Question 7:
Solve the following questions from the Discrete Math zyBook:
a) Exercise 6.1.5, sections b-d

A 5-card hand is dealt from a perfectly shuffled deck of playing cards. What is the probability of each of the following events?

b) (b) What is the probability that the hand is a three of a kind? A three of a kind has three cards of the same rank. The other two cards do not have the same rank as each other and do not have the same rank as the three with the same rank. For example, {4♠, 4♦, 4♣, J♠, 8♥} is a three of a kind hand.

{13 \choose 1}{4 \choose 3}{12 \choose 2}{4 \choose 1}{4 \choose 1}

$$\frac{\binom{13}{1}\cdot\binom{4}{3}\cdot\binom{12}{2}\binom{4}{1}^2}{\binom{52}{5}}$$

c) (c) What is the probability that all 5 cards have the same suit?

$$\frac{4\cdot\binom{13}{5}}{\binom{52}{5}}$$

d) (d) What is the probability that the hand is a two of a kind? A two of a kind has two cards of the same rank (called the pair). Among the remaining three cards, not in the pair, no two have the same rank and none of them have the same rank as the pair. For example, {4♠, 4♦, J♠, K♣, 8♥} is a two of a kind.

$$\frac{\binom{13}{1}\binom{4}{2}\binom{12}{1}\binom{11}{1}\binom{10}{1}\binom{4}{1}^3}{\binom{52}{5}}$$

b) Exercise 6.2.4, sections a-d

A 5-card hand is dealt from a perfectly shuffled deck of playing cards. What is the probability of each of the following events?

a) (a) The hand has at least one club. ≈ 0.7784663866

$p(C_i)$ = probability that the hand has i clubs

$$p(C_1 \cup C_2 \cup C_3 \cup C_4 \cup C_5) = \frac{\binom{13}{1}\binom{39}{4}}{\binom{52}{5}} + \frac{\binom{13}{2}\binom{39}{3}}{\binom{52}{5}} + \frac{\binom{13}{3}\binom{39}{2}}{\binom{52}{5}} + \frac{\binom{13}{4}\binom{39}{1}}{\binom{52}{5}} + \frac{\binom{13}{5}}{\binom{52}{5}}$$

b) (b) The hand has at least two cards with the same rank. ≈ 0.4929171669

Complement: there are no 2 cards with the same rank (all 5 cards are a different rank)

$\frac{\binom{13}{5}\binom{4}{1}^5}{\binom{52}{5}}$ = ≈ 0.50708283313

1 minus complement = $1 - 0.5070828331 = 0.4929171669$

c) (c) The hand has exactly one club or exactly one spade. ≈ 0.82183873549

Probability of 1 club:

$P(C) = \frac{\binom{13}{1}\binom{39}{4}}{\binom{52}{5}}$

$P(S) =$ the probability of 1 spade is the same.

Probability of the intersection of 1 club and 1 spade:

3 cards need to be picked, and they cant be club or spade

$P(C \cap S) = \binom{26}{3}$

$P(C + S) - P(C \cap S)$ = $\frac{(\binom{13}{1}\binom{39}{4}) + (\binom{13}{1}\binom{39}{4}) - \binom{26}{3}}{\binom{52}{5}}$ = 0.82183873549

d) (d) The hand has at least one club or at least one spade. ≈ 0.974689876

1 minus probability of no clubs or spades

$P(C \cap S) = \frac{\binom{26}{5}}{\binom{52}{5}}$

$1 - \binom{26}{3}\binom{52}{5}$ = 0.974689876

Question 8:
Solve the following questions from the Discrete Math zyBook:
a) Exercise 6.3.2, sections a-e

The letters $\{a, b, c, d, e, f, g\}$ are put in a random order. Each permutation is equally likely. Define the following events:

A: The letter b falls in the middle (with three before it and three after it)
B: The letter c appears to the right of b, although c is not necessarily immediately to the right of b. For example, "$agbdcef$" would be an outcome in this event.
C: The letters "def" occur together in that order (e.g. "$gdefbca$")

a) (a) Calculate the probability of each individual event. That is, calculate $p(A)$, $p(B)$, and $p(C)$. $\frac{1}{(42)}$

p(A) = $\frac{1}{7}$ since b is in a fixed position, you can define the probability as $\frac{6!}{7!}$.

$$\frac{\cancel{6}\cdot\cancel{5}\cdot\cancel{4}\cdot\cancel{3}\cdot\cancel{2}\cdot\cancel{1}}{7\cdot\cancel{6}\cdot\cancel{5}\cdot\cancel{4}\cdot\cancel{3}\cdot\cancel{2}\cdot\cancel{1}} = \frac{1}{7}$$

p(B) = $\frac{1}{2}$ because all possible ways to line them up are $7!$, but two letters are in fixed positions relative to each other, so we can divide out that impact by dividing by 2 because one will always be on the right of the other).

Therefore the probability is $\frac{\cancel{7}!}{(2)(\cancel{7}!)}$ = $\frac{1}{2}$

p(C) = $(\frac{1}{42})$ because we can combine def as a single element that takes up 3 spaces. So now there are a total of 5 positions to choose from ([def], [a], [b], [c], [g]) So $\frac{5!}{7!}$ is the probability $\frac{1}{42}$

b) (b) What is $p(A|C)$? $= \frac{1}{10}$

cardinality of $A \cap C$ = $2 \cdot 3!$
cardinality of $C = 5!$

$\frac{|A \cap C|}{|C|} = \frac{2 \cdot 3!}{5!}$ = .1

c) (c) What is $p(B|C)$? $= 0.5$

independent

$\frac{p(B \cap C)}{p(C)} = \frac{\frac{1}{2} \cdot \frac{1}{42}}{\frac{1}{42}}$ = .5

can also do the cardinality of B intersect C (60) — $\frac{5!}{2}$
divided by cardinality of C (120) – $5!$, which equals ½ as well

d) (d) What is $p(A|B)$? ≈ 0.1428571429

independent

$$p(A\cap B) = \frac{1}{7}\cdot\frac{1}{2} = \frac{1}{14}$$

$$\frac{p(A\cap B)}{p(B)} = \frac{\frac{1}{14}}{\frac{1}{2}} = \frac{1}{7}$$

can also do the cardinality of A intersect B (360) — $3\cdot 5!$ divided by cardinality of B (5040) – $7!$, which equals $\frac{1}{7}$ as well

e) (e) Which pairs of events among A, B, and C are independent?

p(A) is 1/7 // p(B) is ½ // p(C) is 1/42

A and C are dependent

$$p(A\cap C) = \frac{1}{7}\cdot\frac{1}{42} = \frac{1}{294}$$

$$\frac{p(A\cap C)}{p(C)} = \frac{\frac{1}{294}}{\frac{1}{42}} \neq \frac{1}{7}$$

B and C are independent

$$p(B\cap C) = \frac{1}{2}\cdot\frac{1}{42} = \frac{1}{84}$$

$$\frac{p(B\cap C)}{p(C)} = \frac{\frac{1}{84}}{\frac{1}{42}} = \frac{1}{2}$$

A and B are independent

$$p(A\cap B) = \frac{1}{7}\cdot\frac{1}{2} = \frac{1}{14}$$

$$\frac{p(A\cap B)}{p(B)} = \frac{\frac{1}{14}}{\frac{1}{2}} = \frac{1}{7}$$

b) Exercise 6.3.6, sections b, c

A biased coin is flipped 10 times. In a single flip of the coin, the probability of heads is $\frac{1}{3}$ and the probability of tails is $\frac{2}{3}$. The outcomes of the coin flips are mutually independent. What is the probability of each event?

b) (b) The first 5 flips come up heads. The last 5 flips come up tails. $(\frac{1}{3})^5 \cdot (\frac{2}{3})^5$

$p(H) = \frac{1}{3}$ and $p(T) = \frac{2}{3}$

$(\frac{1}{3})^5 \cdot (\frac{2}{3})^5$

c) (c) The first flip comes up heads. The rest of the flips come up tails.

$(\frac{1}{3}) \cdot (\frac{2}{3})^9$

c) Exercise 6.4.2, section a

(a) Assume that you have two dice, one of which is fair, and the other is biased toward landing on six, so that 0.25 of the time it lands on six, and 0.15 of the time it lands on each of $1, 2, 3, 4$, and 5. You choose a die at random, and roll it six times, getting the values $4, 3, 6, 6, 5, 5$. What is the probability that the die you chose is the fair die? The outcomes of the rolls are mutually independent.

a)

≈ 0.0544629349

p(1), p(2), p(3), p(4), p(5) == $\frac{1}{15}$, p(6) = $\frac{1}{4}$

$p(F) = \frac{1}{2}$

$p(\overline{F}) = \frac{1}{2}$

Out of 6 rolls, 2 were a six and 4 were not.

if it's loaded $p(X|\overline{F})$ then its $(\frac{1}{4})^2$ for a 6 and $(\frac{1}{15})^4$ for all the other rolls: $(\frac{1}{4})^2(\frac{1}{15})^4$

if it's fair $p(X|F)$ then its $(\frac{1}{6})^6$ for all rolls

$p(F|X)$ == $\frac{p(X|F)p(F)}{p(X|F)p(F) + p(X|\overline{F})p(\overline{F})}$

$\frac{(\frac{1}{4})^2 \cdot (\frac{1}{15})^4 \cdot \frac{1}{2}}{(\frac{1}{6})^6 \cdot \frac{1}{2} + (\frac{1}{4})^2 \cdot (\frac{1}{15})^4 \cdot \frac{1}{2}}$ = 0.0544629349

Question 9:

Solve the following questions from the Discrete Math zyBook:

a) Exercise 6.5.2, sections a, b

A hand of 5 cards is dealt from a perfectly shuffled deck of playing cards. Let the random variable A denote the number of aces in the hand.

a) (a) What is the range of A?

$\{0, 1, 2, 3, 4\}$

b) (b) Give the distribution over the random variable A.

$\binom{4}{0}$ = 1 $\binom{4}{1}$ = 4 $\binom{4}{2}$ = 6 $\binom{4}{3}$ = 4 $\binom{4}{4}$ = 1 // 16 variations

$\{(0, \frac{1}{16}), (1, \frac{1}{4}), (2, \frac{3}{8}), (3, \frac{1}{4}), (4, \frac{1}{16})\}$

b) Exercise 6.6.1, section a

a) (a) Two student council representatives are chosen at random from a group of 7 girls and 3 boys. Let G be the random variable denoting the number of girls chosen. What is $E[G]$?

$= 1.4$

Size of sample space = $\binom{10}{2} = 45$

Probability of picking 2 boys: $\frac{3}{45}$

Probability of picking 2 girls: $\frac{21}{45}$

Probability of picking 1 boy, 1 girl: $\frac{21}{45}$

$E[G] = \frac{21}{45} \cdot 3$ = 1.4

c) Exercise 6.6.4, sections a, b

(a) A fair die is rolled once. Let X be the random variable that denotes the square of the number that shows up on the die. For example, if the die comes up 5, then $X = 25$. What is $E[X]$?

a)

$= 15.1666667$

1 = 1 // 2 = 4 // 3 = 9 // 4 = 16 // 5 = 25 // 6 = 36

$$(1 + 1 + 4 + 9 + 16 + 25 + 36) \cdot \frac{1}{6} = \frac{91}{6}$$

= 15.1666667

(b) A fair coin is tossed three times. Let Y be the random variable that denotes the square of the number of heads. For example, in the outcome HTH, there are two heads and $Y = 4$. What is $E[Y]$?

b)

$E[Y] = 3$

Sample size = 8

Number of heads:

0^2 = 1 (1 instance)

1^2 = 1 (3 instances)

2^2 = 4 (3 instances)

3^2 = 9 (1 instance)

$$0 \cdot \frac{1}{8} + 1 \cdot \frac{3}{8} + 4 \cdot \frac{3}{8} + 9 \cdot \frac{1}{8} = 3$$

d) Exercise 6.7.4, section a

(a) A class of 10 students hang up their coats when they arrive at school. Just before recess, the teacher hands one coat selected at random to each child. What is the expected number of children who get his or her own coat?

a)

$= 1$

Suppose S = students and C = coats

Number of possible outcomes is 10^{10} (S^C)

Probability of the coat being assigned (1) = $\frac{1}{S} \cdot 1$

Probability of the coat not being assigned (0) = $(1 - \frac{1}{S}) \cdot 0$

$\frac{1}{10} \cdot 1 + (1 - \frac{1}{10}) \cdot 0$ = $\frac{1}{10}$

$= \frac{1}{10} \cdot 10$

Question 10:

Solve the following questions from the Discrete Math zyBook:

a) Exercise 6.8.1, sections a-d

The probability that a circuit board produced by a particular manufacturer has a defect is 1%. You can assume that errors are independent, so the event that one circuit board has a defect is independent of whether a different circuit board has a defect.

(a) What is the probability that out of 100 circuit boards made *exactly* 2 have defects?

a)

≈ 0.1848648188

$\binom{100}{98} \cdot (.99)^{98} \cdot (.01)^{2}$

=0.1848648188

(b) What is the probability that out of 100 circuit boards made *at least* 2 have defects?

b)

≈ 0.6302703624

Complement of at least 2 is: exactly 1 or exactly 0

$\binom{100}{1} \cdot (.99)^{1} \cdot (.01)^{99}$

$\binom{100}{0} \cdot (.99)^{0} \cdot (.01)^{100}$

$1-(\binom{100}{1} \cdot (.99)^{99} \cdot (.01)^{1})+(\binom{100}{0} \cdot (.99)^{0} \cdot (.01)^{100})$ = 0.6302703624

(c) What is the expected number of circuit boards with defects out of the 100 made?

c)

$=1$

np

n = 100

p = .01

= 1

(d) Now suppose that the circuit boards are made in batches of two. Either both circuit boards in a batch have a defect or they are both free of defects. The probability that a batch has a defect is 1%. What is the probability that out of 100 circuit boards (50 batches) at least 2 have defects? What is the expected number of circuit boards with defects out of the 100 made? How do your answers compared to the situation in which each circuit board is made separately?

d)

≈ 0.3949939329

each batch either has a 0% defect rate or a 100% defect rate

probability of exactly 2 (k) successes in a sequence of 50 independent trials with probability of success (p) .99 and probability of failure (q).01

The batches are treated as a singularity, where 50 batches = 100 boards

Complement of the probability of at least 2 is the probability of exactly 0

$1-\binom{50}{0} \cdot (0.99^{50}) \cdot (0.01^{0}) = 0.3949939329$

expected number of successes = np where q (since we are talking about defects) = .01 and n = 50, therefore number of defects is $.05$
$np \cdot 2$ (np shows the number of successes per n, which makes 2 boards per batch. So so the expected number needs to be doubled to $E[D] = 1$)

Based on my calculations, it appears the defect probability goes down when the batch size goes down, even it means that the entire batch is a defect

b) Exercise 6.8.3, section b

A gambler has a coin which is either fair (equal probability heads or tails) or is biased with a probability of heads equal to 0.3. Without knowing which coin he is using, you ask him to flip the coin 10 times. If the number of heads is at least 4, you conclude that the coin is fair. If the number of heads is less than 4, you conclude that the coin is biased.

b) (b) What is the probability that you reach an incorrect conclusion if the coin is biased?

≈ 0.1829812907

Biased when X < 4

Fair when X >= 4

$$\binom{10}{0}(\frac{1}{3})^0(\frac{7}{10})^{10}$$
$$\binom{10}{1}(\frac{1}{3})^1(\frac{7}{10})^{9}$$
$$\binom{10}{2}(\frac{1}{3})^2(\frac{7}{10})^{8}$$
$$\binom{10}{3}(\frac{1}{3})^3(\frac{7}{10})^{7}$$

=0.8170187093

Therefore, the probability of reaching an incorrect conclusion is

$$1-\binom{10}{0}(\frac{1}{3})^0(\frac{7}{10})^{10} + \binom{10}{1}(\frac{1}{3})^1(\frac{7}{10})^{9} + \binom{10}{2}(\frac{1}{3})^2(\frac{7}{10})^{8} + \binom{10}{3}(\frac{1}{3})^3(\frac{7}{10})^{7}$$

=0.1829812907