Laboratory 2

Rules

- You can submit this assignment in groups of up to four people.
- You are free to use the programming language of your liking.
- All answers must be thoroughly justified and must be submitted in a single .pdf file.
- Due date: 24 February 2024

Numerical solution of the transport equation

In this laboratory we are going to learn how to do a *numerical simulation* of the transport equation by means of finite differences.
The PDE under consideration is the transport equation

$$u_t + cu_x = 0, \quad \text{for } 0 < x < L \text{ and } 0 < t < T.$$

With initial data given by

$$u(x, 0) = f(x).$$

Task 1: Find the exact analytical solution of the initial value problem.

Numerical method (One-sided method)

Consider a rectangular mesh (t_j, x_m) with uniform time step size $\Delta t = t_{j+1} - t_j$ and space mesh size $\Delta x = x_{m+1} - x_m$, for $m = 0, 1, \dots, M$ and $j = 0, 1, \dots, J$. As before, we denote $u_{j,m} = u(t_j, x_m)$ to denote our numerical approximation to the solution $u(t, x)$ at the indicated point. The simplest numerical scheme is obtained by replacing the time and space derivatives by their first-order (forward) finite difference approximations, so:

$$\frac{\partial u}{\partial t}(t_j, x_m) = \frac{u_{j+1,m} - u_{j,m}}{\Delta t}, \tag{1}$$

$$\frac{\partial u}{\partial x}(t_j, x_m) = \frac{u_{j,m+1} - u_{j,m}}{\Delta x}. \tag{2}$$

Substituting these approximations in the PDE yields the following explicit numerical scheme

$$u_{j+1,m} = -\sigma u_{j,m+1} + (\sigma + 1)u_{j,m}, \quad \text{with } \sigma = c\Delta t/\Delta x. \tag{3}$$

Task 2: Verify that you indeed get this equation.

When working on a bounded interval, say $0 \leq x \leq L$, we must to specify a value for the numerical solution at the right end $x = L$. We use $u_{j,M} = 0$, which corresponds to imposing the boundary condition $u(t, L) = 0$.

Task 3: Consider $L = 1$, plot the numerical solutions at $t = 0.1, 0.2, 0.3$ from the initial data

$$f(x) = 0.4e^{-300(x-0.5)^2} + 0.1e^{-300(x-0.65)^2}, \tag{4}$$

using step sizes $\Delta t = \Delta x = 0.005$ and the following values of $c = -1.5, -1, -0.5, 0.5$.

Numerical method (Lax-Wendroff method)

The Lax-Wendroff method is a popular alternative based on *second order* finite differences. The numerical scheme is

$$u_{j+1,m} = \frac{1}{2}\sigma(\sigma - 1)u_{j,m+1} - (\sigma^2 - 1)u_{j,m} + \frac{1}{2}\sigma(\sigma + 1)u_{j,m-1}. \tag{5}$$

Task 5: Repeat task 3 using the Lax-Wendorff method instead and compare the numerical simulations with the analytical solution. Conclude.

Task 6: Use the one-sided method and the Lax-Wendroff method to simulate the initial solution using the following initial value

$$f(x) = \begin{cases} 1 - \frac{|x-0.7|}{0.1}, & \text{for } |x - 0.7| \leq 0.1 \\ 0, & \text{othewise.} \end{cases} \tag{6}$$

Compare the numerical simulations and conclude. Pay attention at the behaviour of the amplitude and smoothness of the numerical solution.