Question 5:
Use the definition of Θ in order to show the following:

a. $5n^3 + 2n^2 + 3n = \Theta(n^3)$

In order to show that $5n^3 + 2n^2 + 3n = \Theta(n^3)$ we must show that $5n^3 + 2n^2 + 3n = O(n^3)$ and $5n^3 + 2n^2 + 3n = \Omega(n^3)$. Therefore we will prove each below, given the following:

$f(n) = 5n^3 + 2n^2 + 3n$
$g(n) = n^3$

Proof that $5n^3 + 2n^2 + 3n = O(n^3)$(Big Oh)

Given the following values $c = 10$ and $n_0 = 1$, we will show that for any $n \geq 1$ that $f(n) \leq 10 \cdot g(n)$
Since $n \geq 1$, we know that $n^2 \leq n^3$ is also true. So furthermore, we also know the following is true:
$5n^3 + 2n^2 + 3n \leq 5n^3 + 2n^3 + 3n^3$
Looking on the right hand side, $5n^3 + 2n^3 + 3n^3$ is equal to $10n^3$ which bears relation to $g(n)$. In order to replicate $f(n) \leq 10 \cdot g(n)$, we put the two inequalities together. Thus we get
$5n^3 + 2n^2 + 3n \leq 10n^3 = 10 \cdot g(n)$
Therefore we confirm that for $n \geq 1$, $f(n) \leq 10 \cdot g(n)$ which means that $f = Og(n)$ where $g(n)$ is n^3. ■

Proof that $5n^3 + 2n^2 + 3n = \Omega(n^3)$ (Big Omega)
Given the following values $c = 5$ and $n_0 = 1$, we will show that for any $n \geq 1$ that $f(n) \geq 5 \cdot g(n)$.
Since $n \geq 1$, then we know that $3n \geq 0$. We can also prove $2n^2$ is greater than 0. If you extract n from $2n^2$ you get $n(2n)$. Since $n \cdot 2n$ is greater than $3n$ we can add $2n^2$ to the inequality and know that the following statement is still true: $2n^2 + 3n \geq 0$. Add $5n^3$ to both sides and we get $5n^3 + 2n^2 + 3n \geq 5n^3$
Therefore we conclude that for $n \geq 1$, $f(n) \geq 5 \cdot g(n)$ which means that f is $\Omega g(n)$ where $g(n)$ is n^3. ■

In conclusion, since $5n^3 + 2n^2 + 3n = O(n^3)$ and $5n^3 + 2n^2 + 3n = \Omega(n^3)$ are both true, then we confirm that $5n^3 + 2n^2 + 3n = \Theta(n^3)$ is true as well.

b. $\sqrt{7n^2 + 2n - 8} = \Theta(n)$

In order to show that $\sqrt{7n^2 + 2n - 8} = \Theta(n)$ we must show that $\sqrt{7n^2 + 2n - 8} = O(n)$ and $\sqrt{7n^2 + 2n - 8} = \Omega(n)$. Therefore we prove each below given the following:

$f(n) = \sqrt{7n^2 + 2n - 8}$
$g(n) = n$

Proof that $\sqrt{7n^2 + 2n - 8} = O(n)$ (Big Oh):

Given the following values $c = 3$ and $n_0 = 1$, we will show that for any $n \geq 1$, $f(n) \leq 3 \cdot g(n)$
First we square both expressions to get: $f(n)^2 = 7n^2 + 2n - 8$ and $9 \cdot g(n)^2 = 9 \cdot n^2$. Starting with the left hand side, $f(n)^2$, we know that $7n^2 + 2n - 8 \leq 7n^2 + 2n$ because the expressions are identical save the -8 on the left hand side, which makes the left hand side a smaller value.
Since $n \geq 1$, we also know that $n \leq n^2$, so the following must be true: $7n^2 + 2n - 8 \leq 7n^2 + 2n^2$
Looking on the right hand side of the above inequality we know that $7n^2 + 2n^2$ is equal to $9n^2$.
$g(n)$ bears resemblance to $9n^2$. In order to replicate the inequality $f(n)^2 \leq 9 \cdot g(n)^2$, we put the two inequalities together to get $7n^2 + 2n - 8 \leq 9n^2 = 9 \cdot g(n)^2$. Then we square both sides to get: $\sqrt{7n^2 + 2n - 8} \leq \sqrt{9n^2}$. This can be further simplified to: $\sqrt{7n^2 + 2n - 8} \leq 3n$.
Therefore, we confirm that for $n \geq 1$, $f(n) \leq 3 \cdot g(n)$ which means that f is $O(g)$ where $g(n)$ is n. ■

Proof that $\sqrt{7n^2 + 2n - 8} = \Omega(n)$(Big Omega)

Given the following values $c = 1$ and $n_0 = 3$, we will show that for any $n \geq 3$, $f(n) \geq 1 \cdot g(n)$.
First we square both expressions to get: $f(n)^2 = 7n^2 + 2n - 8$ and $g(n)^2 = 1 \cdot g(n)^2$
Plugging in the definitions of $f(n)^2$ and $g(n)^2$, the goal is to show that $7n^2 + 2n - 8 \geq n^2$
Since $n \geq 3$ then it is also true that $n \geq 1$. We will start with the inequality $n \geq 1$. Multiply both sides by -8 and flip the inequality to get $-8n \leq -8$. Add $7n^2 + 2n$ to both sides. This gets us:
$7n^2 + 2n - 8n \geq 7n^2 + 2n - 8$. The left hand side can be simplified to $7n^2 - 6n$.
We will need to show that that $7n^2 - 6n$ is greater than n^2. To do so we will take the inequality $n \geq 3$, and plug in the lowest permitted value of n into $7n^2 - 6n \geq n^2$. We get $45 \geq 9$. And so we have proved that $7n^2 - 6n$ is greater than n^2

Now we can put the inequalities together to show that $7n^2 + 2n - 8 \geq 7n^2 - 6n \geq n^2$. We drop the interim expression in the middle, so we can just say $7n^2 + 2n - 8 \geq n^2$. n^2 is the same thing as $1 \cdot g(n)^2$ This shows that $f(n)^2 \geq 1 \cdot g(n)^2$. To show that $f(n) \geq c \cdot g(n)$, then we square root both sides, to get

$\sqrt{7n^2 + 2n - 8} \geq n$. Therefore we can confirm that for all, $n \geq 3$, $f(n) \geq 1 \cdot g(n)$ which means that f is $\Omega g(n)$ where $g(n)$ = n

In conclusion, since $\sqrt{7n^2 + 2n - 8} = O(n)$ and $\sqrt{7n^2 + 2n - 8} = \Omega(n)$ are both true, then we confirm that $\sqrt{7n^2 + 2n - 8} = \Theta(n)$ is true as well. ■