Question 5:
a) Solve the following questions from the Discrete Math zyBook:

1. Exercise 1.12.2, sections b, e

b)

(b)
$$\begin{array}{l} p \rightarrow (q \land r) \\ \underline{\neg q} \\ \therefore \neg p \end{array}$$

#	Step	Explanation
1	$p \rightarrow (q \land r)$	Hypothesis
2	$p \rightarrow q$	Simplification, 1
3	$\neg q$	Hypothesis
4	$p \rightarrow q$ $\neg q$ $\therefore \neg p$	Modus tollens, 2, 3
5	$\neg p$	Conclusion, 4

e)

(e)
$$\begin{array}{l} p \lor q \\ \neg p \lor r \\ \underline{\neg q} \\ \therefore r \end{array}$$

#	**Step**	**Explanation**
1	$p \lor q$	Hypothesis
2	$\neg p \lor r$	Hypothesis
3	$q \lor r$	Resolution 1, 2
4	$\neg q$	Hypothesis
5	$q \lor r$ $\neg q$ $\therefore r$	Disjunctive syllogism 3,4
6	r	Conclusion, 5

2. Exercise 1.12.3, section c

(c) One of the rules of inference is disjunctive syllogism:

$$\begin{array}{l} p \vee q \\ \neg p \\ \hline \therefore q \end{array}$$

Prove that disjunctive syllogism is valid using the laws of propositional logic and any of the other rules of inference besides disjunctive syllogism. (Hint: you will need one of the conditional identities from the laws of propositional logic).

c)

#	**Step**	**Explanation**
1	$p \vee q$	Hypothesis
2	$\neg p \rightarrow q$	Conditional, 1
3	$\neg p$	Hypothesis
4	$\neg p \rightarrow q$ $\neg p$ $\therefore q$	Modus Ponens 2, 3
5	q	Conclusion, 4

3. Exercise 1.12.5, sections c, d

(c) I will buy a new car and a new house only if I get a job.
I am not going to get a job.

$\therefore$ I will not buy a new car.

c)

c = buy a new car h = buy a new house j = get a job	$H_1 = (c \wedge h) \rightarrow j$ $H_2 = \neg j$ $\therefore \neg c$

#	Step	Explanation
1	$(c \wedge h) \rightarrow j$	Hypothesis
2	$\neg(c \wedge h) \vee j$	Conditional: 1
3	$j \vee \neg(c \wedge h)$	Commutative: 2
4	$\neg j \rightarrow \neg(c \wedge h)$	Conditional: 3
5	$\neg j$	Hypothesis
6	$\neg j \rightarrow \neg(c \wedge h)$ $\neg j$ $\therefore \neg(c \wedge h)$	Modus Ponens: 4, 5
7	$\neg(c \wedge h)$ $\therefore \neg(c)$	Simplification: 6
8	$\neg c$	Conclusion, 7

(d)

I will buy a new car and a new house only if I get a job.
I am not going to get a job.
I will buy a new house.

$\therefore$ I will not buy a new car.

d)

c = buy a new car h = buy a new house j = get a job	$H_1 = (c \land h) \to j$ $H_2 = \neg j$ $H_3 = h$ $\therefore \neg c$

#	Step	Explanation
1	$(c \land h) \to j$	Hypothesis
2	$\neg(c \land h) \lor j$	Conditional: 1
3	$j \lor \neg(c \land h)$	Commutative: 2
4	$\neg j \to \neg(c \land h)$	Conditional: 3
5	$\neg j$	Hypothesis
6	$\neg j \to \neg(c \land h)$ $\neg j$ $\neg(c \land h)$	Modus Ponens: 4, 5
7	$\neg c \lor \neg h$	De Morgan's: 6
8	$\neg h \lor \neg c$	Commutative: 7
9	$h \to \neg c$	Conditional: 8
10	h	Hypothesis
11	$h \to \neg c$ h $\neg c$	Modus Ponens 9, 10
12	$\neg c$	Conclusion, 11

b) Solve the following questions from the Discrete Math zyBook:
1. Exercise 1.13.3, section b

(b)
$$\begin{array}{l} \exists x(P(x) \lor Q(x)) \\ \exists x \neg Q(x) \\ \hline \therefore \exists x P(x) \end{array}$$

<table>
<tr><td></td><td>P</td><td>Q</td><td>$\exists x(P(x) \lor Q(x))$
$\exists x \neg Q(x)$
―――――
$\therefore \exists x P(x)$</td><td>Conclusion</td></tr>
<tr><td>a</td><td>F</td><td>T</td><td>Hypothesis 1 : F $\lor$ T = T
Hypothesis 2: F</td><td rowspan="2">$\exists x(P(x) \lor Q(x))$ = T
Hypothesis 1 is true for domain item “a”:
$P(a) \lor Q(a)$ = T

$\exists x \neg Q(x)$ = T
Hypothesis 2 is true for domain item “b”
$\neg Q(b)$ = T

$\therefore \exists x \neg P(x)$
$\neg P(a) =$ F
$\neg P(b) =$F</td></tr>
<tr><td>b</td><td>F</td><td>F</td><td>Hypothesis: F $\lor$ F = F
Hypothesis 2: $\neg$ F = T</td></tr>
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2. Exercise 1.13.5, sections d, e

d)

(d)

Every student who missed class got a detention.
Penelope is a student in the class.
Penelope did not miss class.

Penelope did not get a detention.

$D(x)$ = x student got a detention $M(x)$ = x student missed class U = set of all students in the class	$H_1 = \forall x(M(x) \rightarrow D(x))$ $H_2 = Penelope$ is in U $H_3 = \neg M(Penelope)$ --- $\therefore \neg D(Penelope)$

#	**Step**	**Explanation**
1	$Penelope$ is in U	Hypothesis
2	$\forall x(M(x) \rightarrow D(x))$	Hypothesis
3	$(M(Penelope) \rightarrow D(Penelope))$	Universal Instantiation: 1, 2
4	$\neg M(Penelope)$	Hypothesis
5	$(M(Penelope) \rightarrow D(Penelope))$ $\neg M(Penelope)$ $\therefore \neg D(Penelope)$	Modus Tollens: 3,4

**The argument in Line 5 cannot be reconciled, therefore I will show that there exists a scenario where H_1, H_2, and H_3 are all true, but the conclusion is false

x	M	D	$(M(Penelope) \rightarrow D(Penelope))$ $\neg M(Penelope)$ --- $\therefore \neg D(Penelope)$	Conclusion
$Penelope$	F	T	Hypothesis 1 : F $\rightarrow$ T = T Hypothesis 2: T	$(M(Penelope) \rightarrow D(Penelope))$ = T $\neg M(Penelope)$= T $\therefore \neg D(Penelope)$ = F

(e)

Every student who missed class or got a detention did not get an A.
Penelope is a student in the class.
Penelope got an A.

Penelope did not get a detention.

e)

$D(x)$ = x student got a detention $M(x)$ = x student missed class $A(x)$ = x student got an A	$H_1 = \forall x(M(x) \vee D(x)) \to \neg A(x))$ $H_2 = Penelope$ is in U $H_3 = (A(Penelope))$ --- $\therefore \neg D(Penelope)$

#	Step	Explanation
1	$Penelope$ is in U	Hypothesis
2	$\forall x(M(x) \vee D(x)) \to \neg A(x))$	Hypothesis
3	$(M(Penelope) \vee D(Penelope)) \to \neg A(Penelope)$	Universal Instantiation: 1, 2
4	$A(Penelope)$	Hypothesis
5	$(M(Penelope) \vee D(Penelope)) \to \neg A(Penelope)$ $A(Penelope)$ $\therefore \neg(M(Penelope) \vee D(Penelope))$	Modus Tollens 3,4
6	$(\neg M(Penelope) \wedge \neg D(Penelope))$	De Morgan's, 5
7	$\neg M(Penelope) \wedge \neg D(Penelope)$ $\therefore \neg D(Penelope)$	Simplification, 6
8	$\therefore \neg D(Penelope)$	Conclusion, 5

Question 6:
Solve Exercise 2.4.1, section d

d) (d) The product of two odd integers is an odd integer.

	Explanation
Statement to Prove	The product of two odd integers is an odd integer.
Proof	• Without loss of generality, assume that x and y are 2 odd integers. By means of a direct proof, we will show that xy is an odd integer. • Since x is odd, there is an integer k which can be expressed as $2k+1$, which can therefore represent x and y respectively as $x = 2k+1$ and $y = 2k+1$ • Take the statement xy and plug $(2k+1)$ in for x and y to get this expression: $(2k+1)(2k+1)$ • Expand the polynomial expression to get $= 4k^2 + 4k + 1$ • Then transform it again to get $2(2k_2 + 2k) + 1$ • Since k is an integer then $2k_2 + 2k$ is an integer m. • It can then be deduced then that integer m is odd as $(2(2k_2 + 2k) + 1) = (2k+1)$
Conclusion	In conclusion $2(m) + 1$ is odd, and therefore the product of two integers (represented as xy) which is the original equation from which $2m+1$ was derived, is odd. ■

Exercise 2.4.3, section b, from the Discrete Math zyBook:

b) (b) If x is a real number and $x \leq 3$, then $12 - 7x + x^2 \geq 0$.

	Explanation
Statement to Prove	If x is a real number and $x \leq 3$, then $12 - 7x + x^2 \geq 0$ $x \leq 3$ means "x is at most 3"
Proof	• Let x be a real number, and $x \leq 3$. By means of direct proof, we will prove that $12 - 7x + x^2 \geq 0$ • We can reformat $x \leq 3$ to learn that $x - 3 \leq 0$. • If $12 - 7x + x^2 \geq 0$ and $x - 3 \leq 0$ then we know that the following is true: $12 - 7x + x^2 \geq 0 \geq x - 3$ • $12 - 7x + x^2 \geq 0$ can be factored to $(x-3)(x-4) \geq 0$ therefore, $x \geq 4$ or $x \leq 3$. Since we have already established that $x \leq 3$ is the same thing as $x - 3 \leq 0$ we will replace $x \leq 3$ with that. And $x \geq 4$ can be reformatted to $x - 4 \geq 0$. • Thus we know that $x - 3 \leq 0 \leq x - 4$ • The only way for this statement to be true is if $x \leq 3$
Conclusion	Therefore we confirm via direct proof that $12 - 7x + x^2 \geq 0$ if $x \leq 3$ ■

Question 7:
Solve Exercise 2.5.1, section d

d) (d) For every integer n, if $n^2 - 2n + 7$ is even, then n is odd.

	Explanation
Statement to Prove	For every integer n, if $n^2 - 2n + 7$ is even, then n is odd.
Proof	• Let n be an even integer. By means of proof by contrapositive, we will prove that for some integer n if n is even, then $n^2 - 2n + 7$ is odd. • Since n is even, there is an integer k such that $n = 2k$ • Plug in $n = 2k$ for n in $n^2 - 2n + 7$ to get this expression: $(2k)^2 - 2(2k) + 7$ • This can be reduced to the following: $4k^2 - 4k + 7$ • This can further be reformulated to $2(2k^2 - 2k) + 7$ • Let's assign the value $2k^2 - 2k$ to the integer m, since k is an integer. • Now we have $2m + 7$. $2m$ is even as it is a number multiplied by 2, and 7 is odd. We know 7 is odd because it is not evenly divisible by 2. • The sum of any even and an odd number is always odd.
Conclusion	So we confirm in this proof of contrapositive that when n is even, then $n^2 - 2n + 7$ is odd. ■

Exercise 2.5.4, sections a, b

a) For every pair of real numbers x and y, if $x^3 + xy^2 \leq x^2y + y^3$, then $x \leq y$.

	Explanation
Statement to Prove	For every integer x and y if $x > y$ then $x^3 + y^2 > x^2y + y^3$
Proof	• By means of contrapositive proof, let x and y be integers where $x > y$. We will show that $x^3 + y^2 > x^2y + y^3$ • If you reformat the inequality so that you extract one x from the left, and one y from the left of the inequality you are left with the following inequality: $x(x^2 + y^2) > y(x^2 + y^2)$ • Thus we can multiply each side $x > y$ from the hypothesis by $(x^2 + y^2)$, to be left with $x(x^2 + y^2) > y(x^2 + y^2)$ • Multiplying in the x and the y on each side results in $x^3 + y^2 > x^2y + y^3$
Conclusion	Therefore we conclude by means of contrapositive proof that as long as $x > y$ then $x^3 + y^2 > x^2y + y^3$ ■

b) For every pair of real numbers x and y, if $x + y > 20$, then $x > 10$ or $y > 10$.

	Explanation
Statement to Prove	For integer x and y if $x + y > 20$ then $x > 10$ or $y > 10$
Proof	• Let us assume for the purpose of this proof by contrapositive that x and y are integers where $x \leq 10$ and $y \leq 10$. We will show then that $x + y \leq 20$ is true. • When we represent $$x$$ and $$y$$ as $x \leq 10$ and $y \leq 10$, if we add x and y according to the values of these inequalities, we get: $x + y \leq 20$.
Conclusion	Therefore we have confirmed by means of proof by contrapositive that if $x \leq 10$ and $y \leq 10$, then $x + y \leq 20$. This also means that x and y must both be greater than 10 in order for, then $x + y > 20$ ■

Exercise 2.5.5, section c, from the Discrete Math zyBook:

c) For every non-zero real number x, if x is irrational, then $\frac{1}{x}$ is also irrational.

	Explanation
Statement to Prove	If $\frac{1}{x}$ is rational, then x is rational
Proof	• By means of contrapositive proof, we will show that x is a real number that is not 0 • Let's assume that $x > 0$ and $x < 0$ then $x \neq 0$, and that any rational statement can be expressed as the ratio of two integers. Therefore in $\frac{1}{x}$, the x must represent an integer. Since we know that $x \neq 0$, then we can confirm that in $\frac{1}{x}$, the denominator is not 0. Therefore the fraction $\frac{1}{x}$ is in fact a real and rational number. • If $\frac{1}{x}$ is rational, then it can be represented as $\frac{1}{\frac{f}{g}}$ where $\frac{f}{g}$ represents the rational nature of x. Since $\frac{f}{g}$ represents x, then we know that $f \neq 0$ because that would make $x = 0$ which we already know is not true. We also know that $g \neq 0$ because that would make the rational representation of x (which is $\frac{1}{x}$) invalid. • Knowing now that $f \neq 0$ and $g \neq 0$, then $\frac{1}{\frac{f}{g}}$ can be transformed into $\frac{g}{f}$ • If $\frac{g}{f}$ is rational, then f and g are both integers • We've already established that $\frac{g}{f}$ can replace $\frac{1}{x}$, but $\frac{g}{f}$ can also transform x (from the conclusion) to $\frac{f}{g}$ (the multiplicative inverse of $\frac{1}{x}$) • And so, we can conclude that if g and f are both integers and $\frac{g}{f}$ is rational, then $\frac{f}{g}$ is rational as well.
Conclusion	Therefore we confirm by means of contrapositive proof that confirm that x is rational ■

Question 8:
Solve Exercise 2.6.6, sections c, d, from the Discrete Math zyBook:

c) The average of three real numbers is greater than or equal to at least one of the numbers.

	Explanation
Statement to Prove	The average of 3 real numbers is greater than or equal to at least 1 of the numbers
Proof	• By means of proof by contradiction, we will assume that it is false that the average of 3 real numbers is greater than or equal to at least one of the numbers. We will attempt to show there exists a set of 3 real numbers where the average is less than each of the 3 numbers. • Instead of testing an infinite number of numbers, we can use the definition of an average to help us with this proof. Average formula can be represented as $\frac{a + a_2 + a_3 ... a_n}{n}$ therefore, to represent the average of the three numbers we can use the formula $\frac{n_1 + n_2 + n_3}{3}$. If there are 3 items being summed, and they are then going to be divided by 3, then each number can essentially be represented as equal portions of the sum in the numerator of the average formula. Therefore, as long as we are going to be finding their average (dividing by 3) we can represent the sum of these 3 numbers as , $\frac{x}{3}$ for each value of n. • If $\frac{x}{3}$ represents n then the average of these 3 numbers can be represented as: $\frac{\frac{x}{3} + \frac{x}{3} + \frac{x}{3}}{3}$ • If we are trying to show that the average of 3 real numbers is less than each one of the 3 numbers then we can represent it with this equation. $\frac{\frac{x}{3} + \frac{x}{3} + \frac{x}{3}}{3} < \frac{x}{3}$ • Multiply both sides by 3 which cancels out both of the denominators leaving just $\frac{x}{3} + \frac{x}{3} + \frac{x}{3} < x$ This can be further simplified down to $\frac{3x}{3} < x$ • After one final simplification we are left with $x < x$ • This is not possible as the only operator that would cause this inequality to evaluate to true is $\leq$, $\geq$, or it would have to be an equality statement with =
Conclusion	Therefore can conclude that the assumption that the average of 3 numbers is less than each of the numbers must be false ■

d) There is no smallest integer.

	Explanation
Statement to Prove	Assume there is an integer i that is the smallest integer.
Proof	• By means of a proof by contradiction, we will attempt to prove that there is indeed an integer i that is the smallest integer. • As an integer, i can be represented as $\frac{i}{1}$ • If i is an integer smaller than another integer, then $i < \frac{i}{1}$ must be true • Multiply both sides by i to arrive at $i < i$ • The only way this expression could be true is if the inequality operator was $\geq$, or $\leq$, or if the statement was an equality statement with =
Conclusion	Therefore, by means of proof by contradiction, we have concluded that the assumption that there is an integer i that is the smallest integer is false ■

Question 9:
Solve Exercise 2.7.2, section b, from the Discrete Math zyBook:

b) If integers x and y have the same parity, then $x + y$ is even.

	Explanation
Statement to Prove	When x and y are even, then $x + y$ is even, and when x and y are odd, then $x + y$ is also even
Proof	• Without loss of generality, let us have two variables x and y which are either both odd, or both even • We will attempt to show via proof by cases, that in the two separate cases assuming that x and y are either both even or odd, that when x and y share a parity, the sum of the two integers is even. CASE 1: • Assume without loss of generality that x and y are even. We will use x as the example to apply across to y as well. • If x is even, then x can be represented as $2k$ where k is some integer. • Plug in $2k$ into $x + y$. It can be plugged in to both x and y because both are even in this case. This gives us $2k + 2k$ • $2k + 2k$ can be reformatted to $4k$, which can be further reformatted to $2(2k)$ • Since k is an integer, then $2(2k)$ is also an integer • Since $x + y$ has been re-represented as $2(2k)$ which is equal to $2m$ where $m = 2k$, then $x + y$ is even. • Thus, Case 1 proves that when x and y are even then $x + y$ is even. CASE 2: • Assume without loss of generality x and y are odd. We will use x as the example to apply across to y as well. • If x is odd, then x can be represented as $2k + 1$ where k is some integer. • Plug in $2k + 1$ into $x + y$. It can be plugged in to both x and y because both are odd in this case. This gives us $(2k + 1) + (2k + 1)$ • $(2k + 1) + (2k + 1)$ can be reformatted to $4k + 2$, which can be further reformatted to $2(2k + 1)$ • Since k is an integer, then $2(2k + 1)$ is also an integer • Since $x + y$ has been re-represented as $2(2k + 1)$ which is equal to $2m$ where $m = 2k + 1$, then $x + y$ is even. • Thus, Case 2 proves that when x and y are odd then $x + y$ is even.
Conclusion	Therefore through a proof by two cases, we can conclude since both cases show that $x + y$ is even, that when x and y share a parity that $x + y$ is even. ■