FN-TOPSIS: Fuzzy Networks for Ranking Traded Equities

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***Abstract*—Fuzzy systems consisting of networked rule bases, called fuzzy networks, capture various types of imprecision inherent in financial data and in the decision-making processes on them. This paper introduces a novel extension of the technique for ordering of preference by similarity to ideal solution (TOPSIS) method and uses fuzzy networks to solve multicriteria decision-making problems where both benefit and cost criteria are presented as subsystems. Thus, the decision maker evaluates the performance of each alternative for portfolio optimization and further observes the performance for both benefit and cost criteria. This approach improves significantly the transparency of the TOPSIS methods, while ensuring high effectiveness in comparison with established approaches. The proposed method is further tested to solve the problem of selection/ranking of traded equity covering developed and emergent financial markets. The ranking produced by the method is validated using Spearman rho rank correlation. Based on the case study, the proposed method outperforms the existing TOPSIS approaches in terms of ranking performance.**

***Index Terms*—Fuzzy networks (FNs), multicriteria decision making (MCDM), portfolio selection, ranking performance, Spearman rho correlation, technique for ordering of preference by similarity to ideal solution, type 1 fuzzy numbers, type 2 fuzzy numbers, Z-numbers.**

I. Introduction

Multicriteria decision-making (MCDM) problems are often observed in reality, and decision makers (DMs) are faced with the challenge of the presence of multiple criteria. The focus is on identifying the best performing solution among feasible alternatives assessed by a group of DMs and evaluated through multiple criteria [1]. In portfolio optimization, investors target high returns and affordable risks. Typically, assets with the potential for high returns also carry a high market risk [2]. Structuring complex problems well and considering multiple criteria explicitly lead to more informed and better decisions. There have been important advances in the field since the start of the modern MCDM discipline in the early 1960s. Various MCDM techniques have been developed with the overall objective to assist DMs solve complex decision problems in a systematic, consistent, and more productive way.

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TOPSIS is an MCDM technique for ranking and selection of alternatives [15]. The TOPSIS analysis considers two reference points—a positive ideal solution (PIS) and a negative ideal solution (NIS)—as well as the distances to both PIS and NIS. The preference order is ranked according to the closeness of PIS and NIS, and according to a combination of the two distance measures. TOPSIS is considered as one of the major decision-making techniques and, in recent years, has been effectively applied to the areas of human resource management [3], transportation [4], product design [5], manufacturing [6], water management [7], quality control [8], military [9], tourism [10], food science [11], and location analysis [12].

TOPSIS is used in this research due to its stability and simplicity of use with cardinal information [13]. TOPSIS has been successfully applied in MCDM problems as one of the most frequent methods used. The main advantage of the TOPSIS methods is that they are easily implemented and understood, as they directly define values based on experts' opinions in order to calculate final results [14].

Fuzzy TOPSIS was introduced to approach uncertainty in linguistic judgment. Initial research on fuzzy TOPSIS was conducted in [15], where TOPSIS is extended to type-1 fuzzy environments; this extended version used type-1 fuzzy linguistic value (represented by type-1 fuzzy number [16]) as a substitute for the directly given crisp value in grade assessment. Overall, the type-1 fuzzy TOPSIS problem is to find the most desirable alternative(s) from a set of n feasible alternatives, according to the decision information by DMs about attribute weights and attribute values. There is no solution satisfying all attributes simultaneously, as attributes are conflicting to some extent. Thus, the solution is a set of noninferior solutions, or a compromise solution according to the DM's preferences [17]. However, the existing fuzzy MCDM methods are only based on type-1 fuzzy sets [18]. In order to offer better care for the problems of vagueness, another discovery, type-2 fuzzy set was provided by Mendel, John, and Liu [19]. This concept looks to comprehensively represent uncertainties, compared with type 1 fuzzy set, due to the ability of providing more flexible spaces [18].

Zadeh introduced the concept of type-2 fuzzy set [20], which is a generalization of the concept of fuzzy set. This concept is illustrated by a fuzzy membership function, where each element of this set is a fuzzy set in [0, 1], unlike a type-1 fuzzy set where the membership grade is a crisp number in [0, 1] [21]. The membership functions of type 2 fuzzy set are 3-D and include a footprint of uncertainty (FOU) as the new third dimension, which can be described as the union of the primary memberships [22]. The FOU provides additional degrees of freedom to directly model and process uncertainties, and type-2 fuzzy set is more comprehensive compared with fuzzy set in providing more flexibility spaces to represent uncertainties [23]. The challenges in computational volume have led to the development of interval type-2 fuzzy set in 2000 by Mendel and Liang [24]. It can be viewed as a special case, as all values of secondary membership are equal to 1 [25]. Currently, interval type-2 fuzzy set is widely used and successfully applied in perceptual computing [26], [27], control systems [28]–[31], and the MCDM field. One of the MCDM methods incorporating interval type 2 fuzzy set is the interval type-2 fuzzy TOPSIS (T2-TOPSIS), which was first established in [18]. The authors introduced a T2-TOPSIS method to approach fuzzy MCDM problems.

Most recently, Z-number has been the newest fuzzy number presented in the literature of fuzzy sets. Z-number is introduced in [32] as an extension of type-1 fuzzy number but is completely different from type-2 fuzzy number. Although both Z -number and type-2 fuzzy number are extensions of type-1 fuzzy number, the former is capable of measuring the reliability of the decision made while the latter is not. Since fuzzy numbers are the medium of quantitative representation for natural language, Z- number enhances the capability of both type-1 and type-2 fuzzy numbers by taking into account the reliability of the numbers used [32]. According to [33], Z-number is represented by two embedded type-1 fuzzy numbers, where one of them plays the role to define the reliability of the first one. Research on utilizing Z-number in decision-making applications is inadequate as compared with other fuzzy numbers, as it is a new concept developed in the theory of fuzzy sets. One of the MCDM methods that implemented Z-number is called Z-TOPSIS and was first established in [34]. The authors presented a Z-TOPSIS method to handle fuzzy MCDM problems, in order to give a meaningful structure for formalizing information in decision-making problems, as it takes into account DMs' reliability. Z-number uncertainty relates to fuzziness of class boundaries. Possibility theory is rooted in uncertainty of type-1; however, over the years, possibility theory has moved in the direction of extending its domain to accommodate uncertainty of type-2 [35].

Fuzzy systems are vital within the armory of fuzzy tools and applicable to real-life decision-making environments. There are three types of fuzzy systems introduced in the literature: systems with a single rule base, systems with multiple rule bases, and systems with networked rule bases. Systems with a single rule base are characterized with a black box nature, where the inputs are mapped directly to the output without considering any internal connection. Systems with multiple rule bases are characterized with a white box nature, where the inputs are mapped to the outputs through interval variables as connections. This type of systems is also termed chained fuzzy systems or hierarchical fuzzy systems. The third type of fuzzy systems incorporates networked rule bases and is termed fuzzy networks (FN). FNs are introduced as a theoretical concept in [36] and are characterized with a white box nature, where the inputs are mapped to the outputs through intermediate variables.

According to [37], the accuracy of single rule base is moderate, but the level of transparency is low, while multiple rule bases are regarded as having low accuracy in dealing with complex process management. While in most decision-making studies, single rule bases and multiple rule bases are common approaches [38], in this research, we focus on FNs as they are both well transparent and accurate. A node represents each subsystem in an FN, whereby the interactions among subsystems are the connections between nodes. Therefore, FNs consider explicitly the interaction among subsystems [37]. An FN is more transparent than a single-rule-based fuzzy system for decision making because it considers separately benefit-related and cost-related criteria. This network takes into account explicitly the internal structure of the modeled process by representing each group of criteria as a node and the interactions among different groups as connections. This network-based approach allows the modeled process to be presented as a white box in contrast with the existing system-based approaches that use a black-box presentation. In this case, the white-box presentation improves significantly the transparency of the model due to the explicit and adequate reflection of the internal structure of the modeled process. This ability brings considerable benefits to modeling complex processes, and although FNs have been introduced recently, a significant volume of work has been done and dedicated to the theoretical development and applications of FNs [36], [37], [39], [40].

On the other hand, the reliability of decision information and the experience of experts are still in need of better incorporation into modeling complex decision-making processes. For example, how confident in their choices are investors as DMs, and how much experience experts as financial analysts have in relevant asset classes and markets [41]. Furthermore, existing TOPSIS methods have a very low transparency level and, therefore, are not able to track the performance of benefit and cost criteria [42]. In decision-making processes, it is important that DMs are aware of how the multiple criteria are performing. Based on [43], in a decision-making environment, it is essential to track the performance of criteria, in order to take control and not underestimate or overestimate uncertainty of the criteria. The proposed method represents a systematic TOPSIS approach to estimating the strengths and weaknesses of alternatives that satisfy transactions, activities, or functional requirements for a business. In addition to that, tracking of criteria allows DMs to determine if it is a sound investment/decision (justification/feasibility) and provides a basis for comparing alternatives. In this case, it involves comparing the total expected cost criteria of each alternative against the total expected benefit criteria, to see whether the benefits outweigh the costs and by how much. The inefficiencies described above bring the motivation of this study.

This paper proposes a novel FN-based modeling method that represents an extension of fuzzy set theory. The method has been validated comparatively against established fuzzy-system-based modeling methods for a case study on ranking traded equities. The main advantages of the proposed method in the context of this case study are its higher transparency and accuracy. This paper is structured as follows. Section II briefly reviews the concepts of fuzzy sets and fuzzy systems, and the operation of FNs. The novel methodology of TOPSIS using FNs with merging rule base FN-TOPSIS is formulated in Section III. Section IV illustrates the application of FN-TOPSIS to the problem of ranking equities traded on the major stock exchanges in a developed and a developing financial market. Further discussion and analysis of the FN-TOPSIS ranking performance are provided in Section V. The main conclusions are summarized in Section VI.

II. Theoretical Preliminaries

A. Fuzzy Sets

Definition 1 (Type-1 fuzzy sets) [20]: A type-1 fuzzy set A is defined on a universe X and is denoted as

$$A = \{(x, \mu_A(x)) | x \in X\}$$

where $\mu_A(x) : X \rightarrow [0, 1]$ is the membership function of A.

The membership $\mu_A(x)$ describes the degree of belongingness of $x \in X$ in A. Throughout this paper, type-1 and type-2 fuzzy numbers and Z-numbers are presented through trapezoidal membership functions. The good coverage of trapezoidal membership functions is a good compromise between efficiency and effectiveness.

Definition 2 (Type-1 fuzzy numbers) [44]: A trapezoidal type-1 fuzzy number is represented by the following membership function:

$$\mu_A(x) = (a_1, a_2, a_3, a_4) = \begin{cases} \dfrac{x - a_1}{a_2 - a_1}, & \text{if} \quad a_1 \le x \le a_2 \\ 1, & \text{if} \quad a_2 \le x \le a_3 \\ \dfrac{a_4 - x}{a_4 - a_3}, & \text{if} \quad a_3 \le x \le a_4 \\ 0, & \text{otherwise.} \end{cases}$$

Definition 3 (Type-2 fuzzy sets) [19]: A type-2 fuzzy set $\tilde{A}$ in the universe of discourse X is represented by a type-2 membership function $\mu_{\tilde{A}}$ as follows:

$$\tilde{A} = \{((x, u), \mu_{\tilde{A}}(x, u)) | \forall u \in J_X \subseteq [0, 1], 0 \le \mu_{\tilde{A}}(x, u) \le 1\}$$

where J_X denotes an interval in [0, 1]. A type-2 fuzzy set $\tilde{A}$ can also be represented as

$$\tilde{A} = \int_{x \in X} \int_{u \in J_x} \frac{\mu_{\tilde{A}}(x, u)}{(x, u)}$$

where $J_X \subseteq [0, 1]$ and $\iint$ denotes the union over all admissible x and u.

Definition 4 (Interval type-2 fuzzy numbers) [19]: A trapezoidal interval type-2 fuzzy number is represented by

$$\tilde{A} = \left(\tilde{A}^U, \tilde{A}^L\right) = \left(\left(a_1^U, a_2^U, a_3^U, a_4^U; \mu_1\left(\tilde{A}^U\right), \mu_2\left(\tilde{A}^U\right)\right),\right.$$
$$\left. a_1^L, a_2^L, a_3^L, a_4^L; \mu_1\left(\tilde{A}^L\right), \mu_2\left(\tilde{A}^L\right)\right)$$

where $\tilde{A}^U$ and $\tilde{A}^L$ are type-1 fuzzy numbers, while $a_1^U, a_2^U, a_3^U, a_4^U, a_1^L, a_2^L, a_3^L$, and a_4^L are the reference points of the interval type-2 fuzzy number $\tilde{A}$. In addition, $\mu_j(\tilde{A}^U)$ denotes the membership value of the element $a_{(j+1)}^U$ in the upper trapezoidal membership function $\tilde{A}^U$ for $1 \le j \le 2$, and $\mu_j(\tilde{A}^L)$ denotes the membership value of the lower trapezoidal membership function $\tilde{A}^L$ for $1 \le j \le 2$. Here, $\mu_1(A^U) \in [0, 1]$, $\mu_2(A^U) \in [0, 1]$, $\mu_1(A^L) \in [0, 1]$, and $\mu_2(A^L) \in [0, 1]$, for $1 \le i \le n$.

Definition 5 (Z-numbers) [32]: Z-number is an ordered pair of type-1 fuzzy numbers denoted as $Z = (\tilde{A}, \tilde{B})$. The first component $\tilde{A}$, a restriction on the values, is a real-valued uncertain variable. The second component $\tilde{B}$ is a measure of reliability for the first component.

The concept of a Z-number $Z = (\tilde{A}, \tilde{B})$ provides a basis for computation with fuzzy numbers that have various reliabilities. The second component $\tilde{B}$ may be interpreted as a response to the question: How confident are DMs that $\tilde{X}$ is $\tilde{A}$.

B. Fuzzy Systems

A fuzzy system consists of a single rule base where inputs are processed simultaneously without taking into account the connections and the structure of the system. For this type of system, the rules are derived based on expert knowledge about the process. The results are normally quite accurate, but the poor transparency of the system can be an obstacle to understanding complex processes.

C. Fuzzy Networks

An FN is a new type of fuzzy system, which consists of networked rule bases (nodes) and deals with inputs sequentially, while taking into account the connections and structure of the system. The rules for both fuzzy systems and FNs are derived from knowledge and data. A networked fuzzy system is transparent and fairly accurate at the same time due to its hybrid nature, which facilitates the understanding and management of complex processes.

There are four formal models for FNs characterized in [36], namely: 1) IF–THEN rule and integer tables; 2) block schemes and topological expressions; 3) incidence and adjacency matrices; and 4) Boolean matrices and binary relations. Here, we

$$\begin{aligned} &Rule\,1: \text{ If is } S_{11} \text{ and } \cdots \text{and}\, p_m \text{ is } S_{m1}, \text{then } q_1 \text{ is } T_{11} \text{and} \cdots \text{ and}\, q_l \text{ is } T_{n1} \\ &\quad\vdots \qquad\qquad \vdots \qquad\qquad \vdots \\ &Rule\, r: \text{ If is } S_{1r} \text{ and } \cdots \text{and}\, p_m \text{ is } S_{mr}, \text{then } q_1 \text{ is } T_{1r} \text{ and} \cdots \text{ and}\, q_l \text{ is } T_{nr}. \end{aligned} \tag{1}$$

employ IF–THEN rules and Boolean matrices, in order to represent the fuzzy rules. Hence, the properties of such models will be reviewed briefly. The choice is justified by the ability of these formal models to work with any number of nodes in FNs.

A fuzzy system with r rules, m inputs $p_1, \ldots, p_m$ taking linguistic terms from the sets $\{s_{11}, \ldots, s_{1r}\}, \ldots, \{s_{m1}, \ldots, s_{mr}\}$, and n outputs $q_1, \ldots, q_n$ taking linguistic terms from the output sets $\{T_{11}, \ldots, T_{1r}\}, \ldots, \{T_{n1}, \ldots, T_{nr}\}$, can be described by the following rule base from (1), as shown at the bottom of the previous page.

A rule base is incorporated as a node within the FN. A generalized Boolean matrix compresses information from a rule base represented by a node. The row and column labels of the Boolean matrix are all possible permutation of linguistics terms of the inputs and outputs for this rule base. The elements of the Boolean matrix are either "0"s or "1"s, where each "1" reflects a present rule. The Boolean matrix representation of the rule base from (1) is given as

$$\begin{array}{c|ccc} & T_{11}\cdots T_{n1} & \cdots & T_{1r}\cdots T_{nr} \\ \hline S_{11}\cdots S_{m1} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ S_{1r}\cdots S_{mr} & 0 & \cdots & 1 \end{array}. \tag{2}$$

Boolean matrices are very suitable for formal representation of FNs [40]. They describe FNs at a lower level of abstraction with respect to individual nodes. Boolean matrices also lend themselves easily to manipulation for the purpose of simplifying FNs to linguistically equivalent fuzzy systems, using the linguistics composition approach. In the next subsection, we briefly review two Boolean matrix operations, as these two are involved in the FN-TOPSIS.

Basic operations: Horizontal merging is a binary operation that can be applied to a pair of sequential nodes in FN. This operation combines the operand nodes from the pair into a single product node. The operation can be applied when the output from the first node is fed forward as an input to the second node in the form of an intermediate variable. The product node has the input from first operand node and the output from the second operand node, whereas the intermediate variable does not appear in the product node.

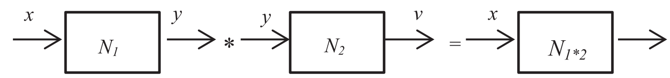

Fig. 1. Horizontal merging of nodes.

Therefore, if the first operand node is the rule base in (1) that is represented by the Boolean matrix in (2), and the second operand node is the rule base in (3), shown at the bottom of the page, that is represented by the generalized Boolean matrix in (4), see (3):

Then, the generalized Boolean matrix of (3) is described as

$$\begin{array}{c|ccc} & R_{11}\cdots R_{g1} & \cdots & R_{1r}\cdots R_{gr} \\ \hline T_{11}\cdots T_{n1} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ T_{1r}\cdots T_{nr} & 0 & \cdots & 1 \end{array}. \tag{4}$$

The product node is the rule base in (5), as shown at the bottom of the page, and its generalized Boolean matrix of (5) is constructed as

$$\begin{array}{c|ccc} & R_{11}\cdots R_{g1} & \cdots & R_{1r}\cdots R_{gr} \\ \hline S_{11}\cdots S_{m1} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ S_{1r}\cdots S_{mr} & 0 & \cdots & 1 \end{array}. \tag{6}$$

The fuzzy system described by the rule base in (3) is with r rules, n inputs $q_1, \ldots, q_n$ taking linguistic terms from the input sets $\{T_{11}, \ldots, T_{1r}\}, \ldots, \{T_{n1}, \ldots, T_{nr}\}$, and g outputs $w_1, \ldots, w_g$ taking linguistic terms from the set of outputs $\{R_{11}, \ldots, R_{1r}\}, \ldots, \{R_{g1}, \ldots, R_{gr}\}$. Similarly, the fuzzy system described by the rule base in (5) is with r rules, m inputs $p_1, \ldots, p_m$ taking linguistic terms from the input sets $\{S_{11}, \ldots, S_{1r}\}, \ldots, \{S_{m1}, \ldots, S_{mr}\}$, and g outputs $w_1, \ldots, w_g$ taking linguistic terms from the set of outputs $\{R_{11}, \ldots, R_{1r}\}, \ldots, \{R_{g1}, \ldots, R_{gr}\}$. In general, the operand rule bases may have a different number of rules, but the number of rules in the product rule base is always equal to the number of rules in the first operand rule base. For simplicity, the notations used in Fig. 1 are in a vector form where the vectors $x, y, \text{and } v$ are of dimensions $n, m, \text{and } g$, respectively.

$$\begin{array}{l} \mathit{Rule}\,1: \text{ If } q_1 \text{ is } T_{11} \text{ and} \cdots \text{and } q_m \text{ is } T_{n1}, \text{then } w_1 \text{ is } R_{11} \text{and} \cdots \text{and } w_g \text{ is } R_{g1} \\ \quad\vdots \qquad\qquad \vdots \qquad\qquad \vdots \\ \mathit{Rule}\,r: \text{ If } q_1 \text{ is } T_{1r} \text{ and} \cdots \text{and } q_m \text{ is } T_{nr}, \text{then } w_1 \text{ is } R_{1r} \text{ and} \cdots \text{and } w_g \text{ is } R_{gr}. \end{array} \tag{3}$$

$$\begin{array}{l} \mathit{Rule}\,1: \text{ If } p_1 \text{ is } S_{11} \text{ and} \cdots \text{and } p_m \text{ is } S_{m1}, \text{then } w_1 \text{ is } R_{11} \text{and} \cdots \text{and } w_g \text{ is } R_{g1} \\ \quad\vdots \qquad\qquad \vdots \qquad\qquad \vdots \\ \mathit{Rule}\,r: \text{ If } p_1 \text{ is } S_{1r} \text{ and} \cdots \text{and } p_m \text{ is } S_{mr}, \text{then } w_1 \text{ is } R_{1r} \text{ and} \cdots \text{and } w_g \text{ is } R_{gr} \end{array} \tag{5}$$

Vertical merging is a binary operation that can be applied to a pair of parallel nodes in an FN. The inputs to the product node represent the union of the inputs to the operand nodes, and the outputs from the product node represent the union of the output from the operand nodes.

Therefore, if the first operand node is the rule base in (1) that is represented by the Boolean matrix in (2), and the second operand is the rule base in (7), shown at the bottom of the page, that is represented by the generalized Boolean matrix in (8): then the generalized Boolean matrix of (8) is described with (9):

$$\begin{array}{c|cccc} & Q_{11}\cdots Q_{h1} & \cdots & Q_{1s}\cdots Q_{hs} \\ \hline R_{11}\cdots R_{g1} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ R_{1r}\cdots R_{gs} & 0 & \cdots & 1 \end{array} \quad . \tag{8}$$

The product node is the rule in the following equation:

$$\begin{aligned} &Rule\,1: \text{ If } p_1 \text{ is } S_{11} \text{ and } \cdots \text{ and } p_g \text{ is } S_{m1}\; w_1 \\ &\text{is } R_{11} \text{ and } \cdots \text{ and } w_g \text{ is } R_{g1} \text{ then } q_1 \text{ is and } \cdots \text{ and} \\ &q_n \text{ is } T_{n1} \text{ and } y_1 \text{ is } Q_{11} \text{ and } \cdots \text{ and } y_h \text{ is} \\ &\qquad\vdots \qquad\qquad \vdots \qquad\qquad \vdots \\ &Rule\,r\cdot s: \text{ If } p_1 \text{ is } S_{1r} \text{ and } \cdots \text{ and } p_m \text{ is } S_{mr} \text{ and } w_1 \\ &\text{is } R_{1s} \text{ and } \cdots \text{ and } w_g \text{ is } R_{gs} \text{ then } q_1 \text{ is } T_{1r} \text{ and} \\ &\cdots \text{ and } q_n \text{ is } T_{nr} \text{ and } y_1 \text{ is } Q_{1s} \text{ and } \cdots \text{ and } y_h \text{ is } Q_{hs} \end{aligned} \tag{9}$$

and the generalized Boolean matrix of (9) is constructed as

$$\begin{array}{c|ccc} & \begin{matrix} T_{11}\cdots T_{n1} \\ Q_{11}\cdots Q_{h1} \end{matrix} & \cdots & \begin{matrix} T_{1r}\cdots T_{nr} \\ Q_{1s}\cdots Q_{hs} \end{matrix} \\ \hline \begin{matrix} S_{11}\cdots S_{m1} \\ R_{11}\cdots R_{g1} \end{matrix} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \begin{matrix} S_{1r}\cdots S_{mr} \\ R_{gr}\cdots R_{gs} \end{matrix} & 0 & \cdots & 1 \end{array} \quad . \tag{10}$$

In this case, the fuzzy system described by the rule base in (8) has s rules, g inputs $w_1, \ldots, w_g$ taking linguistic terms from the input sets $\{R_{11}, \ldots, R_{1s}\}, \ldots, \{R_{g1}, \ldots, R_{gs}\}$, and h outputs $y_1, \ldots, y_h$ taking linguistic terms from the output sets $\{Q_{11} \cdots Q_{1s}\}, \ldots, \{Q_{h1} \cdots Q_{hs}\}$. However, the fuzzy system described by the rule base in (10) is with $r \cdot s$ rules, $m + g$ inputs $x_1, \ldots, x_m, w_1, \ldots, w_g$ taking linguistic terms from the input sets

$$\{S_{11}, \ldots, S_{1r}\}, \ldots, \{S_{m1}, \ldots, S_{mr}\},$$
$$\{R_{11}, \ldots, R_{1s}\}, \ldots, \{R_{g1}, \ldots, R_{gs}\},$$

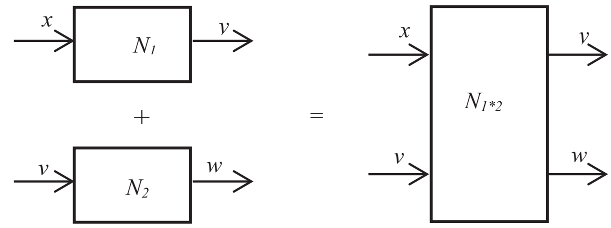

Fig. 2. Vertical merging of nodes.

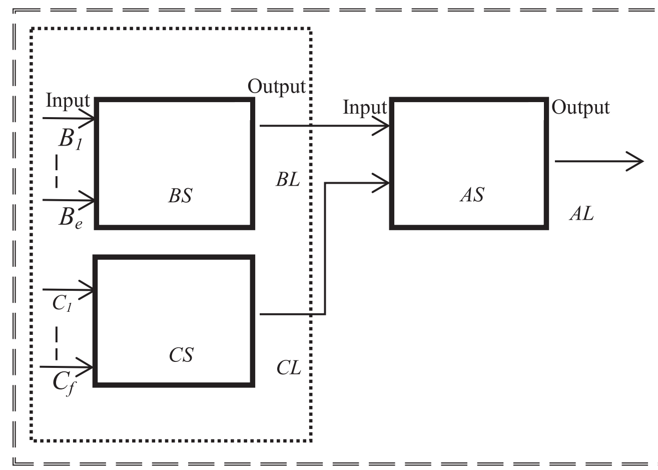

Fig. 3. FN model for TOPSIS.

and $n + h$ outputs $q_1, \ldots, q_g, y_1, \ldots, y_h$ taking linguistic terms from the output sets

$$\{T_{11}, \ldots, T_{1r}\}, \ldots, \{T_{n1}, \ldots, T_{nr}\},$$
$$\{Q_{11}, \ldots, Q_{1s}\}, \ldots, \{Q_{h1}, \ldots, Q_{hs}\}.$$

The number of rules in the product rule base is equal to the product of the number of rules in the operand rule bases. For simplicity, the notations used in Fig. 2 are in a vector form, where the vectors $x, y, v, \text{and}\, w$ have dimensions $n, m, g, \text{and}\, h$, respectively.

III. Method Formulation

In this approach, the DMs' opinions are evaluated independently, since they may have different influence degrees, depending on their experience in the area. Furthermore, criteria are categorized into benefit criteria or cost criteria. Each category generates correspondingly benefit fuzzy systems or cost fuzzy systems, where the output of the systems is benefit levels (BLs) or cost levels (CLs), representing the performance of each category. Fig. 3 illustrates the proposed generalized FN model

$$\begin{aligned} &Rule\,1: \text{ If } w_1 \text{ is } R_{11} \text{ and } \cdots \text{ and } w_g \text{ is } R_{g1}\text{, then } y_1 \text{ is } Q_{11} \text{ and } \cdots \text{ and } w_g \text{ is } Q_{h1} \\ &\qquad\qquad\vdots \qquad\qquad \vdots \qquad\qquad\qquad \vdots \\ &Rule\,s: \text{ If } w_1 \text{ is } R_{1s} \text{ and } \cdots \text{ and } w_g \text{ is } R_{gs}\text{, then } y_1 \text{ is } Q_{1s} \text{ and } \cdots \text{ and } w_g \text{ is } Q_{hs} \end{aligned} \tag{7}$$

TABLE I
LINGUISTIC TERMS FOR THE IMPORTANCE WEIGHT OF EACH CRITERION

Linguistic Terms		Trapezoidal Fuzzy Number
Very Low (VL)	1	(0.00, 0.00, 0.00, 0.10)
Low (L)	2	(0.00, 0.10, 0.10, 0.25)
Medium Low (ML)	3	(0.15, 0.30, 0.30, 0.45)
Medium (M)	4	(0.35, 0.50, 0.50, 0.65)
Medium High (MH)	5	(0.55, 0.70,0.70, 0.85)
High (H)	6	(0.80, 0.90, 0.90, 1.00)
Very High (VH)	7	(0.90, 1.00, 1.00, 1.00)

TABLE II
LINGUISTIC TERMS FOR THE RATING OF EACH ALTERNATIVE

Linguistic Terms		Trapezoidal Fuzzy Number
Very Poor (VP)	1	(0, 0, 0, 1)
Poor (P)	2	(0, 1, 1,3)
Medium Poor (MP)	3	(1, 3, 3, 5)
Fair (F)	4	(3, 5, 5, 7)
Medium Good (MG)	5	(5, 7, 7, 9)
Good (G)	6	(7, 9, 9, 10)
Very Good (VG)	7	(9, 10, 10, 10)

TABLE III
LINGUISTIC TERMS FOR THE LEVEL OF ALTERNATIVES

Linguistic Terms		Trapezoidal Fuzzy Number
Very Bad (VB)	1	(0.00, 0.00, 0.00,0.25)
Bad (B)	2	(0.00, 0.25, 0.25, 0.50)
Regular (R)	3	(0.25, 0.50, 0.50, 0.75)
Good (G)	4	(0.50, 0.75, 0.75, 1.00)
Very Good (VG)	5	(0.75, 1.00, 1.00, 1.00)

for TOPSIS, where benefit system (BS), cost system (CS) and alternative systems (AS) are incorporated in the form of FN nodes. The inputs are the benefit criteria $B_1, \ldots, B_e$ and the cost criteria $C_1, \ldots, C_f$. At the end of the process, alternatives levels (AL) are determined. The dotted frame represents the vertical merging of rule bases, and the dashed frame illustrates the horizontal merging of rule bases.

The next subsections illustrate systematically the implementation of type-1, type-2, and Z-fuzzy numbers to FN-TOPSIS.

A. Type-1 Fuzzy Number Implementation

Tables I and II are used by DMs to evaluate the rating of alternatives and the importance of criteria, and Table III is used to determine the AL as the output, in generating fuzzy rule bases.

The following are the procedures involved in implementing an FN with merging rule bases to TOPSIS, based on type-1fuzzy numbers. Steps 1–6 are adopted from [15] and [45], while steps 7–10 are introduced as part of the proposed method in this paper.

Step 1: Construct decision matrices where each DM opinion is evaluated independently, and categorize into two criteria categories as benefit criteria and cost criteria defined through a BS and a CS.

In the decision matrices D_k^B and D_k^C and weight matrices W_k^B and W_k^C $(k = 1, \ldots, K)$, it is assumed that e is the number of benefit criteria, f is the number of cost criteria, and k is the number of the DM, as shown in the following equation:

$$
\begin{aligned}
D_k^B &= \begin{array}{c} B_1 \\ B_2 \\ \vdots \\ B_e \end{array} \begin{bmatrix} x_{11,k} & x_{12,k} & \cdots & x_{1m,k} \\ x_{21,k} & x_{22,k} & \cdots & x_{2m,k} \\ \vdots & \vdots & \ddots & \vdots \\ x_{e1,k} & x_{e2,k} & \cdots & x_{em,k} \end{bmatrix} \\
D_k^C &= \begin{array}{c} C_1 \\ C_2 \\ \vdots \\ C_f \end{array} \begin{bmatrix} y_{11,k} & y_{12,k} & \cdots & y_{1m,k} \\ y_{21,k} & y_{22,k} & \cdots & y_{2m,k} \\ \vdots & \vdots & \ddots & \vdots \\ y_{f1,k} & y_{f2,k} & \cdots & y_{fm,k} \end{bmatrix} \\
W_k^B &= \begin{bmatrix} g_{1,k} & g_{2,k} & \cdots & g_{e,k} \end{bmatrix} \\
W_k^C &= \begin{bmatrix} h_{1,k} & h_{2,k} & \cdots & h_{f,k} \end{bmatrix}, \quad \text{for } k = 1, \ldots, K
\end{aligned}
\tag{11}
$$

where $x_{ij,k}$ are type-1 fuzzy sets representing the rating of alternatives A_j $(j = 1, \ldots, m)$ with respect to benefit criteria B_i $(i = 1, \ldots, e)$ according to the kth DM, and $g_{i,k}$ are type-1 fuzzy sets representing the weights of benefit criteria $B_{i,k}$ $(i = 1, \ldots, e)$ according to the kth DM, where $k = 1, \ldots, K$. In addition, $y_{ij,k}$ are type-1 fuzzy sets describing the rating of alternatives A_j $(j = 1, \ldots, m)$ with respect to cost criteria C_i $(i = 1, \ldots, f)$ according to the kth DM, and $h_{i,k}$ are type-1 fuzzy sets describing the weights of cost criteria C_i $(i = 1, \ldots, f)$ according to the kth DM, where $k = 1, \ldots, K$.

Step 2: Construct weighted and normalized decision matrices.

The fuzzy rating and weight of each criterion are variables described with type-1 trapezoidal fuzzy numbers. The ratings of alternatives A_j $(j = 1, \ldots, m)$ are described with the type-1 trapezoidal fuzzy numbers $x_{ij,k} = (a_{ij,k}^x, b_{ij,k}^x, c_{ij,k}^x, d_{ij,k}^x)$ and $y_{ij,k} = (a_{ij,k}^y, b_{ij,k}^y, c_{ij,k}^y, d_{ij,k}^y)$, while the importance of benefit criteria B_i $(i = 1, \ldots, e)$ and cost criteria C_i $(i = 1, \ldots, f)$ are, respectively, represented by $g_{i,k} = (a_{i,k}^g, b_{i,k}^g, c_{i,k}^g, d_{i,k}^g)$ and $h_{i,k} = (a_{i,k}^h, b_{i,k}^h, c_{i,k}^h, d_{i,k}^h)$, for $k = 1, \ldots, K$. The normalized fuzzy decision matrices R_k and weight normalized fuzzy decision matrices V_k are calculated as

$$
R_k = [r_{ij,k}]_{(e+f)\times m} \tag{12}
$$

where

$$
\begin{aligned}
r_{ij,k} &= \begin{cases} r_{ij.k}^B = \left(\dfrac{a_{ij,k}^x}{d_{i,k}^{x*}}, \dfrac{b_{ij,k}^x}{d_{i,k}^{x*}}, \dfrac{c_{ij,k}^x}{d_{i,k}^{x*}}, \dfrac{d_{ij,k}^x}{d_{i,k}^{x*}} \right), & \text{for } B_i \in B \\[2ex] r_{ij.k}^C = \left(\dfrac{a_{i,k}^{y*}}{d_{ij,k}^y}, \dfrac{a_{i,k}^{y*}}{c_{ij,k}^y}, \dfrac{a_{i,k}^{y*}}{b_{ij,k}^y}, \dfrac{a_{i,k}^{y*}}{a_{ij,k}^y} \right), & \text{for } C_i \in C \end{cases} \\
d_{i,k}^{x*} &= \max_j d_{ij,k}^x \quad (i = 1, \ldots, e), (j = 1, \ldots, m) \\
a_{i,k}^{y*} &= \min_j a_{ij,k}^y \quad (i = 1, \ldots, f), (j = 1, \ldots, m).
\end{aligned}
$$

B and C are the sets of benefit criteria and cost criteria, respectively:

$$V_k = [v_{ij,k}]_{(e+f)\times m}$$

where

$$v_{ij,k} = \begin{cases} v_{ij.k}^{B} = r_{ij,k}(\cdot)\, g_{i,k}, & \text{for } B_i \in B \\ v_{ij.k}^{C} = r_{ij,k}(\cdot)\, h_{i,k}, & \text{for } C_i \in C \end{cases}$$

and $v_{ij,k} = (a_{ij,k}^{v}, b_{ij,k}^{v}, c_{ij,k}^{v}, d_{ij,k}^{v})$ are type-1 fuzzy sets, for $k = 1, \ldots, K$.

Step 3: Find the fuzzy positive ideal solution (FPIS) and fuzzy negative ideal solution (FNIS) for each alternative, and the distance between each alternative to FPIS and FNIS.

The FPIS and FNIS solutions are correspondingly $A_k^{+} = (v_{1,k}^{+}, v_{2,k}^{+}, \ldots, v_{(e+f),k}^{+})$ and $A_k^{-} = (v_{1,k}^{-}, v_{2,k}^{-}, \ldots, v_{(e+f),k}^{-})$, where $v_{ij,k}^{+} = (1\ 1\ 1\ 1)$ and $v_{ij,}^{-} = (0\ 0\ 0\ 0)$ are type-1 fuzzy sets, for $k = 1, \ldots, K$. The distance for benefit criteria of each alternative j from A_k^{+} is $\Delta_{j,k}^{B+}$, calculated as (13), as shown at the bottom of the page. The distance for benefit criteria of each alternative from $A_{j,k}^{-}$ is $\Delta_{j,k}^{B-}$, calculated as (14), as shown at the bottom of the page. The distance for cost criteria of each alternative from A_k^{+} is$\Delta_{j,k}^{C+}$, calculated as (15), as shown at the bottom of the page. Finally, the distance for cost criteria of each alternative from A_k^{-} is $\Delta_{j,k}^{C-}$, calculated as (16), as shown at the bottom of the page.

Step 4: Find the closeness coefficients for both the BS and CS.

The closeness coefficients $CC_{j,k}^{B}$ for the BSs and the closeness coefficients $CC_{j,k}^{C}$ for the CSs are calculated as

$$CC_{j,k}^{B} = \frac{\Delta_{j,k}^{B-}}{\Delta_{j,k}^{B+} + \Delta_{j,k}^{B-}}, \quad CC_{j,k}^{C} = \frac{\Delta_{j,k}^{C-}}{\Delta_{j,k}^{C+} + \Delta_{j,k}^{C-}}$$
$$\text{for } j = 1, \ldots, m \text{ and } k = 1, \ldots, K. \tag{17}$$

Step 5: Derive the influenced closeness coefficients (ICC) by no applying the influence degree of each DM. Then, find the normalized ICC (NICC), dividing the ICC by the maximum value of ICC.

Let θ_k denote the influence degree, between 0 (uninfluential) and 10 (very influential), of DM k, where $k = 1, \ldots, K$. Next, let σ_k stands for the normalized influence degree of the $k\mathrm{th}$ DM, $k = 1, \ldots, K$, as evaluated with

$$\sigma_k = \frac{\theta_k}{\sum_{l=1}^{K} \theta_l}, \quad \text{for } k = 1, \ldots, K. \tag{18}$$

$$\Delta_{j,k}^{B+} = \sum_{i=1}^{e} \Delta_k^{B}\left(v_{ij,k}^{B}, v_{i,k}^{+}\right)$$
$$\text{where } \Delta_k^{B}\left(v_{ij,k}^{B}, v_{i,k}^{+}\right) = \sqrt{\frac{1}{3}\left[\left(a_{ij,k}^{v,B} - 1\right)^2 + \left(b_{ij,k}^{v,B} - 1\right)^2 + \left(c_{ij,k}^{v,B} - 1\right)^2 + \left(d_{ij,k}^{v,B} - 1\right)^2\right]},$$
$$\text{for } j = 1, \ldots, m \text{ and } B_i \in B \text{ and } k = 1, \ldots, K. \tag{13}$$

$$\Delta_{j,k}^{B+} = \sum_{i=1}^{e} \Delta_k^{B}\left(v_{ij,k}^{B}, v_{i,k}^{+}\right)$$
$$\text{where } \Delta_k^{B}\left(v_{ij,k}^{B}, v_{i,k}^{+}\right) = \sqrt{\frac{1}{3}\left[\left(a_{ij,k}^{v,B} - 1\right)^2 + \left(b_{ij,k}^{v,B} - 1\right)^2 + \left(c_{ij,k}^{v,B} - 1\right)^2 + \left(d_{ij,k}^{v,B} - 1\right)^2\right]},$$
$$\text{for } j = 1, \ldots, m \text{ and } B_i \in B \text{ and } k = 1, \ldots, K. \tag{14}$$

$$\Delta_{j,k}^{C+} = \sum_{i=1}^{f} \Delta_k^{C}\left(v_{ij,k}^{C}, v_{i,k}^{+}\right)$$
$$\text{where } \Delta_k^{C}\left(v_{ij,k}^{C}, v_{i,k}^{+}\right) = \sqrt{\frac{1}{3}\left[\left(a_{ij,k}^{v,C} - 1\right)^2 + \left(b_{ij,k}^{v,C} - 1\right)^2 + \left(c_{ij,k}^{v,C} - 1\right)^2 + \left(d_{ij,k}^{v,C} - 1\right)^2\right]}$$
$$\text{for } j = 1, \ldots, m \text{ and } C_i \in C \text{ and } k = 1, \ldots, K. \tag{15}$$

$$\Delta_{j,k}^{C-} = \sum_{i=1}^{f} \Delta_k^{C}\left(v_{ij,k}^{C}, v_{i,k}^{-}\right)$$
$$\text{where } \Delta_k^{C}\left(v_{ij,k}^{C}, v_{i,k}^{-}\right) = \sqrt{\frac{1}{3}\left[\left(a_{ij,k}^{v,C} - 0\right)^2 + \left(b_{ij,k}^{v,C} - 0\right)^2 + \left(c_{ij,k}^{v,C} - 0\right)^2 + \left(d_{ij,k}^{v,C} - 0\right)^2\right]}$$
$$\text{for } j = 1, \ldots, m \text{ and } C_i \in C \text{ and } k = 1, \ldots, K. \tag{16}$$

Equation (19) evaluates the influence closeness coefficients $ICC_{j,k}^{B}$ and $ICC_{j,k}^{C}$ for each DM *k*, respectively, along the benefit and cost criteria:

$$ICC_{j,k}^{B} = \sigma_k * CC_{j,k}^{B} \text{ and } ICC_{j,k}^{C} = \sigma_k * CC_{j,k}^{C}, \quad \text{for } j = 1, \ldots, m \text{ and } k = 1, \ldots, K. \tag{19}$$

It is further necessary to normalize the coefficients, in order to ensure that their values vary between 0 and 1. Equation (20) evaluates the normalized coefficients, where $NICC_{j,k}^{B}$ and $NICC_{j,k}^{C}$ are, respectively, the normalized influence closeness coefficients for the BS and CS, as related to the *k*th DM

$$NICC_{j,k}^{B} = \frac{ICC_{j,k}^{B}}{\max_j ICC_{j,k}^{B}} \text{ and } NICC_{j,k}^{C} = \frac{ICC_{j,k}^{C}}{\max_j ICC_{j,k}^{C}} \quad \text{for } j = 1, \ldots, m \text{ and } k = 1, \ldots, K. \tag{20}$$

Both $NICC_{j,k}^{B}$ and $NICC_{j,k}^{C}$ will take linguistic terms from Table III for the level of alternatives performance.

Step 6: Construct the antecedent matrices and the consequent matrices for the BS and CS, based on DM opinions and the values of the NICC coefficients.

Having the opinions D_k^B and D_k^C of all DMs $(k = 1, \ldots, K)$ on each alternative $j(j = 1, \ldots, m)$ in respect to each benefit criterion $i(i = 1, \ldots, e)$ and each cost criterion $i(i = 1, \ldots, f)$ [see (11)], we can define the BS antecedent matrix X_k and the CS antecedent matrix Y_k for each DM *k*, as introduced in

$$X_k = \begin{bmatrix} x_{11,k} & x_{12,k} & \cdots & x_{1m,k} \\ x_{21,k} & x_{22}, & \cdots & x_{2m,k} \\ \vdots & \vdots & \ddots & \vdots \\ x_{e1,k} & x_{e2,k} & \cdots & x_{em,k} \end{bmatrix}$$

$$Y_k = \begin{bmatrix} y_{11,k} & y_{12,k} & \cdots & y_{1m,k} \\ y_{21,k} & y_{22,k} & \cdots & y_{2m,k} \\ \vdots & \vdots & \ddots & \vdots \\ y_{f1,k} & y_{f2,k} & \cdots & y_{fm,k} \end{bmatrix}, \quad \text{for } k = 1, \ldots, K \tag{21}$$

where $x_{ij,k}$ and $y_{ij,k}$ are linguistic terms describing DMs' opinions. Having determined the $NICC_j^{B,k}$ and $NICC_j^{C,k}$ coefficients for all DMs $(k = 1, \ldots, K)$, next, the benefit consequent matrix Λ_k and the cost consequent matrix Ψ_k are defined as

$$\Lambda_k = [\lambda_{1,k} \;\; \lambda_{2,k} \;\; \cdots \;\; \lambda_{m,k}]$$
$$\Psi_k = [\psi_{1,k} \;\; \psi_{2,k} \;\; \cdots \;\; \psi_{m,k}], \quad \text{for } k = 1, \ldots, K \tag{22}$$

where $\lambda_{i,k}$ and $\psi_{i,k}$ are linguistic terms representing the output of the BS and CS, based, respectively, on the values of $NICC_{j,k}^{B}$ and $NICC_{j,k}^{C}$. FPIS represents the compromise solution, while FNIS represents the worst possible solution. The range is within the closed interval [0, 1]. NICC equation illustrates the distance of each alternative from FPIS and FNIS with the nearest value to compromise solution which is 1. The closer the NICC to 1, the higher the priority of the alternatives. The scalar is translated into linguistic term to which the scalar has the highest membership degree. The BS consists of *K* matrix decision rules presented as

$$\text{If } X_k = \begin{bmatrix} x_{11,k} & x_{12,k} & \cdots & x_{1m,k} \\ x_{21,k} & x_{22}, & \cdots & x_{2m,k} \\ \vdots & \vdots & \ddots & \vdots \\ x_{e1,k} & x_{e2,k} & \cdots & x_{em,k} \end{bmatrix}, \text{ then}$$
$$\Lambda_k = [\lambda_{1,k} \;\; \lambda_{2,k} \;\; \cdots \;\; \lambda_{m,k}] \quad \text{for } k = 1, \ldots, K \tag{23}$$

and can be described with the rule bases as

$$\begin{aligned} &\text{Rule } 1: \text{If } B_1 \text{ is } x_{11,k} \text{ and } \cdots \text{ and } B_e \text{ is } x_{e1,k}, \text{ then } BL \text{ is } \lambda_{1,k} \\ &\vdots \qquad\qquad \vdots \qquad\qquad \vdots \\ &\text{Rule } m: \text{If } B_1 \text{ is } x_{1m,k} \text{ and } \cdots \text{ and } B_e \text{ is } x_{em,k}, \text{ then BL is } \lambda_{m,k} \end{aligned} \tag{24}$$

where BL is the benefit level of alternatives, for $j = 1, \ldots, m$ and for $k = 1, \ldots, K$. The CS consists of *K* matrix decision rules presented in

$$\text{If } Y_k = \begin{bmatrix} y_{11,k} & y_{12,k} & \cdots & y_{1m,k} \\ y_{21,k} & y_{22,k} & \cdots & y_{2m,k} \\ \vdots & \vdots & \ddots & \vdots \\ y_{f1,k} & y_{f2,k} & \cdots & y_{fm,k} \end{bmatrix}, \text{ then}$$
$$\Psi_k = [\psi_{1,k} \;\; \psi_{2,k} \;\; \cdots \;\; \psi_{m,k}], \quad \text{for } k = 1, \ldots, K \tag{25}$$

and can be described with the rule bases as

$$\begin{aligned} &\text{Rule } 1: \text{If } C_1 \text{ is } y_{11,k} \text{ and } \cdots \text{ and } C_f \; y_{f1,k}, \text{ then } CL_1 \text{ is } \psi_{1,k} \\ &\vdots \qquad\qquad \vdots \qquad\qquad \vdots \\ &\text{Rule } m: \text{If } C_1 \text{ is } y_{f1,k} \text{ and } \cdots \text{ and } C_f \; y_{fm,k}, \text{ then } CL_m \text{ is } \psi_{m,k} \end{aligned} \tag{26}$$

where CL is the cost level of alternatives, for $j = 1, \ldots, m$ and $k = 1, \ldots, K$.

Step 7: Construct the antecedent matrices and consequent matrices for the AS.

The AS antecedent matrices M_k are based on the BLs Λ_k and CLs Ψ_k, which are the outputs of the BS and CS correspondingly. The antecedent matrix of a system with two inputs, i.e., BL and CL, each taking m possible values, will be usually of size $2 \times (m \cdot m)$, as presented in the following:

$$M_k = \begin{array}{c} \text{BL} \\ \text{CL} \end{array} \begin{bmatrix} \lambda_{1,k} & \cdots & \lambda_{1,k} & \cdots & \lambda_{m,k} & \cdots & \lambda_{m,k} \\ \psi_{1,k} & \cdots & \psi_{m,k} & \cdots & \psi_{1,k} & \cdots & \psi_{m,k} \end{bmatrix}, \quad \text{for } k = 1, \ldots, K. \tag{27}$$

However, in this case, each tuple of inputs $(\lambda_{j,k}, \psi_{j,k})$ stands for the assessed levels of the same alternative j through two types of criteria—benefits and costs. Therefore, the AS antecedent matrices M_k are of size $2 \times m$, as constructed in the following:

$$M_k = \begin{array}{c} \mathrm{BL} \\ \mathrm{CL} \end{array} \begin{bmatrix} \lambda_{1,k} & \lambda_{2,k} & \lambda_{3,k} & \cdots & \lambda_{m,k} \\ \psi_{1,k} & \psi_{2,k} & \psi_{3,k} & \cdots & \psi_{m,k} \end{bmatrix}, \quad \text{for } k = 1, \ldots, K. \tag{28}$$

The AS consequent matrices are derived as follows:

i) Calculate the aggregation $\xi_{j,k}$ of weighted $NICC_{j,k}^B$ and $NICC_{j,k}^C$; the division by two in (29) reflects the equal importance of each of the two subsystems by means of a weighted mean:

$$\xi_{j,k} = \frac{NICC_{j,k}^B \times \left(\frac{e}{e+f}\right) + NICC_{j,k}^C \times \left(\frac{f}{e+f}\right)}{2}, \quad \text{for } j = 1, \ldots, m \text{ and } k = 1, \ldots, K. \tag{29}$$

ii) Normalize the values of $\xi_{j,k}$ to ensure they lie within $[0, 1]$, as calculated in the following:

$$N\xi_{j,k} = \frac{\xi_{j,k}}{\max_j \xi_{j,k}}, \quad \text{for } j = 1, \ldots, m \text{ and } k = 1, \ldots, K. \tag{30}$$

iii) For $N\xi_{j,k}$, take linguistic terms from Table III for the ALs. The translation of scalars to linguistic terms is done in the same way as in (22). Then, the *K* for AS consequent matrices, in this case of size $1 \times m$ rather than $1 \times m \cdot m$, are described as

$$N_k = \mathrm{AL} \left[N\xi_{1,k} \; N\xi_{2,k} \; \cdots \; N\xi_{m,k} \right], \quad \text{for } k = 1, \ldots, K \tag{31}$$

where AL is the level of alternatives.

Therefore, the AS is presented with *K* matrix decision rules, as constructed in the following:

$$\text{if } M_k = \begin{array}{c} \mathrm{BL} \\ \mathrm{CL} \end{array} \begin{bmatrix} \lambda_{1,k} & \lambda_{1,k} & \cdots & \lambda_{m,k} \\ \psi_{1,k} & \psi_{2,k} & \cdots & \psi_{m,k} \end{bmatrix}, \text{then}$$

$$N_k = \mathrm{AL} \left[N\xi_{1,k} \; N\xi_{2,k} \; \cdots \; N\xi_{m,k} \right], \quad \text{for } k = 1, \ldots, K \tag{32}$$

and can be described with the rule bases as

$$\begin{aligned} &\text{Rule 1 : If BL is } \lambda_{1,k} \text{ and CL is } \psi_{1,k}, \text{ then AL is } N\xi_{1,k} \\ &\qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\ &\text{Rule } m \text{ : If BL is } \lambda_{m,k} \text{ and CL is } \psi_{m,k} \text{ then AL} \\ &\text{is } N\xi_{m,k}, \quad \text{for } k = 1, \ldots, K \end{aligned} \tag{33}$$

where BL is the level of benefits, CL is the level of costs, and AL is the level of alternatives.

Step 8: Construct the generalized Boolean matrix representing the overall system.

Having derived the rules for the three systems—BS, CS, and AS—we can now translate these rules into Boolean matrix form. The generalized BS Boolean matrix for each alternative *j* is constructed in (34), based on the opinions of all *K* DMs:

$$\begin{array}{c|ccc} & \lambda_{j,1} & \cdots & \lambda_{j,K} \\ \hline x_{1j,1} \cdots x_{ej,1} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ x_{1j,K} \cdots x_{ej,K} & 0 & \cdots & 1 \end{array}, \quad \text{for } j = 1, \ldots, m \tag{34}$$

where the row and column labels of the Boolean matrix are all possible permutation for the BS rule base of the linguistics terms for the input 1–7 as in Tables I and II, and of the linguistic terms for the output 1–5 as in Table III.

The generalized CS Boolean matrix for each alternative *j* is constructed in (35) based on the opinions of all *K* DMs:

$$\begin{array}{c|ccc} & \psi_{j,1} & \cdots & \psi_{j,K} \\ \hline y_{1j,1} \cdots y_{fj,1} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ y_{1j,K} \cdots y_{fj,K} & 0 & \cdots & 1 \end{array}, \quad \text{for } j = 1, \ldots, m \tag{35}$$

where the row and column labels of the Boolean matrix are all possible permutation for the CS rule base of the linguistics terms for the input 1–7 as in Tables I and II, and of the linguistic terms for the output 1–5 as in Table III.

The vertical merging of the BS and CS generalized Boolean matrices will produce the generalized Boolean matrix constructed as

$$\begin{array}{c|ccc} & \begin{array}{c}\lambda_{j,1} \\ \psi_{j,1}\end{array} & \cdots & \begin{array}{c}\lambda_{j,K} \\ \psi_{j,K}\end{array} \\ \hline \begin{array}{c} x_{1j,1} \cdots x_{ej,1} \\ y_{1j,1} \cdots y_{fj,1} \end{array} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \begin{array}{c} x_{1j,K} \cdots x_{ej,K} \\ y_{1j,K} \cdots y_{fj,K} \end{array} & 0 & \cdots & 1 \end{array}, \quad \text{for } j = 1, \ldots, m. \tag{36}$$

Next, the AS generalized Boolean matrix for each alternative *j* is introduced in (37) based on the opinions of all *K* DMs:

$$\begin{array}{c|ccc} & N\xi_{j,1} & \cdots & N\xi_{j,K} \\ \hline \lambda_{j,1} \; \psi_{j,1} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{j,K} \; \psi_{j,K} & 0 & \cdots & 1 \end{array}, \quad \text{for } j = 1, \ldots, m. \tag{37}$$

Then, the resultant generalized Boolean matrix for the overall system for each alternative *j* is produced in (38) based on the opinions of all *K* DMs:

$$\begin{array}{c|ccc} & N\xi_{j,1} & \cdots & N\xi_{j,K} \\ \hline \begin{array}{c} x_{1j,1} \cdots x_{ej,1} \\ y_{1j,1} \cdots y_{fj,1} \end{array} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \begin{array}{c} x_{1j,K} \cdots x_{ej,K} \\ y_{1m,k} \cdots y_{fm,k} \end{array} & 0 & \cdots & 1 \end{array}, \quad \text{for } j = 1, \ldots, m \tag{38}$$

Step 9: Derive the rules for the alternatives based on the generalized Boolean matrix from (40), as shown below for $j = 1, \ldots, m$:

$$\begin{aligned} &\text{Rule 1 : If } B_1 \text{ is } x_{1j,1} \text{ and } \cdots \text{ and } B_e \text{ is } x_{ej,1} \text{ and } C_1 \text{ is } y_{1j,1} \\ &\text{and } \cdots \text{ and } C_f \text{ is } y_{fj,1}, \text{ then AL is } N\xi_{j,1} \\ &\qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\ &\text{Rule } n_j \text{ : If } B_1 \text{ is } x_{1j,K} \text{ and } \cdots \text{ and } B_e \text{ is } x_{ej,K} \text{ and } C_1 \text{ is } y_{1j,K} \\ &\text{and } \cdots \text{ and } C_f \text{ is } y_{fj,K}, \text{ then AL is } N\xi_{j,K}. \end{aligned}$$

TABLE IV
LINGUISTIC TERMS FOR THE IMPORTANCE WEIGHT OF EACH CRITERION

Linguistic Terms		Trapezoidal Type 2 Fuzzy Number
Very Low (VL)	1	(0.00,0.00,0.00,0.10,1,1)(0.00,0.00,0.00,0.10,1,1)
Low (L)	2	(0.00,0.10,0.10,0.25,1,1)(0.00,0.10,0.10,0.25,1,1)
Medium Low (ML)	3	(0.15,0.30,0.30,0.45,1,1)(0.15,0.30,0.30,0.45,1,1)
Medium (M)	4	(0.35,0.50,0.50,0.65,1,1)(0.35,0.50,0.50,0.65,1,1)
Medium High (MH)	5	(0.55,0.70,0.70,0.85,1,1)(0.55,0.70,0.70,0.85,1,1)
High (H)	6	(0.80,0.90,0.90,1.00,1,1)(0.80,0.90,0.90,1.00,1,1)
Very High (VH)	7	(0.90,1.00,1.00,1.00,1,1)(0.90,1.00,1.00,1.00,1,1)

TABLE V
LINGUISTIC TERMS FOR RATING OF ALL ALTERNATIVES

Linguistic Terms		Trapezoidal Type 2 Fuzzy Number
Very Poor (VP)	1	(0,0,0, 1,1) (0,0,0, 1,1)
Poor (P)	2	(0,1,1,3,1,1) (0,1,1,3,1,1)
Medium Poor (MP)	3	(1,3,3,5,1,1) (1,3,3,5,1,1)
Fair (F)	4	(3,5,5,7,1,1) (3,5,5,7,1,1)
Medium Good (MG)	5	(5,7,7,9,1,1) (5,7,7,9,1,1)
Good (G)	6	(7,9,9,10,1,1) (7,9,9,10,1,1)
Very Good (VG)	7	(9,10,10,10,1,1)(9,10,10,10,1,1)

Step 10: Derive a final score for each alternative.

In order to produce a final score Γ_j for each alternative *j*, take the average aggregate membership value of the consequent part of the n_j rules in (38). Then, multiply with the influence multiplier based on the *K* DMs' average influence degree for alternative *j*. This is shown as

$$\Gamma_j = \frac{\sum_{\mathrm{Rule}=1}^{n}\sum_{k=1}^{K} N\xi_{j,k}\cdot\left(NICC^B_{j,k}+NICC^C_{j,k}\right)}{n\cdot K},\quad \text{for } j=1,\ldots,m. \tag{39}$$

Thus, the ranking order of all alternatives can be determined: the better alternatives j have higher values of Γ_j. The alternatives we have developed the above ranking approach for are stock exchange traded equities. We have considered application to a developing financial market and are currently extending the application to comparison of performance in developing and developed financial markets.

B. Interval Type-2 Fuzzy Number Implementation

In this implementation of FN-TOPSIS, we use interval type-2 fuzzy number, as detailed in Tables IV–VI, for rating of alternatives and weighting the importance of criteria. All linguistics terms are written in the form of trapezoidal type-2-fuzzy numbers.

In terms of steps involved in the implementation of type-2 fuzzy numbers in FN-TOPSIS, the concept of ranking trapezoidal interval type-2 fuzzy numbers is relevant to step 3 prior to finding the distance of alternatives from PISs and NISs. The other steps are the same as type-1 fuzzy sets implementation discussed in Section III-A:

Step 3: Find the FPIS and FNIS for each alternative, and the distance between each alternative to FPIS and FNIS.

In order to construct the ranking weighted decision matrices, for $j=1,\ldots,m$ and $k=1,\ldots,m$, we need to calculate the ranking value of each interval type-2 fuzzy numbers $v_{ij,k}$, i.e., $\mathrm{Rank}(v_{ij,k})$.The maximum number n of edges in the upper membership function $v^U_{ij,k}$ and the lower membership function $v^L_{ij,k}$ are first defined, where $i=1,\ldots,e+f$ and $j=1,\ldots,m$. If n is an odd number and $n\geq 3$, then $r=n+1$. If n is an even number and $n\geq 4$, then $r=n$. The $\mathrm{Rank}(v_{ij,k})$ of an interval type-2 fuzzy numbers is presented as

$$\begin{aligned}\mathrm{Rank}(v_{ij,k}) &= \sum_{l\in\{U,L\}} M_1\left(v^l_{ij,k}\right)+\sum_{l\in\{U,L\}} M_2\left(v^l_{ij,k}\right)+\cdots\\ &+\sum_{l\in\{U,L\}} M_{r-1}\left(v^l_{ij,k}\right)-\frac{1}{r}\Bigg(\sum_{l\in\{U,L\}} S_1\left(v^l_{ij,k}\right)+\sum_{l\in\{U,L\}} S_2\left(v^l_{ij,k}\right)\\ &+\cdots+\sum_{l\in\{U,L\}} S_r\left(v^l_{ij,k}\right)\Bigg)+\sum_{l\in\{U,L\}}\mu_1\left(v^l_{ij,k}\right)\\ &+\sum_{l\in\{U,L\}}\mu_2\left(v^l_{ij,k}\right)+\cdots+\sum_{l\in\{U,L\}}\mu_{r-2}\left(v^l_{ij,k}\right).\end{aligned} \tag{40}$$

Here, $M_p(v^l_{ij,k})$ denotes the average of the elements $a^{v,l}_{ij,k,p}$ and $a^{v,l}_{ij,k,(p+1)}$, i.e., $M_p(v^l_{ij,k}) = \left(a^{v,l}_{ij,k,p}+a^{v,l}_{ij,k,(p+1)}\right)/2$, for $p=1,\ldots,r-1$. In addition, $S_p(v^l_{ij,k})$ denotes the standard deviation of elements $a^{v,l}_{ij,k,1}, a^{v,l}_{ij,k,2},\ldots,a^{v,l}_{ij,k,p}$, i.e.,

$$S_p(v^l_{ij,k}) = \sqrt{\frac{1}{p}\sum_{t=1}^{p}\left(a^{v,l}_{ij,k,t}-\frac{1}{p}\sum_{t=1}^{p}a^{v,l}_{ij,k,t}\right)^2},$$

for $p=1,\ldots,r$. Finally, $\mu_p(v^l_{ij,k})$ denotes the membership value of the element $a^{v,l}_{ij,k,(p+1)}$ for $p=1,\ldots,r-2$, where $l\in\{U,L\}$ and r is an even number.

The FPIS $A^+_k = (v^+_{1,k}, v^+_{2,k},\ldots,v^+_{(e+f),k})$ and the FNIS $A^-_k = (v^-_{1,k}, v^-_{2,k},\ldots,v^-_{(e+f),k})$ are defined as

$$\begin{aligned}A^+_k &= \left(v^+_{1,k}, v^+_{2,k},\ldots,v^+_{(e+f),k}\right)\\ A^-_k &= \left(v^-_{1,k}, v^-_{2,k},\ldots,v^-_{(e+f),k}\right)\end{aligned} \tag{41}$$

where

$$v^+_{i,k} = \begin{cases}\max\limits_{1\leq j\leq e+j}\left\{\mathrm{Rank}\left(v^B_{ij,k}\right)\right\}, & B_i\in B\\ \min\limits_{1\leq j\leq e+f}\left\{\mathrm{Rank}\left(v^C_{ij,k}\right)\right\}, & C_i\in C\end{cases}$$

and

$$v^-_{i,k} = \begin{cases}\min\limits_{1\leq j\leq e+f}\left\{\mathrm{Rank}\left(v^B_{ij,k}\right)\right\}, & B_i\in B\\ \max\limits_{1\leq j\leq e+f}\left\{\mathrm{Rank}\left(v^C_{ij,k}\right)\right\}, & C_i\in C.\end{cases}$$

Here, B denotes the set of benefit criteria, C denotes the set of cost criteria, and $i=1,\ldots,m$. The distance $\Delta^+_{j,k}$ between

TABLE VI
LINGUISTIC TERMS FOR AL

Linguistic Terms		Trapezoidal Type 2 Fuzzy Number
Very Bad(VB)	1	(0.00,0.00,0.00,0.25,1,1)(0.00,0.00,0.00,0.25,1,1)
Bad (B)	2	(0.00,0.25,0.25,0.50,1,1)(0.00,0.25,0.25,0.50,1,1)
Regular (R)	3	(0.25,0.50,0.50,0.75,1,1)(0.25,0.50,0.50,0.75,1,1)
Good (G)	4	(0.50,0.75,0.75,1,1,1) (0.50, 0.75, 0.75, 1,1,1)
Very Good (VG)	5	(0.75,1.00,1.00,1.00,1,1) (0.75,1.00,1.00,1.00,1,1)

TABLE VII
LINGUISTIC TERMS FOR EXPERT RELIABILITY

Linguistic Terms	Trapezoidal Fuzzy Number
Strongly Unlikely (SUL)	(0.00, 0.00, 0.00, 0.10)
Unlikely (UL)	(0.00, 0.10, 0.10, 0.25)
Somewhat Unlikely (SWU)	(0.15, 0.30, 0.30, 0.45)
Neutral (N)	(0.35, 0.50, 0.50, 0.65)
Somewhat Likely (SWL)	(0.55, 0.70,0.70, 0.85)
Likely (L)	(0.80, 0.90, 0.90, 1.00)
Strongly Likely (SL)	(0.90, 1.00, 1.00, 1.00)

each alternative $A_{j,k}$ and the FPIS A_k^+ is calculated with (42):

$$\Delta_{j,k}^{+} = \sqrt{\sum_{i=1}^{e+f} \left(\text{Rank}\,(v_{ij,k}) - v_{i,k}^{+}\right)^2}, \quad \text{for } j = 1,\ldots,m \text{ and } k = 1,\ldots,K. \tag{42}$$

The distance $\Delta_{j,k}^-$ between each alternative $A_{j,k}$ and the FNIS A_k^- is calculated as

$$\Delta_{j,k}^{-} = \sqrt{\sum_{i=1}^{e+f} \left(\text{Rank}\,(v_{ij,k}) - v_{i,k}^{-}\right)^2}, \quad \text{for } j = 1,\ldots,m \text{ and } k = 1,\ldots,K. \tag{43}$$

C. Z-Number Implementation

For the Z-number implementation of TOPSIS-FN, Tables I –III from Section III-A are used, with an additional Table VII for the linguistic terms representing DM reliability.

Here, the reliability of experts is taken into consideration during the decision-making process. The experts are advised to use the linguistic terms in Table VII to evaluate the confidence in their decision. DMs are not supposed to use negative weight to represent their opinion. Otherwise, this would imply the use of unreliable information, which is undesirable. This applies at the start of step 1 of the algorithm described in type-1 fuzzy number implementation of FN-TOPSIS. The other steps are the same as the implementation discussed in Section III-A.

Step 1: Use the information from Table VII to derive the second component B of the Z-number, and then, convert the Z-number to type-1 fuzzy number.

Let $Z = (\tilde{A}, \tilde{B})$ is a Z-number, where $\{\tilde{A} = (x, \mu_{\tilde{A}})|x \in [0, 1]\}, \{\tilde{B} = (x, \mu_{\tilde{B}})|x \in [0, 1]\}$, and $\mu_{\tilde{A}}$ and $\mu_{\tilde{B}}$ are trapezoidal membership functions. The second part (reliability) needs to convert into a crisp number using fuzzy expectation, as

$$\alpha = \frac{\int x\mu_{\tilde{B}}\,dx}{\int \mu_{\tilde{B}}\,dx} \tag{44}$$

where $\int$ denotes an algebraic integration. Then, add the weight of the second part (reliability) to the first part (restriction). Weighted Z-numbers can be denoted as

$$\tilde{Z}^{\alpha} = \{(x, \mu_{\tilde{A}^{\alpha}})\,|\mu_{\tilde{A}^{\alpha}}(x) = \alpha\mu_{\tilde{A}}(x), x \in [0, 1]\}.$$

These can be represented with type-1 fuzzy numbers as

$$\tilde{Z}' = \left\{< x, \mu_{\tilde{Z}^{\alpha}}(x) > |\mu_{\tilde{Z}^{\alpha}}(x) = \mu_{\tilde{A}}\left(\frac{x}{\sqrt{\alpha}}\right), x \in [0, 1]\right\}.$$

It is proven in [33] that $\tilde{Z}'$ has the same fuzzy expectation as $\tilde{Z}^{\alpha}$. The remaining steps of the algorithm are the same as for the type-1 fuzzy sets implementation. The next section systematically illustrates the application of type-1 fuzzy sets of the proposed FN-TOPSIS method to solve the problem of selection/ranking of traded equity.

IV. RANKING OF TRADED EQUITY

We study the problem of ranking traded equity in developing financial markets within a crisis period, in order to illustrate the applicability and validity of the proposed FN methodology in a realistic scenario. DMs with different levels of experience evaluate 25 equities listed on the main board of the Kuala Lumpur Stock Exchange (KLSE) on November 30, 2007. A set of financial ratios for the equities are considered toward the benefit and cost criteria in the FN-TOPSIS algorithm. These include the following: 1) Market value of firm (B1), defined as market value of firm-to-earnings before amortization, interest, and taxes. This is one of the critical financial indicators, and the lower the ratio, the better the equity [46]. 2) Return on equity (ROE) (B2), which evaluates how much the company earns on the investment of its shareholders. ROE is measured as net income divided by stockholder funds. Portfolio managers examine ROE when deciding whether to trade (buy or sell) equities. The higher values of the ratio indicate healthier companies. 3) Debt-to-equity ratio (C1), belonging to long-term solvency ratios that are intended to address the firm's long run ability to meet its obligations. It is considered by DMs that the lower the ratio, the better [47]. 4) Current ratio (B3), which measures liquidity of companies and explains the ability of a business to meet its current obligations when fall due. The higher the ratio, the more liquid is the company and, therefore, in a better position.[48]. 5) Market value-to-net sales (B4) is market value ratios of particular interest to investors. The lower the ratio, the better the equity [49]. The lower this ratio is, the better the equity. 6) Price/earnings ratio (C2) measures the ratio of market price of each share of common stock to the earnings per share, the lower this ratio, the better.

In this study, the processes of ranking equities follow the proposed methods in Section III. Fig. 4 illustrates the FN model for the problem of selection/ranking of traded equity and includes four benefit criteria and two cost criteria.

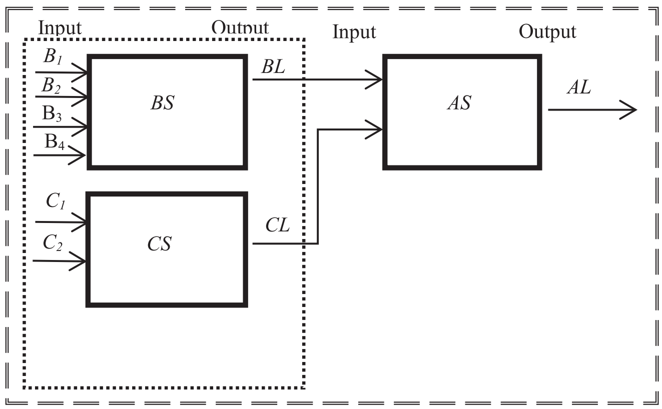

Fig. 4. FN for the FN-TOPSIS application to ranking traded equity.

Step 1: Based on the information provided by experts and using (11), the decision matrices for the BS and CS can be constructed. The rating of each criterion for each equity and the importance of criteria are based on DMs' opinions.

Step 2: Considering the BS, the normalized decision matrix R_k^B and the weight normalized decision matrix V_k^B can be constructed for each *k*, using (12) correspondingly.

For example, the calculations for *E1* using the opinion of DM1 are as follows:

$$\begin{aligned} g_{1,1} &= (0.9,\ 1,\ 1,\ 1)\,;x_{11,1} = (9,\ 10,\ 10,\ 10)\,;\, d_{1,1}^{x*} = 10 \\ r_{11,1}^B &= (9/10,\ 10/10,\ 10/10,\ 10/10) = (0.9,\ 1,\ 1,\ 1) \\ v_{11,1}^B &= (0.9\times 0.9,\ 1\times 1,\ 1\times 1,\ 1\times 1) = (0.81,\ 1,\ 1,\ 1)\,. \end{aligned}$$

This step is then repeated for the CS, in order to calculate the normalized decision matrix R_k^C and the weight normalized decision matrix V_k^C.

Step 3: The FPIS and the FNIS for each equity based on both systems, and the distances between the rating of criteria for each equity and the FPIS and FNIS, can be evaluated as follows.

FPIS and FNIS are determined as

$$\begin{aligned} A_k^+ &= \left[(1,1,1,1)_{1,k},\ (1,1,1,1)_{2,k},\ \ldots,\ (1,1,1,1)_{25,k}\right] \\ A_k^- &= \left[(0,0,0,0)_{1,k},\ (0,0,0,0)_{2,k},\ \ldots,\ (0,0,0,0)_{25,k}\right]. \end{aligned}$$

The distances $\Delta_{j,k}^{B+}$ and $\Delta_{j,k}^{B-}$, between the rating according to DM *k* of benefit criteria $i = 1,\ldots,4$ for each equity $j(j = 1,\ldots,25)$ and the FPIS A_k^+ or FNIS A_k^- are calculated using (13) and (14). For example, the distance between the first equity *E1* according to DM1 and the FPIS A_1^+ is calculated using (13) for $j = 1$ and $k = 1$, as follows:

$$\Delta_1^{B+}\left(v_{11,1},\ v_{1,1}^+\right) = \sqrt{\frac{1}{3}\left[(0.81-1)^2+\cdots+(1-1)^2\right]} = 0.11.$$

Similarly

$$\begin{aligned} \Delta_1^{B+}\left(v_{21,1},\ v_{2,1}^+\right) &= 0.409 \\ \Delta_1^{B+}\left(v_{31,1},\ v_{3,1}^+\right) &= 0.668 \\ \Delta_1^{B+}\left(v_{41,1},\ v_{4,1}^+\right) &= 0.298 \end{aligned}$$

and to produce overall

$$\begin{aligned} \Delta_{1,1}^{B+} = \sum_{i=1}^{4}\Delta_1^{B+}\left(v_{i1,1}, v_{i,1}^+\right) &= 0.11 + 0.409 + 0.668 \\ &+ 0.298 = 1.4841. \end{aligned}$$

Next, using (14) for $j = 1$ and $k = 1$, the distance between *E1* according to DM1 and the FPIS A_1^- is calculated as

$$\Delta_1^{B-}(v_{11,1},\ v_{1,1}^-) = \sqrt{\frac{1}{3}[(0.81-0)^2+\cdots+(1-0)^2]} = 1.373;$$

similarly, $\Delta_2^{B-}(v_{21,1},\ v_{2,1}^-) = 1.063$, $\Delta_3^{B-}(v_{31,1},\ v_{3,1}^-) = 0.789$, and $\Delta_4^{B-}(v_{41,1},\ v_{4,1}^-) = 1.242$, producing overall

$$\begin{aligned} \Delta_{1,1}^{B-} = \sum_{i=1}^{4}\Delta_1^{B-}\left(v_{i1,1}, v_{i,1}^-\right) &= 1.373 + 1.063 + 0.789 \\ &+ 1.242 = 4.4671. \end{aligned}$$

Now, the distances $\Delta_{j,k}^{C+}$ and $\Delta_{j,k}^{C-}$ between the rating according to DM *k* of cost criteria $i = 1,\ldots,2$ for each equity *j* $(j = 1,\ldots,25)$ and the FPIS A_k^+ or FNIS A_k^- are calculated using (15) and (16). For example, the distance between the first equity *E1* according to DM1 and the FPIS A_1^+ is calculated using (15) for $j = 1$ and $k = 1$, as follows:

$$\Delta_1^{C+}\left(v_{11,1},\ v_{1,1}^+\right) = \sqrt{\frac{1}{3}\left[(0.39-1)^2+\cdots+(0.85-1)^2\right]} = 0.49$$

and similarly

$$\Delta_k^{C+}\left(v_{ij,k},\ v_{i,k}^+\right) = \Delta_1^{C+}\left(v_{21,1},\ v_{2,1}^+\right) = 1.12$$

producing overall

$$\Delta_{1,1}^{C+} = \sum_{i=1}^{2}\Delta_1^{C+}\left(v_{i1,1}, v_{i,1}^+\right) = 0.49 + 1.12 = 1.61.$$

Next, using (16) for $j = 1$ and $k = 1$, the distance between *E1* according to DM1 and the FPIS A_1^- is calculated as

$$\begin{aligned} = \Delta_1^{C-}\left(v_{11,1},\ v_{1,1}^-\right) &= \sqrt{\frac{1}{3}\left[(0.39-0)^2+\cdots+(0.85-0)^2\right]} \\ &= 1.017 \end{aligned}$$

and similarly

$$\Delta_k^{C-}\left(v_{ij,k},\ v_{i,k}^-\right) = \Delta_1^{C-}\left(v_{21,1},\ v_{2,1}^-\right) = 0.339$$

producing overall

$$\Delta_{1,1}^{C-} = \sum_{i=1}^{2}\Delta_1^{C-}\left(v_{i1,1}, v_{i,1}^-\right) = 1.017 + 0.339 = 1.358.$$

Step 4: Find the closeness coefficients for the BS $CC_{j,k}^B$ and for the CS $CC_{j,k}^C$, using (17) for each equity E_j, $j = 1,\ldots,25$. For example, the closeness coefficient for *E1* in the BS under the first DM $k = 1$ is calculated using (19) as follows:

$$CC_{1,1}^B = \frac{\Delta_{1,1}^{B-}}{\Delta_{1,1}^{B+} + \Delta_{1,1}^{B-}} = \frac{4.4671}{1.4841 + 4.4671} = 0.751$$

and the closeness coefficient in the CS

$$CC^C_{1,1} = \frac{\Delta^{C-}_{1,1}}{\Delta^{C+}_{1,1} + \Delta^{C-}_{1,1}} = \frac{1.358}{1.61 + 1.358} = 0.457.$$

Step 5: The ICCs $ICC^B_{j,k}$ and $ICC^C_{j,k}$ for each DM *k* are derived by applying the influence degree θ_k of each DM, using (18) and (19). Then, the normalized coefficients $NICC^B_{j,k}$ and $NICC^C_{j,k}$ are calculated with (20).

For example, the influence degree of DM1 is $\theta_1 = 8$, and using (18), his normalized expertise is

$$\sigma_1 = \frac{\theta_1}{\sum_{l=1}^{3} \theta_l} = \frac{8}{8 + 10 + 7} = 0.32.$$

Then, the ICC $ICC^B_{1,1}$ for the BS for equity *E1* according to DM1 is calculated with (19) as

$$ICC^B_{1,1} = \sigma_1 * CC^B_{1,1} = 0.32 * 0.751 = 0.2403.$$

Similarly, the corresponding ICC for the CS $ICC^C_{1,1}$ is produced as

$$ICC^C_{1,1} = \sigma_1 * CC^C_{1,1} = 0.32 * 0.457 = 0.1462.$$

Next, the ICCs have to be normalized prior to matching the coefficients to the linguistic variable in Table III. Using (20), $NICC^B_{1,1}$ and $NICC^C_{1,1}$ are calculated as

$$NICC^B_{1,1} = \frac{ICC^B_{1,1}}{\max_j ICC^B_{j,k}} = \frac{0.2403}{0.2403} \text{ and}$$
$$NICC^C_{1,1} = \frac{ICC^C_{1,1}}{\max_j ICC^C_{1,1}} = \frac{0.1462}{0.1659}.$$

Finally, the normalized coefficients are matched to the variable in Table III:

$$NICC^B_{1,1} = 1 \cong VG, \quad NICC^C_{1,1} = 0.8812 \cong VG.$$

Step 6: The antecedent matrices X_k for the BS are constructed using (21) for $k = 1, \ldots, K$, based on DM *k* opinions. Each DM has a separate benefit antecedent matrix. The consequent matrices Λ_k for the BS are constructed using (22) for $k = 1, \ldots, K$, based on the values of $NICC^B_{j,k}$ calculated at step 5 above and matched to the linguistic terms in Table III. Each DM has a separate benefit antecedent matrix. Similarly, the antecedent matrices Y_k and the consequent matrices Ψ_k are produced for the CS. Thus, the antecedent and consequent matrices for the benefit and cost rule bases are generated in this step.

For example, using (21), and according to the first DM $k = 1$, the antecedent matrix X_1 for the BS is

$$X_1 = \begin{array}{c} \\ B_1 \\ B_2 \\ B_3 \\ B_4 \end{array} \begin{array}{c} \begin{array}{cccc} E_1 & E_2 & \cdots & E_{25} \end{array} \\ \begin{bmatrix} x_{1,1,1} & x_{1,2,1} & \cdots & x_{1,25,1} \\ x_{2,1,1} & x_{2,2,1} & \cdots & x_{2,25,1} \\ x_{3,1,1} & x_{3,2,1} & \ddots & x_{3,25,1} \\ x_{4,1,1} & x_{4,2,1} & \cdots & x_{4,25,1} \end{bmatrix} \end{array}$$

$$= \begin{array}{c} \\ B_1 \\ B_2 \\ B_3 \\ B_4 \end{array} \begin{array}{c} \begin{array}{cccc} E_1 & E_2 & \cdots & E_{25} \end{array} \\ \begin{bmatrix} VG & MG & \cdots & MG \\ VG & VG & \cdots & F \\ VG & M & \ddots & MP \\ G & G & \cdots & G \end{bmatrix} \end{array}$$

where B_i are the four benefit criteria. Then, using (22), the consequent matrix Λ_1 is

$$\Lambda_1 = \text{BL} \begin{array}{c} \begin{array}{cccc} E_1 & E_2 & \cdots & E_{25} \end{array} \\ \begin{bmatrix} \lambda_{1,1} & \lambda_{2,1} & \cdots & \lambda_{25,1} \end{bmatrix} \end{array}$$

$$= \text{BL} \begin{array}{c} \begin{array}{cccc} E_1 & E_2 & \cdots & E_{25} \end{array} \\ \begin{bmatrix} VG & VG & \cdots & G \end{bmatrix} \end{array}.$$

Next, using (22), and according to the first DM $k = 1$, the antecedent matrix Y_1 for the CS is

$$Y_1 = \begin{array}{c} \\ C_1 \\ C_2 \end{array} \begin{array}{c} \begin{array}{cccc} E_1 & E_2 & \cdots & E_{25} \end{array} \\ \begin{bmatrix} y_{1,1,1} & y_{1,2,1} & \cdots & y_{1,25,1} \\ y_{2,1,1} & y_{2,2,1} & \cdots & y_{2,25,1} \end{bmatrix} \end{array}$$

$$= \begin{array}{c} \\ C_1 \\ C_2 \end{array} \begin{array}{c} \begin{array}{cccc} E_1 & E_2 & \cdots & E_{25} \end{array} \\ \begin{bmatrix} G & F & \cdots & F \\ F & G & \cdots & G \end{bmatrix} \end{array}.$$

Then, using (22), the consequent matrix Ψ_1 is

$$\Psi_1 = \text{CL} \begin{array}{c} \begin{array}{ccc} E_1 & E_2 \cdots & E_{25} \end{array} \\ [\Psi_{1,1} \quad \Psi_{2,1} \quad \Psi_{25,1}] \end{array} = \text{CL} \begin{array}{c} \begin{array}{cccc} E_1 & E_2 & \cdots & E_{25} \end{array} \\ [VG \quad G \quad \cdots \quad G] \end{array}.$$

The rule base of the BS for DM1 is constructed using (23) and (24) as if

$$X_1 = \begin{array}{c} \\ B_1 \\ B_2 \\ B_3 \\ B_4 \end{array} \begin{array}{c} \begin{array}{cccc} E_1 & E_2 & \cdots & E_{25} \end{array} \\ \begin{bmatrix} VG & MG & \cdots & MG \\ VG & VG & \cdots & F \\ VG & M & \ddots & MP \\ G & G & \cdots & G \end{bmatrix} \end{array},$$

then $\Lambda_1 = \text{BL} \begin{array}{c} \begin{array}{cccc} E_1 & E_2 & \cdots & E_{25} \end{array} \\ [VG \quad VG \quad \cdots \quad G] \end{array}.$

Rule 1: If B_1 is *VG* and B_2 is *VG* and B_3 is *VG* and B_4 is *G*, then the output BL is *VG*

Rule 2: If B_1 is *MG* and B_2 is *VG* and B_3 is *M* and B_4 is *G*, then the output BL is *VG*

$$\vdots \qquad \vdots \qquad \vdots$$

Rule 25: If B_1 is *MG* and B_2 is *F* and B_3 is *MP* and B_4 is *G*, then the output BL is *G*

TABLE VIII
RANKING BASED ON TYPE-1, TYPE-2, AND Z FUZZY NUMBER IMPLEMENTATION OF THE PROPOSED FN-TOPSIS METHOD

	Type-1 Implementation		Type-2 implementation		Z implementation	
Equity	Final Score	Rank	Final Score	Rank	Final Score	Rank
E1	0.7900	6	0.6836	8	0.6931	5
E2	0.8090	3	0.7198	4	0.5712	10
E3	0.8813	1	0.8701	1	0.8221	1
E4	0.4283	20	0.2717	20	0.2615	20
E5	0.2735	22	0.1861	23	0.2267	23
E6	0.7871	7	0.7684	3	0.6355	7
E7	0.4652	14	0.3891	14	0.3029	17
E8	0.4388	18	0.2799	19	0.2936	19
E9	0.1730	25	0.1628	24	0.1152	25
E10	0.4555	16	0.3357	16	0.3490	16
E11	0.5084	12	0.4600	12	0.4169	13
E12	0.4528	17	0.2920	18	0.3640	15
E13	0.3661	21	0.2561	21	0.2457	21
E14	0.7506	9	0.6441	9	0.6376	6
E15	0.7936	5	0.6981	5	0.7237	3
E16	0.8467	2	0.8370	2	0.7571	2
E17	0.2551	24	0.1595	25	0.2409	22
E18	0.4308	19	0.3568	15	0.3746	14
E19	0.6536	10	0.5532	11	0.4993	11
E20	0.4629	15	0.2987	17	0.2977	18
E21	0.7761	8	0.6907	7	0.7029	4
E22	0.2616	23	0.1982	22	0.1849	24
E23	0.7956	4	0.6958	6	0.5940	9
E24	0.6338	11	0.5574	10	0.6315	8
E25	0.4899	13	0.3928	13	0.4937	12

By analogy, the rule base for the CS is constructed.

Step 7: The AS in this application is the equity system (ES), and the antecedent matrices M_k of each DM *k* for ES are constructed using (28) based on the BL and CL, which are the outputs of the BS and CS, respectively. Each DM has a separate equity antecedent matrix M_k. Next, the ES consequent matrices N_k are derived using (29)–(31), while calculating the aggregations $\xi_{j,k}$ of weighted coefficients $NICC^B_{j,k}$ and $NICC^C_{j,k}$ for each equity *j* ($j = 1, \ldots, 25$), then producing the normalized aggregations $N\xi_{j,k}$, and constructing the ES consequent matrices N_k based on $N\xi_{j,k}$. Each DM *k* has a separate equity consequent matrix N_k.

For example, based on the BL and CL evaluated in step 6 above and using (27), the ES antecedent matrix M_1 according to DM1 is evaluated as

$$M_1 = \begin{array}{c} \\ \text{BL} \\ \text{CL} \end{array} \begin{array}{c} \begin{array}{cccc} E_1 & E_2 & \cdots & E_{25} \end{array} \\ \begin{bmatrix} \lambda_{1,1} & \lambda_{2,1} & \cdots & \lambda_{25,1} \\ \psi_{1,1} & \psi_{2,1} & \cdots & \psi_{25,1} \end{bmatrix} \end{array}$$

$$= \begin{array}{c} \\ \text{BL} \\ \text{CL} \end{array} \begin{array}{c} \begin{array}{cccc} E_1 & E_2 & \cdots & E_{25} \end{array} \\ \begin{bmatrix} VG & VG & \cdots & G \\ VG & G & \cdots & G \end{bmatrix} \end{array}.$$

Next, the ES consequent matrix N_1 according to DM1 is derived through:

i) calculating the aggregated closeness coefficient $\xi_{j,1}$ for each equity $j = 1, \ldots, 25$, with (28) and based on the normalized closeness coefficients $NICC^B_{j,1}$ and $NICC^C_{j,1}$ according to DM1, e.g., for $j = 1$:

$$\xi_{1,1} = \frac{NICC^B_{1,1} \times \left(\frac{4}{4+2}\right) + NICC^C_{1,1} \times \left(\frac{2}{4+2}\right)}{2} =$$

$$\xi_{1,1} = \frac{1.00 \times \left(\frac{2}{3}\right) + 0.8812 \left(\frac{1}{3}\right)}{2} = 0.480$$

ii) calculating the normalized aggregated closeness coefficients $N\xi_{j,1}$ for each equity $j = 1, \ldots, 25$ with (29) and based on the values $\xi_{j,1}$ produced in Step 7(i) above, e.g., for $j = 1$:

$$N\xi_{1,1} = \frac{\xi_{1,1}}{\max_j \xi_{j,1}} = \frac{0.48}{0.50} = 0.96$$

and the value of $N\xi_{1,1}$ is matched to the linguistic variable for equity levels in Table III:

$$N\xi_{1,1} = 0.960 \cong VG$$

iii) the ES consequent matrix N_1 for DM1 is constructed using (30) and based on the values $N\xi_{j,1}$ for each equity *j* produced in Step 7(ii) above, e.g., for $j = 1$:

$$N_1 = \text{EL} \begin{array}{c} \begin{array}{cccc} E_1 & E_2 & \cdots & E_{25} \end{array} \\ \begin{bmatrix} N\xi_{1,1} & N\xi_{2,1} & \cdots & N\xi_{25,1} \end{bmatrix} \end{array}$$

$$= \begin{array}{c} \begin{array}{cccc} E_1 & E_2 & \cdots & E_{25} \end{array} \\ \begin{bmatrix} VG & VG & \cdots & G \end{bmatrix} \end{array}.$$

Therefore, the ES rule base according to DM1 is evaluated using (25) and (26) as

$$\text{if } M_1 = \begin{array}{c} \\ \text{BL} \\ \text{CL} \end{array} \begin{array}{c} \begin{array}{cccc} E_1 & E_2 & \cdots & E_{25} \end{array} \\ \begin{bmatrix} VG & VG & \cdots & G \\ VG & G & \cdots & G \end{bmatrix} \end{array}, \text{ then}$$

$$N_1 = \text{EL} \begin{array}{c} \begin{array}{cccc} E_1 & E_2 & \cdots & E_{25} \end{array} \\ \begin{bmatrix} VG & VG & G \end{bmatrix} \end{array}.$$

$$\begin{aligned} &\text{Rule 1 : If BL is VG and CL is VG, then EL is VG} \\ &\text{Rule 2 : If BL is VG and CL is VG, then EL is VG} \\ &\vdots \qquad\qquad \vdots \qquad\qquad \vdots \\ &\text{Rule 25 : If BL is } G \text{ and CL is } G, \text{ then EL is } G. \end{aligned}$$

Step 8: Having list of rules for three systems—BS, CS, ES—we now present these rules in the Boolean matrix form. The Boolean matrices for each equity are constructed based on the opinions from all DMs. For example, using (33), the Boolean matrix of the BS for *E1* is produced in (45).The row and column labels of the Boolean matrix are all possible permutations of linguistics variable for the input 1–7 as in Table I and the linguistic variable for the output 1–5 as in Table III, for the benefit

TABLE IX
ALTERNATIVE RANKING BASED ON ESTABLISHED TOPSIS METHODS (EM) AND PROPOSED FN-TOPSIS METHODS (PM)

Equity	Actual	Conventional TOPSIS Approach (EM)	Non-Rule Based Fuzzy Fuzzy TOPSIS Approach (EM)			FN-TOPSIS Approach (PM)		
			T-1	T-2	Z	T-1	T-2	Z
E1	2	2	4	3	7	6	8	5
E2	4	7	3	5	9	3	4	10
E3	1	1	1	1	1	1	1	1
E4	21	21	20	18	20	20	20	20
E5	19	24	24	23	24	22	23	23
E6	11	6	6	6	8	7	3	7
E7	17	11	12	12	17	14	14	17
E8	24	14	18	17	18	18	19	19
E9	23	25	25	24	25	25	24	25
E10	22	15	16	14	14	16	16	16
E11	8	20	14	15	13	12	12	13
E12	13	12	17	16	16	17	18	15
E13	25	23	22	22	22	21	21	21
E14	9	10	9	10	10	9	9	6
E15	3	8	8	8	3	5	5	3
E16	5	3	2	2	2	2	2	2
E17	18	18	21	21	21	24	25	22
E18	12	19	19	19	19	19	15	14
E19	15	13	11	11	11	10	11	11
E20	16	17	15	13	15	15	17	18
E21	7	4	7	7	4	8	7	4
E22	20	22	23	20	23	23	22	24
E23	6	5	5	4	5	4	6	9
E24	14	9	10	9	6	11	10	8
E25	10	16	13	25	12	13	13	12

rule base

$$\begin{array}{cccccc} & 1 & 2 & 3 & 4 & 5 \\ 1111 & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 6576 & 0 & 0 & 0 & 0 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 7655 & 0 & 0 & 0 & 0 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 7776 & 0 & 0 & 0 & 0 & 1 \\ 7777 & 0 & 0 & 0 & 0 & 0 \end{array}. \tag{45}$$

Next, using (34), the Boolean matrix of the CS for *E1* is defined as

$$\begin{array}{cccccc} & 1 & 2 & 3 & 4 & 5 \\ 11 & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 44 & 0 & 0 & 0 & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 64 & 0 & 0 & 0 & & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 76 & 0 & 0 & 0 & 0 & 0 \\ 77 & 0 & 0 & 0 & 0 & 0 \end{array}. \tag{46}$$

Step 9: Vertical merging is performed to merge the BS and CS Boolean matrices for each equity, then horizontal merging performed to merge the Boolean matrix obtain from the vertical merging operation with the ES Boolean matrix for each equity. For example, applying vertical merging of the BS and CS Boolean matrices for *E1*, the resultant Boolean matrix is constructed as

$$\begin{array}{cccccc} & 11 & \cdots & \cdots & 55 & 55 \\ 1111/11 & 0 & \cdots & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 6576/44 & 0 & \cdots & \cdots & 1 & 0 \\ 6576/64 & 0 & \cdots & \cdots & 0 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 7655/44 & 0 & \cdots & \cdots & 1 & 0 \\ 7655/64 & 0 & \cdots & \cdots & 0 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 7776/44 & 0 & \cdots & \cdots & 1 & 0 \\ 7776/64 & 0 & \cdots & \cdots & 0 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 7777/77 & 0 & \cdots & \cdots & 0 & 0 \end{array}. \tag{47}$$

The ES Boolean matrix for *E1* is evaluated as

$$\begin{array}{cccccc} & 1 & 2 & 3 & 4 & 5 \\ 11 & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 33 & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 44 & 0 & & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 54 & 0 & 0 & 0 & 0 & 1 \\ 55 & 0 & 0 & 0 & 0 & 1 \end{array}. \tag{48}$$

Next, the resultant Boolean matrix for the overall system is produced as shown in (49), through horizontal merging between the Boolean matrices in (47) and (48):

$$\begin{array}{cccccc} & 1 & 2 & 3 & 4 & 5 \\ 1111/11 & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 6576/44 & 0 & 0 & 0 & 0 & 1 \\ 6576/64 & 0 & 0 & 0 & 0 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 7655/44 & 0 & 0 & 0 & 0 & 1 \\ 7655/64 & 0 & 0 & 0 & 0 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 7776/44 & 0 & 0 & 0 & 0 & 1 \\ 7776/64 & 0 & 0 & 0 & 0 & 1 \\ 7777/77 & 0 & 0 & 0 & 0 & 0 \end{array} \tag{49}$$

where only the rows containing 1 are shown, along with the first and last rows.

TABLE X
SPEARMAN RHO CORRELATION COEFFICIENT FOR ALL TOPSIS METHODS

Equity	Conventional TOPSIS Approach (EM)		Non-Rule Based TOPSIS Approach (EM)						FN-TOPSIS Approach (PM)					
			T-1		T-2		Z		T-1		T-2		Z	
	∂_i	∂_i^2	∂_i	∂_i^2	∂_i	∂_i^2	∂_i	∂_i^2	∂_i	∂_i^2	∂_i	∂_i^2	∂_i	∂_i^2
E1	0	0	−2	4	−1	1	−5	25	−4	16	−6	36	−3	9
E2	−3	9	1	1	−1	1	−5	25	1	1	0	0	−6	36
E3	0	0	0	0	0	0	0	0	0	0	0	0	0	0
E4	0	0	1	1	3	9	1	1	1	1	1	1	1	1
E5	−5	25	−5	25	−4	16	−5	25	−3	9	−4	16	−4	16
E6	5	25	5	25	5	25	3	9	4	16	8	64	4	16
E7	6	36	5	25	5	25	0	0	3	9	3	9	0	0
E8	10	100	6	36	7	49	6	36	6	36	5	25	5	25
E9	−2	4	−2	4	−1	1	−2	4	−2	4	−1	1	−2	4
E10	7	49	6	36	8	64	8	64	6	36	6	36	6	36
E11	−12	144	−6	36	−7	49	−5	25	−4	16	−4	16	−5	25
E12	1	1	−4	16	−3	9	−3	9	−4	16	−5	25	−2	4
E13	2	4	3	9	3	9	3	9	4	16	4	16	4	16
E14	−1	1	0	0	−1	1	−1	1	0	0	0	0	3	9
E15	−5	25	−5	25	−5	25	0	0	−2	4	−2	4	0	0
E16	2	4	3	9	3	9	3	9	3	9	3	9	3	9
E17	0	0	−3	9	−3	9	−3	9	−6	36	−7	49	−4	16
E18	−7	49	−7	49	−7	49	−7	49	−7	49	−3	9	−2	4
E19	2	4	4	16	4	16	4	16	5	25	4	16	4	16
E20	−1	1	1	1	3	9	1	1	1	1	−1	1	−2	4
E21	3	9	0	0	0	0	3	9	−1	1	0	0	3	9
E22	−2	4	−3	9	0	0	−w3	9	−3	9	−2	4	−4	16
E23	1	1	1	1	2	4	1	1	2	4	0	0	−3	9
E24	5	25	4	16	5	25	8	64	3	9	4	16	6	36
E25	−6	36	−3	9	−15	225	−2	4	−3	9	−3	9	−2	4
	0	556	0	362	0	630	0	404	0	332	0	362	0	320
ρ	**0.786**		**0.861**		**0.758**		**0.845**		**0.872**		**0.861**		**0.877**	
Method Ranking According Performance	6		3–4		7		5		2		3-4		1	

From the Boolean matrix in (49), the rule basis for equity *E1* is derived as

$$\begin{array}{lllllllll} \text{Rule 1} : 6576/44/5 & 6 & 5 & 7 & 6 & 4 & 4 & 5 \\ \text{Rule 2} : 6576/64/5 & 6 & 5 & 7 & 6 & 6 & 4 & 5 \\ \text{Rule 3} : 7655/44/5 & 7 & 6 & 5 & 5 & 4 & 4 & 5 \\ \text{Rule 4} : 7655/64/5 & 7 & 6 & 5 & 5 & 6 & 4 & 5 \\ \text{Rule 5} : 7776/44/5 & 7 & 7 & 7 & 6 & 4 & 4 & 5 \\ \text{Rule 6} : 7776/64/5 & 7 & 7 & 7 & 6 & 6 & 4 & 5. \end{array} \tag{50}$$

The rules in (50) with six inputs and one output can be represented in linguistic terms as

Rule 1 : If B1 is G, and B2 is MG and B3 is VG and B4 is G and $C1$ is F and C2 is F, then E1 is VG

Rule 2 : If $B1$ is G, and $B2$ is MG and $B3$ is VG and $B4$ is G and $C1$ is G and $C2$ is F, then $E1$ is VG

Rule 3 : If $B1$ is VG, and $B2$ is G and $B3$ is MG and $B4$ is MG and $C1$ is F and $C2$ is F, then $E1$ is VG

Rule 4 : If $B1$ is VG, and $B2$ is G and $B3$ is MG and $B4$ is MG and $C1$ is G and $C2$ is F, then $E1$ is VG

Rule 5 : If $B1$ is VG, and $B2$ is VG and $B3$ is VG and $B4$ is G and $C1$ is F and $C2$ is F, then $E1$ is VG

Rule 6 : If $B1$ is VG, and $B2$ is VG and $B3$ is VG and $B4$ is G and $C1$ is G and $C2$ is F, then $E1$ is VG. (51)

Step 10: The final score for each alternative $j = 1, \ldots, 25$ is derived with (39), by taking average of the aggregate membership value of the consequent part of all active rules in the overall system for equity *j* and then multiplying with the influence multiplier based on the average influence degree across all *K* DMs for each equity *j*.

For example, there are six active rules for *E1* generated from the Boolean matrix operation. Equation (39) is used in order to obtain final score for *E1*; the average aggregate membership value for the output of the six rules is calculated and, then, multiplied with the influence multiplier for *E1* across all DMs

$$= \Gamma_1 = \frac{\sum_{\text{Rule}=1}^{6} \sum_{k=1}^{3} N\xi_{i,k} \cdot \left(NICC_{j,k}^{B} + NICC_{j,k}^{C} \right)}{6.3}$$

$$= \frac{0.9(0.94) + 0.9(0.79) + 0.9(0.91)}{18}$$

$$+\frac{0.9(0.94)+0.9(0.79)+0.9(0.91)}{18}$$
$$+\frac{0.9(0.94)+0.9(0.79)+0.9(0.91)}{18}$$
$$+\frac{0.9(0.94)+0.9(0.79)+0.9(0.91)}{18}$$
$$+\frac{0.9(0.94)+0.9(0.79)+0.9(0.91)}{18}$$
$$+\frac{0.9(0.94)+0.9(0.79)+0.9(0.91)}{18}=0.79.$$

The final score and ranking positions for all 25 equities considered in this case study, and based on type-1, type 2, and Z fuzzy number implementation of the proposed FN-TOPSIS method, are provided in Table VIII.

V. Analysis of Results

For the validation of the proposed rule-based FN-TOPSIS, the authors consider established TOPSIS methods, as the nonfuzzy TOPSIS [13] and the nonrule-based fuzzy TOPSIS approaches: T1-TOPSIS [15], T2-TOPSIS [18], and Z-TOPSIS [34]. All these methods are applied to evaluate the final ranking of the equities as shown in Table IX, based on case study in Section IV and compared with the performance of FN-TOPSIS. The actual monthly equity returns in November 2007, based on trading the shares of the 25 companies on the KLSE and holding for a month, are used for benchmarking. The rankings are compared using the Spearman rho correlation coefficient ρ, where ρ measures the strength of association between two ranked variables. This comparison approach is intuitively interpretable and less sensitive to bias due to the effect of outliers [50]. The Spearman's Rank coefficient is evaluated as

$$\rho = 1-\frac{6\sum \partial_i^2}{n^3-n} \tag{52}$$

where ∂_i represents the difference between the ranks, and n is the number of considered alternatives.

The coefficient ρ takes values between $+1$ and –1. Perfect positive relationship of ranks is indicated with $\rho = 1$, and $\rho = -1$ indicates perfect negative association of ranks, while $\rho = 0$ shows no relationship.

Considering the criteria set used, i.e., B1, B2, B3, B4, C1, and C2 of traded equity described in Section IV, the three proposed FN TOPSIS methods (PM) outperform the four established TOPSIS methods (EM), as shown in the last row of Table X.

VI. Conclusion

This paper introduces a novel TOPSIS method—FN-TOPSIS—extending the capabilities of rule-based FNs within MCDM analysis. FN-TOPSIS uses type-1, type-2, and Z-fuzzy numbers and incorporates experts' knowledge into decision analysis as well as experts' degree of experience and influence. At the same time, the approach improves transparency of decision analysis, particularly, in the TOPSIS process, by explicitly taking into account all subsystems and interactions among them. FN-TOPSIS not only provides an effective way to process imperfect information in decision-making practice in a more flexible and intelligent manner, but also presents expert knowledge more accurately. The performance of the proposed method is validated using a benchmark and comparing against a set of competitive approaches. The results show that the proposed method outperforms the existing nonrule-based TOPSIS methods in terms of ranking performance. We have successfully applied FN-TOPSIS to the problem of ranking equities traded in a developing financial market during a crisis period. This study continues research on hybrid approaches, implementing fuzzy set theory in equity ranking and investment decisions, in a developed market (U.K.) during a precrisis period [51]–[54]. The next objective is to implement and analyze the performance of the approach within developing and developed financial markets during a postcrisis period.

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