Perceptron Convergence Theorem-Numerical Illustration

We work in $\mathbb{R}^2$ and construct data that is linearly separable by a "teacher" hyperplane (w^*, b^*).

In[1]:=
```
ClearAll[generateSeparableData]
```

In[2]:=
```
generateSeparableData[nSamples_, dim_, margin_] :=
 Module[{wStar, bStar, x, m, label, X = {}, y = {}}, (*Random but fixed separator*)
  wStar = Normalize[RandomReal[NormalDistribution[0, 1], dim]];
  bStar = RandomReal[NormalDistribution[0, 0.1]];
  While[Length[X] < nSamples, x = RandomVariate[NormalDistribution[0, 1], dim];
   m = wStar.x + bStar;
   If[Abs[m] < margin, Continue[]];
   label = If[m > 0, 1, -1];
   AppendTo[X, x];
   AppendTo[y, label];];
  <|"X" → X, "y" → y, "wStar" → wStar, "bStar" → bStar|>]
```

In[3]:=
```
data=generateSeparableData[60, 2, 0.01]
```

Out[3]=
```
<|X → {{0.249334, -1.13311}, {-0.608842, -1.6313}, {-0.583875, -0.183894},
   {0.788876, 0.311926}, {0.674122, -1.35311}, {-1.39333, -0.310924},
   {0.911679, -0.186661}, {-0.381773, -0.260198}, {-0.051436, 0.141476},
   {-0.256701, -1.51856}, {-0.655449, 1.35051}, {-1.30726, 0.680499},
   {-0.329172, -0.902252}, {-0.241739, 0.240388}, {-0.787827, 0.319604},
   {-0.342625, 1.05775}, {0.33018, -2.2407}, {0.063146, 0.170597},
   {-1.56693, -1.58887}, {-1.23681, -0.611086}, {0.774065, 1.52022},
   {-0.117869, 1.46047}, {-0.232091, 1.10023}, {0.615981, -2.84505},
   {-1.37428, -0.403954}, {-1.19532, 0.579547}, {0.782907, -0.331854},
   {0.904582, 0.957991}, {1.02436, -0.297771}, {1.419, -0.196001},
   {-0.315283, -1.16479}, {0.1668, 0.790905}, {1.09682, 0.0996373},
   {0.179655, -1.31645}, {-0.677519, 0.0278771}, {-0.0899046, 1.08644},
   {2.7171, 0.30213}, {0.167918, 0.580165}, {-0.170277, 0.304542},
   {0.733948, -1.42656}, {1.11088, 0.654935}, {0.491511, -1.01011},
   {-0.0363295, -0.615061}, {0.262571, 0.438612}, {-0.675055, 1.23667},
   {0.206511, -0.519051}, {0.197743, 1.5819}, {-1.2968, -2.22199},
   {-0.38051, 0.134161}, {-0.339336, -1.63332}, {1.77593, 0.469068},
   {1.66424, 0.308808}, {-0.16415, -1.21688}, {1.18555, -0.313483},
   {1.74789, -0.176835}, {1.54578, 0.588618}, {-0.181552, 0.518609},
   {-0.332239, 0.151707}, {-1.35165, -0.451836}, {0.0297235, -2.37345}},
 y → {-1, -1, 1, -1, -1, 1, -1, 1, 1, -1, 1, 1, -1, 1, 1, 1, -1, 1, 1, 1, 1,
   1, 1, -1, 1, 1, -1, -1, -1, -1, -1, 1, -1, -1, 1, 1, -1, 1, 1, -1, -1,
   -1, -1, 1, 1, -1, 1, -1, 1, -1, -1, -1, -1, -1, -1, -1, 1, 1, 1, -1},
 wStar → {-0.804466, 0.593999}, bStar → 0.0587749|>
```

In[4]:=
```
X=data["X"]
```

Out[4]=
```
{{0.249334, -1.13311}, {-0.608842, -1.6313}, {-0.583875, -0.183894},
 {0.788876, 0.311926}, {0.674122, -1.35311}, {-1.39333, -0.310924},
 {0.911679, -0.186661}, {-0.381773, -0.260198}, {-0.051436, 0.141476},
 {-0.256701, -1.51856}, {-0.655449, 1.35051}, {-1.30726, 0.680499},
 {-0.329172, -0.902252}, {-0.241739, 0.240388}, {-0.787827, 0.319604},
 {-0.342625, 1.05775}, {0.33018, -2.2407}, {0.063146, 0.170597}, {-1.56693, -1.58887},
 {-1.23681, -0.611086}, {0.774065, 1.52022}, {-0.117869, 1.46047},
 {-0.232091, 1.10023}, {0.615981, -2.84505}, {-1.37428, -0.403954},
 {-1.19532, 0.579547}, {0.782907, -0.331854}, {0.904582, 0.957991},
 {1.02436, -0.297771}, {1.419, -0.196001}, {-0.315283, -1.16479}, {0.1668, 0.790905},
 {1.09682, 0.0996373}, {0.179655, -1.31645}, {-0.677519, 0.0278771},
 {-0.0899046, 1.08644}, {2.7171, 0.30213}, {0.167918, 0.580165},
 {-0.170277, 0.304542}, {0.733948, -1.42656}, {1.11088, 0.654935},
 {0.491511, -1.01011}, {-0.0363295, -0.615061}, {0.262571, 0.438612},
 {-0.675055, 1.23667}, {0.206511, -0.519051}, {0.197743, 1.5819},
 {-1.2968, -2.22199}, {-0.38051, 0.134161}, {-0.339336, -1.63332},
 {1.77593, 0.469068}, {1.66424, 0.308808}, {-0.16415, -1.21688}, {1.18555, -0.313483},
 {1.74789, -0.176835}, {1.54578, 0.588618}, {-0.181552, 0.518609},
 {-0.332239, 0.151707}, {-1.35165, -0.451836}, {0.0297235, -2.37345}}
```

In[5]:=
```
y=data["y"]
```

Out[5]=
```
{-1, -1, 1, -1, -1, 1, -1, 1, 1, -1, 1, 1, -1, 1, 1, 1, -1, 1, 1, 1,
 1, 1, 1, -1, 1, 1, -1, -1, -1, -1, -1, 1, -1, -1, 1, 1, -1, 1, 1, -1,
 -1, -1, -1, 1, 1, -1, 1, -1, 1, -1, -1, -1, -1, -1, -1, -1, 1, 1, 1, -1}
```

In[6]:=
```
wStar=data["wStar"]
```

Out[6]=
```
{-0.804466, 0.593999}
```

In[7]:=
```
bStar=data["bStar"]
```

Out[7]=
```
0.0587749
```

In[8]:=
```
Dimensions[X]
```

Out[8]=
```
{60, 2}
```

In[9]:=
```
Counts[y]
```

Out[9]=
```
<|-1 → 31, 1 → 29|>
```

Plot of linearly separable data

In[10]:=
```
pos = Pick[X, UnitStep[y], 1]
```

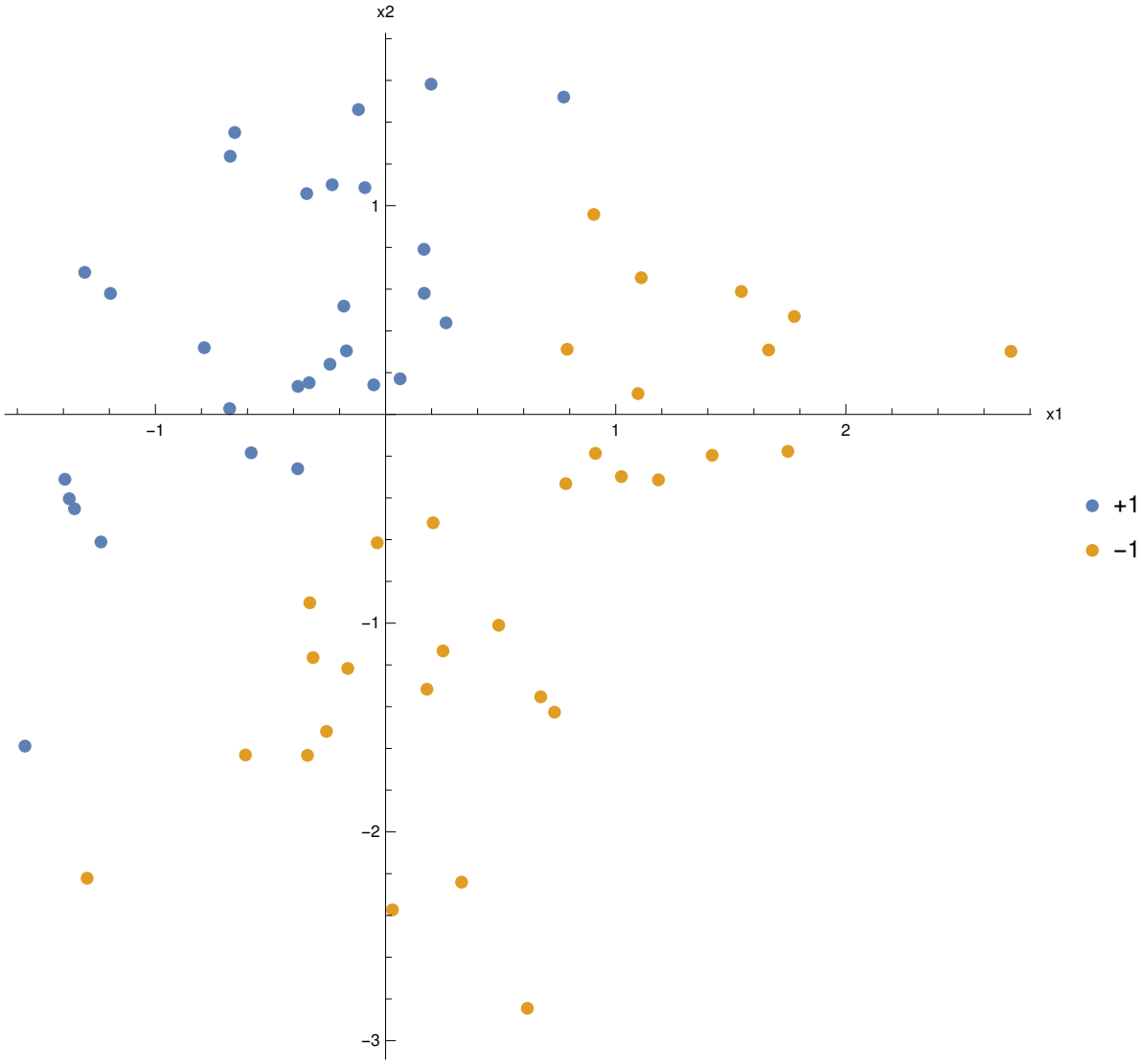
Out[10]=
```
{{-0.583875, -0.183894}, {-1.39333, -0.310924},
 {-0.381773, -0.260198}, {-0.051436, 0.141476}, {-0.655449, 1.35051},
 {-1.30726, 0.680499}, {-0.241739, 0.240388}, {-0.787827, 0.319604},
 {-0.342625, 1.05775}, {0.063146, 0.170597}, {-1.56693, -1.58887},
 {-1.23681, -0.611086}, {0.774065, 1.52022}, {-0.117869, 1.46047},
 {-0.232091, 1.10023}, {-1.37428, -0.403954}, {-1.19532, 0.579547},
 {0.1668, 0.790905}, {-0.677519, 0.0278771}, {-0.0899046, 1.08644},
 {0.167918, 0.580165}, {-0.170277, 0.304542}, {0.262571, 0.438612},
 {-0.675055, 1.23667}, {0.197743, 1.5819}, {-0.38051, 0.134161},
 {-0.181552, 0.518609}, {-0.332239, 0.151707}, {-1.35165, -0.451836}}
```

In[11]:=
```
neg = Pick[X, UnitStep[-y], 1]
```

Out[11]=
```
{{0.249334, -1.13311}, {-0.608842, -1.6313}, {0.788876, 0.311926},
 {0.674122, -1.35311}, {0.911679, -0.186661}, {-0.256701, -1.51856},
 {-0.329172, -0.902252}, {0.33018, -2.2407}, {0.615981, -2.84505},
 {0.782907, -0.331854}, {0.904582, 0.957991}, {1.02436, -0.297771},
 {1.419, -0.196001}, {-0.315283, -1.16479}, {1.09682, 0.0996373},
 {0.179655, -1.31645}, {2.7171, 0.30213}, {0.733948, -1.42656}, {1.11088, 0.654935},
 {0.491511, -1.01011}, {-0.0363295, -0.615061}, {0.206511, -0.519051},
 {-1.2968, -2.22199}, {-0.339336, -1.63332}, {1.77593, 0.469068},
 {1.66424, 0.308808}, {-0.16415, -1.21688}, {1.18555, -0.313483},
 {1.74789, -0.176835}, {1.54578, 0.588618}, {0.0297235, -2.37345}}
```

In[12]:=
```
ListPlot[
 {
  pos,
  neg
 },
 PlotStyle → {PointSize[Large], PointSize[Large]},
 PlotLegends → {"+1", "-1"},
 AxesLabel → {"x1", "x2"},
 PlotRange → All,
 ImageSize → Large,
 AspectRatio → 1
]
```

Out[12]=



Perceptron algorithm with tracking

In[13]:=
```
ClearAll[perceptronTrain]
```

In[14]:=
```
perceptronTrain[X_List, y_List, maxEpochs_Integer : 1000, wStar_: None, bStar_: None] :=
 Module[{nSamples, dim, w, b, epoch, i, xi, yi, margin, mistakes,
   historyW = {}, historyB = {}, historyWDotWStar = {}, historyWNormSq = {},
   mistakesPerEpoch = {}, totalUpdates = 0, wDot},

  nSamples = Length[X];
  dim = Length[First[X]];
  w = ConstantArray[0., dim];
  b = 0.;

  For[epoch = 1, epoch ≤ maxEpochs, epoch++,
   mistakes = 0;
   For[i = 1, i ≤ nSamples, i++,
    xi = X[[i]];
    yi = y[[i]];
    margin = yi (w . xi + b);
    If[margin ≤ 0,
     w = w + yi xi;
     b = b + yi;
     mistakes++;
     totalUpdates++;

     If[wStar =!= None,
      wDot = w . wStar;
      AppendTo[historyWDotWStar, wDot];
      ,
      AppendTo[historyWDotWStar, Missing["NotAvailable"]];
     ];
     AppendTo[historyWNormSq, w . w];
     AppendTo[historyW, w];
     AppendTo[historyB, b];
    ];
   ];
   AppendTo[mistakesPerEpoch, mistakes];
   If[mistakes == 0, Break[]];
  ];

  <|
   "wFinal" → w,
   "bFinal" → b,
   "historyW" → historyW,
   "historyB" → historyB,
   "historyWDotWStar" → historyWDotWStar,
```

```
    "historyWNormSq" → historyWNormSq,
    "mistakesPerEpoch" → mistakesPerEpoch,
    "totalUpdates" → totalUpdates
   |>
  ]
```

In[15]:=
```
result = perceptronTrain[X, y, 100, wStar, bStar]
```

Out[15]=
```
<|wFinal → {-3.47586, 2.51536}, bFinal → 0.,
 historyW → {{-0.249334, 1.13311}, {-0.83321, 0.94922}, {-2.40014, -0.639654},
   {-1.62608, 0.880568}, {-2.40898, 1.21242}, {-2.0937, 2.37721}, {-3.66063,
    0.788338}, {-2.88657, 2.30856}, {-3.79115, 1.35057}, {-3.47586, 2.51536}},
 historyB → {-1., 0., 1., 2., 1., 0., 1., 2., 1., 0.},
 historyWDotWStar → {0.87365, 1.23412, 1.55088, 1.83118,
   2.65812, 3.09637, 3.41312, 3.69343, 3.85209, 4.29033},
 historyWNormSq → {1.34612, 1.59526, 6.16983, 3.41952, 7.27316,
   10.0347, 14.0217, 13.6617, 16.1968, 18.4087},
 mistakesPerEpoch → {6, 4, 0}, totalUpdates → 10|>
```

In[16]:=
```
wFinal = result["wFinal"]
```

Out[16]=
```
{-3.47586, 2.51536}
```

In[17]:=
```
bFinal = result["bFinal"]
```

Out[17]=
```
0.
```

In[18]:=
```
totalUpdates = result["totalUpdates"]
```

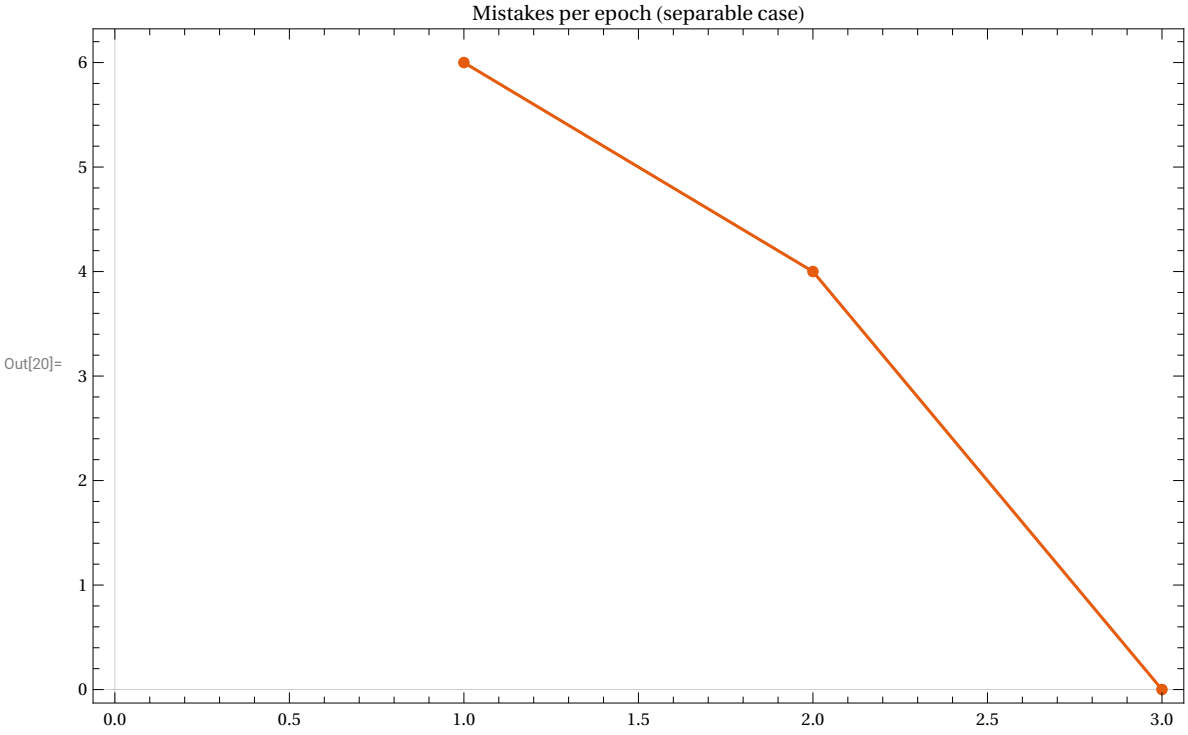
Out[18]=
```
10
```

In[19]:=
```
mistakesPerEpoch = result["mistakesPerEpoch"]
```

Out[19]=
```
{6, 4, 0}
```

In[20]:=
```
ListLinePlot[
 mistakesPerEpoch,
 AxesLabel → {"epoch", "mistakes"},
 PlotMarkers → Automatic,
 PlotTheme → "Scientific",
 ImageSize → Large,
 PlotLabel → "Mistakes per epoch (separable case)"
]
```

Out[20]=



Decision boundary in $\mathbb{R}^2$

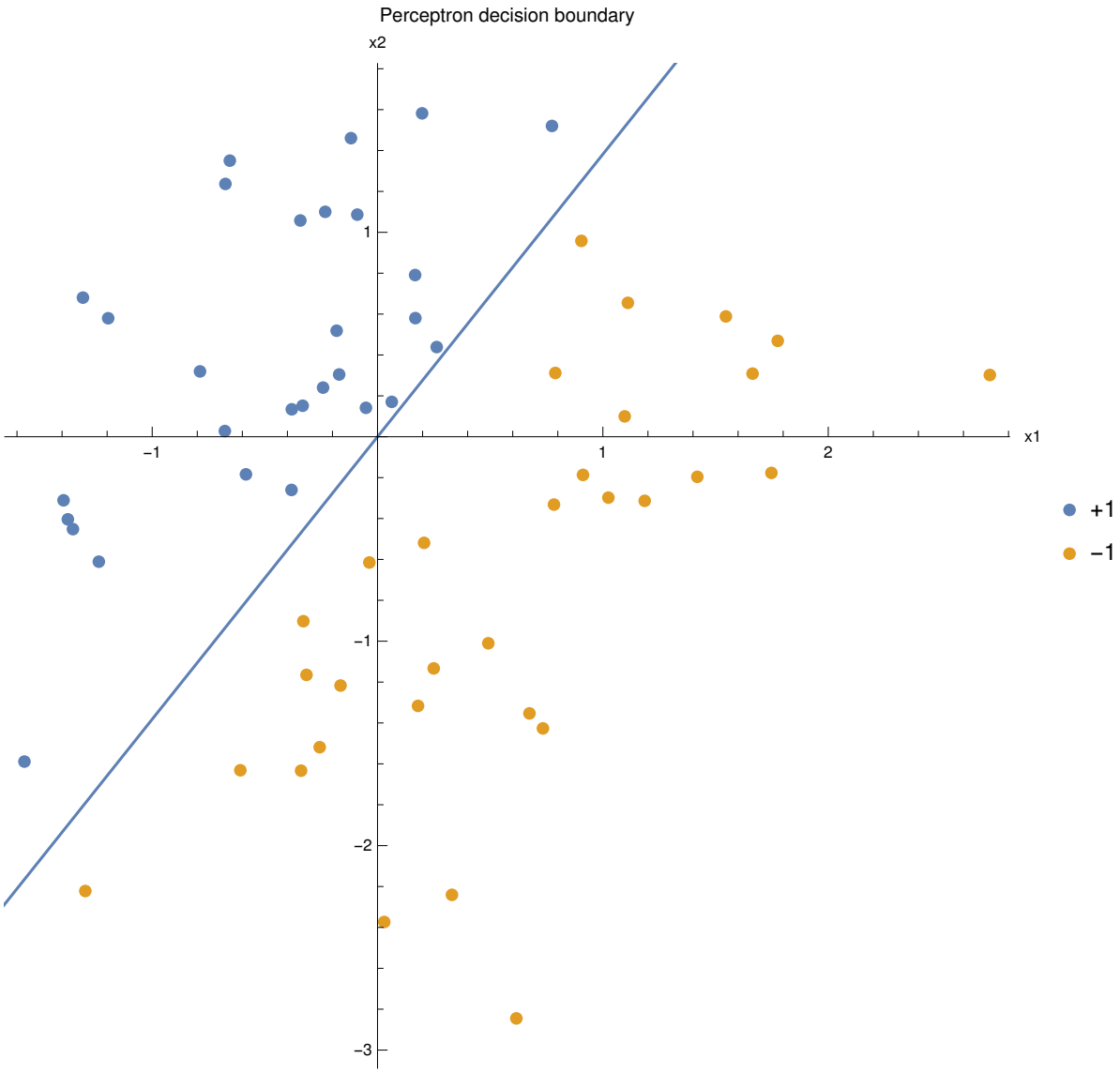
In[21]:=
```
ClearAll[plotDecisionBoundary]
```

In[22]:=
```
plotDecisionBoundary[w_, b_, X_, y_] :=
 Module[{pos, neg, line},
  pos = Pick[X, UnitStep[y], 1];
  neg = Pick[X, UnitStep[-y], 1];

  Show[
   ListPlot[
    {pos, neg},
    PlotStyle → {PointSize[Large], PointSize[Large]},
    PlotLegends → {"+1", "-1"},
    AxesLabel → {"x1", "x2"},
    PlotRange → All,
    AspectRatio → 1
   ],
   (* line w.x + b == 0 => x2 = -(w1 x1 + b)/w2 *)
   If[w[[2]] ≠ 0,
    line = Plot[
      -(w[[1]] x + b)/w[[2]],
      {x, Min[X[[All, 1]]] - 1, Max[X[[All, 1]]] + 1}
     ];
    line,
    {}
   ],
   ImageSize → Large,
   PlotLabel → "Perceptron decision boundary"
  ]
 ]
```

In[23]:=
```
plotDecisionBoundary[wFinal, bFinal, X, y]
```

Out[23]=



R, gamma, and theoretical bound

In[24]:=
```
norms = Norm /@ X
```

Out[24]=
```
{1.16022, 1.74121, 0.61215, 0.848306, 1.51174, 1.4276, 0.930592, 0.462011,
 0.150536, 1.5401, 1.50116, 1.47378, 0.960424, 0.340916, 0.850187,
 1.11186, 2.26489, 0.181908, 2.23155, 1.37953, 1.70595, 1.46521, 1.12444,
 2.91097, 1.43242, 1.32841, 0.850335, 1.31758, 1.06676, 1.43247, 1.20671,
 0.808303, 1.10133, 1.32865, 0.678093, 1.09015, 2.73384, 0.603977,
 0.348912, 1.60429, 1.28957, 1.12334, 0.616133, 0.511199, 1.40892,
 0.558624, 1.59422, 2.57273, 0.403468, 1.66819, 1.83683, 1.69264,
 1.2279, 1.22629, 1.75681, 1.65406, 0.549469, 0.365236, 1.42517, 2.37364}
```

In[25]:=
```
R = Max[norms]
```

Out[25]=
```
2.91097
```

In[26]:=
```
marginsTeacher = MapThread[#1 (#2 . wStar + bStar) &, {y, X}]
```

Out[26]=
```
{0.814875, 0.420422, 0.41925, 0.390565, 1.28728, 0.994976, 0.785516, 0.211341,
 0.18419, 0.636738, 1.38826, 1.51464, 0.212354, 0.396036, 0.882399, 0.962708,
 1.53781, 0.10931, 0.375528, 0.690758, 0.339076, 1.02111, 0.89902, 2.12672,
 0.924387, 1.36462, 0.768168, 0.0998851, 0.942164, 1.19918, 0.379474,
 0.394387, 0.764392, 0.867722, 0.620375, 0.776444, 1.94757, 0.268308,
 0.376654, 1.37903, 0.445856, 0.936631, 0.277345, 0.10808, 1.33641,
 0.415672, 0.839346, 0.217858, 0.444574, 0.638428, 1.09127, 1.09661,
 0.531999, 1.08117, 1.45238, 0.835115, 0.51288, 0.416163, 0.877742, 1.37496}
```

In[27]:=
```
gamma = Min[marginsTeacher]
```

Out[27]=
```
0.0998851
```

In[28]:=
```
bound = If[gamma > 0, (R/gamma)^2, Infinity]
<|"R" → R, "gamma" → gamma, "bound" → bound, "observedUpdates" → totalUpdates|>
```

Out[28]=
```
849.324
```

Out[29]=
```
<|R → 2.91097, gamma → 0.0998851, bound → 849.324, observedUpdates → 10|>
```

In[30]:=
```
wDotWStar = DeleteMissing[result["historyWDotWStar"]]
```

Out[30]=
```
{0.87365, 1.23412, 1.55088, 1.83118,
 2.65812, 3.09637, 3.41312, 3.69343, 3.85209, 4.29033}
```

In[31]:=
```
wNormSq = result["historyWNormSq"]
```

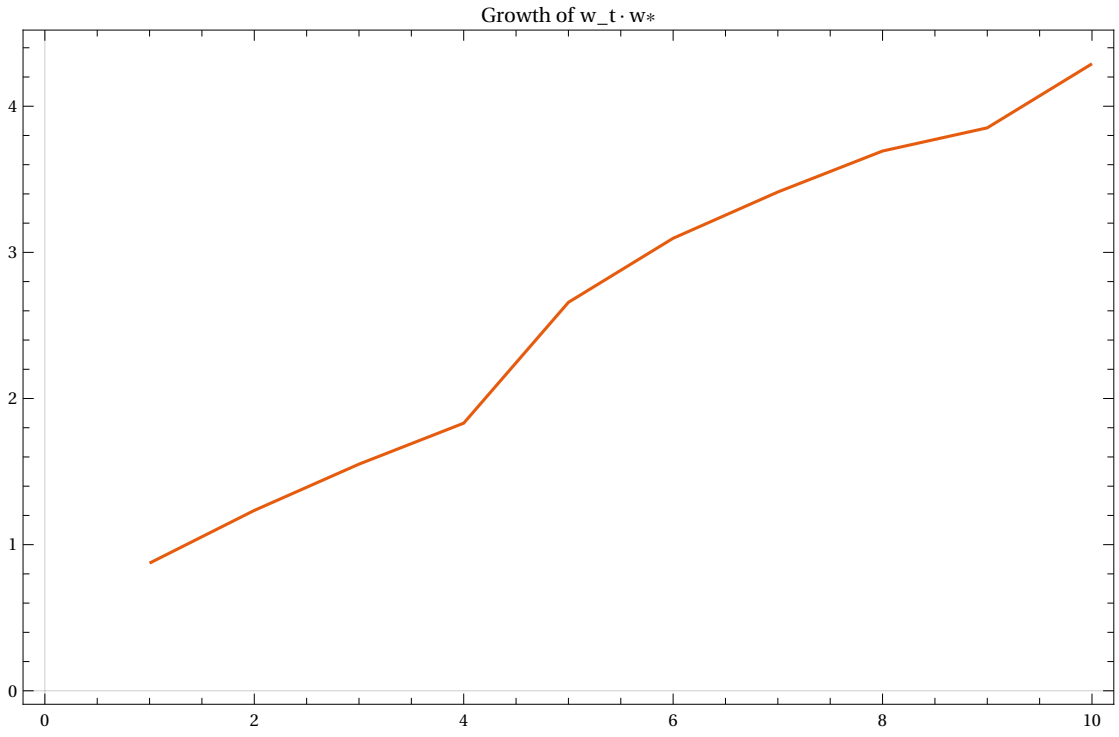
Out[31]=
```
{1.34612, 1.59526, 6.16983, 3.41952,
 7.27316, 10.0347, 14.0217, 13.6617, 16.1968, 18.4087}
```

In[32]:=
```
steps = Range[Length[wNormSq]]
```

Out[32]=
```
{1, 2, 3, 4, 5, 6, 7, 8, 9, 10}
```

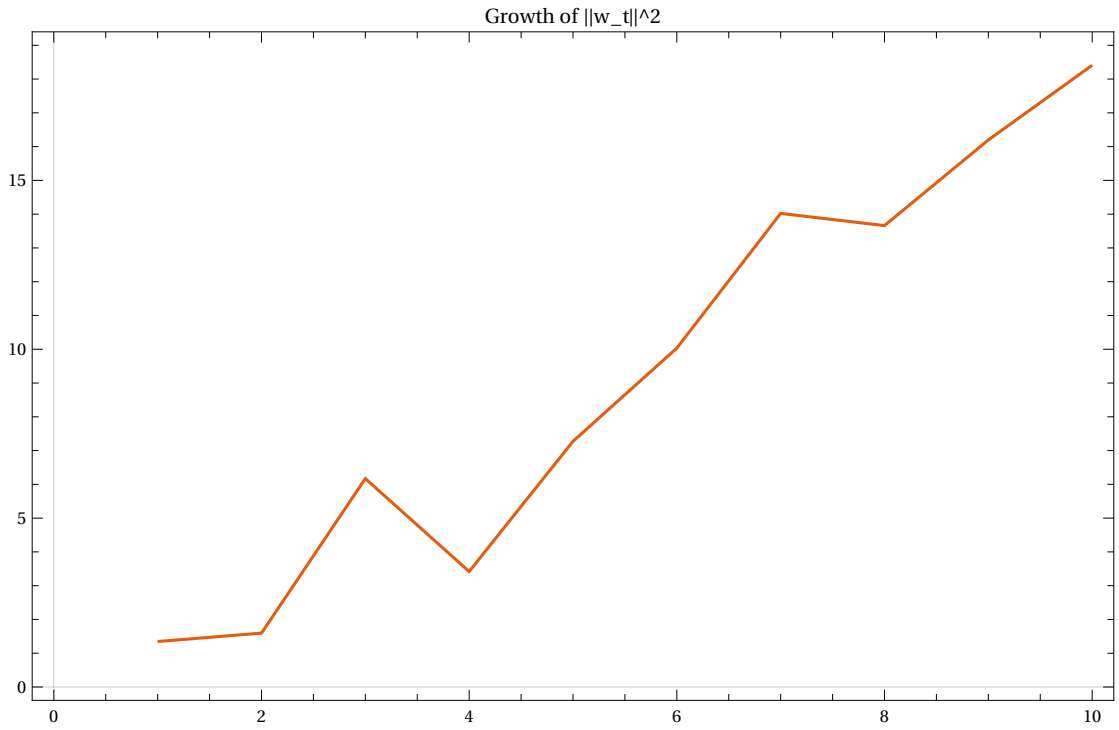
In[33]:=
```
ListLinePlot[
 wDotWStar,
 AxesLabel → {"update step t", "w_t · w*"},
 PlotTheme → "Scientific",
 PlotLabel → "Growth of w_t · w*",
 ImageSize → Large
]
```

Out[33]=



In[34]:=
```
ListLinePlot[
 wNormSq,
 AxesLabel → {"update step t", "||w_t||^2"},
 PlotTheme → "Scientific",
 PlotLabel → "Growth of ||w_t||^2",
 ImageSize → Large
]
```

Out[34]=



Non-separable data: XOR

In[35]:=
```
Xxor = {
  {1., 1.},
  {1., -1.},
  {-1., 1.},
  {-1., -1.}
}
```

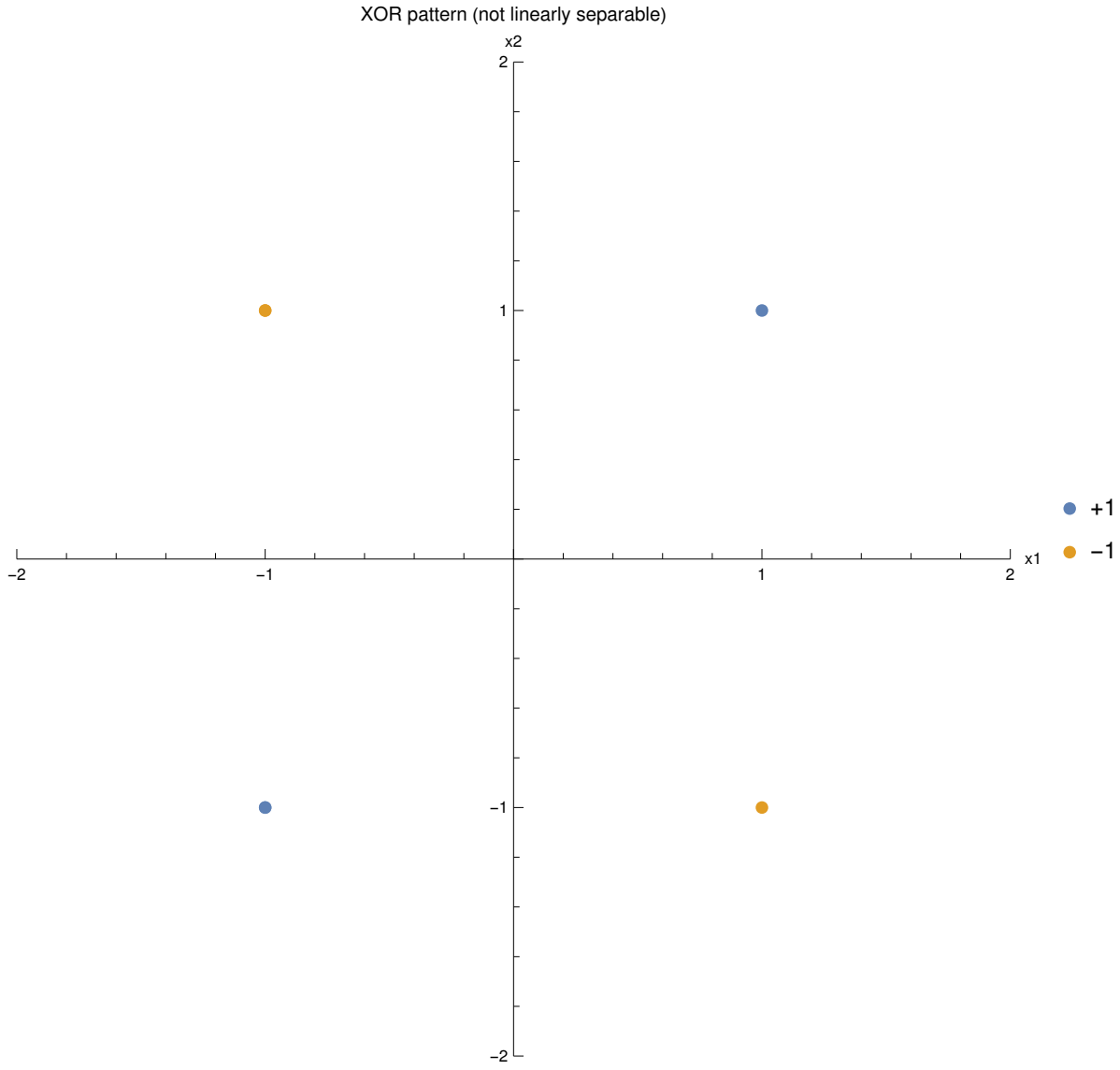
Out[35]= {{1., 1.}, {1., −1.}, {−1., 1.}, {−1., −1.}}

In[36]:=
```
yxor = {1, -1, -1, 1}
```

Out[36]= {1, −1, −1, 1}

In[37]:=
```
ListPlot[
 {
  Pick[Xxor, UnitStep[yxor], 1],
  Pick[Xxor, UnitStep[-yxor], 1]
 },
 PlotStyle → {PointSize[Large], PointSize[Large]},
 PlotLegends → {"+1", "-1"},
 AxesLabel → {"x1", "x2"},
 PlotRange → {{-2, 2}, {-2, 2}},
 PlotLabel → "XOR pattern (not linearly separable)",
 ImageSize → Large,
 AspectRatio → 1
]
```

Out[37]=



In[38]:=
```
resultXor = perceptronTrain[Xxor, yxor, 50]
```

Out[38]=
```
<|wFinal → {0., 0.}, bFinal → 0.,
 historyW → {{1., 1.}, {0., 2.}, {1., 1.}, {0., 0.}, {1., 1.}, {0., 2.}, {1., 1.}, {0., 0.},
```

```
{1., 1.}, {0., 2.}, {1., 1.}, {0., 0.}, {1., 1.}, {0., 2.}, {1., 1.}, {0., 0.}, {1., 1.},
{0., 2.}, {1., 1.}, {0., 0.}, {1., 1.}, {0., 2.}, {1., 1.}, {0., 0.}, {1., 1.}, {0., 2.},
{1., 1.}, {0., 0.}, {1., 1.}, {0., 2.}, {1., 1.}, {0., 0.}, {1., 1.}, {0., 2.}, {1., 1.},
{0., 0.}, {1., 1.}, {0., 2.}, {1., 1.}, {0., 0.}, {1., 1.}, {0., 2.}, {1., 1.}, {0., 0.},
{1., 1.}, {0., 2.}, {1., 1.}, {0., 0.}, {1., 1.}, {0., 2.}, {1., 1.}, {0., 0.}, {1., 1.},
{0., 2.}, {1., 1.}, {0., 0.}, {1., 1.}, {0., 2.}, {1., 1.}, {0., 0.}, {1., 1.}, {0., 2.},
{1., 1.}, {0., 0.}, {1., 1.}, {0., 2.}, {1., 1.}, {0., 0.}, {1., 1.}, {0., 2.}, {1., 1.},
{0., 0.}, {1., 1.}, {0., 2.}, {1., 1.}, {0., 0.}, {1., 1.}, {0., 2.}, {1., 1.}, {0., 0.},
{1., 1.}, {0., 2.}, {1., 1.}, {0., 0.}, {1., 1.}, {0., 2.}, {1., 1.}, {0., 0.}, {1., 1.},
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{1., 1.}, {0., 2.}, {1., 1.}, {0., 0.}, {1., 1.}, {0., 2.}, {1., 1.}, {0., 0.},
{1., 1.}, {0., 2.}, {1., 1.}, {0., 0.}, {1., 1.}, {0., 2.}, {1., 1.}, {0., 0.}},
historyB → {1., 0., -1., 0., 1., 0., -1., 0., 1., 0., -1., 0., 1., 0., -1.,
0., 1., 0., -1., 0., 1., 0., -1., 0., 1., 0., -1., 0., 1., 0., -1., 0., 1., 0.,
-1., 0., 1., 0., -1., 0., 1., 0., -1., 0., 1., 0., -1., 0., 1., 0., -1., 0., 1.,
0., -1., 0., 1., 0., -1., 0., 1., 0., -1., 0., 1., 0., -1., 0., 1., 0., -1.,
0., 1., 0., -1., 0., 1., 0., -1., 0., 1., 0., -1., 0., 1., 0., -1., 0., 1., 0.,
-1., 0., 1., 0., -1., 0., 1., 0., -1., 0., 1., 0., -1., 0., 1., 0., -1., 0., 1.,
0., -1., 0., 1., 0., -1., 0., 1., 0., -1., 0., 1., 0., -1., 0., 1., 0., -1.,
0., 1., 0., -1., 0., 1., 0., -1., 0., 1., 0., -1., 0., 1., 0., -1., 0., 1., 0.,
-1., 0., 1., 0., -1., 0., 1., 0., -1., 0., 1., 0., -1., 0., 1., 0., -1., 0.,
1., 0., -1., 0., 1., 0., -1., 0., 1., 0., -1., 0., 1., 0., -1., 0., 1., 0.,
-1., 0., 1., 0., -1., 0., 1., 0., -1., 0., 1., 0., -1., 0., 1., 0., -1., 0.},
historyWDotWStar → {Missing[NotAvailable], Missing[NotAvailable],
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  historyWNormSq → {2., 4., 2., 0., 2., 4., 2., 0., 2., 4., 2., 0., 2., 4., 2.,
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    2., 4., 2., 0., 2., 4., 2., 0., 2., 4., 2., 0., 2., 4., 2., 0., 2., 4., 2., 0., 2.,
    4., 2., 0., 2., 4., 2., 0., 2., 4., 2., 0., 2., 4., 2., 0., 2., 4., 2., 0., 2., 4.,
    2., 0., 2., 4., 2., 0., 2., 4., 2., 0., 2., 4., 2., 0., 2., 4., 2., 0., 2., 4., 2.,
    0., 2., 4., 2., 0., 2., 4., 2., 0., 2., 4., 2., 0., 2., 4., 2., 0., 2., 4., 2., 0.,
    2., 4., 2., 0., 2., 4., 2., 0., 2., 4., 2., 0., 2., 4., 2., 0., 2., 4., 2., 0.,
    2., 4., 2., 0., 2., 4., 2., 0., 2., 4., 2., 0., 2., 4., 2., 0., 2., 4., 2., 0.,
    2., 4., 2., 0., 2., 4., 2., 0., 2., 4., 2., 0., 2., 4., 2., 0., 2., 4., 2., 0.},
  mistakesPerEpoch → {4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4,
    4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4,
    4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4}, totalUpdates → 200|>
```

In[39]:=
```
wXor = resultXor["wFinal"]
```

Out[39]=
```
{0., 0.}
```

In[40]:=
```
bXor = resultXor["bFinal"]
```

Out[40]=
```
0.
```

In[41]:=
```
mistakesXor = resultXor["mistakesPerEpoch"]
```

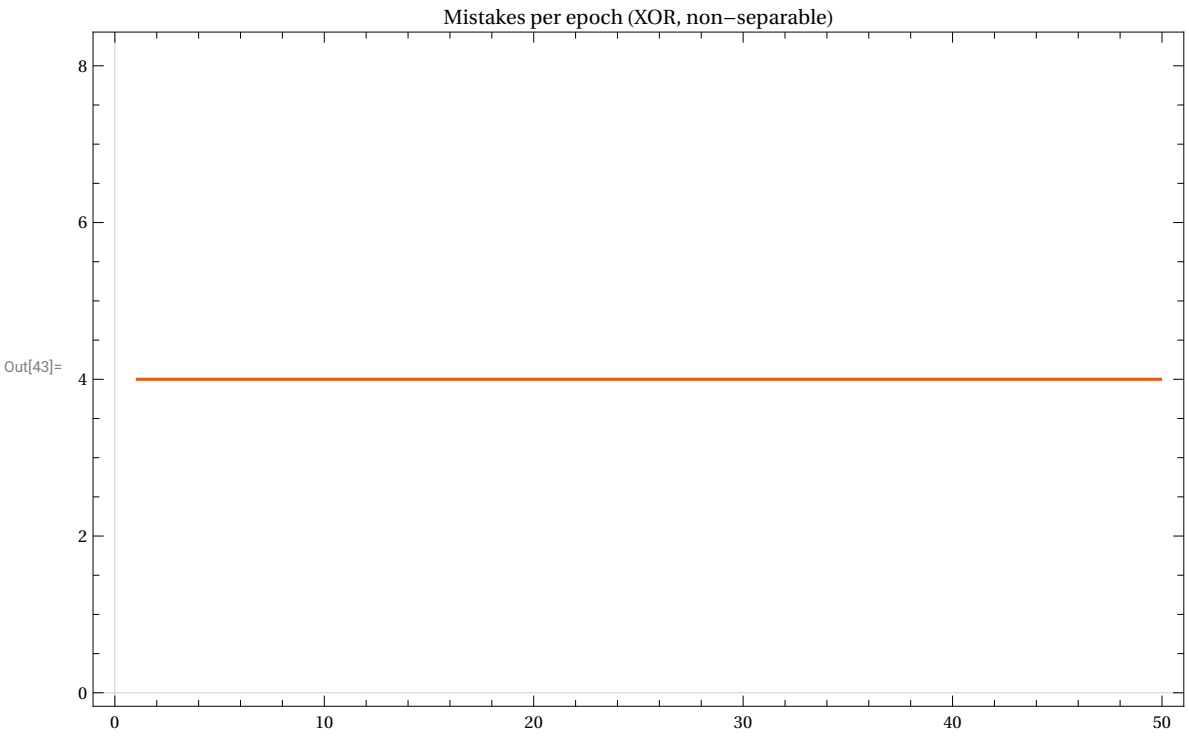
Out[41]=
```
{4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4,
 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4}
```

In[42]:=
```
totalUpdatesXor = resultXor["totalUpdates"]
```

Out[42]=
```
200
```

In[43]:=
```
ListLinePlot[
 mistakesXor,
 AxesLabel → {"epoch", "mistakes"},
 PlotTheme → "Scientific",
 PlotLabel → "Mistakes per epoch (XOR, non-separable)",
 ImageSize → Large
]
```

Out[43]=



In[44]:=
```
plotDecisionBoundary[wXor, bXor, Xxor, yxor]
```

Out[44]=