Toward Entailment Checking: Explore Eigenmarking Search

Tatpong Katanyukul
Deparment of Computer Engineering
Khon Kaen University
Khon Kaen, Thailand
tatpong@kku.ac.th
ORCID 0000-0003-3586-475X

***Abstract*—Logic entailment is essential to reasoning, but entailment checking has the worst-case complexity of an exponential of the variable size. With recent development, quantum computing when mature may allow an effective approach for various combinatorial problems, including entailment checking. Grover algorithm uses Grover operations, selective phase inversion and amplitude amplification to address a search over unstructured data with quadratic improvement from a classical method. Its original form is intended to a single-winner scenario: exactly one match is promised. Its extension to multiple-winner cases employs probabilistic control over a number of applications of Grover operations, while a no-winner case is handled by time-out.**

Our study explores various schemes of "eigenmarking" approach. Still relying on Grover operations, but the approach introduces additional qubits to tag the eigenstates. The tagged eigenstates are to facilitate an interpretation of the measured results and enhance identification of a no-winner case (related to no logic violation in entailment context).

Our investigation experiments three variations of eigenmarking on a two-qubit system using an IBM Aer simulator. The results show strong distinguishability in all schemes with the best relative distinguishabilities of 19 and 53 in worst case and in average case, respectively. Our findings reveal a viable quantum mechanism to differentiate a no-winner case from other scenarios, which could play a pivot role in entailment checking and logic reasoning in general.

***Index Terms*—quantum algorithm, quantum search, entailment, reasoning, SAT problem, computational science, eigenmarking**

I. Introduction

Entailment is crucial for logic reasoning. It allows truth inference of an entailed sentence from its entailer. However, in the worst-case [1, pp. 240], entailment checking is exponential of the number of logical symbols involved.

With a prospect on recent quantum computing development [2], we explore an extension to Grover search [3] to address the entailment checking.

Grover search is used extensively in many quantum algorithms [5]. It addresses a search over unstructured data with quadratic improvement [4] from a classical method. Its speed up is achieved through quantum mechanical properties such as superposition and entanglement, which together allow computational parallism. Its key mechanism employs selective phase inversion and inversion about mean. These two operations are often jointly called "Grover operation," designed to amplify the amplitude (and consequently probability) of the answer for that we are searching.

Grover search in its original form is intended to a single-winner scenario: exactly one answer is promised. Its extension [4] to multiple-winner cases employs probabilistic control over a number of applications of Grover operations, while a no-winner case is handled by time-out.

Entailment checking is distinct from a general search problem in that task is mainly to distinguish a no-winner case (i.e., no entailment violation) from other cases (i.e., there is at least some violation). To address this issue efficiently, we explore an approach using additional qubit(s), facilitating handling of various cases from no-winner, one-winner, multiple-winner all the way to all-winner scenarios. This approach is collectively called "eigenmarking." Three variations are studied, particularly for differentiation between no-winner case and other cases. Although eigenmarking is intended for entailment checking, its applicability could go beyond this goal and may in part advance the development of quantum search, its applications, and quantum algorithms in general.

II. Background

a) Entailment: Sentence (or knowledge base) α entails sentence β, written with notation $\alpha \vDash \beta$, means that every model[1] that is true in the entailer α must be true in the entailee β. There are two main approaches for entailment checking [1]: *theorem proving* and *model checking*. Theorem proving applies rules of inference to the entailer to eventually derive the entailee. Model checking goes through combinations of truth values for all logical symbols to validate the relation of an entailer and its entailee, i.e., there is no model that is true

[1]In logical reasoning context, a model means a specific set of truth values for all logical symbols in both sentences.

in α but false in β. It is equivalent to test unsatisfiability of $\alpha \wedge \neg\beta$.

b) Quantum computation: Exploiting quantum mechanical properties in computation is the main characteristic of quantum computation. Quantum mechanical system evolves in two distinct modes [6], [7]. (1) Linear evolution mode, its state $|\psi\rangle$ evolves according to Schrödinger equation (SE) based on Hamiltonian $\hat{H}$ (energy operator) of the system. I.e., $i\hbar\frac{\partial|\psi\rangle}{\partial t} = \hat{H}|\psi\rangle$ where $i = \sqrt{-1}$ and $\hbar$ is a reduced Planck constant. This is equivalent to *unitary transformation*, where the unitary operator corresponds to the Hamiltonian. I.e., the solution to SE, $|\psi\rangle = |\psi_0\rangle \exp(-\frac{i}{\hbar}\hat{H}t)$ where $|\psi_0\rangle$ is an initial state. Let unitary operator $U \equiv \exp(-\frac{i}{\hbar}\hat{H}t)$ of specific time t, we have $|\psi\rangle = U|\psi_0\rangle$.

Quantum state $|\psi\rangle$ can be represented in a summation form, e.g., a one-qubit state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, or in a matrix form, e.g., $|\psi\rangle = [\alpha, \beta]^T$ (the eigenstates $|0\rangle$ and $|1\rangle$ are implicit), where α and β are (probability) amplitudes.

With precise control on Hamiltonian, at least in theory, we can have any unitary operator we want. There are common unitary operators. E.g., not operator, $X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Hadamard operator, $H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$.

(2) Measurement mode, its state collapses to one of its eigenstates upon measurement. The eigenstates correspond to the measurement operator, which is applied to measure the system. A probability of an eigenstate to which the state collapses is a squared modulus of the amplitude of that eigenstate in the state before the measurement. E.g., given a state $|\psi\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$, after the measurement in a standard basis, the state after measurement $|\psi'\rangle$ has probability of $|\frac{1}{\sqrt{2}}|^2 = 0.5$ to be in $|0\rangle$ and similarly probability of 0.5 to be in $|1\rangle$.

A quantum computer can be implemented using many technologies, e.g., superconducting circuit [8], and to control qubits can be done by changing Hamiltonian (consequently unitary transformation, U), e.g., through microwave pulses [8].

III. Grover algorithm

Grover algorithm addresses a search problem: given a function $f : \{0,1\}^n \to \{0,1\}$ with a promise that there is exactly one answer $\mathbf{x}'$ such that $f(\mathbf{x}) = 1$ if $\mathbf{x} = \mathbf{x}'$ and $f(\mathbf{x}) = 0$ otherwise. A binary input $\mathbf{x}$ has n qubits. Thus, there are $N = 2^n$ possible combinations to search.

Suppose the underlying function is given as a corresponding unitary operator U_f, which takes $n+1$ qubits (n for input $\mathbf{x}$ and 1 for ancillary y) and rotates y qubit by π (phase inversion) only when $f(\mathbf{x}) = 1$. This could be done by phase kickback [7], but for simplicity here we formulate it as a direct controlled phase rotation, i.e., $U_f \equiv |\mathbf{x}'\rangle\langle\mathbf{x}'| \otimes R_z(\pi) + \sum_{\mathbf{x}\in\tilde{X}} |\mathbf{x}\rangle\langle\mathbf{x}| \otimes I$ where $\tilde{X}$ is a set of non-winning states; identity $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and rotation $R_z(\theta) = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{bmatrix}$.

Grover algorithm (Fig. 1) is as follows.

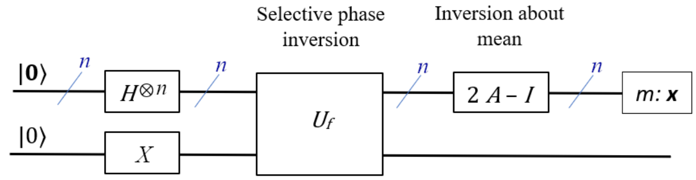

Fig. 1. Original Grover search.

- (1) Prepare $|\mathbf{x}\rangle$ in a ground state, i.e., $|\mathbf{x}_0\rangle = |\mathbf{0}\rangle \equiv |\underbrace{00\ldots0}_{n}\rangle$.
- (2) Apply $H^{\otimes n}$, where $H^{\otimes n} = \underbrace{H \otimes H \otimes \cdots \otimes H}_{n}$ is a tensor product of H. It is equivalent to apply H independently to each qubit of $|\mathbf{x}_0\rangle$, i.e., $|\mathbf{x}_1\rangle = H^{\otimes n}|\mathbf{x}_0\rangle$.
- (3) Apply the phase inversion, i.e., $|\psi_2\rangle = U_f|\mathbf{x}_1, 1\rangle$.
- (4) Apply the inversion about the mean to the input part, i.e., segment $|\mathbf{x}_2\rangle|y\rangle = |\psi_2\rangle$ and have $|\mathbf{x}_3\rangle = (2A-I)|\mathbf{x}_2\rangle$, where I is an identity matrix and an average matrix $A = \frac{1}{N}\begin{bmatrix} 1 & 1 & \ldots & 1 \\ 1 & 1 & \ldots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \ldots & 1 \end{bmatrix}$.
- (5) Grover operation (steps 3 and 4) should be applied for J time(s), where[2] $J = \arg\min_j \frac{1}{N-1}\cos^2\left((2j+1)\sin^{-1}\sqrt{\frac{1}{N}}\right)$ or simply $J = \text{round}\left(\frac{\pi}{4}\sqrt{N} - \frac{1}{2}\right)$ when N is large.
- (6) Measure the qubits.

The answer, which is the single winning state, then has the highest probability and the measured state is very likely to be the sought-after answer $\mathbf{x}'$.

For example, a case of $n = 2$ and the answer $\mathbf{x}' = 01$, Grover search can be conducted as follows.

- (1) Prepare $|\mathbf{x}_0\rangle = |00\rangle = [1, 0, 0, 0]^T$.
- (2) Apply Hadamard operator $|\mathbf{x}_1\rangle = H^{\otimes 2}|\mathbf{x}_0\rangle = [\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}]^T$.
- (3) Apply the phase inversion, i.e., $|\psi_2\rangle = U_f|\mathbf{x}_1, 1\rangle = U_f[0, \frac{1}{2}, 0, \frac{1}{2}, 0, \frac{1}{2}, 0, \frac{1}{2}]^T = [0, \frac{1}{2}, 0, -\frac{1}{2}, 0, \frac{1}{2}, 0, \frac{1}{2}]^T$.
- (4) Apply the inversion about the mean to the input part, i.e., segment $|\mathbf{x}_2\rangle = [\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}]^T$ (and $|y\rangle = [0, 1]^T$) and $|\mathbf{x}_3\rangle = (2A - I)|\mathbf{x}_2\rangle = [0, 1, 0, 0]^T$.
- (5) $J = \text{round}\left(\frac{\pi}{4}\sqrt{4} - \frac{1}{2}\right) = 1$.
- (6) Here, when measured, $|01\rangle$ will be observed with probability of 1.

As analyzed by [4], the state can be written in term of the single answer (winning state) $|\mathbf{x}'\rangle$ and any non-answer $|\mathbf{x}\rangle$ as,

[2] Derived from [4].

$$
\begin{aligned}
|\mathbf{x}_1\rangle &= H^{\otimes n}|\mathbf{x}_0\rangle \\
&= \frac{1}{\sqrt{2^n}}\left(|\mathbf{x}'\rangle + \sum_{\mathbf{x}\in\tilde{X}}|\mathbf{x}\rangle\right)
\end{aligned} \tag{1}
$$

$$
\begin{aligned}
|\psi_2\rangle &= U_f|\mathbf{x},1\rangle \\
&= \frac{1}{\sqrt{2^n}}\left(-|\mathbf{x}',1\rangle + \sum_{\mathbf{x}\in\tilde{X}}|\mathbf{x},1\rangle\right) \\
&= \underbrace{\frac{1}{\sqrt{2^n}}\left(-|\mathbf{x}'\rangle + \sum_{\mathbf{x}\in\tilde{X}}|\mathbf{x}\rangle\right)}_{|\mathbf{x}_2\rangle}\underbrace{|1\rangle}_{|y\rangle}
\end{aligned} \tag{2}
$$

Let $-k$ and ℓ be the probability amplitudes of the winning state $|\mathbf{x}'\rangle$ and any non-winning state $|\mathbf{x}\rangle$ respectively. Therefore,

$$
\begin{aligned}
|\mathbf{x}_2\rangle &= \underbrace{-\frac{1}{\sqrt{2^n}}}_{-k}|\mathbf{x}'\rangle + \underbrace{\frac{1}{\sqrt{2^n}}}_{\ell}\sum_{\mathbf{x}\in\tilde{X}}|\mathbf{x}\rangle \\
&= -k|\mathbf{x}'\rangle + \ell\sum_{\mathbf{x}\in\tilde{X}}|\mathbf{x}\rangle.
\end{aligned} \tag{3}
$$

Next, inversion about the mean

$$
|\mathbf{x}_3\rangle = (2A - I)|\mathbf{x}_2\rangle.
$$

Recall that A is to average the probability amplitudes: there is only one winning state $\mathbf{x}'$ (with amplitude $-k$), but there are $N-1$ non-winning states (each with amplitude ℓ), thus

$$
A|\mathbf{x}_2\rangle = \frac{-k+(N-1)\ell}{N}\left(|\mathbf{x}'\rangle + \sum_{\mathbf{x}\in\tilde{X}}|\mathbf{x}\rangle\right) \tag{4}
$$

$$
2A|\mathbf{x}_2\rangle = \frac{-2k+(2N-2)\ell}{N}\left(|\mathbf{x}'\rangle + \sum_{\mathbf{x}\in\tilde{X}}|\mathbf{x}\rangle\right) \tag{5}
$$

$$
\begin{aligned}
(2A-I)|\mathbf{x}_2\rangle &= \left(\frac{-2k+(2N-2)\ell}{N} + k_2\right)|\mathbf{x}'\rangle \\
&\quad + \left(\frac{-2k+(2N-2)\ell}{N} - \ell_2\right)\sum_{\mathbf{x}\in\tilde{X}}|\mathbf{x}\rangle \\
&= \left(\frac{(N-2)k+(2N-2)\ell}{N}\right)|\mathbf{x}'\rangle \\
&\quad + \left(\frac{-2k+(N-2)\ell}{N}\right)\sum_{\mathbf{x}\in\tilde{X}}|\mathbf{x}\rangle
\end{aligned} \tag{6}
$$

That is, given state $|s\rangle = k|\mathbf{x}'\rangle + \ell|\mathbf{x}\rangle$, Grover iteration (phase inversion and inversion about the mean) evolves the state to:

$$
\begin{aligned}
|\hat{s}\rangle &= \left(\frac{N-2}{N}k + \frac{2(N-1)}{N}\ell\right)|\mathbf{x}'\rangle \\
&\quad + \left(-\frac{2}{N}k + \frac{N-2}{N}\ell\right)\sum_{\mathbf{x}\in\tilde{X}}|\mathbf{x}\rangle
\end{aligned} \tag{7}
$$

To analyze an optimal number of Grover operations, let denote k_j and ℓ_j for amplitudes k and ℓ after j application(s). For $N = 2^n$ and n being a word length (a number of qubits in a solution state $|\mathbf{x}\rangle$),

- $k_0 = \frac{1}{\sqrt{N}}$ and $\ell_0 = \frac{1}{\sqrt{N}}$.
- $k_j = \frac{N-2}{N}k_{j-1} + \frac{2(N-1)}{N}\ell_{j-1}$ and $\ell_j = -\frac{2}{N}k_{j-1} + \frac{N-2}{N}\ell_{j-1}$ for $j = 1, 2, \ldots$

Figure 2 shows how probability amplitudes of the winning state evolve over grover iterations under different settings. Notice that their evolutions show strong sinusoidal patterns with frequencies directly associate to numbers of qubits. These patterns are apparently realized in [4],

$$
k_j = \sin((2j+1)\theta) \tag{8}
$$

$$
\ell_j = \frac{1}{\sqrt{\eta - 1}}\cos((2j+1)\theta) \tag{9}
$$

where $\theta = \sin^{-1}(\frac{1}{\sqrt{\eta}})$. With equation 9, it is apparent that to minimize the chance of observing the non-answer state, $\cos \sim 0 \Rightarrow \theta \sim \frac{\pi}{2}$, i.e., $(2j^*+1)\theta = \frac{\pi}{2}$ or $j^* \approx \frac{\pi}{4\theta} - \frac{1}{2} \approx \frac{\pi\sqrt{N}}{4} - \frac{1}{2}$ for a large N (since $\theta = \sin^{-1}\frac{1}{\sqrt{N}} \approx \frac{1}{\sqrt{N}}$).

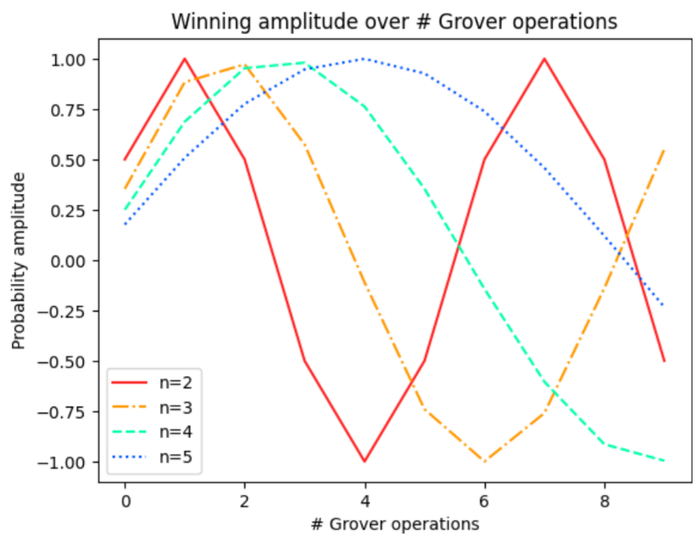

Fig. 2. Evolution of a probability amplitude of the winning state per grover iterations at different numbers of qubits N.

IV. Quantum Eigenmarking

Three eigenmarking schemes are explored here. Despite the detail, they share the common mechanism: using additional qubit(s) to facilitate differentiation particularly of a non-winner case from other cases.

To be concise, while refering to the first scheme simply as eigenmarking (Fig. 3), we will refer to eigenmarking with null marking (Fig. 4) and with subtle marking (Fig. 5) as null and subtle marking, respectively.

a) Eigenmarking: This scheme modifies Grover search by (1) having two additional qubits $|t_1 t_0\rangle$ each has controlled-phase on the ancillary $|y\rangle$, (2) having the U_f shift phase by only $\pi/2$ (instead of full phase inversion, equivalent to π), and (3) extending the inversion about the mean to average over all qubits (not just the input qubits).

Specifically,

- (1) have $U_f \equiv |\mathbf{x}'\rangle\langle\mathbf{x}'| \otimes R_z(\pi/2) + \sum_{\mathbf{x}\in\tilde{X}} |\mathbf{x}\rangle\langle\mathbf{x}| \otimes I$;
- (2) prepare $|\psi_0\rangle = |\mathbf{x}, y\rangle = |\mathbf{0}, 0\rangle$;
- (3) do $|\psi_1\rangle = H^{\otimes(n+1)}|\psi_0\rangle$;
- (4) apply Grover selection $|\psi_2\rangle = U_f|\psi_1\rangle$;
- (5) apply marking: let $|t_1, t_0\rangle = H^{\otimes 2}|0, 0\rangle$ and segment $|\mathbf{x}_2\rangle|y_2\rangle = |\psi_2\rangle$, then $|t_0', y_3\rangle = \mathbf{CR}(\pi/2)|t_0, y_2\rangle$ and $|t_1', y_4\rangle = \mathbf{CR}(-\pi/2)|t_1, y_3\rangle$ where controlled phase rotation $\mathbf{CR}(\theta) = I \otimes |0\rangle\langle 0| + R_z(\theta) \otimes |1\rangle\langle 1|$;
- (6) apply inversion about the mean to all qubits: aggregate $|\psi_4\rangle \equiv |t_1', t_0', \mathbf{x}_2, y_4\rangle$ and $|\psi_5\rangle = (2A - I)|\psi_4\rangle$;
- (7) measure the qubits.

The rationale behind this design is that (1) with two additional qubits, the number of states will be quadruple: even with a scenario of all winning states, the winning states are still minority and their amplitudes would be effectively amplified by inversion about the mean. (2) With phase rotation by prefix, there will always be some states stand out, even in a scenario of no winning states. Those stand-out states are amplified only by their prefix, therefore they always show up roughly in unity and should be easily identified.

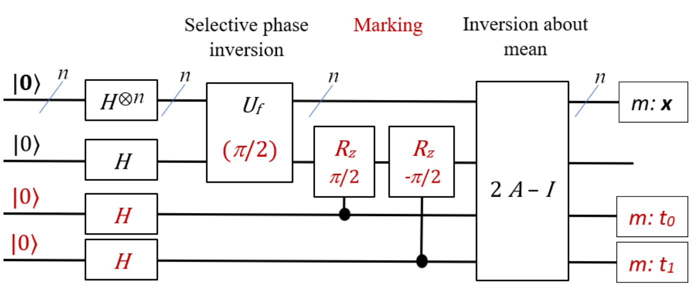

Fig. 3. Grover search with eigenmarking. The averaging in operation $2A - I$ is performed over every qubit, indicated by an enlarged block, c.f., one in original Grover search.

b) Null marking: Similar to the first scheme, but picked a state to represent a null or no-winner scenario, it uses phase rotation with multiple controls along with Not operators to select state $|t_1, t_0, \mathbf{x}\rangle = |10\underbrace{1\ldots1}_{\mathbf{x}}\rangle$ as a null state.

c) Subtle marking: This scheme is more like a minimalist version of the null marking: only one additional qubit is employed and no modification to the U_f (full phase inversion, just as in the original Grover algorithm).

Specifically, the evolution is as follows.

- (1) Prepare system in the ground state: $|\psi_0\rangle = |\mathbf{0}\rangle$, where $|\psi_0\rangle = |t_0, \mathbf{x}, y\rangle$.
- (2) Apply Hadamard: $|\psi_1\rangle = H^{\otimes n+2}|\psi_0\rangle$.

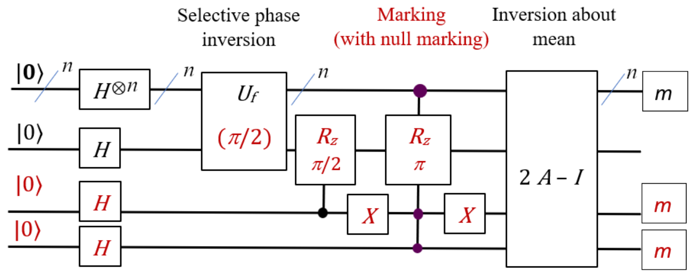

Fig. 4. Eigenmarking with null marking.

- (3) Apply Grover selection: $|\psi_2\rangle = (I \otimes U_f)|\psi_1\rangle$.
- (4) Apply marking: $|\psi_3\rangle = \text{MCR}(\pi)|\psi_2\rangle$, where multiple-qubit control phase $\text{MCR}(\theta) = |1, \mathbf{1}\rangle\langle 1, \mathbf{1}| \otimes R_z(\theta) + \sum_{\mathbf{x}\neq|1,\mathbf{1}\rangle} |\mathbf{x}\rangle\langle\mathbf{x}| \otimes I$.
- (5) Do inversion about the mean: $|\psi_4\rangle = (2A - I)|\psi_3\rangle$.
- (6) Measure the qubits.

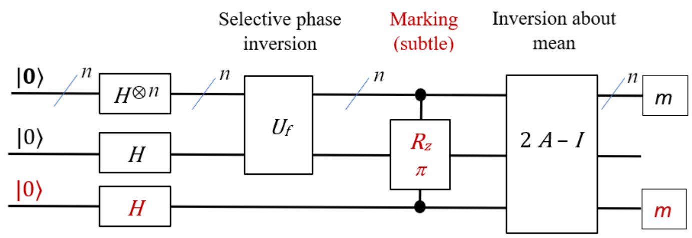

Fig. 5. Eigenmarking with subtle marking.

V. Methodology

The three eigenmarking schemes are tested on all possible scenarios of a two-qubit system using Qiskit (version 1.1.1) and Qiskit Aer Simulator (version 0.15.1). Each treatment is repeated for 40 times, while each time the simulator simulates it for 1024 runs. The main results are presented in Tables I and II with visualization in Fig. 6 and 7. All the detailed experimental results are shown in the Appendix.

a) Marking factor:

$$M = \frac{w - w_0}{w + w_0}, \tag{10}$$

where the definitions of w and w_0 depend on the marking scheme.

For eigenmarking, $w = \max C_{01}$; $w_0 = \max C_{10}$ and C_{xx} is a count of measured states with prefix `xx`. The prefix `01` marks the answers, while `10` marks complementary states. The maximization is performed over states with the specified prefix. E.g., $C_{01} = [84, 192, 71, 69]$ and $\max C_{01} = 192$ when the measured results are $\{\ldots, \underbrace{84}_{\text{'01 00'}}, \underbrace{192}_{\text{'01 01'}}, \underbrace{71}_{\text{'01 10'}}, \underbrace{69}_{\text{'01 11'}}, \ldots\}$ from 1024 runs.

For null marking, $w = \max C_{01}$ (same as eigenmarking) and $w_0 = C_{1011}$, which is simply the count of measurement on state `1011` (the null marking state).

For subtle marking, $w = \max C_0$, which is the maximum count among prefix `0` (answer marking prefix). E.g., for two-qubit input, C_0 is a set of counts of state `000` to state `011`. The null count, $w_0 = C_{111}$, is the count of measurement on state `111` (the null marking state for this scheme).

The larger difference in marking factors between no-winner case and others indicates the effectiveness of the corresponding scheme.

TABLE I
MARKING FACTOR: MEAN(STANDARD DEVIATION)

Scheme	Scenario: # winner(s) 0	1	2	3	4
Eigen.	0.01(0.05)	0.44(0.05)	0.46(0.05)	0.45(0.05)	0.95(0.01)
Null.	-0.42(0.04)	0.08(0.14)	0.16(0.15)	0.18(0.10)	0.10(0.07)
Subtle.	-0.73(0.02)	0.19(0.33)	0.48(0.47)	0.70(0.39)	0.33(0.06)

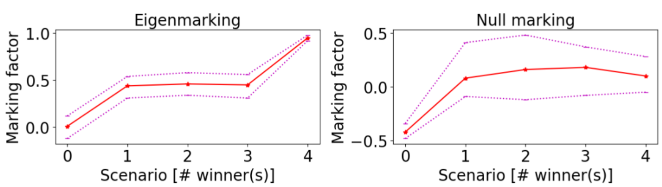

Fig. 6. Marking factors using eigenmarking (left) and null marking (right).

b) Winning margin: relative winning margin is the relative difference between minimal count of the winning state and maximal count of the non-winning state. We explore two versions: global version whose non-winning state can be any non-winning state and local version whose candiates of the non-winning states are taken only those with the answer prefix. Specifically,

$$W = \frac{c - c'}{c'}, \tag{11}$$

where $c = \min_x C_x$ s.t. $x \in \Omega_w$; Ω_w is a set of winning states; and $c' = \max_x C_x$ s.t. $x \notin \Omega_w$ for a global margin or $c' = \max_{x \in \Omega_p} C_x$ s.t. $x \notin \Omega_w$ for a local margin, where Ω_p is a set of states with the answer prefix, i.e., prefix `01` for eigenmarking and null marking and prefix `0` for subtle marking. Note that no-winner and all-winner scenarios are excluded from computing the winning margins, since these scenarios have no counts of either winner or non-winner states.

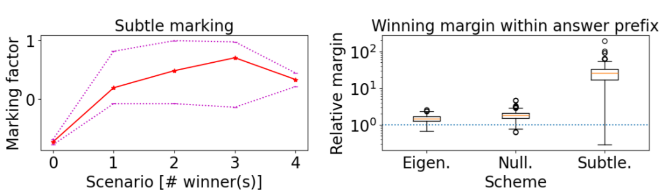

Fig. 7. Marking factors using subtle marking (left) and winning margins (right).

TABLE II
WINNING MARGIN: [MIN., MEAN, MAX.](STD. DEV.)

Scheme	Relative winning margin: Global	Local	Local prefix
Eigen.	[0.57, **1.10**, 1.76](0.2)	[0.67, **1.49**, 2.60](0.3)	`01`
Null.	[-0.44, **-0.09**, 0.27](0.1)	[0.62, **1.82**, 4.74](0.5)	`01`
Subtle.	[-0.37, **0.31**, 6.12](1.4)	[0.28, **25.72**, 197.00](15.8)	`0`

c) Distinguishability: To compare effectiveness of the three schemes, distinguishabilities, defined as follows, are presented in Table III.

Worst-case distinguishability

$$D = \frac{\min\{\min(M_i)\}_{i>0} - \max(M_0)}{|\max(M_0)|}, \tag{12}$$

where M_i is a marking factor of the i-winner(s) case.

Average-case distinguishability

$$d == \sqrt[K]{\Pi_{i>0}(\bar{M}_i - \bar{M}_0)}, \tag{13}$$

where $\bar{M}_i$ is an average marking factor of the i-winner(s) case and $K = N - 1$.

TABLE III
DISTINGUISHABILITY

Scheme	Worst-case D	Worst-case $D/\lvert\bar{M}_0\rvert$	Average-case d	Average-case $d/\lvert\bar{M}_0\rvert$
Eigen.	0.190	19.000	0.532	53.188
Null.	0.220	0.524	0.548	1.306
Subtle.	0.550	0.753	1.140	1.561

Subtle marking seems to provide better distinguishability for both worst and average cases in an absolute sense: its corresponding D's and d's are the largest among ones of the three schemes. However, relative values $D/|\bar{M}_0|$ and $d/|\bar{M}_0|$ reveal another perspective: eigenmarking seems to provide the best distinguishability: the difference between the no-winner and some-winner is most noticeable when compared to the size of no-winner marking factor.

VI. Conclusion and Discussion

As shown in Table I, the marking factors in no-winner scenarios have shown to be significantly different from ones in other scenarios in all marking schemes. In addition, the winning margins — Table II showing how well the answers (if exist) will stand out — are also significantly large when using local versions.

Overall, all schemes seem to do well in both differentiation of non-winner case and finding the answers if exist. Eigenmarking scheme is the only one having the non-zero global winning margin, which means that the answers if exist always stand out regardless of prefix. Comparing all three schemes, eigenmarking and subtle marking have shown to provided strong distinguishability, with eigenmarking having the edge on the relative view.

However, the simplicity of the subtle marking is also beneficial in many aspects: (1) using a fewer extra qubit (less prone to error and noise when used in a real hardware); (2) easier analysis since there are only two sets of resulting amplitudes; (3) less prone to noise (c.f. two previous schemes) since it uses full rotation (π radian c.f. $\pi/2$). and (4) minimal change to the original Grover algorithm allows re-use of many techniques and analyses intended for Grover algorithm.

Our findings here have shown a great prospect on this eigenmarking approach. The approach has potential much beyond being a crucial part in entailment checking. It could have profound effects on development of quantum algorithms in general. However, with this preliminary experiment on a two-qubit system, there are still many issues that remain unanswered, including scalability (how this mechanism works in a larger system), formal analysis (how this mechanism works theoretically), and applicability in a real quantum machine (how robust this mechanism is when run on a quantum computer in our current noisy intermediate-scale quantum era).

Code: `https://github.com/tatpongkatanyukul/Publication/tree/main/QEigenMarking`

References

[1] S. Russell and P. Norvig, "Artificial intelligence: a modern approach," 4th ed. global ed. Pearson, 2022.

[2] Y. Kim, A. Eddins, S. Anand, et al., "Evidence for the utility of quantum computing before fault tolerance," Nature 618, 2023.

[3] L. Grover, "Quantum mechanics helps in searching for a needle in a haystack," Phys. Rev. Lett. 79(325), 1997.

[4] M. Boyer, G. Brassard, P. Høyer, and A. Tapp, "Tight bounds on quantum search," Fortschritte der Physik, 46, 1998.

[5] A. Ambainis, "Quantum search algorithms," ACM Sigact News, 35(2), 2004, pp. 22–35.

[6] M. A. Nielsen and I. L. Chuang, "Quantum computation and quantum information," CUP, 2016.

[7] N. S. Yanofsky and M. A. Mannucci, "Quantum computing for computer scientists," CUP, 2008.

[8] P. Krantz, M. Kjaergaard, F. Yan, T. P. Orlando, S. Gustavsson, and W. D. Oliver, "A quantum engineer's guide to superconducting qubits," Appl. Phys. Rev. 6(021318), 2019.

Appendix

a) Eigenmarking: Fig. 8 to 11 show the count results from 40 repeated experiments (each with 1024 runs) using eigenmarking. The strength of this scheme is that the answers always have higher probabilities than other states, when there is at least one answer state. The downside is that when there is no answer state, the indicator is not as obvious as ones of other schemes.

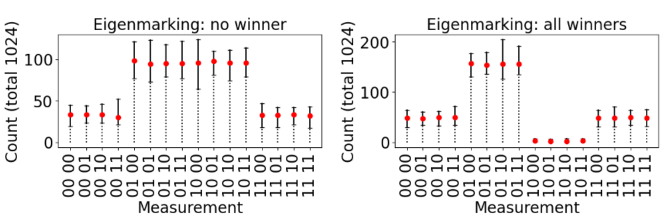

Fig. 8. Eigenmarking results: $\lambda = 0$ (no winner, left); $\lambda = 4/16$ (all winners, right). With additional two qubits, $\lambda = 4/16$ is the largest winning fraction.

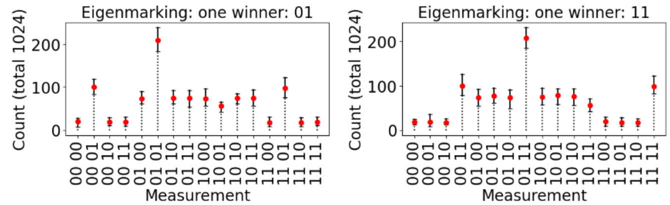

Fig. 9. Eigenmarking results: $\lambda = 1/16$ (one winner), arbitrarily selected.

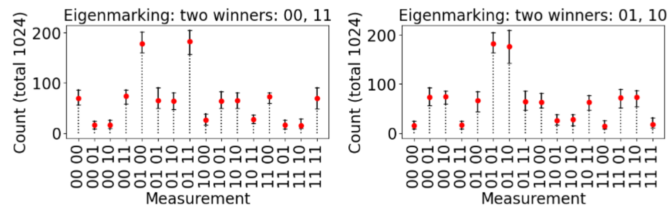

Fig. 10. Eigenmarking results: $\lambda = 2/16$ (two winners), arbitrarily selected.

b) Null marking: Fig. 12 to 15 show the count results from 40 repeated experiments using eigenmarking with null marking. Note that states with prefix `11` mirror the marked ones (prefix `01`). These mirroring pairs could be utilized for complementary checking against noise.

c) Subtle marking: Fig. 16 to 19 show the count results from 40 repeated experiments using eigenmarking with subtle marking.

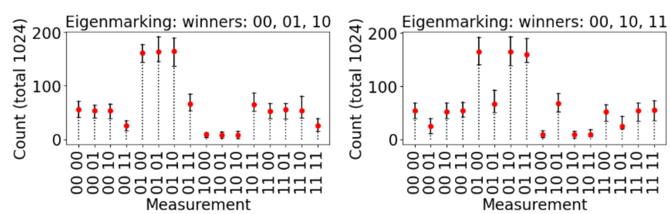

Fig. 11. Eigenmarking results: $\lambda = 3/16$ (three winners), arbitrarily selected.

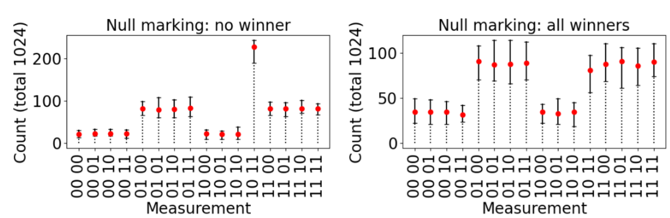

Fig. 12. Null marking results: $\lambda = 0$ (no winner, left); $\lambda = 4/16$ (all winners, right). With null marking, the state `1011` stands out when there is no winner.

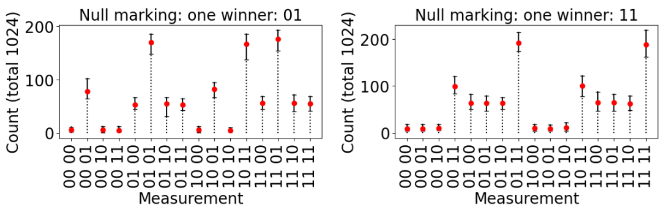

Fig. 13. Null marking results: $\lambda = 1/16$ (one winner), arbitrarily selected. With null marking, the state `1011` is suppressed when the `11` is a winning state, but the answer (prefix `01`) still shows up fine.

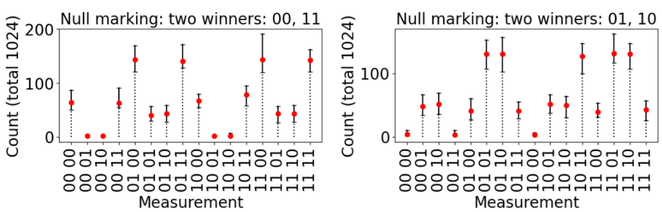

Fig. 14. Null marking results: $\lambda = 2/16$ (two winners), arbitrarily selected.

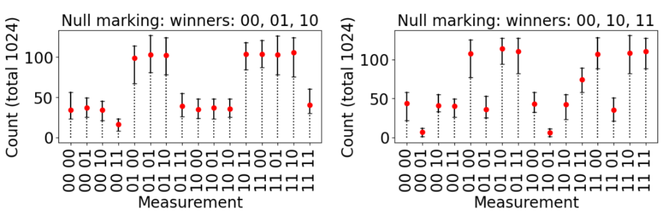

Fig. 15. Null marking results: $\lambda = 3/16$ (three winners), arbitrarily selected.

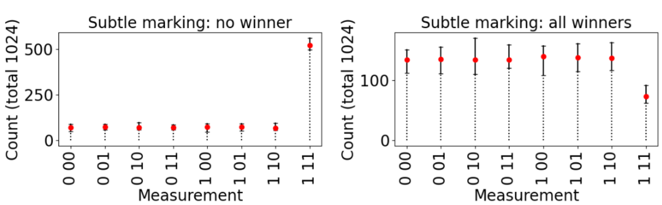

Fig. 16. Subtle marking results: $\lambda = 0$ (no winner, left); $\lambda = 4/16$ (all winners, right).

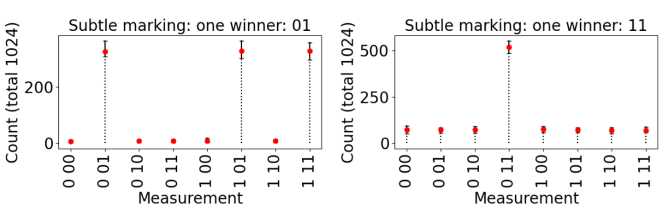

Fig. 17. Subtle marking results: $\lambda = 1/16$ (one winner), arbitrarily selected.

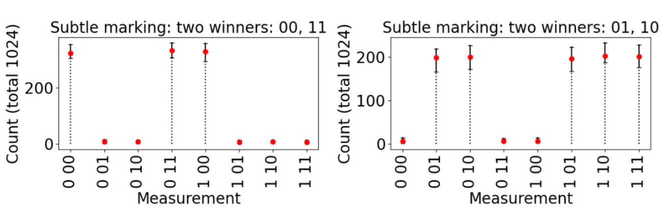

Fig. 18. Subtle marking results: $\lambda = 2/16$ (two winners), arbitrarily selected.

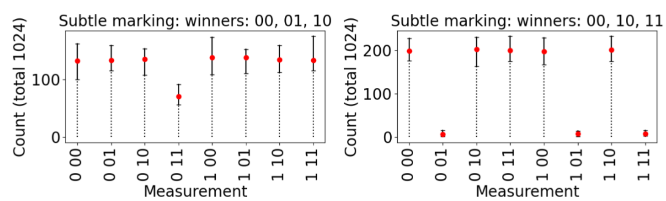

Fig. 19. Subtle marking results: $\lambda = 3/16$ (three winners), arbitrarily selected.