Multiobjective Investment Policy for a Nonlinear Stochastic Financial System: A Fuzzy Approach

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***Abstract*—The financial market always suffers from continuous and discontinuous (jump) changes and can be regarded as a nonlinear stochastic jump diffusion system. Most investors expect their investment policies to be not only higher benefits but also lower risk as a multiobjective optimization problem (MOP). In this study, a multiobjective H_2/H_∞ fuzzy investment is proposed for nonlinear stochastic jump diffusion financial systems to achieve the desired target with minimum investment cost and risk in Pareto optimal sense, simultaneously. The Takagi–Sugeno (T–S) fuzzy model is used to approximate the nonlinear stochastic jump diffusion financial system to simplify the multiobjective H_2/H_∞ investment policy design procedure. By the help of the T–S fuzzy model, the multiobjective H_2/H_∞ fuzzy investment policy problem of nonlinear stochastic financial system can be transformed to a linear-matrix-inequality-constrained (LMI-constrained) MOP to avoid solving the annoying Hamilton–Jacobi inequalities. Because the LMI-constrained MOP is not easy to directly calculate its Pareto optimal solutions, an indirect method is proposed to solve this MOP for the multiobjective H_2/H_∞ fuzzy investment policy design of nonlinear stochastic jump diffusion financial systems. An LMI-constrained multiobjective evolution algorithm (LMI-constrained MOEA) is also developed to efficiently solve the Pareto optimal solutions of the LMI-constrained MOP for the multiobjective H_2/H_∞ fuzzy investment policy design of nonlinear stochastic jump diffusion financial systems. When the Pareto optimal regulation solutions are solved by the proposed LMI-constrained MOEA, investors can select one investment policy to achieve their desired target with minimum investment cost and risk according to his/her own preference.**

***Index Terms*—Linear-matrix-inequality (LMI)-constrained multiobjective evolution algorithm (MOEA), multiobjective H_2/H_∞ fuzzy investment policy, nonlinear stochastic jump diffusion system, Pareto optimality, stochastic financial system, Takagi–Sugeno (T–S) fuzzy model.**

I. Introduction

In the fields of finance, stocks, and social economics, because of the interaction between nonlinear factors, all kinds of economic problems become more and more complicated today. Different types of mathematical model for the financial systems have been widely studied. In recent years, researchers focus on the system dynamical model to describe the real economic and financial systems [1]–[5]. However, in practical cases, the financial dynamic system is a nonlinear stochastic system and may suffer from continuous and discontinuous parametric fluctuations due to national and international situation change, oil price change, the surplus between investment and savings, the variable of interest rate, false economy policy, etc. Thus, using a nonlinear stochastic dynamical model to describe a real financial system would be more appealing [6]–[9]. The stochastic parametric fluctuations can be decomposed as continuous state-dependent fluctuations and discontinuous (jump) state-dependent fluctuations. Both of them will influence the stability of the stochastic financial dynamical system. Moreover, the external investment disturbance due to the unpredictable investment-environmental changes or a worldwide event, such as war, natural disaster, fatal epidemic disease, etc., can also affect the fluctuations of financial system. Thus, in this study, a nonlinear stochastic jump diffusion system with internal random fluctuation and external disturbance is employed to describe a nonlinear stochastic financial system.

Managers and investors always expect their investment policies to be with not only higher return on investment (ROI) but also lower risk. Because the ROI and lower investment risks normally conflict with each other, investment policy can be regarded as a multiobjective optimization problem (MOP). Thus, the multiobjective investment policies appear naturally and are also widely applied in financial systems [10]. For a financial system, the multiobjective H_2/H_∞ investment policy design for the nonlinear stochastic jump diffusion financial system can be seen as how to search a management policy to maximize ROI (H_2 management policy) and minimize investment risk (H_∞ management policy), simultaneously.

Robust regulation control is to design a state feedback controller to ensure that the trajectories of the controlled system gradually converge to some desired trajectories (target) despite intrinsic noise and extrinsic disturbance [11], [19], [24], [25], [33]. It is particularly appropriate to be employed in financial system. The most two important topics of robust optimal regulation control are how to minimize the regulation error to achieve the desired target despite intrinsic fluctuation and how to increase the system robustness to reject the effect due to environmental disturbance. The optimal H_2 management policy is proposed to minimize the regulation error to achieve the desired reference target despite intrinsic fluctuation [13]–[15]. The robust H_∞ management policy is introduced to reduce the effects of external disturbances on the regulation performance. For a financial system, the H_2 optimal regulation control can be regarded as how to design an investment policy to ensure

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that the financial system can gradually converge to the expectant trajectories of managers or investors and minimize the investment cost, i.e., a higher ROI policy. On the other hand, the robust H_∞ regulation control for a financial system can be seen as how to design a manageable policy to minimize the investment risk due to intrinsic continuous and discontinuous random fluctuations and external disturbance. However, both of optimal H_2 and robust H_∞ management policy need to solve a difficult Hamilton-Jacobi inequality (HJI) (a second-order nonlinear partial differential equation) to achieve a desired steady state (target) of nonlinear stochastic financial system. To overcome this difficult problem, i.e., solving HJI, the Takagi–Sugeno (T–S) fuzzy model is employed to simplify the multiobjective H_2/H_∞ investment policy problem of nonlinear stochastic financial systems in this study.

Recently, the T–S fuzzy model has been widely used to efficiently approximate nonlinear systems [17]–[20], and a T–S fuzzy controller has also been successfully applied to the stabilization control design for nonlinear system. To avoid solving the annoying HJIs, the T–S fuzzy model is employed to approximate the nonlinear stochastic jump diffusion financial system by interpolating several local linear stochastic jump diffusion financial systems at different operation points. When a nonlinear stochastic jump diffusion financial system was represented by an interpolation of a set of linear stochastic jump diffusion financial systems, a T–S fuzzy model-based regulation method can also be developed to achieve the multiobjective H_2/H_∞ investment policy. Thus, the HJIs can be replaced by a set of linear matrix inequalities (LMIs). These LMIs can be efficiently solved by a convex optimization method to complete the multiobjective H_2/H_∞ fuzzy investment policy design. Therefore, the multiobjective H_2/H_∞ investment policy problem of the nonlinear stochastic jump diffusion financial system becomes to how to solve an LMI-constrained MOP.

Although a number of multiobjective evolutionary algorithms (MOEAs) have been discussed for their ability to efficiently search the Pareto optimal solutions of MOP in a single run [27]–[30], [32], [41], most of them focus on the multiobjective problem with algebraic functional systems or algebraic constraints. Few of these studies discuss the system dynamical constrained MOP for the nonlinear stochastic financial system, in which robust stability must be guaranteed beforehand. More effort is still needed to apply the MOEA to solve the MOP for multiobjective H_2/H_∞ investment policy of nonlinear stochastic financial systems. Based on the T–S fuzzy model, the multiobjective H_2/H_∞ investment policy problem of the nonlinear stochastic jump diffusion financial system can be transformed to an LMI-constrained MOP. In general, there exists no unique solution for LMI-constrained MOP. The multiple solutions of MOP are called Pareto optimal solutions. To efficiently solve the multiobjective H_2/H_∞ investment policy problem of the nonlinear financial system, an indirect method is introduced to help us approach the Pareto optimal solutions at the Pareto front of the LMI-constrained MOP by gradually decreasing the upper bound of H_2 and H_∞ performance indices, i.e., nondominating searching scheme is employed to find Pareto front for Pareto optimal solutions of management policy without violating the stability of nonlinear stochastic system, simultaneously.

The proposed LMI-constrained MOEA combines the LMI technique with MOEA together to solve the LMI-constrained MOP for multiobjective H_2/H_∞ investment policy of nonlinear stochastic jump diffusion financial systems. The LMI-constrained MOEA search algorithm via a number of genetic operation, e.g., selection, mutation, and crossover, is proposed with the help of LMI toolbox in MATLAB to efficiently search for Pareto front of Pareto optimal solutions and its corresponding fuzzy regulation gains $\{K_i\}$ from the LMI constraints to meet the robust stability and multiobjective H_2/H_∞ policy optimization of nonlinear stochastic jump diffusion financial systems, simultaneously. As long as the Pareto front can be obtained, the multiobjective H_2/H_∞ investment policy problem for a nonlinear stochastic jump diffusion financial system can also be solved, and the manager(or investor) can select a mutual benefit policy according to his/her own preference.

Notation:

A^T: the transpose of matrix A; $A \geq 0$ $(A > 0)$: symmetric positive semidefinite (symmetric positive definite) matrix A; $I_{n\times n}$: n-dimentional identity matrix; $\|\boldsymbol{x}\|_2$: the Euclidean norm for the given vector $\boldsymbol{x} \in \mathbb{R}^n$; C^2: the class of functions $V(\boldsymbol{x})$ twice continuously differential with respect to $\boldsymbol{x}$; $f_{\boldsymbol{x}}$: the gradient column vector of n_x-dimensional twice continuously differentiable function $\boldsymbol{f}(\boldsymbol{x})$ (i.e., $\frac{\partial f(\boldsymbol{x})}{\partial \boldsymbol{x}}$); $f_{\boldsymbol{xx}}$: the Hessian matrix with elements of second partial derivatives of n_x-dimensional twice continuously differentiable function $\boldsymbol{f}(\boldsymbol{x})$, (i.e., $\frac{\partial^2 f(\boldsymbol{x})}{\partial \boldsymbol{x}^2}$); $\mathcal{L}^2_{\mathcal{F}}(\mathbb{R}^+, \mathbb{R}^{n_y})$: the space of nonanticipative stochastic processes $y(t) \in \mathbb{R}^l$ with respect to an increasing σ-algebras $\mathcal{F}_t (t \geq 0)$ satisfying $\|\boldsymbol{y}(t)\|_{\mathcal{L}^2(\mathbb{R}^+;\mathbb{R}^{n}\boldsymbol{y})} \triangleq \boldsymbol{E}\left\{\int_0^\infty \boldsymbol{y}^T(t)\boldsymbol{y}(t)dt\right\}^{\frac{1}{2}}$; $\|\boldsymbol{y}(t)\|_{\mathcal{L}^2(\mathbb{R}^+;\mathbb{R}^{n}\boldsymbol{y},Q)} \triangleq \boldsymbol{E}\left\{\int_0^\infty \boldsymbol{y}^T(t)Q\boldsymbol{y}(t)dt\right\}^{\frac{1}{2}}$; $\mathcal{B}(\Theta)$ is the Borel algebra generated by Θ; $\boldsymbol{E}$: the expectation operator; $\bar{\tau}(M)$: the maximum eigenvalue of real-value matrix M; ϕ: the empty set.

II. Preliminaries

In [1] and [5], a financial dynamical model is given to illustrate the interaction between interesting rate, x, investment demand, y, and the price index, z of a financial system as follows:

$$\begin{cases} \dot{x}(t) = z(t) + (y(t) - a)\,x(t) \\ \dot{y}(t) = 1 - by(t) - (x(t))^2 \\ \dot{z}(t) = -x(t) - cz(t) \end{cases} \tag{1}$$

where the parameters $a \geq 0$ is the saving amount, $b \geq 0$ is the per-investment cost, and $c \geq 0$ is the elasticity of demands of commercials. In the nonlinear financial model (1), the interest rate $x(t)$ can be influenced by the surplus between investment and saving as well as adjustments of the prices. The investment demand $y(t)$ is proportional to the rate of investment and inversely proportional to the cost of investment and the interest rate. The price index $z(t)$ depends on the difference between supply and demand in the market, and it is also influenced by the inflation rate.

However, in practice, most financial systems are stochastic systems, and the dynamical model in (1) may suffer from parametric fluctuations due to the surplus between investment and savings, oil price, or government policy changes. To mimic the real financial system, the dynamical model in (1) should be modified by continuous and discontinuous intrinsic random

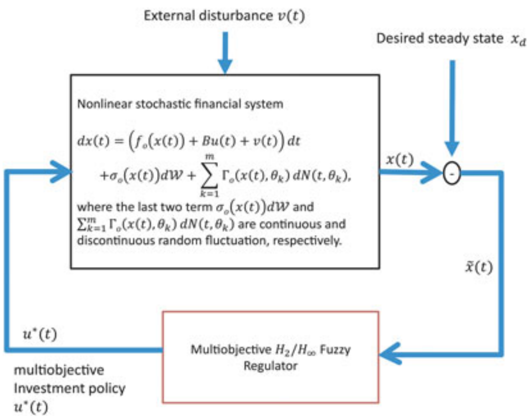

Fig. 1. System diagram for the multiobjective investment policy problem of the nonlinear financial system. The nonlinear stochastic system in (3) is to be controlled to achieve the desired steady state $\boldsymbol{x}_d$ by the multiobjective H_2/H_∞ regulator. In order to achieve the desired steady state $\boldsymbol{x}_d$, the original nonlinear stochastic financial system in (3) is shifted to $\boldsymbol{x}_d$ so that the regulation problem to $\mathbf{x}_d$ becomes a stabilization problem of the nonlinear stochastic system in (5). When the shifted stochastic financial system in (5) is completely approximated by the T–S fuzzy model, the multiobjective H_2/H_∞ investment policy problem in (18) can be regarded as an LMI-constrained MOP in (20) by employing the proposed indirect method in Lemma 2. Since the LMI-constrained MOP in (20) is difficult to solve by directly calculating, we proposed the LMI-constrained MOEA to efficiently solve the LMI-constrained MOP in (20). Thus, the multiobjective H_2/H_∞ investment policy problem for a nonlinear stochastic financial system can be efficiently solved by commercial software MATLAB.

fluctuations and environmental disturbance. In the following, the Wiener process (i.e., Brownian motion) and the marked Poisson process are introduced to model the continuous and discontinuous random fluctuations in real financial system (see Fig. 1). More precisely speaking, a financial market, which refers to the buy and sell behavior, can be described as a stochastic dynamical financial system. Thus, the stochastic dynamical system can be employed to describe the price changes of the objects of transaction (like stocks, bonds, futures, and oil) or economic index changes (investment rate, price index, and investment demand) [36], [38]. One of the most famous stochastic financial systems is the stochastic stock price system [36].

Let $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathcal{P})$ be a filtration probability space with $\mathcal{F} = \{\mathcal{F}_t : t \geq 0\}$. $\mathcal{F}$ is generated by the following two mutually independent stochastic processes: 1) One-dimensional standard Wiener process $\mathcal{W}(t)$; and 2) the marked Poisson processes $\boldsymbol{N}(t; \theta_k)$, for $k = 1, 2, .., m$, and the filtration $\mathcal{F}_t$ is also generated by σ-algebra of $\mathcal{W}(s)$ and σ-algebra of marked Poisson processes $\boldsymbol{N}(s; \theta_k)$, where $k = 1, 2, .., m$, for $s < t$.

The nonlinear stochastic jump diffusion financial system with continuous and discontinuous random fluctuations in (1) can be described as follows:

$$\begin{cases} dx(t) = (z(t) + (y(t) - a)\, x(t))\, dt + \sigma_1 (x(t), y(t), z(t)) \\ \times\, d\mathcal{W} + \sum_{k=1}^{m} \gamma_1 (x(t), y(t), z(t), \theta_k)\, d\boldsymbol{N}(t; \theta_k) \\ dy(t) = \left(1 - by(t) - (x(t))^2\right) dt + \sigma_2 (x(t), y(t), z(t)) \\ \times\, d\mathcal{W} + \sum_{k=1}^{m} \gamma_2 (x(t), y(t), z(t), \theta_k)\, d\boldsymbol{N}(t; \theta_k) \\ dz(t) = (-x(t) - cz(t))\, dt + \sigma_3 (x(t), y(t), z(t))\, d\mathcal{W} \\ +\, \sum_{k=1}^{m} \gamma_3 (x(t), y(t), z(t), \theta_k)\, d\boldsymbol{N}(t; \theta_k)\, . \end{cases} \tag{2}$$

Thus, the stochastic nonlinear autonomous controlled system for the financial system in (2) can be written as follows (see system diagram in Fig. 1):

$$\begin{aligned} d\boldsymbol{x}(t) = &\ (f_o(\boldsymbol{x}(t)) + B\boldsymbol{u}(t) + \boldsymbol{v}(t))\, dt + \sigma_o(\boldsymbol{x}(t))\, d\mathcal{W} \\ &+ \sum_{k=1}^{m} \Gamma_o(\boldsymbol{x}(t), \theta_k)\, d\boldsymbol{N}(t; \theta_k) \end{aligned} \tag{3}$$

with

$$\begin{aligned} &\boldsymbol{x}(t) = [x(t), y(t), z(t)]^T, \boldsymbol{u}(t) = [u_1(t), u_2(t), u_3(t)]^T \\ &\boldsymbol{v}(t) = [v_1(t), v_2(t), v_3(t)]^T \\ &f_1(\boldsymbol{x}(t)) = (z(t) + (y(t) - a)\, x(t)) \\ &f_2(\boldsymbol{x}(t)) = 1 - by(t) - (x(t))^2 \\ &f_3(\boldsymbol{x}(t)) = (-x(t) - cz(t)) \\ &f_o(\boldsymbol{x}(t)) = [f_1(\boldsymbol{x}(t)), f_2(\boldsymbol{x}(t)),\ f_3(\boldsymbol{x}(t))]^T \\ &\sigma_o(\boldsymbol{x}(t)) = [\sigma_1(\boldsymbol{x}(t)), \sigma_2(\boldsymbol{x}(t)), \sigma_3(\boldsymbol{x}(t))]^T \\ &\Gamma_o(\boldsymbol{x}(t), \theta_k) = [\gamma_1(\boldsymbol{x}(t), \theta_k), \gamma_2(\boldsymbol{x}(t), \theta_k), \gamma_3(\boldsymbol{x}(t), \theta_k)]^T \end{aligned}$$

where f_o: $\mathbb{R}^3 \to \mathbb{R}^3$, σ_o: $\mathbb{R}^3 \to \mathbb{R}^3$, and Γ_o: $\mathbb{R}^3 \times \Theta \to \mathbb{R}^3$ are nonlinear Borel measurable continuous functions, which are satisfied with Lipschitz continuity. B is a 3×3 real-value constant matrix. The càdlàg process $\boldsymbol{x}(t) \in \mathbb{R}^3$ is the state vector; the initial state vector $\boldsymbol{x}(\mathbf{0}) = \boldsymbol{x}_0$; the input vector $\boldsymbol{u}(t) \in \mathcal{L}^2_{\mathcal{F}}(\mathbb{R}^+; \mathbb{R}^3)$ is the admissible regulation effort (i.e., investment policy) with respect to $\{\mathcal{F}_t\}_{t \geq 0}$; $\boldsymbol{v}(t) \in \mathcal{L}^2_{\mathcal{F}}(\mathbb{R}^+; \mathbb{R}^3)$ is regarded as an unknown finite energy stochastic external disturbance and denotes the external disturbance caused by the international situation like war or natural disaster. Since 1-D standard Wiener process $\mathcal{W}(t)$ is a continuous but nondifferentiable stochastic process, the term $\sigma_o(\boldsymbol{x}(t))d\mathcal{W}(t)$ can be regarded as a continuous state-dependent internal random fluctuation, i.e., the financial system random fluctuation dependent on the current magnitude of investment rate, price index, and investment demand; the term $\Theta = \{\theta_1, \theta_2, ..., \theta_m\} \subset \mathbb{R}^1$ is the mark space of Poisson random processes and denotes the set of all minding financial emergencial incidents, which will cause discontinuous changes of the system in (3); the mark θ_k denotes a financial emergencial incident in the discussed financial system such as financial tsunami, company merger, hot money inflow or the collapse of bank; and the term $\Gamma_o(\boldsymbol{x}(t), \theta_k)$ denotes a suddenly violent change (the magnitude jump) at investment rate, price index, and investment demand of the stochastic financial system in (3) when the financial emergencial incident happens. Thus, the summation term $\sum_{k=1}^{m} \Gamma_o(\boldsymbol{x}(t), \theta_k)\, d\boldsymbol{N}(t; \theta_k)$ denotes all the possible nonlinear stochastic discontinuous changes of the system in (3) at the time instant t, whose jump amplitudes depend on the corresponding jump coefficient functional $\Gamma_o(\boldsymbol{x}(t), \theta_k)$. It is necessary to note that in this paper, we assume all the emergencial incidents are mutually exclusive event (i.e., when the jump occurs at time instant t, there is only one mark θ_k to be assigned). In this study, we use 3-D nonlinear financial system (1) to illustrate the financial multiobjective design. However, the financial stochastic system in (2) or (3) can be extended to any

n-dimensional nonlinear stochastic financial system. At this situation, $\boldsymbol{x}(t) = (x_1(t), x_2(t), ..., x_n(t)) \in \mathbb{R}^n$. At this case, we have $f_o : \mathbb{R}^n \to \mathbb{R}^n$, $\sigma_o : \mathbb{R}^n \to \mathbb{R}^n$, $\Gamma_o : \mathbb{R}^n \times \Theta \to \mathbb{R}^n$ and $B_o \in \mathbb{R}^{n\times p}$ when $\boldsymbol{u}(t) \in \mathcal{L}^2_{\mathcal{F}}(\mathbb{R}^+; \mathbb{R}^p)$.

Some important properties of Poisson jump process are given as follows [7]:

1) $\boldsymbol{E}\{d\boldsymbol{N}(t;\theta_k)\} = \lambda_k dt$ where the finite scalar number $\lambda_k > 0$ is the Poisson jump intensity for mark θ_k.
2) $\boldsymbol{E}\{[d\boldsymbol{N}(t;\theta_k)]\, dt\} = 0$, for all k.
3) $\boldsymbol{E}\{[d\boldsymbol{N}(t;\theta_k)]\, d\mathcal{W}\} = 0$, for all k.
4) $\boldsymbol{E}\{[d\boldsymbol{N}(t;\theta_{k_1})]\,[d\boldsymbol{N}(t;\theta_{k_2})]\} = 0$, for all $k_1 \neq k_2$.
5) $\boldsymbol{E}\{[d\boldsymbol{N}(t;\theta_{k_1})]\,[d\boldsymbol{N}(t;\theta_{k_2})]\} = \lambda_{k_1} dt$, for all $k_1 = k_2$.

In fact, managers or investors always expect that the dynamic behaviors of the invested financial system are satisfied with their expectation. Thus, the multiobjective optimal control theories are particularly suited for the multiobjective investment policy problem of financial systems.

From the system diagram in Fig. 1, the regulation purpose of stochastic financial system in (3) is to design regulation effort $\boldsymbol{u}(t)$ so that 1) the intrinsic random continuous and discontinuous fluctuations $\sigma_o(\boldsymbol{x}(t))\, dW$ and $\sum_{k=1}^{m} \Gamma_o(\boldsymbol{x}(t), \theta_k) d\boldsymbol{N}(t;\theta_k)$ could be tolerated by the stochastic financial system; 2) the effect of external disturbance $\boldsymbol{v}(t)$ on the regulation performance of financial system should be as small as possible, i.e., with a less risk; and 3) the desired target $\boldsymbol{x}_d$ could be finally achieved with a low cost.

In order to regulate $\boldsymbol{x}(t)$ to the desired steady state (target) $\boldsymbol{x}_d$, for the convenience of control design, the origin of nonlinear stochastic financial system in (3) should be shifted to $\boldsymbol{x}_d$ at first. In such a situation, if the shifted nonlinear stochastic financial system is robustly stabilized at the origin, then the robust regulation of $\boldsymbol{x}(t)$ to the desired state(target) $\boldsymbol{x}_d$ will be equivalently achieved. This will simplify the regulation control design procedure of the stochastic nonlinear financial system. Let us denote

$$\tilde{\boldsymbol{x}}(t) = \boldsymbol{x}(t) - \boldsymbol{x}_d. \tag{4}$$

Then, we get the following shifted nonlinear stochastic financial system in (3) as follows:

$$\begin{aligned} d\tilde{\boldsymbol{x}}(t) = &\ (f(\tilde{\boldsymbol{x}}(t)) + B\boldsymbol{u}(t) + \boldsymbol{v}(t))\, dt + \sigma(\tilde{\boldsymbol{x}}(t))\, d\mathcal{W} \\ &+ \sum_{k=1}^{m} \Gamma(\tilde{\boldsymbol{x}}(t), \theta_k)\, d\boldsymbol{N}(t;\theta_k) \end{aligned} \tag{5}$$

where $f(\tilde{\boldsymbol{x}}(t)) = f_o(\tilde{\boldsymbol{x}}(t) + x_d)$, $\sigma(\tilde{\boldsymbol{x}}(t)) = \sigma_o(\tilde{\boldsymbol{x}}(t) + x_d)$, and $\Gamma(\tilde{\boldsymbol{x}}(t), \theta_k) = \Gamma_o(\tilde{\boldsymbol{x}}(t) + x_d, \theta_k)$.

Thus, the origin $\tilde{\boldsymbol{x}}(t) = 0$ of the nonlinear stochastic financial system in (5) is at the desired steady state (target) $\boldsymbol{x}_d$ of the original nonlinear stochastic financial system in (3), i.e., the investment policy problem of regulating nonlinear stochastic financial system in (3) to the desired $\boldsymbol{x}_d$ is transformed to the stabilization problem of the shifted nonlinear stochastic financial system in (5).

In order to achieve the desired state $\boldsymbol{x}_d$ with less regulation effort $\boldsymbol{u}(t)$ in spite of continuous and discontinuous random fluctuation, the H_2 control performance $J_2(\boldsymbol{u}(t))$ index for the nonlinear stochastic jump diffusion system in (5) without consideration of $\boldsymbol{v}(t)$ is defined as follows:

$$J_2(\boldsymbol{u}(t)) = \|\tilde{\boldsymbol{x}}(t)\|^2_{\mathcal{L}^2_{\mathcal{F}}(\mathbb{R}^+;\mathbb{R}^3, Q_1)} + \|\boldsymbol{u}(t)\|^2_{\mathcal{L}^2_{\mathcal{F}}(\mathbb{R}^+;\mathbb{R}^3, R_1)} \tag{6}$$

where $Q_1 > 0$ and $R_1 > 0$ are weighting matrices to tradeoff between regulation error $\tilde{\boldsymbol{x}}(t)$ and investment effort $\boldsymbol{u}(t)$.

To avoid the investment risk from the external disturbance, the H_∞ control performance index $J_\infty(\boldsymbol{u}(t))$ of the nonlinear stochastic financial system in (5) is defined as follows:

$$\begin{aligned} J_\infty(\boldsymbol{u}(t)) = &\sup_{\substack{\boldsymbol{v}(t) \in \mathcal{L}^2_{\mathcal{F}}(\mathbb{R}_+, \mathbb{R}^3), \\ \boldsymbol{v} \neq 0, \boldsymbol{x}_0 = 0}} \\ &\frac{\|\tilde{\boldsymbol{x}}(t)\|^2_{\mathcal{L}^2_{\mathcal{F}}(\mathbb{R}^+;\mathbb{R}^3, Q_2)} + \|\boldsymbol{u}(t)\|^2_{\mathcal{L}^2_{\mathcal{F}}(\mathbb{R}^+;\mathbb{R}^3, R_2)}}{\|\boldsymbol{v}(t)\|^2_{\mathcal{L}^2_{\mathcal{F}}(\mathbb{R}^+;\mathbb{R}^3)}} \\ &\text{if } \tilde{\boldsymbol{x}}(0) = \tilde{\boldsymbol{x}}_0 = 0 \end{aligned} \tag{7}$$

where $Q_2 > 0$, $R_2 > 0$, i.e., the worst-case effect from the exogenous disturbance $\boldsymbol{v}(t) \in \mathcal{L}^2_{\mathcal{F}}(R_+; R^3)$ to the regulation errors $\tilde{\boldsymbol{x}}(t)$ and regulation effort $\boldsymbol{u}(t)$ from the average energy point of view. The H_∞ performance index $J_\infty(\boldsymbol{u}(t))$ in (7) can be considered as the worst-case investment risk from external disturbance. Since the external disturbance is unpredictable, the worst-case effect of external disturbance is considered in the investment risk.

In this study, from the system diagram of nonlinear stochastic financial system in Fig. 1, we want to design a multiobjective H_2/H_∞ robust investment policy for system state $\boldsymbol{x}(t)$ to achieve a desired state $\boldsymbol{x}_d$ from the perspective of minimum H_2 regulation error with minimum investment effort $\boldsymbol{u}(t)$ in (6) and minimum H_∞ investment risk in (7) under intrinsic fluctuations and external disturbance. A more detailed design procedure of the multiobjective H_2/H_∞ robust investment policy $\boldsymbol{u}(t)$ to simultaneously minimize $J_2(\boldsymbol{u}(t))$ in (6) and $J_\infty(\boldsymbol{u}(t))$ in (7) will be discussed in the following.

Remark 1: If the initial condition $\tilde{\boldsymbol{x}}(0) = \tilde{\boldsymbol{x}}_0 \neq 0$, then the H_∞ performance index $J_\infty(\boldsymbol{u}(t))$ should be rewritten as

$$\begin{aligned} J_\infty(\boldsymbol{u}(t)) = &\sup_{\substack{v(t) \in \mathcal{L}^2_{\mathcal{F}}(R_+, \mathbb{R}^3), \\ v \neq 0, x_0 = 0}} \\ &\frac{\|\tilde{\boldsymbol{x}}(t)\|^2_{\mathcal{L}^2_{\mathcal{F}}(R^+;\mathbb{R}^3, Q_2)} + \|\boldsymbol{u}(t)\|^2_{\mathcal{L}^2_{\mathcal{F}}(R^+;\mathbb{R}^3, R_2)} - E\{V(\boldsymbol{x}_0)\}}{\|\boldsymbol{v}(t)\|^2_{\mathcal{L}^2_{\mathcal{F}}(\mathbb{R}^+;\mathbb{R}^3)}} \end{aligned} \tag{8}$$

where $V(\cdot) \in C^2(\mathbb{R}^3)$ and $V(\cdot) \geq 0$, i.e., the effect of initial condition $\tilde{\boldsymbol{x}}_0$ should be deleted in order to obtain the real effect of $\boldsymbol{v}(t)$ on the controlled output.

Lemma 1 (see [7]): Let $V : \mathbb{R}^3 \to \mathbb{R}$, $V(\cdot) \in C^2(\mathbb{R}^3)$ and $V(\cdot) \geq 0$. For the nonlinear stochastic jump diffusion system in (5), the Itô–Lévy formula of $V(\tilde{\boldsymbol{x}}(t))$ is given as follows:

$$dV(\tilde{\boldsymbol{x}}(t)) = \Big[V_{\tilde{\boldsymbol{x}}}^T f(\tilde{\boldsymbol{x}}(t)) + V_{\tilde{\boldsymbol{x}}}^T B\boldsymbol{u}(t) + V_{\tilde{\boldsymbol{x}}}^T \boldsymbol{v}(t) + \frac{1}{2}\boldsymbol{\sigma}^T(\tilde{\boldsymbol{x}}(t)) V_{\tilde{\boldsymbol{x}}\tilde{\boldsymbol{x}}}\boldsymbol{\sigma}(\tilde{\boldsymbol{x}}(t))\Big] dt + V_{\tilde{\boldsymbol{x}}}^T \sigma(\tilde{\boldsymbol{x}}(t)) d\mathcal{W}(t) + \sum_{k=1}^{m} \{V(\tilde{\boldsymbol{x}}(t) + \Gamma(\tilde{\boldsymbol{x}}(t), \theta_k)) - V(\tilde{\boldsymbol{x}}(t))\} d\boldsymbol{N}(t;\theta_k). \quad (9)$$

III. Multiobjective H_2/H_∞ Investment Policy Design for Nonlinear Stochastic Financial Jump System via Fuzzy Interpolation Method

In general, construction of the multiobjective H_2/H_∞ regulation control for nonlinear stochastic system needs to solve an HJI-constrained MOP, which is difficult to be solved analytically or numerically. To overcome this difficult problem, the T–S fuzzy interpolation method is employed here to approximate the nonlinear stochastic jump diffusion financial system in (5). The T–S fuzzy dynamical model is proposed to approximate the nonlinear stochastic system in (5) by interpolating several local linearized stochastic jump diffusion financial systems around some operation points [17]–[20], [34], [39], [40]. This T–S fuzzified model is described by a group of IF–THEN rules and is used to simplify the multiobjective H_2/H_∞ investment policy design problem of the nonlinear stochastic jump diffusion financial system. The ith rule of this T–S fuzzy model for the nonlinear stochastic jump diffusion financial system in (5) is described by

System Rule :

for $i = 1, 2, \ldots l$

if $\boldsymbol{z}_1$ is G_{i1} and $\cdots$ and $\boldsymbol{z}_g$ is G_{ig}

then

$$d\tilde{\boldsymbol{x}}(t) = [A_i\tilde{\boldsymbol{x}}(t) + B\boldsymbol{u}(t) + \boldsymbol{v}(t)] dt + C_i\tilde{\boldsymbol{x}}(t) d\mathcal{W}(t) + \sum_{k=1}^{m} E_i(\theta_k)\tilde{\boldsymbol{x}}(t) d\boldsymbol{N}(t;\theta_k) \quad (10)$$

where l is the number of fuzzy rules, G_{ij} is the fuzzy set, the matrices $A_i, B, C_i \in \mathbb{R}^{3\times 3}$ are constant matrices, $E_i(\theta_k)$ is a constant matrix for $k = 1, 2, ..., m$, and $\boldsymbol{z}_1 \cdots \boldsymbol{z}_g$ are premise variables.

The overall fuzzy system in (10) can be inferred as follows [18]:

$$d\tilde{\boldsymbol{x}}(t) = \sum_{i=1}^{l} h_i(\boldsymbol{z}) \Big\{[A_i\tilde{\boldsymbol{x}}(t) + B\boldsymbol{u}(t) + \boldsymbol{v}(t)] dt + C_i\tilde{\boldsymbol{x}}(t) d\mathcal{W}(t) + \sum_{k=1}^{m} E_i(\theta_k)\tilde{\boldsymbol{x}}(t) d\boldsymbol{N}(t;\theta_k)\Big\} \quad (11)$$

where $\boldsymbol{z} = [\boldsymbol{z}_1^T, \boldsymbol{z}_2^T \cdots, \boldsymbol{z}_g^T]^T$

$$\mu_i(\boldsymbol{z}) = \prod_{j=1}^{g} G_{ij}(\boldsymbol{z}_j) \geq 0, \; h_i(\boldsymbol{z}) = \frac{\mu_i(\boldsymbol{z})}{\sum_{i=1}^{l} \mu_i(\boldsymbol{z})} \geq 0 \quad (12)$$

and $G_{ij}(\boldsymbol{z}_j)$ is the membership grade of $\boldsymbol{z}_j$ in G_{ij}. From the definitions given above, we have

$$\sum_{i=1}^{l} h_i(\boldsymbol{z}) = 1. \quad (13)$$

The physical meaning of fuzzy model in (11) is that the following l locally linearized financial systems

$$d\tilde{\boldsymbol{x}}(t) = [A_i\tilde{\boldsymbol{x}}(t) + B\boldsymbol{u}(t) + \boldsymbol{v}(t)] dt + C_i\tilde{\boldsymbol{x}}(t) d\mathcal{W}(t) + \sum_{k=1}^{m} E_i(\theta_k)\tilde{\boldsymbol{x}}(t) d\boldsymbol{N}(t;\theta_k), \quad \text{for } 1 \leq i \leq l \quad (14)$$

at different operation points (different fuzzy set G_{ij}) are interpolated smoothly via the fuzzy certainty function $h_i(z)$ to approximate the original nonlinear stochastic jump diffusion financial system in (5).

Similarly, the multiobjective H_2/H_∞ investment policy $\boldsymbol{u}(t) = K(\tilde{\boldsymbol{x}}(t))$ for the nonlinear stochastic jump diffusion financial system could be approximated by the following fuzzy investment policy:

Investment Policy i :

for $i = 1, 2, \ldots l$

if $\boldsymbol{z}_1$ is G_{i1} and $\cdots$ and $\boldsymbol{z}_g$ is G_{ig}, then $\boldsymbol{u}(t) = K_i\tilde{\boldsymbol{x}}(t)$. (15)

The overall multiobjective H_2/H_∞ investment policy $\boldsymbol{u}(t)$ for the nonlinear stochastic jump diffusion financial system can be represented by

$$\boldsymbol{u}(t) = \sum_{i=1}^{l} h_i(\boldsymbol{z}) K_i \tilde{\boldsymbol{x}}(t) \quad (16)$$

where $h_i(\boldsymbol{z})$ is designed as (11); K_i is the regulation gain for the ith fuzzy linearized system for $i = 1, 2, \ldots l$.

The fuzzy parameters A_i, C_i, and $E_i(\theta_k)$ of l local linear financial systems in (11) can be easily identified by system identification toolbox in MATLAB. If adequate number of fuzzy rules is used in the T–S fuzzy system in (11), then the fuzzy approximation error could be considered as one kind of parametric fluctuations, and the proposed robust H_2/H_∞ multiobjective investment policy can efficiently override these parametric fluctuations due to fuzzy approximation. Therefore, the multiobjective H_2/H_∞ investment policy problem can be regarded as how to design fuzzy investment policy $\boldsymbol{u}(t)$ in (16) so that the T–S stochastic financial system in (11) could achieve the multiobjective H_2/H_∞ investment policy in (6) and (7).

A higher benefit investment behavior always couples with higher risk and higher capital cost. The optimal H_2 regulation in (6) expects the energy of regulation error $\tilde{\boldsymbol{x}}(t)$ and admissible investment policy $\boldsymbol{u}(t)$ as lower as possible, i.e., use a minimum capital to earn a desired benefit. However, for the robust H_∞ regulation design, a better (less) H_∞ performance index is always coupled with a higher capital cost to reject intrinsic continuous and discontinuous fluctuation as well as external disturbance to achieve the desired target $\boldsymbol{x}_d$ with a less risk. It is clear that the optimal H_2 regulation design and robust H_∞ regulation design are mutually conflicted. Thus, designing a controller (investment policy) to optimize the H_2 performance index $J_2(\boldsymbol{u}(t))$ and the H_∞ performance index $J_\infty(\boldsymbol{u}(t))$ is indeed a MOP of financial regulation systems. The definition of the multiobjective H_2/H_∞ investment policy for the nonlinear stochastic jump diffusion financial system is given as

Definition 1: The multiobjective H_2/H_∞ investment policy of a given nonlinear stochastic Poisson jump diffusion financial system (11) is to design an admissible investment policy $\boldsymbol{u}(t)$ in (16), which could make the H_2 and H_∞ performance indices minimum in the Pareto optimal sense, simultaneously, i.e.

$$\begin{aligned}&\min_{\boldsymbol{u}(t)\in\mathcal{U}} \left(J_2(\boldsymbol{u}(t)), J_\infty(\boldsymbol{u}(t))\right)\\ &s.t.\ (11)\end{aligned} \tag{17}$$

where $\mathcal{U}$ is the set of all the admissible investment policy for the given nonlinear stochastic jump diffusion financial system; the objective functional $J_2(\boldsymbol{u}(t))$ and $J_\infty(\boldsymbol{u}(t))$ are defined in (6) and (7), respectively; the vector of the objective functionals $(J_2(\boldsymbol{u}(t)), J_\infty(\boldsymbol{u}(t)))$ is called objective vector of $\boldsymbol{u}(t)$.

Lemma 2 (see [35]): Suppose α and β are the upper bounds of the H_2 and H_∞ performance indices, respectively, i.e., $J_2(\boldsymbol{u}(t)) \le \alpha$, and $J_\infty(\boldsymbol{u}(t)) \le \beta$. The MOP in (17) is equivalent to the MOP given in the following:

$$\begin{aligned}&\min_{\boldsymbol{u}(t)\in\mathcal{U}} (\alpha, \beta)\\ &s.t.\ J_2(\boldsymbol{u}(t)) \le \alpha \text{ and } J_\infty(\boldsymbol{u}(t)) \le \beta.\end{aligned} \tag{18}$$

Proof: The proof of this lemma is straightforward. One only needs to prove that both inequalities contained in the multiobjective problem in (18) become equal for Pareto optimal solutions. We will show this by contradiction. Given a 3-tuple Pareto optimal solution $(\boldsymbol{u}^*(t), \alpha^*, \beta^*)$ of MOP in (18), we assume that either one of the inequality in (18) remains a strict inequality at the Pareto optimal solution. Without loss of generality, suppose that $J_2(\boldsymbol{u}(t)) < \alpha^*$. As the result, there exists α_1 such that $\alpha_1 < \alpha^*$ and $J_2(\boldsymbol{u}(t)) = \alpha_1$. Now, for the same $\boldsymbol{u}(t)$, the solution (α, β^*) dominates the Pareto optimal solution (α^*, β^*), leading to a contradiction. This implies that both inequality constraints in MOP (18) indeed become equality for Pareto optimal solutions. The optimization problem in (18) is, hence, equivalent to the MOP described in (17).

In this study, Lemma 2 provides an indirect method to solve the multiobjective H_2/H_∞ investment policy problem of nonlinear stochastic jump diffusion financial system.

Lemma 3 (see [22]): For any two real matrices A, B with appropriate dimension, we have

$$A^T B + B^T A \le \gamma^2 A^T A + \frac{1}{\gamma^2} B^T B$$

where γ is any nonzero real number.

Lemma 4 (see [18]): For any matrix M_i with appropriate dimension and the scheduling functions $h_i(\boldsymbol{z})$ with $0 \le h_i(\boldsymbol{z}) \le 1$, for $i \in \mathbb{N}^+$, $1 \le i \le m$, $P > 0$, and $\sum_{i=1}^m h_i(\boldsymbol{z}) = 1$, we have

$$\left(\sum_{j=1}^{l} h_j(\boldsymbol{z}) M_j\right)^T P \left(\sum_{i=1}^{l} h_i(\boldsymbol{z}) M_i\right) \le \sum_{i=1}^{l} h_i(\boldsymbol{z}) M_i^T P M_i.$$

Then, the following theorems will provide the sufficient condition for the fuzzy investment policy $\boldsymbol{u}(t)$ in (16) to solve the multiobjective H_2/H_∞ investment policy problem for nonlinear stochastic jump financial system in (5). In the following, according to two kinds of Poisson noise in stochastic financial systems, i.e., marked Poisson process $\boldsymbol{N}(t;\theta_k)$ and marked compensation Poisson processes $\hat{\boldsymbol{N}}(t;\theta_k)$, the multiobjective H_2/H_∞ investment policy problem will be solved separately in the following.

A. Multiobjective H_2/H_∞ Investment Policy Problem for the Nonlinear Stochastic Jump Diffusion Financial System Driven by the Marked Poisson Process $\boldsymbol{N}(t;\theta_k)$

Theorem 1: If the following LMIs-constrained MOP can be solved:

$$\min_{\{P,,K_1,K_2\ldots,K_l\}} (\alpha, \beta)$$

$$\text{s.t. the following LMIs, for all } i,j = 1,2,...,l, \tag{19}$$

$$P \le \alpha \left[Tr\left(R_{\tilde{\boldsymbol{x}}_0}\right)\right]^{-1} I \tag{20}$$

$$\begin{bmatrix} \Psi_{ij}^2 & W & Y_j^T & W C_i^T & W E_i^T(\theta_1) & \ldots & W E_i^T(\theta_m) \\ \star & -Q_1^{-1} & 0 & 0 & 0 & 0 & 0 \\ \star & \star & -R_1^{-1} & 0 & 0 & 0 & 0 \\ \star & \star & \star & -W & 0 & 0 & \vdots \\ \star & \star & \star & \star & -\lambda_1^{-1} W & 0 & 0 \\ \star & \star & \star & \star & \star & \ddots & 0 \\ \star & \star & \star & \star & \star & \star & -\lambda_m^{-1} W \end{bmatrix} \le 0 \tag{21}$$

$$\begin{bmatrix} \Psi_{ij}^{\infty} & W & Y_j^T & WC_i^T & WE_i^T(\theta_1) & \dots & WE_i^T(\theta_m) \\ \star & -Q_2^{-1} & 0 & 0 & 0 & 0 & 0 \\ \star & \star & -R_2^{-1} & 0 & 0 & 0 & 0 \\ \star & \star & \star & -W & 0 & 0 & \vdots \\ \star & \star & \star & \star & -\lambda_1^{-1}W & 0 & 0 \\ \star & \star & \star & \star & \star & \ddots & 0 \\ \star & \star & \star & \star & \star & \star & -\lambda_m^{-1}W \end{bmatrix} \leq 0 \tag{22}$$

where $W = P^{-1}$, $Y_j = K_j W$, $\Psi_{ij}^2 = A_i W + W A_i^T + BY_j + Y_j^T B^T + \sum_{k=1}^m \lambda_k [W E_i^T(\theta_k) + E_i(\theta_k) W]$, $\Psi_{ij}^{\infty} = A_i W + W A_i^T + BY_j + Y_j^T B^T + \frac{1}{\beta} I + \sum_{k=1}^m \lambda_k [W E_i^T(\theta_k) + E_i(\theta_k) W]$, then the multiobjective H_2/H_∞ investment policy problem for the fuzzy stochastic jump financial systems in (5) can be solved.

Proof: Let $V(\tilde{\boldsymbol{x}}(t)) = \tilde{\boldsymbol{x}}^T(t) P \tilde{\boldsymbol{x}}(t)$ be the Lyapunov function for the nonlinear stochastic jump financial system in (5), where $P = P^T > 0$ is a positive-definite matrix.

We derive the sufficient condition for $J_2(\boldsymbol{u}(t)) \leq \alpha$ of the MOP in (18) first. Add and subtract the term $dV(\tilde{\boldsymbol{x}}(t))$ to the integrand of $\boldsymbol{E}\{\int_0^\infty (\tilde{\boldsymbol{x}}(t)^T Q_1 \tilde{\boldsymbol{x}}(t) + \boldsymbol{u}^T(t) R_2 \boldsymbol{u}(t)) dt\}$. By the fact of $\lim_{t\to\infty} V(\tilde{\boldsymbol{x}}(t)) > 0$ and Lemma 1, we have

$$\begin{aligned} J_2(\boldsymbol{u}(t)) &= \boldsymbol{E}\left\{\int_0^\infty (\tilde{\boldsymbol{x}}(t)^T Q_1 \tilde{\boldsymbol{x}}(t) + \boldsymbol{u}^T(t) R_1 \boldsymbol{u}(t)) dt\right\} \\ &\leq \boldsymbol{E}\{V(\tilde{\boldsymbol{x}}_0)\} + \boldsymbol{E}\left\{\int_0^\infty [(\tilde{\boldsymbol{x}}(t)^T Q_1 \tilde{\boldsymbol{x}}(t) + \boldsymbol{u}^T(t) R_1 \boldsymbol{u}(t)) dt \right. \\ &\left. + dV(\tilde{\boldsymbol{x}})]\right\} = \boldsymbol{E}\{V(\tilde{\boldsymbol{x}}_0)\} + \boldsymbol{E}\left\{\int_0^\infty (\tilde{\boldsymbol{x}}(t)^T Q_1 \tilde{\boldsymbol{x}}(t) + \boldsymbol{u}^T(t) \right. \\ &\times R_1 \boldsymbol{u}(t) + 2\tilde{\boldsymbol{x}}(t)^T P f(\tilde{\boldsymbol{x}}(t)) + 2\tilde{\boldsymbol{x}}(t)^T P B \boldsymbol{u}(t) + \boldsymbol{\sigma}^T(\tilde{\boldsymbol{x}}(t)) \\ &\times P \boldsymbol{\sigma}(\tilde{\boldsymbol{x}}(t)) + \sum_{k=1}^m \lambda_k \{V(\tilde{\boldsymbol{x}}(t) + \Gamma(\tilde{\boldsymbol{x}}(t), \theta_k)) \\ &\left. - V(\tilde{\boldsymbol{x}}(t))\}) dt\right\}. \end{aligned}$$

If both the following Hamilton–Jacobi–Bellman inequality (HJBI)

$$\begin{aligned} &\tilde{\boldsymbol{x}}(t)^T Q_1 \tilde{\boldsymbol{x}}(t) + \boldsymbol{u}^T(t) R_1 \boldsymbol{u}(t) + 2\tilde{\boldsymbol{x}}(t)^T P f(\tilde{\boldsymbol{x}}(t)) \\ &+ 2\tilde{\boldsymbol{x}}(t)^T P B \boldsymbol{u}(t) + \boldsymbol{\sigma}^T(\tilde{\boldsymbol{x}}(t)) P \boldsymbol{\sigma}(\tilde{\boldsymbol{x}}(t)) + \sum_{k=1}^m \lambda_k \\ &\times \{V(\tilde{\boldsymbol{x}}(t) + \Gamma(\tilde{\boldsymbol{x}}(t), \boldsymbol{\theta}_k) - V(\tilde{\boldsymbol{x}}(t))\} \leq 0 \end{aligned} \tag{23}$$

and initial mean inequality

$$\boldsymbol{E}\{V(\tilde{\boldsymbol{x}}_0)\} \leq \alpha \tag{24}$$

hold, we have

$$\begin{aligned} J_2(\boldsymbol{u}(t)) &= \boldsymbol{E}\left\{\int_0^\infty \left[\tilde{\boldsymbol{x}}^T(t) Q_1 \tilde{\boldsymbol{x}}(t) + \boldsymbol{u}^T(t) R_1 \boldsymbol{u}(t)\right] dt\right\} \\ &\leq \boldsymbol{E}\{V(\tilde{\boldsymbol{x}}_0))\} \leq \alpha. \end{aligned}$$

By applying Lemma 4, we obtain the following inequalities:

$$2\tilde{\boldsymbol{x}}^T P f(\boldsymbol{x}(t)) = \sum_{i=1}^l h_i(\boldsymbol{z}) \tilde{\boldsymbol{x}}^T(t) \left[A_i^T P + P A_i\right] \tilde{\boldsymbol{x}}(t) \tag{25}$$

$$2\tilde{\boldsymbol{x}}(t)^T P B \boldsymbol{u}(t) = \sum_{j=1}^l h_j(\boldsymbol{z}) \tilde{\boldsymbol{x}}^T(t) \left[P B K_j + (P B K_j)^T\right] \tilde{\boldsymbol{x}}(t) \tag{26}$$

$$\boldsymbol{\sigma}^T(\tilde{\boldsymbol{x}}(t)) V_{\tilde{\mathrm{x}}\tilde{\mathrm{x}}}(\boldsymbol{x}(t)) \boldsymbol{\sigma}(\tilde{\boldsymbol{x}}(t)) \leq \sum_{i=1}^l h_i(\boldsymbol{z}) \tilde{\boldsymbol{x}}^T(t) C_i^T P C_i \tilde{\boldsymbol{x}}(t) \tag{27}$$

and

$$\begin{aligned} &V(\tilde{\boldsymbol{x}}(t) + \Gamma(\tilde{\boldsymbol{x}}(t), \theta_k)) - V(\tilde{\boldsymbol{x}}(t)) = [\tilde{\boldsymbol{x}}(t) + \Gamma(\tilde{\boldsymbol{x}}(t), \theta_k)]^T \\ &\times P[\tilde{\boldsymbol{x}}(t) + \Gamma(\tilde{\boldsymbol{x}}(t), \theta_k)] - \tilde{\boldsymbol{x}}^T(t) P \tilde{\boldsymbol{x}}(t) \leq \sum_{i=1}^l h_i(\boldsymbol{z}) \tilde{\boldsymbol{x}}^T(t) \\ &\times \left\{\left[E_i^T(\theta_k) P E_i(\theta_k)\right] + \left[E_i^T(\theta_k) P\right] + \left[P E_i(\theta_k)\right]\right\} \tilde{\boldsymbol{x}}(t). \end{aligned} \tag{28}$$

It is clear that

$$\boldsymbol{E}\{V(\tilde{\boldsymbol{x}}_0)\} \leq \bar{\tau}(P) \boldsymbol{E}\left\{Tr\left(\tilde{\boldsymbol{x}}_0^T \tilde{\boldsymbol{x}}_0\right)\right\} = \bar{\tau}(P) Tr(R_{\tilde{\boldsymbol{x}}_0}) \leq \alpha \tag{29}$$

and $P \leq \alpha [Tr(R_{\tilde{\boldsymbol{x}}_0})]^{-1} I$ which is (20).

Thus, the HJBI in (23) can be replaced by

$$\begin{aligned} &\tilde{\boldsymbol{x}}^T(t) Q_1 \tilde{\boldsymbol{x}}(t) + \boldsymbol{u}^T(t) R_1 \boldsymbol{u}(t) + 2\tilde{\boldsymbol{x}}(t)^T P f(\tilde{\boldsymbol{x}}(t)) \\ &+ 2\tilde{\boldsymbol{x}}(t)^T P B \boldsymbol{u}(t) + \boldsymbol{\sigma}(\tilde{\boldsymbol{x}}(t))^T P \boldsymbol{\sigma}(\tilde{\boldsymbol{x}}(t)) + \sum_{k=1}^m \lambda_k \\ &\{V(\tilde{\boldsymbol{x}}(t) + \Gamma(\tilde{\boldsymbol{x}}(t), \theta_k) - V(\tilde{\boldsymbol{x}}(t))\} \leq \sum_{i=1}^l \sum_{j=1}^l \\ &h_i(\boldsymbol{z}) h_j(\boldsymbol{z}) \tilde{\boldsymbol{x}}^T(t) (Q_1 + K_j^T R_1 K_j + A_i^T P + P A_i \\ &+ P B K_j + (P B K_j)^T + C_i^T P C_i + \sum_{k=1}^m \lambda_k [E_i^T(\theta_k) \\ &\times P E_i(\theta_k) + E_i^T(\theta_k) P + P E_i(\theta_k)]) \tilde{\boldsymbol{x}}(t). \end{aligned} \tag{30}$$

Now, We derive the sufficient condition for $J_\infty \leq \beta$ of the MOP in (18).

By using Lemmas 3 and 4, we get

$$\begin{aligned}
&\boldsymbol{E}\left\{\int_0^\infty \left[\tilde{\boldsymbol{x}}^T(t)Q_2\tilde{\boldsymbol{x}}(t)+\boldsymbol{u}^T(t)R_2\boldsymbol{u}(t)\right]dt\right\}\\
&\le \boldsymbol{E}\left\{\int_0^\infty \left(\tilde{\boldsymbol{x}}^T(t)Q_2\tilde{\boldsymbol{x}}(t)+\boldsymbol{u}^T(t)R_2\boldsymbol{u}(t)\right)dt+dV(\tilde{x})\right\}\\
&+\boldsymbol{E}\{V(\tilde{x}_0)\}=\boldsymbol{E}\left\{\int_0^\infty \{(\bar{x}^T(t)\,\tilde{Q}_2\bar{x}^T dt+\boldsymbol{u}^T(t)R_2\boldsymbol{u}(t)\right.\\
&+2\tilde{\boldsymbol{x}}^T(t)Pf(\tilde{\boldsymbol{x}}(t))+2\tilde{\boldsymbol{x}}^T(t)PB\boldsymbol{u}(t)+2\tilde{\boldsymbol{x}}^T(t)P\boldsymbol{v}(t)\\
&+\boldsymbol{\sigma}^T(\tilde{\boldsymbol{x}}(t))P\boldsymbol{\sigma}(\tilde{\boldsymbol{x}}(t))+\sum_{k=1}^m\lambda_k[V(\tilde{\boldsymbol{x}}(t)+\Gamma(\tilde{\boldsymbol{x}}(t),\theta_k))\\
&-V(\tilde{\boldsymbol{x}}(t))])dt\}+\boldsymbol{E}\{V(\tilde{\boldsymbol{x}}_0)\}\le\boldsymbol{E}\{V(\tilde{\boldsymbol{x}}_0)\}\\
&+\boldsymbol{E}\left\{\int_0^\infty[\tilde{\boldsymbol{x}}^T(t)Q_2\tilde{\boldsymbol{x}}(t)+\boldsymbol{u}^T(t)R_2\boldsymbol{u}(t)+2\tilde{\boldsymbol{x}}^T(t)P\right.\\
&\times f(\tilde{\boldsymbol{x}}(t))+2\tilde{\boldsymbol{x}}^T(t)PB\boldsymbol{u}(t)+\frac{1}{\beta}\tilde{\boldsymbol{x}}^T(t)PP\tilde{\boldsymbol{x}}(t)\\
&+\boldsymbol{\sigma}^T(\tilde{\boldsymbol{x}}(t))P\boldsymbol{\sigma}(\tilde{\boldsymbol{x}}(t))+\beta\boldsymbol{v}^T(t)\boldsymbol{v}(t)]dt\\
&\left.+\sum_{k=1}^m\lambda_k\{V(\tilde{\boldsymbol{x}}(t)+\Gamma(\tilde{\boldsymbol{x}}(t),\theta_k))-V(\tilde{\boldsymbol{x}}(t))\})dt\right\}.
\end{aligned}$$

If the following Hamilton–Jacobi–Isaacs inequality is satisfied:

$$\begin{aligned}
&\tilde{\boldsymbol{x}}^T(t)Q_2\tilde{\boldsymbol{x}}(t)+\boldsymbol{u}^T(t)R_2\boldsymbol{u}(t)+2\tilde{\boldsymbol{x}}^T(t)Pf(\tilde{\boldsymbol{x}}(t))\\
&+2\tilde{\boldsymbol{x}}^T(t)PB\boldsymbol{u}(t)+\frac{1}{\beta}\tilde{\boldsymbol{x}}^T(t)PP\tilde{\boldsymbol{x}}(t)+\boldsymbol{\sigma}^T(\tilde{\boldsymbol{x}}(t))\\
&\times P\boldsymbol{\sigma}(\tilde{\boldsymbol{x}}(t))+\sum_{k=1}^m\lambda_k\{V(\tilde{\boldsymbol{x}}(t)+\Gamma(\tilde{\boldsymbol{x}}(t),\theta_k))\\
&-V(\tilde{\boldsymbol{x}}(t))\}\le 0
\end{aligned}\tag{31}$$

then $J_\infty(\boldsymbol{u}(t))\le\beta$ for all possible $\boldsymbol{v}(t)\in\mathcal{L}^2_{\mathcal{F}}(\mathbb{R}^+;\mathbb{R}^3)$.

By introducing the T–S fuzzy model, we obtain the following inequality:

$$\begin{aligned}
&\tilde{\boldsymbol{x}}^T(t)Q_2\tilde{\boldsymbol{x}}(t)+\boldsymbol{u}^T(t)R_2\boldsymbol{u}(t)+2\tilde{\boldsymbol{x}}(t)^TPf(\tilde{\boldsymbol{x}}(t))(t)\\
&+2\tilde{\boldsymbol{x}}(t)^TPB\boldsymbol{u}+\frac{1}{\beta}\tilde{\boldsymbol{x}}^T(t)PP\tilde{\boldsymbol{x}}(t)+\boldsymbol{\sigma}^T(\tilde{\boldsymbol{x}}(t))P\boldsymbol{\sigma}(\tilde{\boldsymbol{x}}(t))\\
&+\sum_{k=1}^m\lambda_k\{V(\tilde{\boldsymbol{x}}(t)+\Gamma(\tilde{\boldsymbol{x}}(t)))-V(\tilde{\boldsymbol{x}}(t))\}\le\sum_{i,j=1}^l h_i(\boldsymbol{z})\\
&h_j(\boldsymbol{z})\tilde{\boldsymbol{x}}^T(t)(Q_2+K_j^TR_1K_j+A_i^TP+PA_i+PBK_j\\
&+(PBK_j)^T+C_i^TPC_i+\frac{1}{\beta}PP+\sum_{k=1}^m\lambda_k[E_i^T(\theta_k)\\
&\times PE_i(\theta_k)+E_i^T(\theta_k)P+PE_i(\theta_k)])\tilde{\boldsymbol{x}}(t).
\end{aligned}\tag{32}$$

From inequalities in (30) and (32), if the following two algebraic Riccati like inequalities are satisfied:

$$\begin{aligned}
&Q_1+K_j^TR_1K_j+A_i^TP+PA_i+PBK_j+(PBK_j)^T\\
&+C_i^TPC_i+\sum_{k=1}^m\lambda_k\left[E_i^T(\theta_k)PE_i(\theta_k)+E_i^T(\theta_k)P\right.\\
&\left.+PE_i(\theta_k)\right]\le 0
\end{aligned}\tag{33}$$

$$\begin{aligned}
&Q_2+K_j^TR_2K_j+A_i^TP+PA_i+PBK_j+(PBK_j)^T\\
&+C_i^TPC_i+\frac{1}{\beta}PP+\sum_{k=1}^m\lambda_k\left[E_i^T(\theta_k)PE_i(\theta_k)\right.\\
&\left.+E_i^T(\theta_k)P+PE_i(\theta_k)\right]\le 0
\end{aligned}\tag{34}$$

then we have $J_2(\boldsymbol{u}(t))\le\alpha$ and $J_\infty(\boldsymbol{u}(t))\le\beta$, respectively.

Let $W=P^{-1}$ and $Y_j=K_jW$; then, the inequalities in (33) and (34) are equivalent to the following two inequalities, respectively:

$$\begin{aligned}
&WQ_1W+Y_j^TR_1Y_j+WA_i^T+A_iW+BY_j+(BY_j)^T\\
&+WC_i^TW^{-1}C_iW+\sum_{k=1}^m\lambda_k[WE_i^T(\theta_k)W^{-1}E_i(\theta_k)W\\
&+WE_i^T(\theta_k)+E_i(\theta_k)W]\le 0
\end{aligned}\tag{35}$$

$$\begin{aligned}
&WQ_2W+Y_j^TR_2Y_j+WA_i^T+A_iW+BY_j+(BY_j)^T\\
&+WC_i^TW^{-1}C_iW+\frac{1}{\beta}I+\sum_{k=1}^m\lambda_k[WE_i^T(\theta_k)W^{-1}\\
&\times E_i(\theta_k)W+WE_i^T(\theta_k)+E_i(\theta_k)W]\le 0.
\end{aligned}\tag{36}$$

By Schur complement [22], the quadratic inequalities are equivalent to the LMIs in (21) and (22).

Definition 2 (see [21]): The nonlinear stochastic jump diffusion financial system in (5) or (11) is said to be exponentially mean square stable, if for some positive constants $A>0$ and $m>0$, the following inequality holds:

$$\boldsymbol{E}\left\{\|\tilde{\boldsymbol{x}}(t)\|_2^2\right\}\le\boldsymbol{A}\exp(-mt).$$

Theorem 2: For the nonlinear stochastic jump diffusion financial system in (5) or (11), if the external noise $\boldsymbol{v}(t)=0$, and $\boldsymbol{u}(t)$ is a feasible solution of the MOP in (19), then $\boldsymbol{u}(t)$ stabilizes the nonlinear stochastic Poisson jump diffusion financial system in (5) or (11) exponentially in the mean square sense, i.e., $\boldsymbol{x}(t)\to\boldsymbol{x}_d(t)$ exponentially in the mean square sense.

Proof: Since the given Lyapunov function $V(\tilde{\boldsymbol{x}}(t))=\tilde{\boldsymbol{x}}^T(t)P\tilde{\boldsymbol{x}}(t)$ is satisfied with the following two inequalities:

$$m_1\|\tilde{\boldsymbol{x}}(t)\|_2^2\le V(\tilde{\boldsymbol{x}}(t))\le m_2\|\tilde{\boldsymbol{x}}(t)\|_2^2\tag{37}$$

where $m_1>0$ and $m_2>0$, by the Itô–Lévy formula of $V(\tilde{\boldsymbol{x}}(t))$ in (9) and the fact that $\boldsymbol{u}(t)$ is a feasible solution of the LMI-constrained MOP in (19), we obtain

$$\begin{aligned} d\boldsymbol{E}\left\{V(\tilde{\boldsymbol{x}}(t))\right\} &= \boldsymbol{E}\left\{dV(\tilde{\boldsymbol{x}}(t))\right\} = \boldsymbol{E}\{(2\tilde{\boldsymbol{x}}^T(t)Pf(\tilde{\boldsymbol{x}}(t)) \\ &+ 2\tilde{\boldsymbol{x}}^T(t)PB\boldsymbol{u}(t) + \boldsymbol{\sigma}^T(\tilde{\boldsymbol{x}}(t))P\boldsymbol{\sigma}(\tilde{\boldsymbol{x}}(t)) + \sum_{k=1}^{m}\lambda_k \\ &[V(\tilde{\boldsymbol{x}}(t) + \Gamma(\tilde{\boldsymbol{x}}(t),\theta_k)) - V(\tilde{\boldsymbol{x}}(t))])dt\} \\ &\le \boldsymbol{E}\left\{-\tilde{\boldsymbol{x}}^T(t)Q_1\tilde{\boldsymbol{x}}(t)\right\}dt \le \boldsymbol{E}\left\{-m_3\left\|\tilde{\boldsymbol{x}}(t)\right\|_2^2\right\}dt \\ &\le \frac{-m_3}{m_2}\boldsymbol{E}\left\{V(\tilde{\boldsymbol{x}}(t))\right\}dt < 0 \end{aligned} \tag{38}$$

where m_3 is the smallest eigenvalue value of positive-definite matrix $Q_1 > 0$.

By using inequalities in (38), we get

$$\frac{d}{dt}\boldsymbol{E}\left\{V(\tilde{\boldsymbol{x}}(t))\right\} \le \frac{-m_3}{m_2}\boldsymbol{E}\left\{V(\tilde{\boldsymbol{x}}(t))\right\} \tag{39}$$

which implies the following two inequalities:

$$\boldsymbol{E}\left\{\|\tilde{\boldsymbol{x}}(t)\|^2\right\} \le \boldsymbol{E}\left\{\frac{V(\tilde{\boldsymbol{x}}(t))}{m_1}\right\} \le \boldsymbol{E}\left\{\frac{V(\tilde{\boldsymbol{x}}_0)}{m_1}\right\}\exp\left(\frac{-m_3 t}{m_2}\right). \tag{40}$$

It is obvious that

$$\lim_{t\longrightarrow\infty}\boldsymbol{E}\left\{\|\tilde{\boldsymbol{x}}(t)\|_2^2\right\} = 0$$

and we obtain $\lim_{t\to\infty}\tilde{\boldsymbol{x}}(t) = 0$ exponentially in the mean square sense.

B. Multiobjective H_2/H_∞ Investment Policy Problem for the Nonlinear Stochastic Jump Diffusion Financial System Driven by Marked Compensation Poisson Processes $\hat{\boldsymbol{N}}(t;\theta_k)$

Moreover, in some situations, the nonlinear stochastic jump diffusion financial systems are driven by compensation Poisson processes [7]:

$$\hat{\boldsymbol{N}}(t,\theta_k) \triangleq \boldsymbol{N}(t;\theta_k) - \lambda_k t, \text{ for } k = 1,2,...,m. \tag{41}$$

Thus, the stochastic nonlinear autonomous controlled system for the financial system in (5) can be replaced as follows:

$$\begin{aligned} d\tilde{\boldsymbol{x}}(t) &= (f(\tilde{\boldsymbol{x}}(t)) + B\boldsymbol{u}(t) + \boldsymbol{v}(t))\,dt \\ &+ \sigma(\tilde{\boldsymbol{x}}(t))\,d\mathcal{W} + \sum_{k=1}^{m}\Gamma(\tilde{\boldsymbol{x}}(t),\theta_k)\,d\hat{\boldsymbol{N}}(t;\theta_k). \end{aligned} \tag{42}$$

The overall fuzzy system in (10) should be modified as

$$\begin{aligned} d\tilde{\boldsymbol{x}}(t) &= \sum_{i=1}^{l}h_i(\boldsymbol{z})\{[A_i\tilde{\boldsymbol{x}}(t) + B\boldsymbol{u}(t) + \boldsymbol{v}(t)]\,dt \\ &+ C_i\tilde{\boldsymbol{x}}(t)d\mathcal{W}(t) + \sum_{k=1}^{m}E_i(\theta_k)\tilde{\boldsymbol{x}}(t)\,d\hat{\boldsymbol{N}}(t;\theta_k)\}. \end{aligned} \tag{43}$$

The Itô–Lévy formula for the nonlinear stochastic jump diffusion financial system driven by compensation Poisson process $d\hat{\boldsymbol{N}}(t,\theta_k)$ is given as

Lemma 5 (see [6]): Let $V:\mathbb{R}^3\to\mathbb{R}$, $V(\cdot)\in C^2(\mathbb{R}^3)$ and $V(\cdot)\ge 0$. For the nonlinear stochastic jump diffusion system in (42), the Itô–Lévy formula of $V(\tilde{\boldsymbol{x}}(t))$ is given as follows:

$$\begin{aligned} dV(\tilde{\boldsymbol{x}}(t)) &= [V_{\tilde{\boldsymbol{x}}}^T f(\tilde{\boldsymbol{x}}(t)) + V_{\tilde{\boldsymbol{x}}}^T B\boldsymbol{u}(t) + V_{\tilde{\boldsymbol{x}}}^T\boldsymbol{v}(t) \\ &+ \frac{1}{2}\boldsymbol{\sigma}(\tilde{\boldsymbol{x}}(t))^T V_{\tilde{\boldsymbol{x}}\tilde{\boldsymbol{x}}}\boldsymbol{\sigma}(\tilde{\boldsymbol{x}}(t)) + \sum_{k=1}^{m}\lambda_k\{V(\tilde{\boldsymbol{x}}(t) + \Gamma(\tilde{\boldsymbol{x}}(t),\theta_k)) \\ &- V(\tilde{\boldsymbol{x}}(t)) - V_{\tilde{\boldsymbol{x}}}^T\Gamma(\tilde{\boldsymbol{x}}(t),\theta_k)\}]dt + V_{\tilde{\boldsymbol{x}}}^T\sigma(\tilde{\boldsymbol{x}}(t))\,d\mathcal{W}(t) \\ &+ \sum_{k=1}^{m}\lambda_k\{V(\tilde{\boldsymbol{x}}(t) + \Gamma(\tilde{\boldsymbol{x}}(t),\theta_k)) - V(\tilde{\boldsymbol{x}}(t))\}d\hat{\boldsymbol{N}}(t;\theta_k). \end{aligned} \tag{44}$$

Thus, the sufficient condition for the fuzzy investment policy $\boldsymbol{u}(t)$ in (16) to solve the multiobjective H_2/H_∞ investment policy problem for the nonlinear stochastic jump diffusion financial system in (42) is given by the following theorem.

Theorem 3: If the following LMI-constrained MOP can be solved:

$$\min_{\{P,,K_1,K_2...,K_l\}}(\alpha,\beta)$$

$$s.t.\text{ the following LMIs, for all } i,j = 1,2,...,l \tag{45}$$

$$P \le \alpha\left[Tr\left(R_{\tilde{\boldsymbol{x}}_0}\right)\right]^{-1}I \tag{46}$$

$$\begin{bmatrix} \hat{\Psi}_{ij}^2 & W & Y_j^T & WC_i^T & WE_i^T(\theta_1) & \ldots & WE_i^T(\theta_m) \\ \star & -Q_1^{-1} & 0 & 0 & 0 & 0 & 0 \\ \star & \star & -R_1^{-1} & 0 & 0 & 0 & 0 \\ \star & \star & \star & -W & 0 & 0 & \vdots \\ \star & \star & \star & \star & -\lambda_1^{-1}W & 0 & 0 \\ \star & \star & \star & \star & \star & \ddots & 0 \\ \star & \star & \star & \star & \star & \star & -\lambda_m^{-1}W \end{bmatrix} \le 0 \tag{47}$$

$$\begin{bmatrix} \hat{\Psi}_{ij}^\infty & W & Y_j^T & WC_i^T & WE_i^T(\theta_1) & \ldots & WE_i^T(\theta_m) \\ \star & -Q_2^{-1} & 0 & 0 & 0 & 0 & 0 \\ \star & \star & -R_2^{-1} & 0 & 0 & 0 & 0 \\ \star & \star & \star & -W & 0 & 0 & \vdots \\ \star & \star & \star & \star & -\lambda_1^{-1}W & 0 & 0 \\ \star & \star & \star & \star & \star & \ddots & 0 \\ \star & \star & \star & \star & \star & \star & -\lambda_m^{-1}W \end{bmatrix} \le 0 \tag{48}$$

where $W = P^{-1}$, $Y_j = K_jW$, $\hat{\Psi}_{ij}^2 = A_iW + WA_i^T + BY_j + Y_j^TB^T$, $\hat{\Psi}_{ij}^\infty = A_iW + WA_i^T + BY_j + Y_j^TB^T + \frac{1}{\beta}I$, then the multiobjective H_2/H_∞ investment policy problem for the fuzzy stochastic jump diffusion financial systems in (42) can be solved.

Proof: This proof is similar to Theorem 1. By applying Lemma 5, the HJIs in (23) and (31) will be replaced as

$$\begin{aligned}&\tilde{\boldsymbol{x}}(t)^T Q_1 \tilde{\boldsymbol{x}}(t) + \boldsymbol{u}^T(t) R_1 \boldsymbol{u}(t) + 2\tilde{\boldsymbol{x}}(t)^T P f(\tilde{\boldsymbol{x}}(t))\\&+ 2\tilde{\boldsymbol{x}}(t)^T P B \boldsymbol{u}(t) + \boldsymbol{\sigma}(\tilde{\boldsymbol{x}}(t))^T P \boldsymbol{\sigma}(\tilde{\boldsymbol{x}}(t)) + \sum_{k=1}^{m} \lambda_k \{ V(\tilde{\boldsymbol{x}}(t)\\&+ \Gamma(\tilde{\boldsymbol{x}}(t), \theta_k) - V(\tilde{\boldsymbol{x}}(t)) - V_{\tilde{\boldsymbol{x}}}^T \Gamma(\tilde{\boldsymbol{x}}(t), \theta_k) \} \le 0 \qquad (49)\end{aligned}$$

and

$$\begin{aligned}&\tilde{\boldsymbol{x}}^T(t) Q_2 \tilde{\boldsymbol{x}}(t) + \boldsymbol{u}^T(t) R_2 \boldsymbol{u}(t) + 2\tilde{\boldsymbol{x}}^T(t) P f(\tilde{\boldsymbol{x}}(t))\\&+ 2\tilde{\boldsymbol{x}}^T(t) P B u + \frac{1}{\beta} \tilde{\boldsymbol{x}}^T(t) P P \tilde{\boldsymbol{x}}(t) + \boldsymbol{\sigma}^T(\tilde{\boldsymbol{x}}(t))\\&\times P \boldsymbol{\sigma}(\tilde{\boldsymbol{x}}(t)) + \sum_{k=1}^{m} \lambda_k \{ V(\tilde{\boldsymbol{x}}(t) + \Gamma(\tilde{\boldsymbol{x}}(t)))\\&- V(\tilde{\boldsymbol{x}}(t)) - V_{\tilde{\boldsymbol{x}}}^T \Gamma(\tilde{\boldsymbol{x}}(t), \theta_k) \} \le 0 \qquad (50)\end{aligned}$$

respectively.

Based on above analyses in (49) and (50), the inequalities in (35) and (36) are replaced by

$$\begin{aligned}&W Q_1 W + Y_j^T R_1 Y_j + W A_i^T + A_i W + B Y_j + (B Y_j)^T + W\\&\times C_i^T W^{-1} C_i W + \sum_{k=1}^{m} \lambda_k [W E_i^T(\theta_k) W^{-1} E_i(\theta_k) W] \le 0\end{aligned} \qquad (51)$$

and

$$\begin{aligned}&W Q_1 W + Y_j^T R_1 Y_j + W A_i^T + A_i W + B Y_j + (B Y_j)^T + W\\&\times W C_i^{T-1} C_i W + \frac{1}{\beta} I + \sum_{k=1}^{m} \lambda_k [W E_i^T(\theta_k) W^{-1} E_i(\theta_k) W]\\&\le 0 \qquad (52)\end{aligned}$$

respectively.

By Schur complement [22], the quadratic inequalities are equivalent to the LMIs in (46)–(48).

Remark 2: In this study, we use the 3-D nonlinear stochastic financial system to illustrate the proposed multiobjective H_2/H_∞ investment theories. However, the proposed theories can be extended to the n-dimensional nonlinear stochastic financial system, i.e., the system state $x(t)$ of the nonlinear stochastic financial system in (5) can be of arbitrary n-dimension. At this situation (n-dimentional nonlinear stochastic financial system), the nonlinear functions are defined as $f : \mathbb{R}^n \to \mathbb{R}^n$, $\sigma : \mathbb{R}^n \to \mathbb{R}^n$, and $\Gamma : \mathbb{R}^n \times \Theta \to \mathbb{R}^n$, which are nonlinear Borel measurable continuous functions. Moreover, these constant matrices A_i, C_i, and $E_i(\theta_k)$ for the T–S fuzzy model in (11) and (43) should be $\mathbb{R}^{n\times n}$ matrices and $B \in \mathbb{R}^{n\times p}$ where $\boldsymbol{u}(t) \in \mathbb{R}^{n\times p}$. Once the n-dimensional nonlinear stochastic financial system can be approximated by the T–S fuzzy model, the proposed LMI-constrained MOEA can efficiently solve the multiobjective H_2/H_∞ investment policy problem, too.

Remark 3: Theorem 3 is similar to Theorem 1, except marked Poisson process $N(t;\theta_k)$ being replaced by marked compensation Poisson processes $\hat{N}(t;\theta_k)$ and some modifications in $\hat{\Psi}_{ij}^2$ and $\hat{\Psi}_{ij}^\infty$, i.e., the term $\sum_{k=1}^{m} \lambda_k \left[W E_i^T(\theta_k) + E_i(\theta_k) W\right]$ is eliminated from $\hat{\Psi}_{ij}^2$ and $\hat{\Psi}_{ij}^\infty$ in Theorem 3.

Remark 4: The existence and uniqueness conditions for the stochastic differential equation in (42) are, respectively, given as follows [6]:

The existence condition is

$$\begin{aligned}&\text{for all } \tilde{\boldsymbol{x}}(t) \in \mathbb{R}^3\\&\|f(\tilde{\boldsymbol{x}}(t))\|^2 + \|\sigma(\tilde{\boldsymbol{x}}(t))\|^2 + \|\boldsymbol{u}(\tilde{\boldsymbol{x}}(t))\|^2\\&+ \sum_{k=1}^{m} \lambda_i \|\Gamma(\tilde{\boldsymbol{x}}(t), \theta_k)\|^2 \le K_1 \left(1 + \|\tilde{\boldsymbol{x}}(t)\|^2\right)\end{aligned}$$

where $K_1 < \infty$.

The uniqueness condition is

$$\begin{aligned}&\text{for all } \tilde{\boldsymbol{x}},\ \tilde{\boldsymbol{y}} \in \mathbb{R}^3\\&\|f(\tilde{\boldsymbol{x}}(t)) - f(\tilde{\boldsymbol{y}}(t))\|^2 + \|\sigma(\tilde{\boldsymbol{x}}(t)) - \sigma(\tilde{\boldsymbol{y}}(t))\|^2\\&+ \|\boldsymbol{u}(\tilde{\boldsymbol{x}}(t)) - \boldsymbol{u}(\tilde{\boldsymbol{y}}(t))\|^2\\&+ \sum_{k=1}^{m} \lambda_i \|\Gamma(\tilde{\boldsymbol{x}}(t), \theta_k) - \Gamma(\tilde{\boldsymbol{y}}, \theta_k)\|^2\\&\le K_2 \|\tilde{\boldsymbol{x}}(t) - \tilde{\boldsymbol{y}}(t)\|^2\end{aligned}$$

where $K_2 < \infty$.

IV. Multiobjective H_2/H_∞ Investment Policy of the Nonlinear Stochastic Financial System Design via LMI-Constrained MOEA Algorithm

The MOEA is a stochastic search method by simulating the natural selection in natural evolution. It is also particularly suitable for solving the multiobjective investment optimization problem of nonlinear stochastic financial systems due to its population-based nature, permitting a set of Pareto optimal solutions to be obtained in a single run. For a multiobjective H_2/H_∞ investment policy problem of nonlinear stochastic jump diffusion financial system, the MOP in Theorems 1 and 3 with LMI constraints to guarantee the robust stability of nonlinear stochastic financial system of (5) or (11) under continuous and discontinuous parametric fluctuations and external disturbance. It is not easy to solve the MOP in Theorem 1 or Theorem 3, directly. In this section, an LMI-constrained MOEA searching algorithm is developed to help us solve the MOP in Theorem 1 or Theorem 3, iteratively. Before further discussion, some important definitions of Pareto optimality of an LMI-constrained MOP for nonlinear stochastic financial system are given as follows:

Definition 3 (see [29] and [31]): Consider the LMI-constrained MOP in (19). A feasible objective vector (α^1, β^1) is said to dominate another feasible objective vector (α^2, β^2) if and only if $\alpha^1 \le \alpha^2$ and $\beta^1 \le \beta^2$ for at least one inequality being a strict inequality.

Definition 4 (see [29] and [31]): Let $(P^1, K_1^1, ..., K_l^1)$ and $(P^2, K_1^2, ..., K_l^2)$ be the feasible solution corresponding to the objective value (α^1, β^1) and (α^2, β^2) subject to the LMIs in (20)–(22) for all $i, j = 1, ..., l$, respectively. $\left(P^1, K_1^1, .., K_l^1\right)$ is said to dominate $\left(P^2, K_1^2, .., K_l^2\right)$ if $\alpha^1 \leq \alpha^2$ and $\beta^1 \leq \beta^2$ for at least one inequality being a strict inequality.

Definition 5 (see [29] and [31]): A solution $(P^*, K_1^*, .., K_l^*)$ with objective value (α^*, β^*) is said to be a Pareto optimal solution of (19), if there does not exist another feasible solution $(P, K_1, .., K_l)$ with objective value (α, β), such that (α, β) dominates (α^*, β^*).

Definition 6 (see [29] and [31]): For the LMI-constrained MOP of nonlinear stochastic financial system in (19), the Pareto front $\mathbb{P}_{\mathbb{F}}$ is defined as

$$\mathbb{P}_{\mathbb{F}} = \begin{cases} (\alpha^*, \beta^*) | (P^*, K_1^*, .., K_l^*) \text{ is a Pareto} \\ \text{optimal solution and } (\alpha^*, \beta^*) \text{ is generated} \\ \text{by } (P^*, K_1^*, .., K_l^*) \text{ subject to the LMIs in} \\ (20), (21) \\ \text{and } (22), \text{for all } i, j = 1, ..., l. \end{cases}$$

The LMI-constrained MOP for the nonlinear stochastic financial system is concerned with evolution algorithm (EA). For the proposed LMI-constrained MOEA, the EA operates on a number of encoding feasible objective vectors called population so that for a feasible objective vector (α^k, β^k), it should be encoded into a chromosome $\mathbf{C}_k$. A feasible chromosome $\mathbf{C}_k$ is defined as a coded feasible objective vector. The chromosome of EA for the LMI-constrained MOEA employs the real-valued representation to avoid long binary string and large searching space in EA for the multiobjective H_2/H_∞ investment policy of the nonlinear stochastic jump diffusion financial system. In general, EA takes population to be the algorithm input and return the chromosomes with fitness performance to be the next population. The LMI-constraints in (20)–(22) to guarantee robust stability of financial system and the two upper bounds α and β are just like the environments in natural evolution. Only the adaptive chromosomes survive. For guaranteeing that all of these chromosomes can be decoded as the feasible objective vectors for the MOP in (19), each chromosome $\mathbf{C}_k$ should be examined by the existence of a feasible solution $\left(P^k, K_1^k, .., K_l^k\right)$ in LMIs in (20)–(22) with the LMI toolbox in MATLAB after mating operation. If some chromosomes are not feasible, these chromosomes need to be canceled from the candidate chromosomes. It is worth to mention that LMI constraints in (20)–(22) impose more restrictions on the searching of the feasible chromosomes than the conventional MOEA approach. Besides, using the LMI toolbox in MATLAB could help efficiently examine whether these chromosomes satisfy the LMIs in (20)–(22), which accelerates the selection speed of the initial populations via the initialization scheme.

Design procedure of multiobjective H_2/H_∞ investment policy of nonlinear stochastic financial systems:

Step 1: Select the searching range $(\alpha_0, \beta_0) \times (\bar{\alpha}, \bar{\beta})$ for the feasible objective vector (α, β) and set the iteration number $\bar{\mathbb{N}}$, the population number $\mathbb{N}_p$, the crossover ration $\mathbb{N}_c$, and the mutation ratio $\mathbb{N}_m$ in the LMI-constrained MOEA.

Step 2: Select $\mathbb{N}_p$ feasible chromosomes from the feasible chromosome set randomly to be the initial population $\mathbb{P}_1$.

Step 3: Set iteration index $\mathbb{N}_i = 1$.

Step 4: Operate the EA with the crossover ratio $\mathbb{N}_c$, the mutation ratio $\mathbb{N}_m$, and generate $2\mathbb{N}_p$ number feasible chromosomes by examining whether their corresponding objective vectors are feasible objective vectors for the LMIs in (20)–(22).

Step 5: Set the iteration index $\mathbb{N}_i = \mathbb{N}_i + 1$ and select $\mathbb{N}_p$ chromosomes from the $2\mathbb{N}_p$ feasible chromosomes in Step 4 through nondominated sorting method to be the population $\mathbb{P}_{\mathbb{N}_i}$.

Step 6: Repeat Steps 4 and 5 until the iteration number $\bar{\mathbb{N}}$ is reached. If the iteration number $\bar{\mathbb{N}}$ is satisfied, then we set $\mathbb{P}_{\mathbb{N}_i} = \mathbb{P}_{\mathbb{F}}$.

Step 7: Select a "preferable" feasible objective individual $\left(\alpha^\dagger, \beta^\dagger\right) \in \mathbb{P}_{\mathbb{F}}$ according to designer own preference. Once the "preferable" feasible objective individual is selected, the corresponding Pareto optimal solution

$$\boldsymbol{\xi}^\dagger = \left\{W^\dagger, K_1^\dagger, K_2^\dagger \ldots, K_l^\dagger\right\}$$

is obtained. By using $\boldsymbol{\xi}^\dagger$, the proposed multiobjective H_2/H_∞ fuzzy investment policy $\boldsymbol{u}(t) = \sum_{i=1}^{l} h_i(\boldsymbol{z}) \left(K_i^\dagger \tilde{\boldsymbol{x}}(t)\right)$ with $K_i^\dagger = Y_i^\dagger W^{\dagger -1}$ in (16) can be constructed and the multiobjective H_2/H_∞ investment policy problem in (5) or (11) can be solved with $J_2 = \alpha^\dagger$ and $J_\infty = \beta^\dagger$, simultaneously.

Note that the system diagram of the multiobjective H_2/H_∞ investment policy problem is given in Fig. 1, and the system diagram of the proposed LMI-constrained MOEA is given in Fig. 2, used in the design procedure.

Remark 5: The computational complexity of the proposed LMI-constrained MOEA algorithm is approximately $O\left(n\,(n+1)\,l\mathbb{N}_p^2\bar{\mathbb{N}}\right)$, including $O\left(\frac{n(n+1)l}{2}\right)$ for solving the LMIs, and $O\left(2\mathbb{N}_p^2\bar{\mathbb{N}}\right)$ for the MOEA, where n is the dimension of system state $\boldsymbol{x}(t)$, $\bar{\mathbb{N}}$ is the iteration number of the MOEA, l is the number of local linear models of the nonlinear stochastic financial system in (11), and $\mathbb{N}_p$ is the population number of the LMI-constrained MOEA.

V. Simulation Results

To illustrate the design procedure and to confirm the performance of the proposed optimal investment policy for nonlinear stochastic jump diffusion financial system, we introduce a nonlinear stochastic jump diffusion financial system to mimic an emerging market in (2). In general, an emerging market always suffer from continuous and discontinuous intrinsic random fluctuations due to national and international situation change, oil price change, the surplus between investment and saving, the variable of interest rate, etc. The external investment disturbance is caused by the unpredictable investment changes or worldwide events such as war, nature disaster, fatal epidemic, etc.

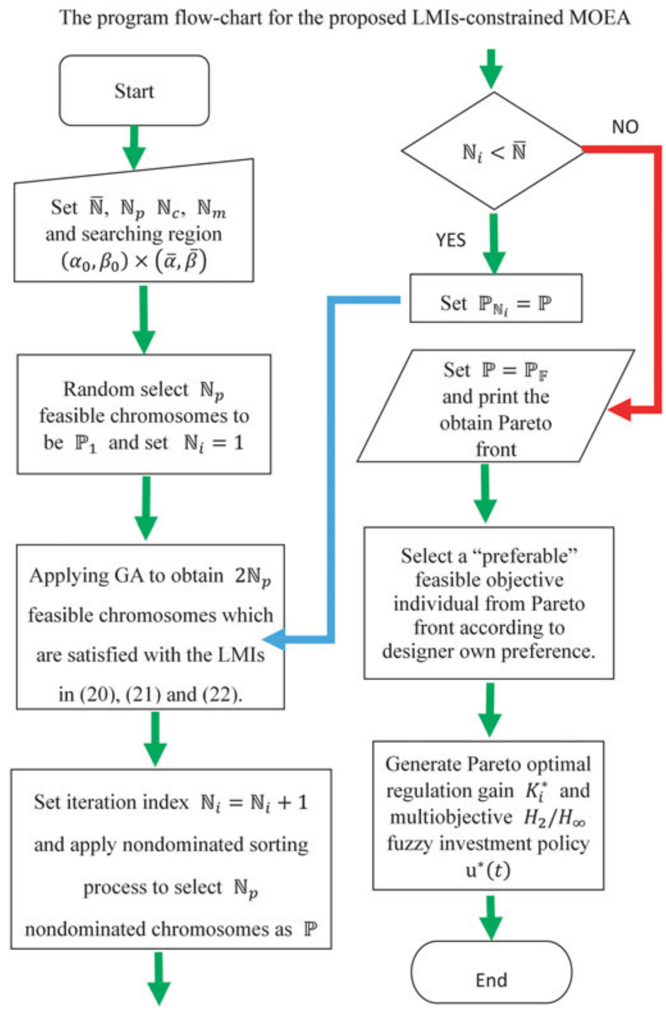

Fig. 2. Flowchart of the proposed LMI-constrained MOEA.

The related parameters and intrinsic continuous and discontinuous fluctuation are, respectively, given as follows:

$$a = 1.5,\ b = 0.2,\ c = 0.25$$

$$\Theta = \{\theta_1, \theta_2, \ldots, \theta_6\}$$

$$\begin{cases} \lambda_i = 0.3, \text{ for } i = 1,2,3,4 \\ \lambda_i = 0.2, \text{ for } i = 5,6 \end{cases}$$

$$\sigma_1 (x(t), y(t), z(t)) = 0.03 \times [z(t) + (y(t) - a)\, x(t)]$$

$$\sigma_2 (x(t), y(t), z(t)) = 0.01 \times \left[1 - by(t) - (x(t))^2\right]$$

$$\sigma_3 (x(t), y(t), z(t)) = 0.02 \times [-x(t) - cz(t)]$$

$$B = I_{3\times 3}$$

$$\gamma_1 (x(t), y(t), z(t), \theta_i) = \begin{cases} 0.3x(t), \text{ if } \theta_i = \theta_1 \\ -0.3x(t), \text{ if } \theta_i = \theta_2 \\ 0, \text{ else} \end{cases}$$

$$\gamma_2 (x(t), y(t), z(t), \theta_i) = \begin{cases} 0.05y(t), \text{ if } \theta_i = \theta_3 \\ -0.05y(t), \text{ if } \theta_i = \theta_4 \\ 0, \text{ else} \end{cases}$$

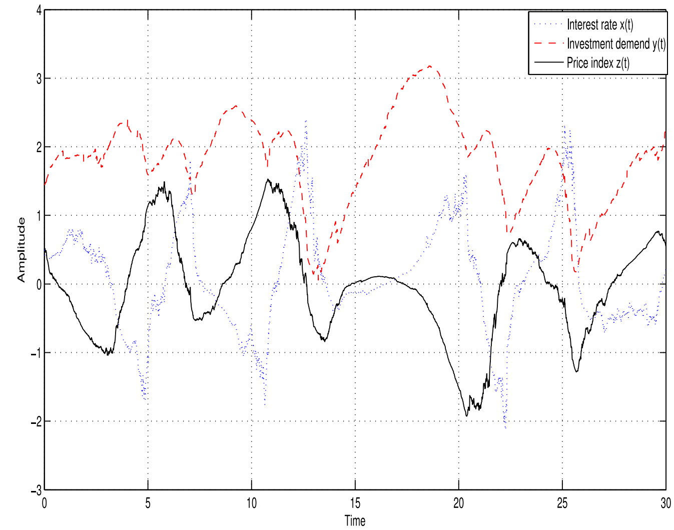

Fig. 3. Trajectories of the interest rate $x(t)$, the investment demand $y(t)$, and the price index $z(t)$ for the nonlinear stochastic jump diffusion system in (2), i.e., the system dynamic behaviors for nonlinear stochastic jump diffusion financial system in (3) without introducing investment policy $\boldsymbol{u}(t)$.

$$\gamma_3 (x(t), y(t), z(t), \theta_i) = \begin{cases} -0.1z(t), \text{ if } \theta_i = \theta_5 \\ 0.1z(t), \text{ if } \theta_i = \theta_6 \\ 0, \text{ else.} \end{cases}$$

Suppose the initial states are given as

$$\tilde{\boldsymbol{x}}_0 = (0.37, -3.06, 0.71).$$

The external investment disturbance is assumed to be

$$\boldsymbol{v}(t) = [0.01 \sin(2t), -0.02 \sin(2t), -0.01 \sin(2t)].$$

Fig. 3 is used to describe the dynamical behaviors of the nonlinear stochastic jump diffusion financial system in (2) with much fluctuations in real situation. It is seen that the three states $x(t)$, $y(t)$, and $z(t)$ of the nonlinear stochastic financial system fluctuate with random jumps. Therefore, the multiobjective H_2/H_∞ investment policy of the MOP in (17) is employed by the government to design an investment policy to regulate the stochastic financial system to achieve the following desired steady state:

$$\boldsymbol{x}_d = (0.1,\ 4.5,\ -0.2) \tag{53}$$

i.e., the government of emerging market expects its financial system to be regulated to achieve a desired financial steady state with interest rate 0.1, investment demand 4.5, and price index -0.2 to coordinate inflation rate, supply and demand in the emerging market to stimulate the recovery of emerging market under continuous and discontinuous random fluctuation.

In this example, we assume the vector of premise variables $\boldsymbol{z} = [x_1(t),\ x_2(t)]^T = [\boldsymbol{z}_1(t),\ \boldsymbol{z}_2(t)]^T$ in (11) and (16) is available in the regulation design. Since there are eight fuzzy sets associated with premise variable $\boldsymbol{z}_1(t)$ and eight fuzzy sets associated with premise variable $\boldsymbol{z}_2(t)$, totally there are 64 fuzzy IF–THEN rules, in the T–S fuzzy financial system. The operation

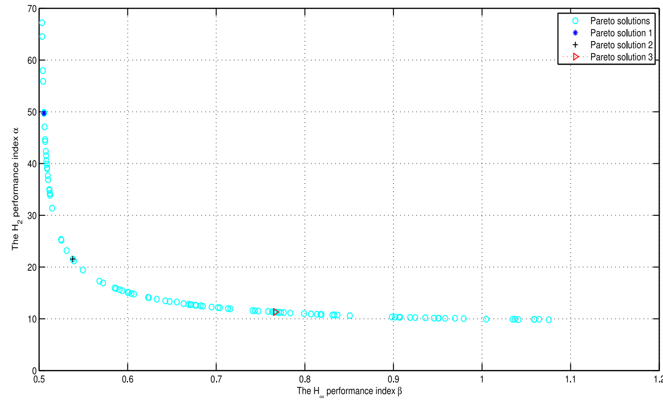

Fig. 4. Pareto front for Pareto optimal solutions of the MOP in (19) can be obtained by the proposed LMI-constrained MOEA. Three marked Pareto optimal solutions are more preferable multiobjective H_2/H_∞ investment policy with their simulations in Figs. 3 and 5, respectively.

TABLE I
PARETO OBJECTIVE VECTORS OF THE THREE CHOSEN INDIVIDUALS IN MEAN SENSE

Pareto solution 1	Pareto solution 2	Pareto solution 3
$(49.7177, 0.5055)$	$(21.5353, 0.5380)$	$(11.3162, 0.7667)$

point of the T–S fuzzy model and their linearized local linear stochastic models are shown in the Appendix. Suppose the weighting matrices in the multiobjective H_2/H_∞ investment policy in (17) is used to regulate the stochastic financial system with the following weighting matrices:

$$Q_1 = I_{3\times 3}, \quad R_1 = I_{3\times 3}, Q_2 = I_{3\times 3}, \quad R_2 = 0.5 I_{3\times 3}.$$

Based on the proposed design procedure of the multiobjective H_2/H_∞ optimal investment policy, the MOEA algorithm is employed to solve the MOP in (19). In the MOEA algorithm, the searching region Γ is set as $\Gamma = [0 \quad 70] \times [0.4 \quad 1.2]$, the maximum number of individuals $\mathbb{N}_p = 80$, iteration number $\mathbb{N}_i = 100$, crossover rate $\mathbb{N}_c = 0.8$, and mutation ratio $\mathbb{N}_m = 0.2$.

Once the iteration number $\bar{\mathbb{N}} = 100$ is achieved, the Pareto front $\mathbb{P}_{\mathbb{F}}$ for the Pareto optimal solutions of the MOP for the investment policy of the nonlinear stochastic jump diffusion financial system of the emerging market in (5) or (11) can be obtained as shown in Fig. 4.

In order to illustrate how to select the preferable solution of the investment policy, we choose three Pareto optimal solutions from the Pareto front for comparison, whose Pareto objective vectors are given in Table I. Moreover, in Figs. 5–7, the simulation results of the nonlinear stochastic jump diffusion financial system for the emerging market are given to illustrate the performance of the multiobjective H_2/H_∞ optimal fuzzy investment policy of the three chosen Pareto optimal solutions for the nonlinear stochastic jump diffusion financial system in (5) or (11). The corresponding P matrices in (20) of the three chosen Pareto solutions are also given in Table II. In Figs. 5–7, the Pareto optimal solution 1 has the best J_∞ performance

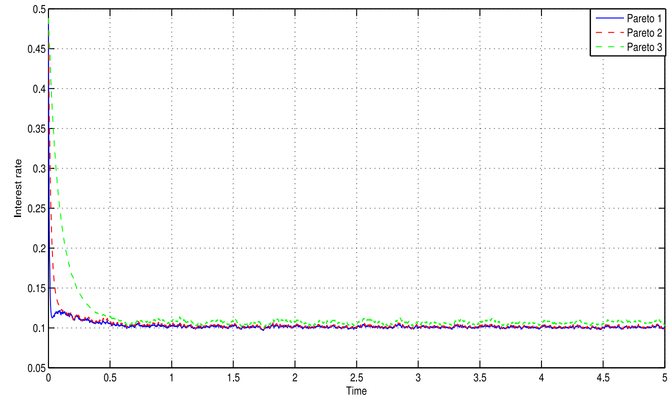

Fig. 5. Interest rate trajectories $x(t)$ of the three chosen Pareto solutions.

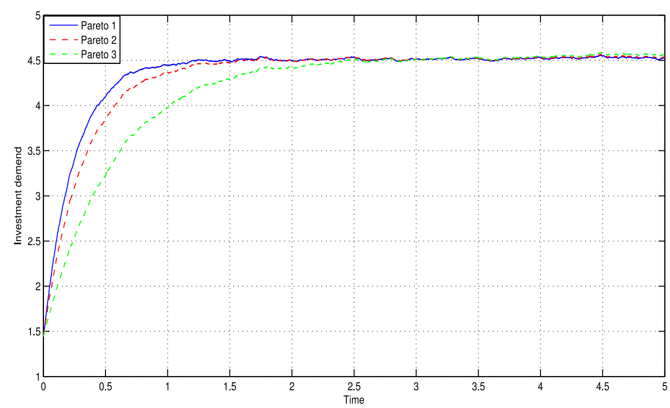

Fig. 6. Investment demand trajectories $y(t)$ of the three chosen Pareto solutions.

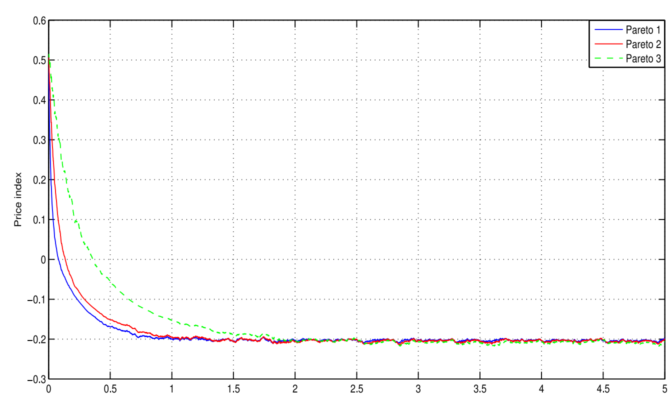

Fig. 7. Price index trajectories $z(t)$ of the three chosen Pareto solutions.

index of the three chosen Pareto optimal solutions so that the trajectory of Pareto optimal solution 1 has the minimum perturbation but has the maximum J_2 performance index. However, for the Pareto optimal solution 3, it has the minimum J_2 performance index, but its trajectory has the maximum perturbation of the three because it has the maximum J_∞ performance index. It is easy to observe that Pareto solution 2 is the preferable regulation solution of the three chosen Pareto solutions because the

TABLE II
POSITIVE-DEFINITE MATRIX OF THE THREE CHOSEN PARETO OPTIMAL SOLUTIONS IN FIG. 2

Pareto solution 1
$P_{\text{solution1}} = \begin{bmatrix} 68.54 & 1.56 & 10.49 \\ 1.56 & 2.28 & 1.99 \\ 10.49 & 1.99 & 24.13 \end{bmatrix}$
Pareto solution 2
$P_{\text{solution2}} = \begin{bmatrix} 31.44 & 0.18 & -0.77 \\ 0.18 & 1.65 & 0.71 \\ -0.77 & 0.71 & 10.35 \end{bmatrix}$
Pareto solution 3
$P_{\text{solution3}} = \begin{bmatrix} 10.57 & -0.21 & -1.94 \\ -0.21 & 1.03 & 0.41 \\ -1.94 & 0.41 & 4.79 \end{bmatrix}$

Pareto solution 2 is a compromise solution for J_2 and J_∞ in the multiobjective investment policy problem of nonlinear stochastic jump diffusion financial system. However, these three Pareto optimal solutions have satisfactory regulation results to achieve their desired steady states of the financial system in spite of intrinsic continuous and discontinuous random fluctuation and external disturbance, respectively.

VI. CONCLUSION

This study has investigated the multiobjective H_2/H_∞ investment policy for nonlinear stochastic jump diffusion financial systems via the T–S fuzzy model interpolation method. Unlike the most multiobjective problems only focusing on algebraic systems by the indirect method in Lemma 2, the proposed multiobjective H_2/H_∞ investment policy could solve the optimal robustness and regulation problems of a nonlinear stochastic jump diffusion financial system to achieve a desired target with a less risk and a less regulation investment effort, simultaneously. To avoid solving nonlinear system dynamic-constrained MOP, the T–S fuzzy model interpolation method is employed such that the nonlinear system dynamic constraints can be replaced by two sets of LMI constraints. If there exist some feasible solutions for the LMI constraints in (20)–(22), the LMI-constrained MOP could be solved easier by the proposed LMI-constrained MOEA algorithm. Thus, the multiobjective H_2/H_∞ investment policy for nonlinear stochastic jump diffusion financial systems can be solved efficiently by the help of the commercial software MATLAB via LMI toolbox. When the Pareto front is obtained for Pareto optimal solutions of the MOP in (19), the manager can select a preferable fuzzy investment policy from the set of Pareto optimal fuzzy investment policy according to his/her own preference and finish the multiobjective H_2/H_∞ investment policy design of the nonlinear stochastic financial system. Finally, an example is given to confirm the satisfactory performance of the proposed multiobjective H_2/H_∞ investment policy design for nonlinear stochastic jump diffusion financial systems through the computer simulation.

APPENDIX

The eight operation points of x_1 are given at

$$\begin{aligned} &x_1^1 = -1.50, \quad x_1^2 = -1.11, \quad x_1^3 = -0.72, \quad x_1^4 = -0.33 \\ &x_1^5 = 0.07, \quad x_1^6 = 0.46, \quad x_1^7 = 0.85, \quad x_1^8 = 1.24 \end{aligned}$$

and the eight operation points of x_2 are given at

$$\begin{aligned} &x_2^1 = -3.30, \quad x_2^2 = -3.10, \quad x_2^3 = -2.90, \quad x_2^4 = -2.70 \\ &x_2^5 = -2.50, \quad x_2^6 = -2.30, \quad x_2^7 = -2.10, \quad x_2^8 = -1.90. \end{aligned}$$

The qth rule of this T–S fuzzy model for the nonlinear stochastic jump diffusion financial system in (5) is described as

$$\begin{aligned} &\text{System Rule } q = 8\,(j-1) + i, \ \text{for } i, j = 1, ..., 8 \\ &\text{if } \boldsymbol{z}_1 \text{ is } x_1^i \text{ and } \boldsymbol{z}_2 \text{ is } x_2^j \\ &\text{then } d\tilde{\boldsymbol{x}}(t) = [A_q \tilde{\boldsymbol{x}}(t) + B\boldsymbol{u}(t) + \boldsymbol{v}(t)]\,dt \\ &+ C_q \tilde{\boldsymbol{x}}(t) d\mathcal{W}(t) + \sum_{k=1}^{m} E_q(\theta_k) \tilde{\boldsymbol{x}}(t)\, d\boldsymbol{N}(t; \theta_k) \end{aligned} \tag{54}$$

where

$$A_q = \begin{bmatrix} M_1^q & x_d & 1 \\ M_2^q & M_3^q & 0 \\ M_4^q & 0 & -c \end{bmatrix}, \quad C_q = \begin{bmatrix} 0.3M_1^q & 0.3x_d & 0.3 \\ 0.1M_2^q & 0.1M_3^q & 0 \\ 0.2M_4^q & 0 & -0.2c \end{bmatrix}$$

$$M_1^q = -a + x_2^j + y_d + \frac{\Delta_1}{x_1^i}, \ M_2^q = -x_1^i - 2x_d$$

$$M_3^q = -b + \frac{\Delta_2}{x_2^j}, \ M_4^q = -1 - \frac{\Delta_3}{x_1^i}, \ \Delta_1 = z_d + x_d y_d - a x_d$$

$$\Delta_2 = 1 - b y_d - (x_d)^2\,, \ \Delta_3 = x_d + c z_d$$

$$E_q(\theta_1) = \text{diag}([0.21, 0, 0]), E_q(\theta_2) = \text{diag}([-0.3, 0, 0])$$

$$E_q(\theta_3) = \text{diag}([0, 0.05, 0]), E_q(\theta_4) = \text{diag}([0, -0.05, 0])$$

$$E_q(\theta_5) = \text{diag}([0, 0, -0.1]), E_q(\theta_6) = \text{diag}([0, 0, 0.15]).$$

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