Question 5:

5a

Proof by induction on n. Let $p(n)$ be the proposition that for all positive integers n, that 3 divides evenly into $n^3 + 2n$.

Base case, where $n = 1$:

$$\frac{3}{n^3 + 2n} = \frac{3}{(1)^3 + 2(1)} = \frac{3}{3} = 1$$

remainder is 0 $\therefore$ where $n = 1$, we confirm that 3 divides evenly.

Inductive case: Assume the induction hypothesis that $p(k)$ is true for any arbitrary positive integer k, that is:

$k^3 + 2k$ is evenly divisible by 3

Now we will show that $p(k+1)$ is also true which can be represented as: $(k+1)^3 + 2(k+1)$

This expression can be expanded into the following: $k^3 + 3k^2 + 3k + 1 + 2k + 2$
By the commutative property of addition, we can reorder the expression to get $(k^3 + 2k) + (3k^2 + 3k + 3)$. The left side of this expression has already been shown to be evenly divisible by 3 via the induction hypothesis. The right side, we can factor out the 3 to get $3(k^2 + k + 1)$. Given that k has already been established as an integer, any integer derivative of k that is being multiplied by 3 is naturally divisible by 3 on the basis that the number is being multiplied by 3. Thus we have 2 addends that are both divisible by 3, and so we confirm that the expression which represents $p(k+1)$ is also evenly divisible by 3. ■

5b.

Proof by induction on n. Let $p(n)$ be the proposition that for all positive integers $n \geq 2$, that n is a product of primes.

Base case, $p(2)$ where $n = 2$. 2 is a prime number, and a product of itself, therefore it is a product of primes.

Induction step (strong induction): We will make an induction hypothesis that for all values of k from 2 to k, that all values of k are products of primes. We will show that given this induction hypothesis is true for all values of k, this hypothesis will also hold true for all values of $k + 1$ as well. For all values of $k + 1$, one of the following cases is true: the number is a prime number, or the number is not a prime number. If the number is prime then as mentioned earlier, a prime number is a product of itself, therefore is a product of primes. If the number is not prime more evaluations must occur to prove that it is a product of primes. We know that all numbers can be composed as a product of two numbers. Therefore we can state that all values of $k+1$ can be expressed as $x \cdot y$. The induction hypothesis already stated that any number from 2 to k a product of primes. Therefore x and y each are both products of primes. A product of two numbers that are both products of primes is also therefore a product of primes. $\therefore$ we confirm that all values of $k+1$ are products of primes. $\blacksquare$

Question 6:

Exercise 7.4.1, sections a-g

$$\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$$

a) Verify that $P(3)$ is true.

$$\sum_{j=1}^{3} 3^2 = \frac{3(3+1)(2(3)+1)}{6}$$

$$\sum_{j=1}^{3} 3^2 = 14$$

$$\frac{3(3+1)(2(3)+1)}{6} = 14$$

Both sides of the equation are 14, therefore $P(3)$ is true

b) Express $P(k)$

$$\sum_{j=1}^{k} j^2 = \frac{k(k+1)(2k+1)}{6}$$

c) Express $P(k+1)$

$$\sum_{j=1}^{k+1} j^2 = \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}$$

this can be reduced to:

$$\sum_{j=1}^{k+1} (j)^2 = \frac{(k+1)((k+2)(2k+3)}{6}$$

d) What must be proven in the base case? Since the claim is for every positive integer n and the first positive integer n is 1, the base case is 1. Therefore, we can express this as:

$$\sum_{j=1}^{1} 1^2 = \frac{1(1+1)(2(1)+1)}{6}$$

In this case, both sides are equal to 1, therefore we confirm that this equation is true.

e) What must be proven in the inductive step? The inductive hypothesis (see section f below) states that the expression is true for all values of k, so for the inductive step we replace k with $k+1$. This can be expressed as:

$$\sum_{j=1}^{k+1}(j+1)^2 = \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}$$

this can be reduced to:

$$\sum_{j=1}^{k+1}(j+1)^2 = \frac{(k+1)((k+2)(2k+3)}{6}$$

f) What would the inductive hypothesis be? The inductive hypothesis is that for all values of k, the sum of their squares is equal to $\frac{k(k+1)(2k+1)}{6}$

g) Prove by induction that for any positive integer n that the following is true:

$$\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$$

Proof by induction on n. Let $P(n)$ be the proposition that for all positive integers n that

$$\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$$

Base case, where $n = 1$ we show the following:

$$\sum_{j=1}^{1} 1^2 = \frac{1(1+1)(2(1)+1)}{6}$$

Both sides are equal to 1 so the base case holds true.

Inductive step. Assume the induction hypothesis that $P(k)$ is true for all arbitrary positive integers k, that is:

$$\sum_{j=1}^{k} j^2 = \frac{k(k+1)(2k+1)}{6}$$

Now we will show that $P(k+1)$ is also true which can be represented as:

$$\sum_{j=1}^{k+1}(j+1)^2 = \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}$$

which can further be reduced down to

$$\sum_{j=1}^{k+1}(j)^2 = \frac{(k+1)((k+2)(2k+3)}{6}$$

If the sum of all squares from $1 - k$ can be represented as $\sum_{j=1}^{k} j^2 = \frac{k(k+1)(2k+1)}{6}$, then the sum of all squares from 1 to k to $k+1$ can be represented as:

$$LHS = \frac{k(k+1)(2k+1)}{6} + (k+1)^2$$
$$RHS = \frac{(k+1)((k+2)(2k+3)}{6} = \frac{2k^3+9k^2+13k+6}{6}$$

Working on the LHS first:

$= \frac{2k^3+3k^2+k}{6} + (k+1)^2$ Simplify the numerator of the fraction

$= \frac{2k^3+3k^2+k \cdot 6(k^2+2k+1)}{6}$ Expand the (k+1)^2 and add it to numerator by multiplying by 6

$= \frac{2k^3+3k^2+k+6k^2+12k+6}{6}$ Multiply across the 6

$= \frac{2k^3+9k^2+13k+6}{6}$ Combine all like expressions to arrive at final expression

The final expression of the LHS matches the final expression of the RHS
$= \frac{2k^3+9k^2+13k+6}{6} = \frac{2k^3+9k^2+13k+6}{6}$

$\therefore$ we confirm that the proposition $\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$ is true. ■

Exercise 7.4.3, section c

c)

For $n \geq 1$, show that $\sum_{j=1}^{n} \frac{1}{j^2} \leq 2 - \frac{1}{n}$

Proof by induction on $n \geq 1$. Let $P(n)$ be the proposition that $2 - \frac{1}{n}$ is equal to the sum of all values $\frac{1}{j^2}$ from 1 to j.

Base case where $n = 1$

$2 - \frac{1}{1} \leq \frac{1}{1^2}$

Both sides of the equation are equal to 1 so we confirm that the base case holds true.

Induction step:

Assume the induction hypothesis that $P(k)$ is true for arbitrary positive integer k greater than 1, that is:

$\sum_{j=1}^{k} \frac{1}{j^2} \leq 2 - \frac{1}{k}$

Now we will show that $p(k+1)$ is also true which can be represented as:

$\sum_{j=1}^{k+1} \frac{1}{(j+1)^2} \leq 2 - \frac{1}{k+1}$

If the sum of all values $\frac{1}{k^2}$ from $1 - k$ are all less than $2 - \frac{1}{k}$ then the sum of all fractions where the denominators are squared up to $k+1$ can be represented as the following:

$2 - \frac{1}{k} \leq (2 - \frac{1}{k}) + (\frac{1}{(k+1)^2})$

$-\frac{1}{k} \leq (-\frac{1}{k}) + (\frac{1}{(k+1)^2})$ subtract 2 from both sides

$1 \geq (1 + (-\frac{k}{(k+1)^2})$ multiply both sides by -k and flip the inequality

$-1 \leq (\frac{k}{(k+1)^2} - 1)$ multiply both sides by -1 and flip the inequality.

$0 \leq (\frac{k}{(k+1)^2})$ add zero to both sides. This leaves you with the final expression.

Since we already know that k is an integer, even as a fraction, any fraction is still greater than 0 so we know this statement holds true. ∴ we confirm that $\sum_{j=1}^{n} \frac{1}{j^2} \leq 2 - \frac{1}{n}$ is true. ■

Exercise 7.5.1, section a

a) Prove that for any positive integer n, 4 divides evenly into $3^{2n} - 1$

Proof by induction on n. Let $P(n)$ be the proposition that 4 divides evenly into $3^{2n} - 1$
Base case, where $n = 1$
$$\frac{3^{2n} - 1}{4} = \frac{8}{4} = 2$$
The expression evaluates to a whole number with no remainder, so we know that the base case holds true.

Induction step: Assume the induction hypothesis that $P(k)$ is true for arbitrary positive integer k greater than 1, that is: $3^{2k} - 1$ divides evenly into 4.

We will show that 4 divides evenly into $3^{2(k+1)} - 1$
$= 3^{2k+2} - 1$ carry the 2 over
$= 3^{2k} \cdot 3^2 - 1$ exponent rule splits added exponents two two values with same bases
$= (3^2)^k \cdot 3^2 - 1$ exponent rule, raises power to a power when exponents are being multiplied
$= (3^2)^k \cdot (3^2)^1 - 1$ any number to the power of 1 is that number
$= (3^2)^{k+1} - 1$ combine same bases
$= (9)^{k+1} - 1$ simplify 3^2

9 can be powered by any number and it will produce a value that when subtracted by 1 will be divisible by 4. For example, 9^2 = 81, subtracted by 1 is 80 which is divisible by 4

$\therefore$ we confirm that 4 divides evenly into $3^{2(k+1)} - 1$. ■