right) - \beta_j \beta_k \quad (23)$$

$$= \mathbb{E}\left(x_j x_k \langle \vec{x}, \vec{\beta} \rangle^2\right) + \mathbb{E}(2x_j x_k \langle \vec{x}, \vec{\beta} \rangle v) - \mathbb{E}(x_j x_k v^2) - \beta_j \beta_k \quad (24)$$

$$= \mathbb{E}\left(x_j x_k \langle \vec{x}, \vec{\beta} \rangle^2\right) - \beta_j \beta_k \quad (25)$$

$$= \mathbb{E}\left(x_j x_k \left(\sum_{i=1}^d x_i \beta_i\right)^2\right) - \beta_j \beta_k \quad (26)$$

$$= \mathbb{E}\left(x_j x_k \sum_{h=1}^d \sum_{i=1}^d x_h x_i \beta_h \beta_i\right) - \beta_j \beta_k \quad (27)$$

$$= 2\mathbb{E}(x_j^2 x_k^2 \beta_j \beta_k) - \beta_j \beta_k \quad (28)$$

$$= 2\beta_j \beta_k - \beta_j \beta_k \quad (29)$$

$$= \beta_j \beta_k \quad (30)$$

■

**Lemma 3.4.** *BetaL2NormEst gives a  $\rho$ -zCDP approximation of  $\|\beta\|_2^2$*

*Proof.* We know that  $y_i \sim \mathcal{N}(0, \|\beta\|_2^2 + 1)$ . So calling **MVCR**ec with  $y_1, \dots, y_n$ , which is 1-dimensional data yields a scalar representing the covariance,  $\tilde{v}$ . Then this estimate should approximately equal  $\|\beta\|_2^2 + 1$ , and therefore  $\|\beta\|_2^2 \approx \tilde{v} - 1$ .

With respect to privacy, we pass  $y_1, \dots, y_n$  into CoinPress's **MVCR**ec, which returns a  $(\sum_{i=1}^t \rho_i)$ -zCDP estimate of  $\Sigma$ . Therefore **BetaL2NormEst** gives a  $\rho$ -zCDP approximation of  $\|\beta\|_2^2$  ■

Once we have pass in the  $z_i$ 's, and pass in the center as a  $d \times 1$  vector of all zeros, and  $r = 100\sqrt{d}$ , then we get an estimate for  $\frac{1}{n} X^T y = \hat{\beta}$ .

**Theorem 3.3 (H).** *Our algorithm, `CoinPressLinReg`, is  $\rho$ -zCDP.*

*Proof.* We know that  $\rho$ -zCDP can be composed with smooth degradation over all of its privacy parameters [BS16]. Since `CoinPressLinReg` composes two  $\rho_i$ -zCDP algorithms, and  $\sum_{i=1}^2 \rho_i = \rho$ , then `CoinPressLinReg` is  $\rho$ -zCDP. ■

## 4 Experimental Results

To test the results of our algorithm, we computed the population loss of our estimated  $\hat{\beta}$ . The population loss is found by

$$\mathbb{E} \left( \left( \langle \vec{x}, \vec{\beta} \rangle - y \right)^2 \right)$$

For all of the following experiments, we generate  $\beta$ , our underlying distribution, as a  $d$ -dimensional vector where each entry is sampled from  $\mathcal{N}(0, 1)$ . We sample the entries in  $X_{n \times d}$  from  $\mathcal{N}(0, 1)$ , and generate the  $n$ -dimensional vector  $y$  by taking the dot product  $\langle X, \beta \rangle$  and adding random noise proportional to  $\mathcal{N}(0, 1)$ . Each experiment is run 50 times, for each value of  $n$  or  $\rho$ . In each iteration we generate new data, find the error between our predicted  $\hat{\beta}$  and  $\beta$  on newly generated  $X$  and  $y$  data with  $n = 1000$ , and then at the end we take the average error over every trial.

We first compared the accuracy of our algorithm at different dimensions. From the graphs below, we can see that in lower dimensions, our algorithm is accurate even with smaller values of  $n$ .

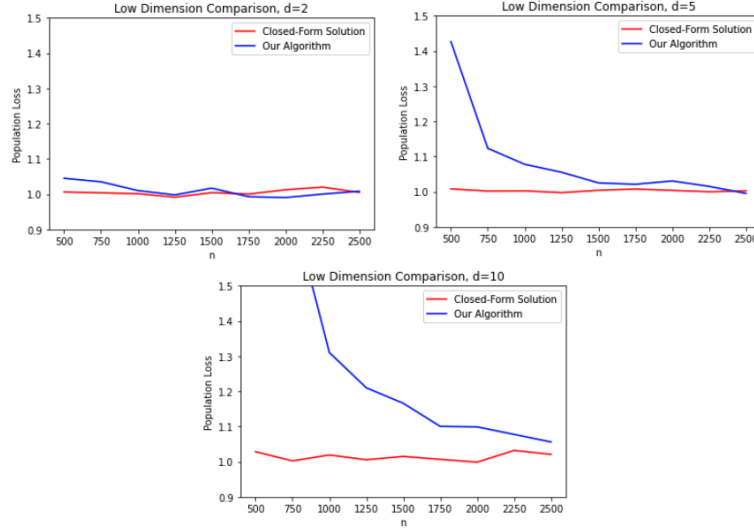


Figure 2: Low Dimensional Comparison ( $\rho = 0.5$ )

We also compared the accuracy of our algorithm at higher dimensions, and saw that for larger values of  $n \geq 4000$ , give good estimates when  $d \leq 35$ .

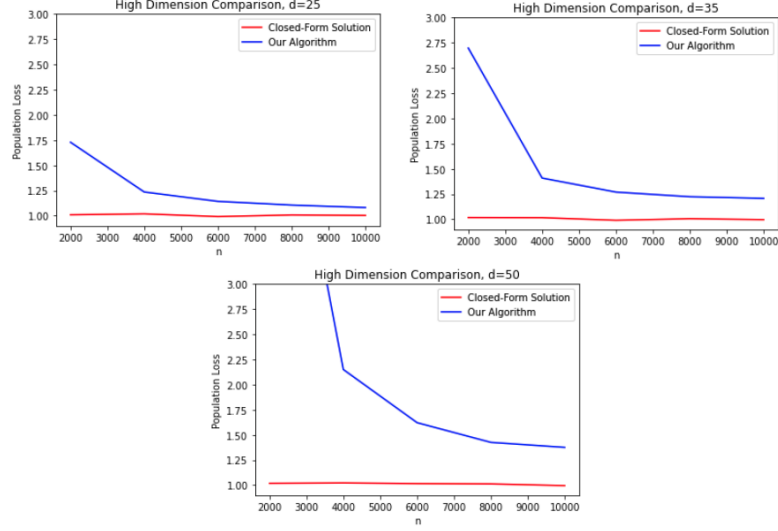


Figure 3: High Dimensional Comparison ( $\rho = 0.5, d = \{25, 35, 50\}$ )

We also compared the accuracy of our algorithm while varying  $\rho$ , the total privacy budget. We saw that with fewer dimensions, our algorithm is still accurate for low values of  $\rho$ .

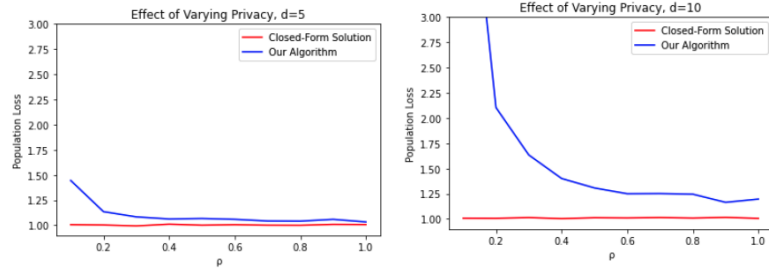


Figure 4: Measuring Effects of Privacy ( $d = \{5, 10\}, n = 1000$ )



We also wanted to analyze different error functions when measuring the accuracy of our algorithm.

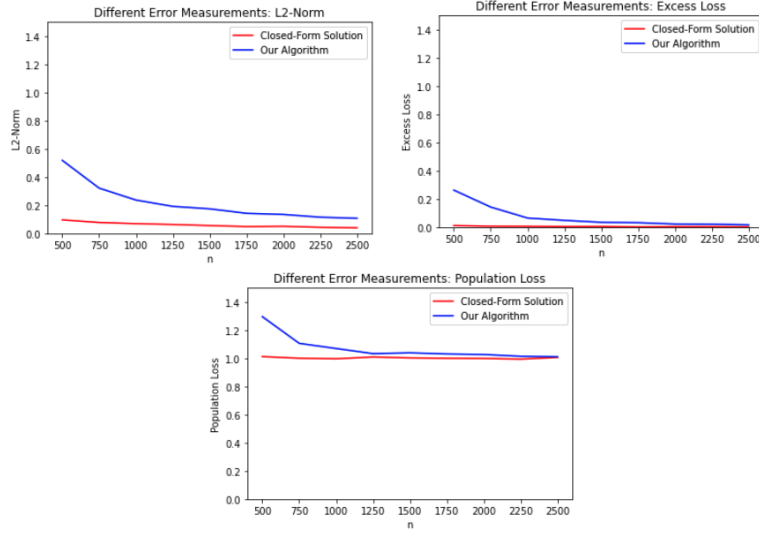


Figure 5: Different Error Functions ( $d = 5, \rho = 0.5$ )

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