Question 3:

a. Solve Exercise 8.2.2, section b from the Discrete Math zyBook.

Give complete proofs for the growth rates of the polynomials below. You should provide specific values for c and n_0 and prove algebraically that the functions satisfy the definitions for O and Ω.

b) (b) $f(n) = n^3 + 3n^2 + 4$. Prove that f is $\Theta(n^3)$.

In order to show that $n^3 + 3n^2 + 4 = \Theta(n^3)$ we must show that $n^3 + 3n^2 + 4 = O(n^3)$ and $n^3 + 3n^2 + 4 = \Omega(n^3)$. Therefore we will prove each below, given the following:

$f(n) = n^3 + 3n^2 + 4$
$g(n) = n^3$

Proof that $n^3 + 3n^2 + 4 = O(n^3)$

Given the following values $c = 8$ and $n_0 = 1$, we will show that for any $n \geq 1$ that $f(n) \leq 8 \cdot g(n)$.

Since $n \geq 1$, we know that $n^2 \leq n^3$ is also true. Based on this, we also know the following is true: $n^3 + 3n^2 + 4 \leq n^3 + 3n^3 + 4n^3$. Looking on the right hand side, we know that $n^3 + 3n^3 + 4n^3 = 8n^3$. $8n^3$ bears relation to $g(n)$. In order to replicate $f(n) \leq 8 \cdot g(n)$ we put the two inequalities together thus we get $n^3 + 3n^2 + 4 \leq 8 \cdot n^3$. Therefore we confirm that for $n \geq 1, f(n) \leq 8g(n)$ which means that $f = Og(n)$ where $g(n)$ is n^3. ■

Proof that $n^3 + 3n^2 + 4 = \Omega(n^3)$

Given the following values of $c = 1$ and $n_0 = 1$, we will show that for any $n \geq 1$ that $f(n) \geq 1 \cdot g(n)$. We will start with $n \geq 1$. Since $n \geq 1$, then $3n^2 \geq 0$. Of course $4 \geq 0$ as well. You can add these inequalities to get $3n^2 + 4 \geq 0$. Add n^3 to both sides to get the following statement: $n^3 + 3n^2 + 4 \geq n^3$. The n^3 is equal to the n^3 in $g(n)$ and $1 \cdot g(n) = g(n)$ Therefore we can state that $n^3 + 3n^2 + 4 \geq n^3 = g(n)$.

So for $n \geq 1$, $f(n) \geq 1 \cdot g(n)$ which means that $f = \Omega g(n)$ where $g(n)$ is n^3 ■

In conclusion, since $n^3 + 3n^2 + 4 = O(n^3)$ and $n^3 + 3n^2 + 4 = \Omega(n^3)$ are both true, then we confirm that $n^3 + 3n^2 + 4 = \Theta(n^3)$ is true as well. ■

b. Solve Exercise 8.3.5, sections a-e from the Discrete Math zyBook

The algorithm below makes some changes to an input sequence of numbers.

MysteryAlgorithm

Input: $a_1, a_2, \ldots, a_n$
n, the length of the sequence.
p, a number.
Output: ??

$i := 1$
$j := n$

While $(i < j)$
 While ($i < j$ and $a_i < p$)
  $i := i + 1$
 End-while
 While ($i < j$ and $a_j \geq p$)
  $j := j - 1$
 End-while
 If $(i < j)$, swap a_i and a_j
End-while

Return($a_1, a_2, \ldots, a_n$)

(a) Describe in English how the sequence of numbers is changed by the algorithm. (Hint: try the algorithm out on a small list of positive and negative numbers with $p = 0$)

a)
This is an algorithm that is putting items that are bigger than P to the right side of the sequence, and items that are less than P to the left side of the sequence. “Left side” and “right side” in this case means that the numbers that are adjacent to each other are all “less than P” or “greater than P”, with the exception of the two numbers that are the “invisible dividing line” between the two sides.

For example, the following 3 sequences are valid final outputs of the algorithm:
$\{-3, -15, -4, 12, 9, 4\}$ where P = 0, and the “split” is between -4 and 12
$\{-15, -12, -18, -20, -1, -6\}$ where P = 0, the “split” is off screen because all numbers are less than 0
$\{-15, -12, -18, -20, -1, 6\}$ where P = 2, and the “split” is between -1 and 6

How it is doing it, is it starts from a_1 on the left side of the sequence (where the “less than P” values will all end up) and compares the a_1 value to the value of P. If it is less than P it moves to the next value in the sequence, since the first value does not need to move sides. It keeps checking, down the sequence, until it comes across a number that is greater than P (therefore needs to be moved to the right side of P). It then stops, and moves to the next while loop, evaluating from the end of the sequence, to see if there is a value that is lower than P on the right side of the sequence that needs to be moved over to the “less than P” side. It employs the same method as the last loop. It starts from the end, checks to see if the value is in fact greater than P, if so, moves on to the next one. Once it encounters the first value from the end that is less than P (therefore on the wrong side), it moves on to the if statement which will swap the last evaluated value of ai with the last evaluated value of aj, thus placing them in their respective sides of P. After the swap, the algorithm runs again starting now where it had last left off with ai and aj, and repeats the process as before. It continues to do this until all items that are less than P are to the left and all items greater than P are to the right. Note that it does not order all the numbers sequentially, it just moves them from one side to the other depending on its compared value to P.

b) (b) What is the total number of times that the lines "$i := i + 1$" or "$j := j - 1$" are executed on a sequence of length n? Does your answer depend on the actual values of the numbers in the sequence or just the length of the sequence? If so, describe the inputs that maximize and minimize the number of times the two lines are executed.

At a maximum, i++ and j-- combined will happen $n - 1$ times. While the outer loop runs n-1 times as well, since the i++ and the j-- don't necessarily run from 1 to n for every iteration of the outer loop, the runtime is not exponential. The whole algorithm will only run through the whole sequence once until i and j are equal to each other. No matter what the sequence is, the algorithm has to run the whole length of the sequence. The ultimate behavior of the algorithm (whether the outer loop is run multiple times, or if the inner loops are run only once, etc etc) will depend on the numbers of the sequence but does not impact the number of times i and j is incremented/decremented because no matter what the algorithm still just has to run n times through the whole sequence.

Imagine a sequence like the following: $\{3, 15, 4, -12, -9, -4\}$ where $p = 0$. In this case all of the numbers will need to be swapped. The outer loop will run 3 times for each swap that needs to occur, but at the end of the day, the i and j will increment/decrement for a total of $n - 1$ times until i and j are equal to each other.

c) (c) What is the total number of times that the swap operation is executed? Does your answer depend on the actual values of the numbers in the sequence or just the length of the sequence? If so, describe the inputs that maximize and minimize the number of times the swap is executed.

The swap will run at a maximum $\frac{n}{2}$ times, because the algorithm will change the position of two values at the same time. The answer does depend on the values of the numbers in the sequence as well as the length of the sequence. If none of the numbers needs to be swapped, then the swap will not happen at all. If all of the numbers need to be swapped, then it will run $\frac{n}{2}$. For example, in an instance where all of the numbers need to be swapped of a 6-number sequence, since the algorithm moves inwards from both sides, it will first swap [indices] 1 and 6, then 2 and 5, then 3 and 4, and then output the results. Even in a similar instance where there is a 7 number sequence, it will first swap 1 and 7, 2 and 6, then 3 and 5, and then 4 will already be either to the left or to the right of whatever "side" it belongs on, so that swap does not have to happen. The outer loop will run one more time though so that the number 4 can be evaluated still, but after that loop runs a swap will not occur. To minimize the number of swaps, all of the items need to be already on the "left" or "right" sides as needed, or the value of P needs to be greater than or less than all of the values in the sequence.

d) (d) Give an asymptotic lower bound for the time complexity of the algorithm. Is it important to consider the worst-case input in determining an asymptotic lower bound (using Ω) on the time complexity of the algorithm? (Hint: argue that the number of swaps is at most the number of times that i is incremented or j is decremented).

The lower bound is $\Omega(n)$. This is because no matter what, the algorithm will go through all of the items in the sequence, even if no swap needs to occur. Every value is evaluated. The worst case input doesn't change the overall proportion of times that i is incremented or j is decremented compared to the input quantity. A bad-case input would be a large sequence where no items need to be swapped. A worst case input would be a very large sequence of numbers where all of the numbers need to be swapped. i and j are still incremented/decremented a total of $n-1$ times. Best case would be if no numbers need to be swapped. This would be the case if the sequence was already in the right format, or if all of the values were already greater than or less than P. In this case, the outer loop would run only once, incrementing and decrementing the i and the j until they converge to equal values. At that point, the condition for the swap cannot be met, nor can the condition to run another outer loop be met so the algorithm ends.

e)

(e) Give a matching upper bound (using O-notation) for the time complexity of the algorithm.

The upper bound is $O(n)$ as well. This is because the algorithm is never running more than $n-1$ times, the length of the sequence.

Question 4:
Solve the following questions from the Discrete Math zyBook:
a) Exercise 5.1.2, section b, c

b) (b) Strings of length $7, 8$, or 9. Characters can be special characters, digits, or letters.
Digits - 10, Letters - 26, Special characters - 4, Total characters: 40
40^7 + 40^8 + 40^9

c) (c) Strings of length $7, 8$, or 9. Characters can be special characters, digits, or letters. The first character cannot be a letter.
14 = all characters, no letters + 40 for the remaining characters in the password
$(14^1 * 40^6) + (14^1 * 40^7) + (14^1 * 40^8)$

b) Exercise 5.3.2, section a

a) (a) How many strings are there over the set $\{a, b, c\}$ that have length 10 in which no two consecutive characters are the same? For example, the string "$\mathbf{abcbcbabcb}$" would count and the strings "$\mathbf{abbbcbabcb}$" and "$\mathbf{aacbcbabcb}$" would not count.
$3 \cdot 2^9$

c) Exercise 5.3.3, sections b, c
License plate numbers in a certain state consist of seven characters. The first character is a digit (0 through 9). The next four characters are capital letters ($\mathbf{A}$ through $\mathbf{Z}$) and the last two characters are digits. Therefore, a license plate number in this state can be any string of the form:

Digit-Letter-Letter-Letter-Letter-Digit-Digit

b) (b) How many license plate numbers are possible if no digit appears more than once?
All variations = 9*26*26*26*26*9*9 = $9^3 * 26^4$ = 333,135,504 … therefore if no digit appears more than once
$26^4 \cdot (10 \cdot 9 \cdot 8)$ = 329,022,720

c) (c) How many license plate numbers are possible if no digit or letter appears more than once?
$10 \cdot 9 \cdot 8 \cdot 26 \cdot 25 \cdot 24 \cdot 23$
10*26*25*24*23*9*8 = 258,336,000

d) Exercise 5.2.3, sections a, b

Let $B = \{0, 1\}$. B^n is the set of binary strings with n bits. Define the set E_n to be the set of binary strings with n bits that have an even number of 1's. Note that zero is an even number, so a string with zero 1's (i.e., a string that is all 0's) has an even number of 1's.

a) (a) Show a bijection between B^9 and E_{10}. Explain why your function is a bijection.

$B^9 = \{0,1\}^9$ – 9 bit strings composed of 0s and 1s

Power set of $\{0,1\}^9$ = 2^9 = 512 total number of strings that are possible for B^9

Number of strings in E_{10} with even number of 1s:

0 ones = $\binom{10}{0}$ = 1 2 ones = $\binom{10}{2}$ = 45 4 ones = $\binom{10}{4}$ = 210

6 ones = $\binom{10}{6}$ = 210 8 ones = $\binom{10}{8}$ = 45 10 ones = $\binom{10}{10}$ = 1

1 + 45 + 210 + 210 + 45 + 1 = 512 total number of strings that are possible for E_{10}

For every 9-bit string option, there is exactly one option that E_{10} can be mapped to (because they have the same cardinality). So the relationship between B^9 and E_{10} is 1 to 1.

All of the strings in B^9 are unique to each other (no one string in the B^9 set matches another string in B^9), and all of the strings in E_{10} are unique to each other in the same way (no one string in E_{10} matches another string in E_{10}. So any function that maps B^9 to E_{10} would be a bijection because there are 512 unique items to map to 512 other unique items. This makes it onto because every item in the codomain (E_{10}) is being mapped to. This also shows that it is one-to-one because no two items in the domain are being mapped to a single item in the codomain.

b) (b) What is $|E_{10}|$? 512

Number of strings in E_{10} with even number of 1s:

0 ones = $\binom{10}{0}$ = 1 2 ones = $\binom{10}{2}$ = 45 4 ones = $\binom{10}{4}$ = 210

6 ones = $\binom{10}{6}$ = 210 8 ones = $\binom{10}{8}$ = 45 10 ones = $\binom{10}{10}$ = 1

í

Question 5:
Solve the following questions from the Discrete Math zyBook:
a) Exercise 5.4.2, sections a, b

At a certain university in the U.S., all phone numbers are 7-digits long and start with either 824 or 825.

a) (a) How many different phone numbers are possible?

$824,\ 825$ = 2, therefore between the two sets of numbers: $2 \cdot 10^4$

b) (b) How many different phone numbers are there in which the last four digits are all different?

$2 \cdot (10^4 \cdot 10^3 \cdot 10^2 \cdot 10^1)$

b) Exercise 5.5.3, sections a-g

How many 10-bit strings are there subject to each of the following restrictions?

a) (a) No restrictions.

$1,024$

$2^{10} = 1024$

b) (b) The string starts with 001.

128

$1^3 * 2^7 = 128$

c) (c) The string starts with 001 or 10.

384

$1^3 * 2^7 = 128$
$1^2 * 2^8 = 256$
$128 + 256$ = 384

d) (d) The first two bits are the same as the last two bits.

256

There are 4 variations of the 2 bits: 00,01,10, 11
For each variation, there are $1^4 * 2^6$ unique strings.
Therefore the total number of options are : 4* 1^4 * 2^6 = 4 * 64 = 256

e) (e) The string has exactly six 0's.

210

$\binom{10}{6}$ = 210

f) (f) The string has exactly six 0's and the first bit is 1.

84

$\binom{9}{6}$ = 84

If you were to imagine a string where 10 choose 6 applies, and add a constant value to the left of it, it still is 10 choose 6. Therefore, reduce the number of bits to 9, but still choose 6.

g) (g) There is exactly one 1 in the first half and exactly three 1's in the second half.

50

Exactly one 1 in the first half = 5 choose 1 = 5

Exactly three 1s in the second half = 5 choose 3 = 10

$5 * 10$ = 50

c) Exercise 5.5.5, section a

a) (a) There are 30 boys and 35 girls that try out for a chorus. The choir director will select 10 girls and 10 boys from the children trying out. How many ways are there for the choir director to make his selection?

$\binom{30}{10} \cdot \binom{35}{10}$

30 choose 10 = 30,045,015

35 choose 10 = 183,579,396

d) Exercise 5.5.8, sections c-f

This question refers to a standard deck of playing cards. If you are unfamiliar with playing cards, there is an explanation in "Probability of an event" section under the heading "Standard playing cards." A five-card hand is just a subset of 5 cards from a deck of 52 cards.

c) (c) How many five-card hands are made entirely of hearts and diamonds?

$$\binom{26}{5} = 65,780$$

Total 5-card hands possible - (52 choose 5) = 2,598,960
Made up of hearts and diamonds - (26 choose 5) = 65,780

d) (d) How many five-card hands have four cards of the same rank?

Imagine there are 13 piles, each are composed of 1 rank (4 cards each, each a different suit) Pick one of the 13 piles that are made up of one rank: (13 choose 1), and you will keep all 4 cards (4 choose 4). Then you need to choose 1 card from the remaining 12 piles of ranks (12 choose 1). You will keep just 1 of those cards (4 choose 1)

Therefore: $\binom{13}{1} \cdot \binom{4}{4} \cdot \binom{12}{1} \cdot \binom{4}{1}$

e) (e) A "full house" is a five-card hand that has two cards of the same rank and three cards of the same rank. For example, $\{\text{queen of hearts, queen of spades, 8 of diamonds, 8 of spades, 8 of clubs}\}$. How many five-card hands contain a full house?

* it doesnt matter what order (4 choose 2) or (4 choose 3) are in because it is multiplication

$$\binom{13}{1}\binom{4}{2}\binom{12}{1}\binom{4}{3}$$

f) (f) How many five-card hands do not have any two cards of the same rank?

Picking 1 card from a different rank. Each pick from a rank is paired with a (4 choose 1) to indicate we are picking 1 card from that rank.For ease of reading I listed (4 choose 1) to the power of 5

$$\binom{13}{1} \cdot \binom{12}{1} \cdot \binom{11}{1} \cdot \binom{10}{1} \cdot \binom{9}{1} \cdot \binom{4}{1}^5$$

e) Exercise 5.6.6, sections a, b

The country has two major political parties, the Democrats and the Republicans. Suppose that the national senate consists of 100 members, 44 of which are Democrats and 56 of which are Republicans.

a) (a) How many ways are there to select a committee of 10 senate members with the same number of Democrats and Republicans?

$$\binom{56}{5} \cdot \binom{44}{5}$$

(b) Suppose that each party must select a speaker and a vice speaker. How many ways are there for the two speakers and two vice speakers to be selected?

b)

Permutation because given (VS, S):

(Person1, Person2) is not the same thing as (Person2, Person1)

Multiplied by the permutation for the other party

$P(56, 2) \cdot P(44, 2)$

Question 6:
Solve the following questions from the Discrete Math zyBook:
a) Exercise 5.7.2, sections a, b

A 5-card hand is drawn from a deck of standard playing cards.

a) (a) How many 5-card hands have at least one club?
Show how many are possible, and then subtract that from the ones that don't have any clubs at all.

$$\binom{52}{5} - \binom{39}{5}$$

b) (b) How many 5-card hands have at least two cards with the same rank? 1616
Show how many 5 card hands are possible and subtract that from the number of hands that have cards that are all different ranks. If one rank is represented at least more than 1 time, then it would meet the "at least two cards with the same rank".

$$\binom{52}{5} - (\binom{13}{1} \cdot \binom{12}{1} \cdot \binom{11}{1} \cdot \binom{10}{1} \cdot \binom{9}{1} \cdot \binom{4}{1}^5)$$

b) Exercise 5.8.4, sections a, b

20 different comic books will be distributed to five kids.

a) (a) How many ways are there to distribute the comic books if there are no restrictions on how many go to each kid (other than the fact that all 20 will be given out)?

5^{20} ways to select kids who get comic books

b) (b) How many ways are there to distribute the comic books if they are divided evenly so that 4 go to each kid?

$$\frac{20!}{4!^5}$$

ABCDE
A repeats 4 times in the 20 item set, leaving 16 places to fill in (20 choose 4)
B repeats 4 times in the 16 item set, leaving 12 places to fill in (16 choose 4)
C repeats 4 times in the 12 item set, leaving 8 places to fill in (12 choose 4)
D repeats 4 times in the 8 item set, leaving 4 places to fill in (8 choose 4)
E repeats 4 times in the 4 item set, leaving 0 places to fill in (4 choose 4)

$$\binom{20}{4}\binom{16}{4}\binom{12}{4}\binom{8}{4}\binom{4}{4} = \frac{20!}{4!^5}$$

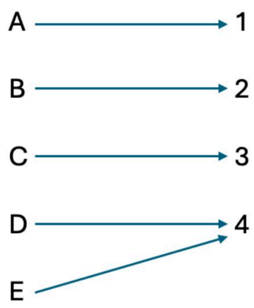
Question 7:

How many one-to-one functions are there from a set with five elements to sets with the following number of elements?

a) 4 elements

Not a one to one function

Can only be a function in the below sort of example, which would no longer be one-to-one:



b) 5 elements

P(5,5) = 120

c) 6 elements

P(6,5) = 720

d) 7 elements

P(7,5) = 2520