Question 3:

Solve the following questions from the Discrete Math zyBook:

a) Exercise 4.1.3, sections b, c

Which of the following are functions from $\mathbb{R}$ to $\mathbb{R}$? If f is a function, give its range.

b) (b) $f(x) = \frac{1}{x^2-4}$

No, not well defined. $f(2)$ does not evaluate to a real value ($\frac{1}{0}$)

c) (c) $f(x) = \sqrt{x^2}$

Yes, well defined. Range is $\{y : (x, y) \in f$ where $f(x) = \mathbb{R}\}$

b) Exercise 4.1.5, sections b, d, h, i, l

Express the range of each function using roster notation.

b) (b) Let $A = \{2, 3, 4, 5\}$.
$f : A \to \mathbb{Z}$, such that $f(x) = x^2$.

$\{4, 9, 16, 25\}$

d) (d) $f : \{0,1\}^5 \to \mathbb{Z}$. For $x \in \{0,1\}^5$, $f(x)$ is the number of 1's that occur in x. For example $f(01101) = 3$, because there are three 1's in the string "01101".

$\{0, 1, 2, 3, 4, 5\}$

h) (h) Let $A = \{1, 2, 3\}$.
$f : A \times A \to \mathbb{Z} \times \mathbb{Z}$, where $f(x, y) = (y, x)$.

	1	2	3
1	(1,1)	(1,2)	(1,3)
2	(2,1)	(2,2)	(2,3)
3	(3,1)	(3,2)	(3,3)

$\{(1,1), (2,1), (3,1), (1,2), (2,2), (3,2), (1,3), (2,3), (3,3)\}$

i) (i) Let $A = \{1, 2, 3\}$.
$f : A \times A \to \mathbb{Z} \times \mathbb{Z}$, where $f(x, y) = (x, y + 1)$.

	1 + 1	2 + 1	3 + 1
1	(1,2)	(1,3)	(1,4)
2	(2,2)	(2,3)	(2,4)
3	(3,2)	(3,3)	(3,4)

$\{(1,2), (2,2), (3,2), (1,3), (2,3), (3,3), (1,4), (2,4), (3,4)\}$

l) (l) Let $A = \{1, 2, 3\}$.
$f : P(A) \to P(A)$. For $X \subseteq A$, $f(X) = X - \{1\}$.

size 0 {}
size 1 {1},{2},{3}
size 2 {1,2}, {1,3}, {2,3}
size 3 {1,2,3}

$X \subseteq A = \emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}$

$X - \{1\} = \emptyset, \{2\}, \{3\}, \{2,3\}$

Question 4:

I. Solve the following questions from the Discrete Math zyBook:

a. Exercise 4.2.2, sections c, g, k

For each of the functions below, indicate whether the function is onto, one-to-one, neither or both. If the function is not onto or not one-to-one, give an example showing why.

c) (c) $h: \mathbb{Z} \to \mathbb{Z}$. $h(x) = x^3$

One to one, but not onto

Not onto example: No integer value will make $h(x) = 2$ true.

g) (g) $f: \mathbb{Z}\times\mathbb{Z} \to \mathbb{Z}\times\mathbb{Z}$, $f(x, y) = (x + 1, 2y)$

One to one, but not onto

Not onto example: $f(x, y)$ will never evaluate to a value where the final result is (even, odd). Example, $(2, 7)$ cannot be mapped to. Additionally will not be true in any instance where y input is negative, regardless of if it is even or odd. For example, it can never map to $(2, -6)$

k) (k) $f: \mathbb{Z}^+\times\mathbb{Z}^+ \to \mathbb{Z}^+$, $f(x, y) = 2^x + y$.

Not one to one and not onto.

Not one to one example: $(4, 1)$ and $(3, 9)$ both map to 17.

Not onto example: 1 and 2cannot be mapped to. Smallest possible target mapping is $f(1, 1) = 2^1 + 1 = 3$

b. Exercise 4.2.4, sections b, c, d, g

For each of the functions below, indicate whether the function is onto, one-to-one, neither or both. If the function is not onto or not one-to-one, give an example showing why.

b) (b) $f: \{0, 1\}^3 \to \{0, 1\}^3$. The output of f is obtained by taking the input string and replacing the first bit by 1, regardless of whether the first bit is a 0 or 1. For example, $f(001) = 101$ and $f(110) = 110$.

Not one to one and not onto.

Not one to one example: Since it doesn't matter if the first bit is 0 or 1, then 110 and 010 both will map to 110.

Not onto: No value that begins with 0 will be mapped to, for example 010.

c) (c) $f : \{0,1\}^3 \to \{0,1\}^3$. The output of f is obtained by taking the input string and reversing the bits. For example $f(011) = 110$.

Both one to one and onto

d) (d) $f : \{0,1\}^3 \to \{0,1\}^4$. The output of f is obtained by taking the input string and adding an extra copy of the first bit to the end of the string. For example, $f(100) = 1001$.

One to one, but not onto

Not onto example: Any instance where the last digit doesn't match the first digit will not be mapped to. For example 0001 or 1110

g) (g) Let A be defined to be the set $\{1,2,3,4,5,6,7,8\}$ and let $B = \{1\}$. $f : P(A) \to P(A)$. For $X \subseteq A$, $f(X) = X - B$. Recall that for a finite set A, $P(A)$ denotes the power set of A which is the set of all subsets of A.

Not one to one, and not onto.

Not one to one example: $\{1, 2, 3\}$ and $\{2, 3\}$ for example will both map to $\{2, 3\}$

Not onto example: since B is being subtracted out, $\{1\}$ is not mapped to.

II. Give an example of a function from the set of integers to the set of positive integers that is:

a. one-to-one, but not onto.

$$f(x) = \begin{cases} 2x + 3 \text{ if } x \geq 0 \\ -2x \text{ if } x < 0 \end{cases}$$

Is one-to-one because: When x is positive or 0, will map to only positive ODD numbers 5, 7, 9 etc. When x is negative, it will map to all positive even numbers.
Is not onto though, because the smallest positive odd number that can be mapped to is 3. Therefore 1 cannot be mapped to.

b. onto, but not one-to-one.

$f(x) = |x| + 1$

Every value ≥ 1 will be mapped to (therefore is onto), but -1 and 1 for example, will both map to 1 (therefore not one-to-one).

c. one-to-one and onto.

$$f(x) = \begin{cases} 2x + 1 \text{ if } x \geq 0 \\ -2x \text{ if } x < 0 \end{cases}$$

For the function to be onto:

When x is positive or 0, will map to all positive ODD numbers … 1, 3, 5, etc
When x is negative, it will map to all positive even numbers.

For the function to be one-to-one, the breakup of positive/negative numbers, ensures no double-mapping

d. neither one-to-one nor onto

$f(x) = x^2$

-2 and 2 both map to 4, therefore is not one to one, and for example 2 is not a perfect square therefore cannot be mapped to (and thus not onto).

Question 5:

Solve the following questions from the Discrete Math zyBook:

a) Exercise 4.3.2, sections c, d, g, i

For each of the following functions, indicate whether the function has a well-defined inverse. If the inverse is well-defined, give the input/output relationship of f^{-1}.

c) (c) $f : \mathbb{R} \to \mathbb{R}. f(x) = 2x + 3$

Inverse is well-defined: $f : \mathbb{R} \to \mathbb{R}. f^{-1}(y) = \frac{y-3}{2}$

(d) Let A be defined to be the set $\{1, 2, 3, 4, 5, 6, 7, 8\}$

$f : P(A) \to \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$

d) For $X \subseteq A$, $f(X) = |X|$. Recall that for a finite set A, $P(A)$ denotes the power set of A which is the set of all subsets of A.

Inverse not well-defined. Multiple values from y will point to different values of x when reversed.

Example where $y = |X| = 1$, we can imagine in an arrow diagram, an arrow pointing to all subsets x with a cardinality of 1.

g) (g) $f : \{0, 1\}^3 \to \{0, 1\}^3$. The output of f is obtained by taking the input string and reversing the bits. For example, $f(011) = 110$

Inverse is well-defined: The definition of the inverse is the same as the definition of the function.

i) (i) $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}, f(x, y) = (x + 5, y - 2)$

$\mathbb{Z}$ being all integers, then the inverse is well-defined:

$f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}, f^{-1}(y, x) = (y - 5, x + 2)$

b) Exercise 4.4.8, sections c, d

The domain and target set of functions f, g, and h are $\mathbb{Z}$. The functions are defined as:

- $f(x) = 2x + 3$
- $g(x) = 5x + 7$
- $h(x) = x^2 + 1$

Give an explicit formula for each function given below.

c) (c) $f \circ h$ = $2x^2 + 5$

$= 2(x^2 + 1) + 3$

$= 2x^2 + 2 + 3$

$=2x^2+5$

d) (d) $h \circ f$ $=4x^2+12x+10$

$=(2x+3)^2+1$
$=4x^2+12x+9+1$
$=4x^2+12x+10$

c) Exercise 4.4.2, sections b-d

Consider three functions f, g, and h, whose domain and target are $\mathbb{Z}$. Let

$$f(x)=x^2 \qquad g(x)=2^x \qquad h(x)=\left\lceil \frac{x}{5} \right\rceil$$

b) (b) Evaluate $(f \circ h)(52)$

$(f \circ h)(52)=121$

$=(\lceil \frac{52}{5} \rceil)^2$
$=11^2$
$=121$

c) (c) Evaluate $(g \circ h \circ f)(4)$

$(g \circ h \circ f)(4)=16$

$f(4)=4^2=16$

$h(f(4))=\lceil \frac{16}{5} \rceil=4$

$g(h(f(4)))=2^4=16$

d) (d) Give a mathematical expression for $h \circ f$.

$h \circ f=\lceil \frac{x^2}{5} \rceil$

d) Exercise 4.4.6, sections c-e

Define the following functions f, g, and h:

- $f : \{0,1\}^3 \to \{0,1\}^3$. The output of f is obtained by taking the input string and replacing the first bit by 1, regardless of whether the first bit is a 0 or 1. For example, $f(001) = 101$ and $f(110) = 110$.
- $g : \{0,1\}^3 \to \{0,1\}^3$. The output of g is obtained by taking the input string and reversing the bits. For example, $g(011) = 110$.
- $h : \{0,1\}^3 \to \{0,1\}^3$. The output of h is obtained by taking the input string x, and replacing the last bit with a copy of the first bit. For example, $h(011) = 010$.

c) (c) What is $(h \circ f)(010)$?

$h \circ f(010) = 111$

$f(010) = 110$

$h(f(010) = h(110) = 111$

d) (d) What is the range of $h \circ f$?

$\{101, 111\}$

Everything where bit 1 and bit 3 are both 1 = 100, 110, 101, 111

e) (e) What is the range of $g \circ f$?

$\{001, 011, 101, 111\}$

Everything where the last bit is 1: 000, 001, 011, 010, 100, 101, 110, 111

e) **Extra Credit:** Exercise 4.4.4, sections c, d

Let $f : X \to Y$ and $g : Y \to Z$ be two functions.

(c) Is it possible that f is not one-to-one and $g \circ f$ is one-to-one? Justify your answer. If the answer is "yes", give a specific example for f and g.

c)

No it is not possible. The targets of f and g are different, but their inputs are linked - whatever goes out of f goes into g. If f is not one to one, this means that the number of items being input into g is reduced. This then makes it impossible for $g \circ f$ to be one-to-one because there aren't enough items from X to be mapped to Z.

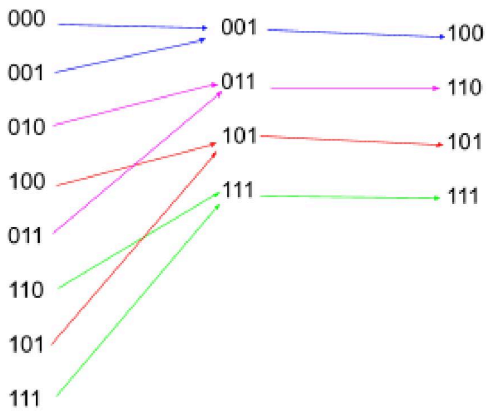
To prove this, we can use strings where $f : \{0,1\}^3 \to \{0,1\}^3$. The output of f is obtained by taking the last digit and replacing it with 1 regardless of if the digit is a 1 or a 0. For example $f(000) = 001$ or $f(001) = 001$

The range of f is $\{001, 011, 101, 111\}$ and the function is not one to one.

The outputs of f (its range) then get fed into g. If g were defined as:

$g : \{0,1\}^3 \to \{0,1\}^3$. The output of g is obtained by reversing the digits. For example $g(001) \to 100$.

The range of g is $\{100, 110, 101, 111\}$ and the function is one to one, but this does not mean that $g \circ f$ is one to one. We can see using the arrow diagram below that the composition is not one-to-one. See the arrow diagram for $g \circ f$ below:



You will see that $g \circ f$ is not one to one because 000 and 001 both map to 100. Therefore no it is not possible for $g \circ f$ to be one-to-one if f is not one-to one.

Let $f : X \to Y$ and $g : Y \to Z$ be two functions.

(d) Is it possible that g is not one-to-one and $g \circ f$ is one-to-one? Justify your answer. If the answer is "yes", give a specific example for f and g.

d)

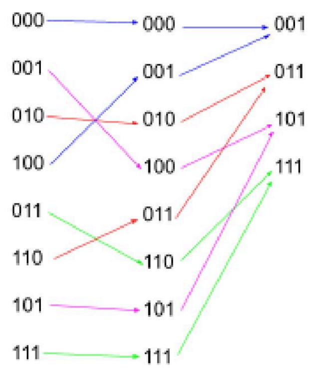
This also is not possible.

Let's take the last example and reverse the definitions of f and g to make the definition of g not one to one.

$f : \{0,1\}^3 \to \{0,1\}^3$. The output of f is obtained by reversing the digits. For example $f(001) \to 100$.

The range of f is the same as its domain (8 unique strings), and the function is one to one. The outputs of f (its range) then get fed into g. If g were defined as:

$g : \{0,1\}^3 \to \{0,1\}^3$. The output of g is obtained by taking the last digit and replacing it with 1 regardless of if the digit is a 1 or a 0. For example $g(000) = 001$ or $g(001) = 001$. The range of g is $\{001, 011, 101, 111\}$ and the function is not one to one.



Since g was not one to one, the number of outputs was reduced. This causes the original inputs of X to be mapped to multiple values in g's output Z. Therefore, no it is not possible for $g \circ f$ to be one-to-one if g is not one-to-one.