Running in Circles

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Initialization

```
Off[General::spell, General::spell1]

LSolve = Last[Solve[##]] &

Last[Solve[##1]] &

PeS = PowerExpand[Simplify[#]] &
FpPeS = FixedPoint[PeS, #] &

PowerExpand[Simplify[#1]] &

FixedPoint[PeS, #1] &

unitsquare = Function[p, Line[p + #/2 & /@ {{1, 1}, {-1, 1}, {-1, -1}, {1, -1}, {1, 1}}]]

Function[p, Line[(p + #1/2 &) /@ {{1, 1}, {-1, 1}, {-1, -1}, {1, -1}, {1, 1}}]]
```

Basic algorithm

Defining equation for a circle

x is a vector, *r* is the radius (well, a sphere in an Length[*x*]-dimensional vector space, actually)

```
circleeqn = Function[{x, r}, x.x - r^2 == 0]

Function[{x, r}, x.x - r^2 == 0]

circleeqn[{x, y}, R]
```

$-R^2 + x^2 + y^2 == 0$

Iterations

```
circleeqn[{x + 1, y}, R]
```

$-R^2 + (1 + x)^2 + y^2 == 0$

```
circleeqn[#, R]⟦1⟧ & /@ {{x + 1, y}, {x, y}}
Simplify[Subtract @@ %]
```

$\{-R^2 + (1 + x)^2 + y^2, -R^2 + x^2 + y^2\}$

$1 + 2x$

```
Simplify[Subtract @@ (circleeqn[#, R]⟦1⟧ & /@ #)] & /@
 {{{x + 1, y}, {x, y}}, {{x, y - 1}, {x, y}}, {{x + 1, y - 1}, {x, y}}}
```

$\{1 + 2x, 1 - 2y, 2(1 + x - y)\}$

Slope

```
circleeqn[{x, y[x]}, R]
D[%, x]
LSolve[%, y'[x]]
```

$-R^2 + x^2 + y[x]^2 == 0$

$2x + 2y[x]\,y'[x] == 0$

$\left\{y'[x] \to -\frac{x}{y[x]}\right\}$

Different iteration step strategies

We can define a drawing algorithm such that the only possible directions are either horizontal or vertical. In this case, we need to split a full circle into 4 quadrants.
The discrimination criteria between quadrants are the signs of the coordinates: {+, +}, {−, +}, {−, −}, {+, −}.

Alternatively, we can define a drawing algorithm such that the possible directions are horizontal, vertical, or diagonal. In this case, we need to split a full circle into 8 octants.
The discrimination criteria between octants are the signs of the coordinates and the relative magnitude of the partial derivatives of the discriminating function with respect to the coordnates:
$\left\{+, +, \left|\frac{\partial\Delta}{\partial x}\right| < \left|\frac{\partial\Delta}{\partial y}\right|\right\}, \left\{+, +, \left|\frac{\partial\Delta}{\partial x}\right| > \left|\frac{\partial\Delta}{\partial y}\right|\right\}, \left\{-, +, \left|\frac{\partial\Delta}{\partial x}\right| > \left|\frac{\partial\Delta}{\partial y}\right|\right\}, \left\{-, +, \left|\frac{\partial\Delta}{\partial x}\right| < \left|\frac{\partial\Delta}{\partial y}\right|\right\},$
$\left\{-, -, \left|\frac{\partial\Delta}{\partial x}\right| < \left|\frac{\partial\Delta}{\partial y}\right|\right\}, \left\{-, -, \left|\frac{\partial\Delta}{\partial x}\right| > \left|\frac{\partial\Delta}{\partial y}\right|\right\}, \left\{+, -, \left|\frac{\partial\Delta}{\partial x}\right| > \left|\frac{\partial\Delta}{\partial y}\right|\right\}, \left\{+, -, \left|\frac{\partial\Delta}{\partial x}\right| < \left|\frac{\partial\Delta}{\partial y}\right|\right\}$

Orthogonal~~only~~ iteration

Assumption: We iterate in mathematically positive sense of orientation.

Quadrant	δx	δy
{+, +}	−	+
{−, +}	−	−
{−, −}	+	−
{+, −}	+	+

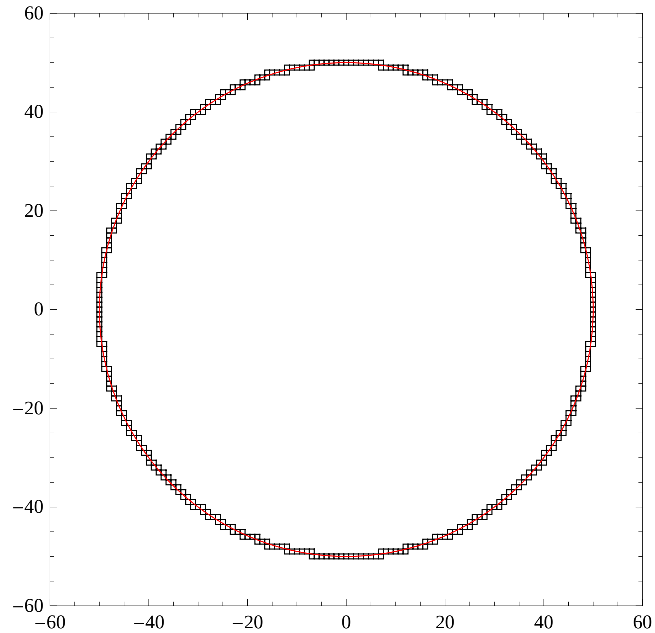
Underlying general rules:
$\text{Sign}[\delta x] = -\text{Sign}[y]$; $\text{Sign}[\delta y] = \text{Sign}[x]$.

We want to choose the alternative that minimizes the error, i.e., gives us min $\{|\Delta + 1 + 2\,x\,\delta x|, |\Delta + 1 + 2\,y\,\delta y|\}$

```
OrthogonalIterate =
 Function[{x, Δ}, Block[{δx1 = If[x⟦2⟧ < 0, 1, -1], δx2 = If[x⟦1⟧ < 0, -1, 1], Δx, Δy},
   Δx1 = 1 + 2 x⟦1⟧ δx1; Δx2 = 1 + 2 x⟦2⟧ δx2;
   If[Abs[Δ + Δx1] < Abs[Δ + Δx2], {x + {δx1, 0}, Δ + Δx1}, {x + {0, δx2}, Δ + Δx2}]]
 ]
```

```
Function[{x, Δ}, Block[{δx1 = If[x⟦2⟧ < 0, 1, -1], δx2 = If[x⟦1⟧ < 0, -1, 1], Δx, Δy},
  Δx1 = 1 + 2 x⟦1⟧ δx1; Δx2 = 1 + 2 x⟦2⟧ δx2;
  If[Abs[Δ + Δx1] < Abs[Δ + Δx2], {x + {δx1, 0}, Δ + Δx1}, {x + {0, δx2}, Δ + Δx2}]]]
```

```
NestList[OrthogonalIterate @@ # &, {{50, 0}, 0}, 399];
Show[
 Graphics[unitsquare[First[#]] & /@ %, Frame → True, AspectRatio → 1,
  PlotRange → 60 {{-1, 1}, {-1, 1}}], Graphics[{Hue[0], Circle[{0, 0}, 50]},
  Frame → True, AspectRatio → 1, PlotRange → 60 {{-1, 1}, {-1, 1}}]]
```



■ Diagonal iteration

Assumption: We iterate in mathematically positive sense of orientation.

Octant	Straight	Diagonal
$\left\{+, +, \left\lvert\frac{\partial\Delta}{\partial x}\right\rvert > \left\lvert\frac{\partial\Delta}{\partial y}\right\rvert\right\}$	$\{0, ++y\}$	$\{--x, ++y\}$
$\left\{+, +, \left\lvert\frac{\partial\Delta}{\partial x}\right\rvert < \left\lvert\frac{\partial\Delta}{\partial y}\right\rvert\right\}$	$\{--x, 0\}$	$\{--x, ++y\}$
$\left\{-, +, \left\lvert\frac{\partial\Delta}{\partial x}\right\rvert < \left\lvert\frac{\partial\Delta}{\partial y}\right\rvert\right\}$	$\{--x, 0\}$	$\{--x, --y\}$
$\left\{-, +, \left\lvert\frac{\partial\Delta}{\partial x}\right\rvert > \left\lvert\frac{\partial\Delta}{\partial y}\right\rvert\right\}$	$\{0, --y\}$	$\{--x, --y\}$
$\left\{-, -, \left\lvert\frac{\partial\Delta}{\partial x}\right\rvert > \left\lvert\frac{\partial\Delta}{\partial y}\right\rvert\right\}$	$\{0, --y\}$	$\{++x, --y\}$
$\left\{-, -, \left\lvert\frac{\partial\Delta}{\partial x}\right\rvert < \left\lvert\frac{\partial\Delta}{\partial y}\right\rvert\right\}$	$\{++x, 0\}$	$\{++x, --y\}$
$\left\{+, -, \left\lvert\frac{\partial\Delta}{\partial x}\right\rvert < \left\lvert\frac{\partial\Delta}{\partial y}\right\rvert\right\}$	$\{++x, 0\}$	$\{++x, ++y\}$
$\left\{+, -, \left\lvert\frac{\partial\Delta}{\partial x}\right\rvert > \left\lvert\frac{\partial\Delta}{\partial y}\right\rvert\right\}$	$\{0, ++y\}$	$\{++x, ++y\}$

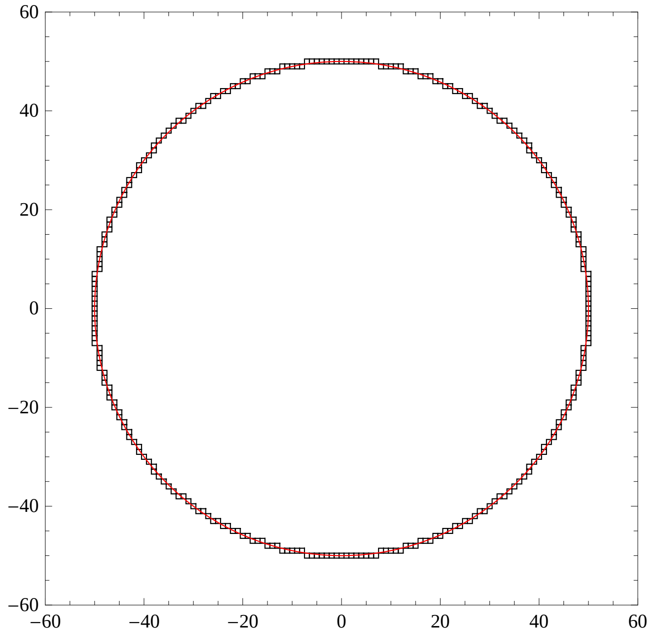
Underlying general rules:
$\mathrm{Sign}[\delta x] = -\mathrm{Sign}[y]$; $\mathrm{Sign}[\delta y] = \mathrm{Sign}[x]$,
straight direction is the direction of the smaller partial derivative.

We want to choose the alternative that minimizes the error.

```
DiagonalIterate =
 Function[{x, Δ}, Block[{δx1 = If[x⟦2⟧ < 0, 1, -1], δx2 = If[x⟦1⟧ < 0, -1, 1], Δx1, Δx2, δs, Δs},
   Δx1 = 1 + 2 x⟦1⟧ δx1; Δx2 = 1 + 2 x⟦2⟧ δx2;
   {δs, Δs} = If[Abs[x⟦1⟧] < Abs[x⟦2⟧], {{δx1, 0}, Δx1}, {{0, δx2}, Δx2}];
   If[Abs[Δ + Δs] < Abs[Δ + Δx1 + Δx2],
    {x + δs, Δ + Δs}, {x + {δx1, δx2}, Δ + Δx1 + Δx2}]
  ]
 ]
```

```
Function[{x, Δ},
 Block[{δx1 = If[x⟦2⟧ < 0, 1, -1], δx2 = If[x⟦1⟧ < 0, -1, 1], Δx1, Δx2, δs, Δs}, Δx1 = 1 + 2 x⟦1⟧ δx1;
  Δx2 = 1 + 2 x⟦2⟧ δx2; {δs, Δs} = If[Abs[x⟦1⟧] < Abs[x⟦2⟧], {{δx1, 0}, Δx1}, {{0, δx2}, Δx2}];
  If[Abs[Δ + Δs] < Abs[Δ + Δx1 + Δx2], {x + δs, Δ + Δs}, {x + {δx1, δx2}, Δ + Δx1 + Δx2}]]]
```

```
NestList[DiagonalIterate @@ # &, {{50, 0}, 0}, 283];
Show[
 Graphics[unitsquare[First[#]] & /@ %, Frame → True, AspectRatio → 1,
  PlotRange → 60 {{-1, 1}, {-1, 1}}], Graphics[{Hue[0], Circle[{0, 0}, 50]},
  Frame → True, AspectRatio → 1, PlotRange → 60 {{-1, 1}, {-1, 1}}]]
```



Generalization to ellipses

■ Basic equation

An ellipse with major and minor axes parallel to the coordinate axes has the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 == 0$.

An ellipse with major axis in the direction $\begin{pmatrix} \xi \\ \upsilon \end{pmatrix}$ has the equation $\frac{\langle x \;\; y | \xi \;\; \upsilon \rangle \langle \xi \;\; \upsilon | x \;\; y \rangle}{a^2 \langle \xi \;\; \upsilon | \xi \;\; \upsilon \rangle} + \frac{\langle x \;\; y | -\upsilon \;\; \xi \rangle \langle -\upsilon \;\; \xi | x \;\; y \rangle}{b^2 \langle \xi \;\; \upsilon | \xi \;\; \upsilon \rangle} - 1 == 0$.

$$\frac{(\{\xi, \upsilon\}.\{x, y\})^2}{a^2 (\{\xi, \upsilon\}.\{\xi, \upsilon\})} + \frac{(\{-\upsilon, \xi\}.\{x, y\})^2}{b^2 (\{\xi, \upsilon\}.\{\xi, \upsilon\})} - 1 == 0$$

```
Simplify[%]
```

$$-1 + \frac{(y\,\xi - x\,\upsilon)^2}{b^2 \left(\xi^2 + \upsilon^2\right)} + \frac{(x\,\xi + y\,\upsilon)^2}{a^2 \left(\xi^2 + \upsilon^2\right)} == 0$$

$$\frac{a^2 \,(y\,\xi - x\,\upsilon)^2 + b^2 \,(x\,\xi + y\,\upsilon)^2}{a^2\, b^2 \left(\xi^2 + \upsilon^2\right)} == 1$$

Equation in matrix form

```
MatrixForm /@ {{x, y}, {{ξ, -υ}, {υ, ξ}}, {{1/a^2, 0}, {0, 1/b^2}}/({ξ, υ}.{ξ, υ}), {{ξ, υ}, {-υ, ξ}}, {x, y}}
```

$$\left\{\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} \xi & -\upsilon \\ \upsilon & \xi \end{pmatrix}, \begin{pmatrix} \frac{1}{a^2 (\xi^2+\upsilon^2)} & 0 \\ 0 & \frac{1}{b^2 (\xi^2+\upsilon^2)} \end{pmatrix}, \begin{pmatrix} \xi & \upsilon \\ -\upsilon & \xi \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix}\right\}$$

```
Simplify[{x, y}.{{ξ, -υ}, {υ, ξ}}.{{1/a^2, 0}, {0, 1/b^2}}/({ξ, υ}.{ξ, υ}).{{ξ, υ}, {-υ, ξ}}.{x, y} == 1]
```

$$\frac{a^2 (y\,\xi - x\,\upsilon)^2 + b^2 (x\,\xi + y\,\upsilon)^2}{a^2\, b^2 (\xi^2 + \upsilon^2)} == 1$$

The matrices that do not depend on the point coordinates $\begin{pmatrix} x \\ y \end{pmatrix}$ can be collected into one matrix:

```
MatrixForm[Simplify[{{ξ, -υ}, {υ, ξ}}.{{1/a^2, 0}, {0, 1/b^2}}/({ξ, υ}.{ξ, υ}).{{ξ, υ}, {-υ, ξ}}]]
```

$$\begin{pmatrix} \frac{b^2 \xi^2+a^2 \upsilon^2}{a^2 b^2 (\xi^2+\upsilon^2)} & \frac{(-a^2+b^2)\,\xi\,\upsilon}{a^2 b^2 (\xi^2+\upsilon^2)} \\ \frac{(-a^2+b^2)\,\xi\,\upsilon}{a^2 b^2 (\xi^2+\upsilon^2)} & \frac{a^2 \xi^2+b^2 \upsilon^2}{a^2 b^2 (\xi^2+\upsilon^2)} \end{pmatrix}$$

```
Collect[{x, y}.{{A, C}, {C, B}}.{x, y}, {x, y}]
```

$A\,x^2 + 2\,C\,x\,y + B\,y^2$

If we want to avoid all divisions, the equation is

```
(ellipsediscriminant =
   Collect[{x, y}.{{ξ, -υ}, {υ, ξ}}.{{b^2, 0}, {0, a^2}}.{{ξ, υ}, {-υ, ξ}}.{x, y} -
     a^2 b^2 {ξ, υ}.{ξ, υ}, {x, y}, Simplify]) == 0
```

$$2\,(-a^2 + b^2)\,x\,y\,\xi\,\upsilon - a^2\,b^2\,(\xi^2 + \upsilon^2) + x^2\,(b^2\,\xi^2 + a^2\,\upsilon^2) + y^2\,(a^2\,\xi^2 + b^2\,\upsilon^2) == 0$$

```
ellipsediscriminant
```

$$2\,(-a^2 + b^2)\,x\,y\,\xi\,\upsilon - a^2\,b^2\,(\xi^2 + \upsilon^2) + x^2\,(b^2\,\xi^2 + a^2\,\upsilon^2) + y^2\,(a^2\,\xi^2 + b^2\,\upsilon^2)$$

```
LSolve[D[0 == ellipsediscriminant /. y → y[x], x], y'[x]] /. y[x] → y
```

$$\left\{y'[x] \to \frac{-b^2\,x\,\xi^2 + a^2\,y\,\xi\,\upsilon - b^2\,y\,\xi\,\upsilon - a^2\,x\,\upsilon^2}{a^2\,y\,\xi^2 - a^2\,x\,\xi\,\upsilon + b^2\,x\,\xi\,\upsilon + b^2\,y\,\upsilon^2}\right\}$$

```
Collect[D[ellipsediscriminant, #], {x, y}, Simplify] & /@ {x, y}
```

$$\left\{2\,(-a^2 + b^2)\,y\,\xi\,\upsilon + 2\,x\,(b^2\,\xi^2 + a^2\,\upsilon^2),\ 2\,(-a^2 + b^2)\,x\,\xi\,\upsilon + 2\,y\,(a^2\,\xi^2 + b^2\,\upsilon^2)\right\}$$

Constant coefficients that can be calculated in advance

```
Append[Limit[ellipsediscriminant/z /.
   {{x^2 → z, y^2 → 0, x y → 0}, {x^2 → 0, y^2 → z, x y → 0}, {x^2 → 0, y^2 → 0, x y → z}}, z → ∞],
  -ellipsediscriminant /. {x^2 → 0, y^2 → 0, x y → 0}]
MapThread[Rule, {%, {A, B, 2 C, D}}]
ellipsediscriminant /. %
D[%, #] & /@ {x, y}
```

$\{b^2 \xi^2 + a^2 \upsilon^2,\ a^2 \xi^2 + b^2 \upsilon^2,\ 2(-a^2 + b^2)\xi \upsilon,\ a^2 b^2 (\xi^2 + \upsilon^2)\}$

$\{b^2 \xi^2 + a^2 \upsilon^2 \to A,\ a^2 \xi^2 + b^2 \upsilon^2 \to B,\ 2(-a^2 + b^2)\xi \upsilon \to 2C,\ a^2 b^2 (\xi^2 + \upsilon^2) \to D\}$

$-D + A x^2 + 2 C x y + B y^2$

$\{2 A x + 2 C y,\ 2 C x + 2 B y\}$

Iterated variables

The discriminant and its gradient need not be computed from scratch for each new point, but can be iterated, exploiting the fact that for continuous lines, each of the coordinate can only change by 0 or ± 1.

Discriminant

```
A x^2 + 2 C x y + B y^2 - D
TableForm[% /. {{x → x + 1}, {x → x + 1, y → y + 1}, {y → y + 1}, {x → x - 1, y → y + 1},
   {x → x - 1}, {x → x - 1, y → y - 1}, {y → y - 1}, {x → x + 1, y → y - 1}}]
TableForm[Collect[#, {x, y}, Simplify] & /@ (% - %%)]
TableForm[Simplify[{-1, 1, -1}.%[[#]]] & /@ {{1, 2, 3}, {3, 4, 5}, {5, 6, 7}, {7, 8, 1}}]
```

$-D + A x^2 + 2 C x y + B y^2$

$-D + A (1 + x)^2 + 2 C (1 + x) y + B y^2$
$-D + A (1 + x)^2 + 2 C (1 + x) (1 + y) + B (1 + y)^2$
$-D + A x^2 + 2 C x (1 + y) + B (1 + y)^2$
$-D + A (-1 + x)^2 + 2 C (-1 + x) (1 + y) + B (1 + y)^2$
$-D + A (-1 + x)^2 + 2 C (-1 + x) y + B y^2$
$-D + A (-1 + x)^2 + 2 C (-1 + x) (-1 + y) + B (-1 + y)^2$
$-D + A x^2 + 2 C x (-1 + y) + B (-1 + y)^2$
$-D + A (1 + x)^2 + 2 C (1 + x) (-1 + y) + B (-1 + y)^2$

$A + 2 A x + 2 C y$
$A + B + 2 C + 2 (A + C) x + 2 (B + C) y$
$B + 2 C x + 2 B y$
$A + B - 2 C + (-2 A + 2 C) x + 2 (B - C) y$
$A - 2 A x - 2 C y$
$A + B + 2 C - 2 (A + C) x - 2 (B + C) y$
$B - 2 C x - 2 B y$
$A + B - 2 C + 2 (A - C) x + (-2 B + 2 C) y$

$2 C$
$-2 C$
$2 C$
$-2 C$

- **Partial derivative of discriminant w.r.t. *x***

```
2 (A x + C y)
TableForm[% /. {{x → x + 1}, {x → x + 1, y → y + 1}, {y → y + 1}, {x → x - 1, y → y + 1},
   {x → x - 1}, {x → x - 1, y → y - 1}, {y → y - 1}, {x → x + 1, y → y - 1}}]
TableForm[Simplify[% - %%]]
```

```
2 (A x + C y)

2 (A (1 + x) + C y)
2 (A (1 + x) + C (1 + y))
2 (A x + C (1 + y))
2 (A (-1 + x) + C (1 + y))
2 (A (-1 + x) + C y)
2 (A (-1 + x) + C (-1 + y))
2 (A x + C (-1 + y))
2 (A (1 + x) + C (-1 + y))

2 A
2 (A + C)
2 C
-2 A + 2 C
-2 A
-2 (A + C)
-2 C
2 (A - C)
```

- **Partial derivative of discriminant w.r.t. *y***

```
2 (C x + B y)
TableForm[% /. {{x → x + 1}, {x → x + 1, y → y + 1}, {y → y + 1}, {x → x - 1, y → y + 1},
   {x → x - 1}, {x → x - 1, y → y - 1}, {y → y - 1}, {x → x + 1, y → y - 1}}]
TableForm[Simplify[% - %%]]
```

```
2 (C x + B y)

2 (C (1 + x) + B y)
2 (C (1 + x) + B (1 + y))
2 (C x + B (1 + y))
2 (C (-1 + x) + B (1 + y))
2 (C (-1 + x) + B y)
2 (C (-1 + x) + B (-1 + y))
2 (C x + B (-1 + y))
2 (C (1 + x) + B (-1 + y))

2 C
2 (B + C)
2 B
2 (B - C)
-2 C
-2 (B + C)
-2 B
2 (-B + C)
```

Gradient of discriminant

```
2 {A x + C y, C x + B y}
Outer[ReplaceAll, %, {{x → x + 1}, {x → x + 1, y → y + 1}, {y → y + 1}, {x → x - 1, y → y + 1},
   {x → x - 1}, {x → x - 1, y → y - 1}, {y → y - 1}, {x → x + 1, y → y - 1}}, 1];
TableForm[Transpose[%], TableDepth → 1]
TableForm[Transpose[Simplify[%% - %%%]], TableDepth → 1]
TableForm[
 Simplify[{-1, 1, -1}.%[[#]]] & /@ {{1, 2, 3}, {3, 4, 5}, {5, 6, 7}, {7, 8, 1}}, TableDepth → 1]
```

```
{2 (A x + C y), 2 (C x + B y)}

{2 (A (1 + x) + C y), 2 (C (1 + x) + B y)}
{2 (A (1 + x) + C (1 + y)), 2 (C (1 + x) + B (1 + y))}
{2 (A x + C (1 + y)), 2 (C x + B (1 + y))}
{2 (A (-1 + x) + C (1 + y)), 2 (C (-1 + x) + B (1 + y))}
{2 (A (-1 + x) + C y), 2 (C (-1 + x) + B y)}
{2 (A (-1 + x) + C (-1 + y)), 2 (C (-1 + x) + B (-1 + y))}
{2 (A x + C (-1 + y)), 2 (C x + B (-1 + y))}
{2 (A (1 + x) + C (-1 + y)), 2 (C (1 + x) + B (-1 + y))}

{2 A, 2 C}
{2 (A + C), 2 (B + C)}
{2 C, 2 B}
{-2 A + 2 C, 2 (B - C)}
{-2 A, -2 C}
{-2 (A + C), -2 (B + C)}
{-2 C, -2 B}
{2 (A - C), 2 (-B + C)}

{0, 0}
{0, 0}
{0, 0}
{0, 0}
```

Setting up initial conditions for the iterations

Identifying quadrants and octants

Orthogonal~~o~~nly iteration

Diagonal iteration

Assumption: We iterate in mathematically positive sense of orientation (counterclockwise)

Octants:

$\frac{\partial\Delta}{\partial x}$	$\frac{\partial\Delta}{\partial y}$	$\left\lvert\frac{\partial\Delta}{\partial x}\right\rvert - \left\lvert\frac{\partial\Delta}{\partial y}\right\rvert$	Straight	Diagonal
> 0	> 0	> 0	$\{0, +1\}$	$\{-1, +1\}$
> 0	> 0	< 0	$\{-1, 0\}$	$\{-1, +1\}$
< 0	> 0	< 0	$\{-1, 0\}$	$\{-1, -1\}$
< 0	> 0	> 0	$\{0, -1\}$	$\{-1, -1\}$
< 0	< 0	> 0	$\{0, -1\}$	$\{+1, -1\}$
< 0	< 0	< 0	$\{+1, 0\}$	$\{+1, -1\}$
> 0	< 0	< 0	$\{+1, 0\}$	$\{+1, +1\}$
> 0	< 0	> 0	$\{0, +1\}$	$\{+1, +1\}$

Underlying general rules:

$\delta x = -\mathrm{Sign}\left[\frac{\partial\Delta}{\partial y}\right]$, $\delta y = \mathrm{Sign}\left[\frac{\partial\Delta}{\partial x}\right]$, the straight direction is the direction for which the change in Δ is smaller.

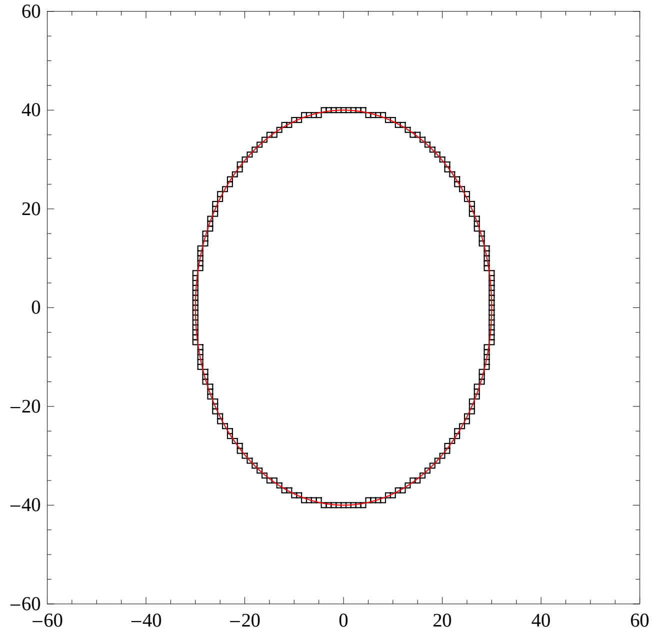
```
EllipseDiagonalIteration = Function[{x, iterationvariables, iterationconstants}, Block[{
     Δ = iterationvariables⟦1⟧,
     dΔ = iterationvariables⟦2⟧,
     A = iterationconstants⟦1⟧,
     B = iterationconstants⟦2⟧,
     C = iterationconstants⟦3⟧,
     δx1, δx2, Δx1, Δx2, δd1, δd2, δs, Δs, δds
    },
    δx1 = If[dΔ⟦2⟧ < 0, 1, -1]; δx2 = If[dΔ⟦1⟧ < 0, -1, 1];
    Δx1 = δx1 dΔ⟦1⟧; Δx2 = δx2 dΔ⟦2⟧;
    δd1 = 2 δx1 {A, C}; δd2 = 2 δx2 {C, B};
    {δs, Δs, δds} = If[Subtract @@ (Abs[dΔ]) < 0,
      {{δx1, 0}, A + Δx1, δd1}, {{0, δx2}, B + Δx2, δd2}];
    If[Abs[Δ + Δs] < Abs[Δ + A + B + 2 Sign[δx1] Sign[δx2] C + Δx1 + Δx2],
     {x + δs, {Δ + Δs, dΔ + δds}, iterationconstants},
     {x + {δx1, δx2},
      {Δ + A + B + 2 Sign[δx1] Sign[δx2] C + Δx1 + Δx2, dΔ + δd1 + δd2}, iterationconstants}]
  ]
 ]
```

```
Function[{x, iterationvariables, iterationconstants},
 Block[{Δ = iterationvariables⟦1⟧, dΔ = iterationvariables⟦2⟧, A = iterationconstants⟦1⟧,
    B = iterationconstants⟦2⟧, C = iterationconstants⟦3⟧, δx1, δx2, Δx1, Δx2,
    δd1, δd2, δs, Δs, δds}, δx1 = If[dΔ⟦2⟧ < 0, 1, -1]; δx2 = If[dΔ⟦1⟧ < 0, -1, 1];
   Δx1 = δx1 dΔ⟦1⟧; Δx2 = δx2 dΔ⟦2⟧; δd1 = 2 δx1 {A, C}; δd2 = 2 δx2 {C, B};
   {δs, Δs, δds} = If[Subtract @@ Abs[dΔ] < 0, {{δx1, 0}, A + Δx1, δd1}, {{0, δx2}, B + Δx2, δd2}];
   If[Abs[Δ + Δs] < Abs[Δ + A + B + 2 Sign[δx1] Sign[δx2] C + Δx1 + Δx2],
    {x + δs, {Δ + Δs, dΔ + δds}, iterationconstants}, {x + {δx1, δx2},
     {Δ + A + B + 2 Sign[δx1] Sign[δx2] C + Δx1 + Δx2, dΔ + δd1 + δd2}, iterationconstants}]]]
```

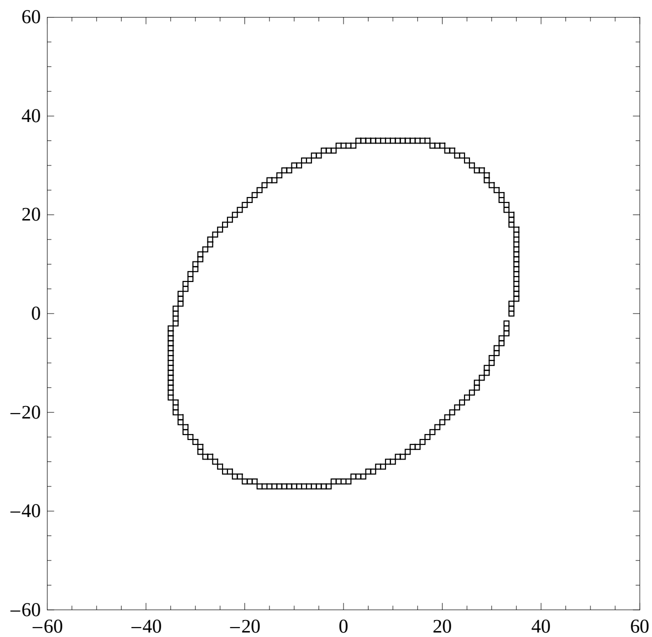
```
ellipsefunction[{30, 0}, {1, 0}, {30, 40}]
```

```
{{30, 0}, {0, {96 000, 0}}, {1600, 900, 0}}
```

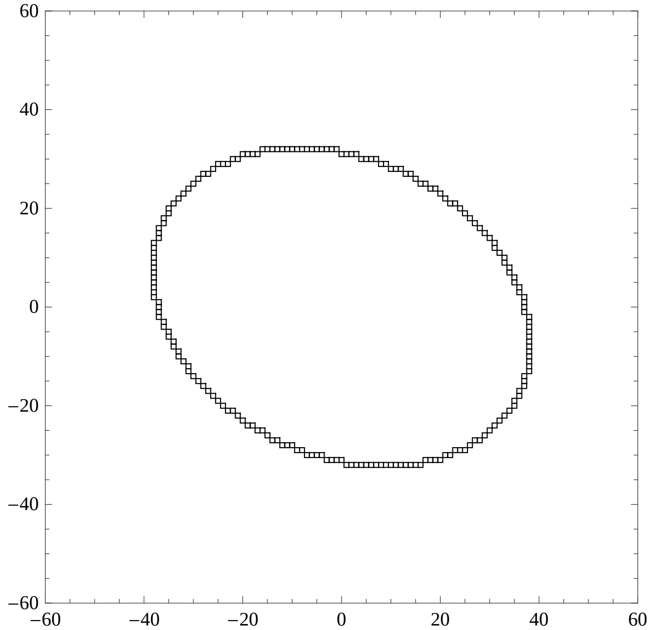
```
NestList[EllipseDiagonalIteration @@ # &, ellipsefunction[{30, 0}, {1, 0}, {30, 40}], 199];
Show[Graphics[unitsquare[First[#]] & /@ %,
  Frame → True, AspectRatio → 1, PlotRange → 60 {{-1, 1}, {-1, 1}}],
 Graphics[{Hue[0], Circle[{0, 0}, {30, 40}]}, Frame → True, AspectRatio → 1,
  PlotRange → 60 {{-1, 1}, {-1, 1}}], DisplayFunction → $DisplayFunction]
```



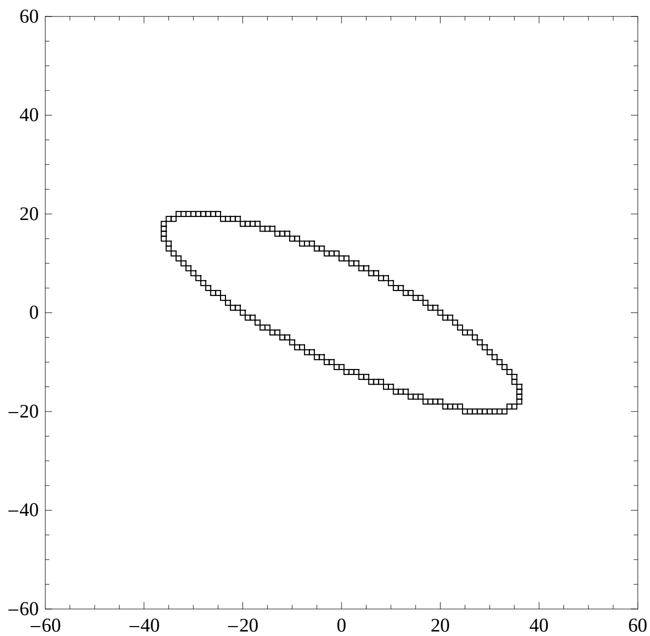

```
NestList[EllipseDiagonalIteration @@ # &, ellipsefunction[{34, 0}, {1, 1}, {40, 30}], 196];
Show[Graphics[unitsquare[First[#]] & /@ %,
  Frame → True, AspectRatio → 1, PlotRange → 60 {{-1, 1}, {-1, 1}}]]
```



```
NestList[EllipseDiagonalIteration @@ # &, ellipsefunction[{37, 0}, {-2, 1}, {40, 30}], 197];
Show[Graphics[unitsquare[First[#]] & /@ %,
  Frame → True, AspectRatio → 1, PlotRange → 60 {{-1, 1}, {-1, 1}}]]
```




```
NestList[EllipseDiagonalIteration @@ # &, ellipsefunction[{20, 0}, {-2, 1}, {40, 10}], 151];
Show[Graphics[unitsquare[First[#]] & /@ %,
  Frame → True, AspectRatio → 1, PlotRange → 60 {{-1, 1}, {-1, 1}}]]
```



Starting and stopping conditions

Finding a parametrization that will set the initial point onto an ellipse arc to be drawn, and that will tell the ellipse iteration program when to stop drawing, is not entirely straightforward.
The most practical approach seems to be to define the initial and final tangent directions (because we happen to keep track of the partial derivatives of the discriminant Δ, anyway). Since every tangent direction occurs twice on the ellipse arc, we need to keep track of a direction, too, to resolve the ambiguity.

The tangent direction $\begin{pmatrix} t \\ u \end{pmatrix}$ is perpendicular to the gradient of the discriminant, i.e., $0 = \left\langle \begin{matrix} t & u \end{matrix} \,\middle|\, \begin{matrix} \frac{\partial\Delta}{\partial x} & \frac{\partial\Delta}{\partial y} \end{matrix} \right\rangle = t\,\frac{\partial\Delta}{\partial x} + u\,\frac{\partial\Delta}{\partial y}$.

For the orientation of the direction, we require that initially, $0 < \left\langle \begin{matrix} t & u \end{matrix} \,\middle|\, \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} \,\middle|\, \begin{matrix} \frac{\partial\Delta}{\partial x} & \frac{\partial\Delta}{\partial y} \end{matrix} \right\rangle = t\,\frac{\partial\Delta}{\partial y} - u\,\frac{\partial\Delta}{\partial x}$.

Stopping condition

Initially, the dot product $\left\langle \begin{matrix} t & u \end{matrix} \,\middle|\, \begin{matrix} \frac{\partial\Delta}{\partial x} & \frac{\partial\Delta}{\partial y} \end{matrix} \right\rangle$ will go positive, i.e., the transition we are looking for as termination condition is a transition from negative to positive.
If we want to be able to use the same tangent vector as starting and stopping condition, we must start testing for the stopping condition *after* the first iteration.

Starting condition

Problem:
Find the point on a discrete grid closest to the point that satisfies
$0 == A\,x^2 + B\,y^2 + 2\,C\,x\,y - D$ subject to $0 == t(A\,x + C\,y) + u(C\,x + B\,y)$ and $0 < t(C\,x + B\,y) - u(A\,x + C\,y)$.

Note the negaive sign, such that $D > 0$, contrary to the definition further above.

```
FpPeS[FullSimplify[Solve[{0 == A x^2 + B y^2 + 2 C x y - D, 0 == t (A x + C y) + u (C x + B y)}, {x, y}]]]
```

$$\left\{\left\{x \to -\frac{\sqrt{D}\,(C\,t + B\,u)}{\sqrt{A\,B - C^2}\,\sqrt{A\,t^2 + u\,(2\,C\,t + B\,u)}},\ y \to \frac{\sqrt{D}\,(A\,t + C\,u)}{\sqrt{A\,B - C^2}\,\sqrt{A\,t^2 + u\,(2\,C\,t + B\,u)}}\right\},\right.$$
$$\left.\left\{x \to \frac{\sqrt{D}\,(C\,t + B\,u)}{\sqrt{A\,B - C^2}\,\sqrt{A\,t^2 + u\,(2\,C\,t + B\,u)}},\ y \to -\frac{\sqrt{D}\,(A\,t + C\,u)}{\sqrt{A\,B - C^2}\,\sqrt{A\,t^2 + u\,(2\,C\,t + B\,u)}}\right\}\right\}$$

$$\sqrt{\frac{D}{(A\,B - C^2)(A\,t^2 + B\,u^2 + 2\,C\,t\,u)}}\begin{pmatrix} \pm(C\,t + B\,u) \\ \mp(A\,t + C\,u) \end{pmatrix}$$

$$\{A \to b^2\,\xi^2 + a^2\,\upsilon^2,\ B \to a^2\,\xi^2 + b^2\,\upsilon^2,\ C \to (-a^2 + b^2)\,\xi\,\upsilon,\ D \to a^2\,b^2\,(\xi^2 + \upsilon^2)\}$$

$$\text{Simplify}\left[\sqrt{\frac{D}{(A\,B - C^2)\,(A\,t^2 + B\,u^2 + 2\,C\,t\,u)}}\ \text{/. \%}\right]$$

```
Simplify[{C t + B u, A t + C u} /. %%]
```

$$\{A \to b^2\,\xi^2 + a^2\,\upsilon^2,\ B \to a^2\,\xi^2 + b^2\,\upsilon^2,\ C \to (-a^2 + b^2)\,\xi\,\upsilon,\ D \to a^2\,b^2\,(\xi^2 + \upsilon^2)\}$$

$$\sqrt{\frac{1}{(\xi^2 + \upsilon^2)\,\left(a^2\,(u\,\xi - t\,\upsilon)^2 + b^2\,(t\,\xi + u\,\upsilon)^2\right)}}$$

$$\{a^2\,\xi\,(u\,\xi - t\,\upsilon) + b^2\,\upsilon\,(t\,\xi + u\,\upsilon),\ a^2\,\upsilon\,(-u\,\xi + t\,\upsilon) + b^2\,\xi\,(t\,\xi + u\,\upsilon)\}$$

$$\text{Collect}\left[\left(a^2\,(u\,\xi - t\,\upsilon)^2 + b^2\,(t\,\xi + u\,\upsilon)^2\right),\ \{u,\ t\}\right]$$

$$t\,u\,(-2\,a^2\,\xi\,\upsilon + 2\,b^2\,\xi\,\upsilon) + t^2\,(b^2\,\xi^2 + a^2\,\upsilon^2) + u^2\,(a^2\,\xi^2 + b^2\,\upsilon^2)$$

$$\{A \to b^2\,\xi^2 + a^2\,\upsilon^2,\ B \to a^2\,\xi^2 + b^2\,\upsilon^2,\ C \to (-a^2 + b^2)\,\xi\,\upsilon,\ D \to a^2\,b^2\,(\xi^2 + \upsilon^2)\}$$

$$\text{Simplify}\left[\sqrt{\frac{D}{A\,B - C^2}}\ \text{/. \%}\right]$$

$$\{A \to b^2\,\xi^2 + a^2\,\upsilon^2,\ B \to a^2\,\xi^2 + b^2\,\upsilon^2,\ C \to (-a^2 + b^2)\,\xi\,\upsilon,\ D \to a^2\,b^2\,(\xi^2 + \upsilon^2)\}$$

$$\sqrt{\frac{1}{\xi^2 + \upsilon^2}}$$

$$\frac{1}{\sqrt{(\xi^2+\upsilon^2)\,(A\,t^2+B\,u^2+2\,C\,t\,u)}}\begin{pmatrix}\pm(C\,t + B\,u)\\ \mp(A\,t + C\,u)\end{pmatrix}$$

Numeric range considerations

We need to make sure that the quantities used in the computations do not overflow.

```
ellipsediscriminant
```

$$2\,(-a^2 + b^2)\,x\,y\,\xi\,\upsilon - a^2\,b^2\,(\xi^2 + \upsilon^2) + x^2\,(b^2\,\xi^2 + a^2\,\upsilon^2) + y^2\,(a^2\,\xi^2 + b^2\,\upsilon^2)$$

$$\text{ellipsematrixeqns} = \{A == b^2\,\xi^2 + a^2\,\upsilon^2,\ B == a^2\,\xi^2 + b^2\,\upsilon^2,\ C == (-a^2 + b^2)\,\xi\,\upsilon,\ D == a^2\,b^2\,(\xi^2 + \upsilon^2)\}$$

$$\{A == b^2\,\xi^2 + a^2\,\upsilon^2,\ B == a^2\,\xi^2 + b^2\,\upsilon^2,\ C == (-a^2 + b^2)\,\xi\,\upsilon,\ D == a^2\,b^2\,(\xi^2 + \upsilon^2)\}$$

```
Collect[D[ellipsediscriminant, #], {x, y}, Simplify] & /@ {x, y}
D[%, #] & /@ {x, y}
```

$$\{2\,(-a^2 + b^2)\,y\,\xi\,\upsilon + 2\,x\,(b^2\,\xi^2 + a^2\,\upsilon^2),\ 2\,(-a^2 + b^2)\,x\,\xi\,\upsilon + 2\,y\,(a^2\,\xi^2 + b^2\,\upsilon^2)\}$$

$$\{\{2\,(b^2\,\xi^2 + a^2\,\upsilon^2),\ 2\,(-a^2 + b^2)\,\xi\,\upsilon\},\ \{2\,(-a^2 + b^2)\,\xi\,\upsilon,\ 2\,(a^2\,\xi^2 + b^2\,\upsilon^2)\}\}$$

```
{C t + B u, A t + C u} /. LSolve[ellipsematrixeqns, {A, B, C, D}]
```

$$\{-(a^2 - b^2)\,t\,\xi\,\upsilon + u\,(a^2\,\xi^2 + b^2\,\upsilon^2),\ -(a^2 - b^2)\,u\,\xi\,\upsilon + t\,(b^2\,\xi^2 + a^2\,\upsilon^2)\}$$

```
A t^2 + B u^2 + 2 C t u /. LSolve[ellipsematrixeqns, {A, B, C, D}]
```

$-2\,(a^2-b^2)\,t\,u\,\xi\,\upsilon + t^2\,(b^2\,\xi^2 + a^2\,\upsilon^2) + u^2\,(a^2\,\xi^2 + b^2\,\upsilon^2)$

In the final iterations, we must scale the matrix elements A, B, C, and D such that the iteration states can't overflow.

- **Using normalization to alleviate dynamic range and roundoff problems**

To help restricting the numerical range of quantities, it helps to normalize the vectors describing the principal axis $\begin{pmatrix}\xi\\ \upsilon\end{pmatrix}$ and the tangent direction $\begin{pmatrix}t\\ u\end{pmatrix}$ to unit magnitude.

```
ellipsematrixeqns /. {ξ^2 + υ^2 → 1}
```

$\{A == b^2\,\xi^2 + a^2\,\upsilon^2,\ B == a^2\,\xi^2 + b^2\,\upsilon^2,\ C == (-a^2 + b^2)\,\xi\,\upsilon,\ D == a^2\,b^2\}$

■ Implementation of ellipse drawing with starting and stopping conditions

This version uses $\{x, \Xi, ab\}$, where
x is the 2-dimensional initial coordinate vector
Ξ is the 2-dimensional direction of the first principal axis
ab is the 2-dimensional vector of principal axis lengths

```
ellipsebegin = Function[{T, Ξ, ab}, Block[{τ = T, ξ = Ξ, A, B, C, D, x, Δ, dΔ},
   A = ab[[2]]^2 ξ[[1]]^2 + ab[[1]]^2 ξ[[2]]^2; B = ab[[1]]^2 ξ[[1]]^2 + ab[[2]]^2 ξ[[2]]^2; C = (ab[[2]]^2 - ab[[1]]^2) ξ[[1]] ξ[[2]];
   D = (ab[[1]] ab[[2]])^2 (ξ.ξ); x = Round[{{0, 1}, {-1, 0}}.{{A, C}, {C, B}}.τ / Sqrt[(ξ.ξ) (τ.{{A, C}, {C, B}}.τ)]];
   {x, {Δ = A x[[1]]^2 + B x[[2]]^2 + 2 C x[[1]] x[[2]] - D, dΔ = 2 {A x[[1]] + C x[[2]], C x[[1]] + B x[[2]]}}, {A, B, C}}]]
```

```
Function[{T, Ξ, ab}, Block[{τ = T, ξ = Ξ, A, B, C, D, x, Δ, dΔ},
  A = ab[[2]]^2 ξ[[1]]^2 + ab[[1]]^2 ξ[[2]]^2; B = ab[[1]]^2 ξ[[1]]^2 + ab[[2]]^2 ξ[[2]]^2; C = (ab[[2]]^2 - ab[[1]]^2) ξ[[1]] ξ[[2]];
  D = (ab[[1]] ab[[2]])^2 ξ.ξ; x = Round[{{0, 1}, {-1, 0}}.{{A, C}, {C, B}}.τ / Sqrt[ξ.ξ τ.{{A, C}, {C, B}}.τ]];
  {x, {Δ = A x[[1]]^2 + B x[[2]]^2 + 2 C x[[1]] x[[2]] - D, dΔ = 2 {A x[[1]] + C x[[2]], C x[[1]] + B x[[2]]}}, {A, B, C}}]]
```

This version iterates $\{x$, iterationvariables, iterationconstants$\}$, where
x is the 2-dimensional coordinate vector
iterationvariables $\equiv \{\Delta, d\Delta\}$, where
Δ is the discriminant
$d\Delta \equiv \left\{\frac{\partial\Delta}{\partial x}, \frac{\partial\Delta}{\partial y}\right\}$
iterationconstants are the matrix elements $\{A, B, C\}$

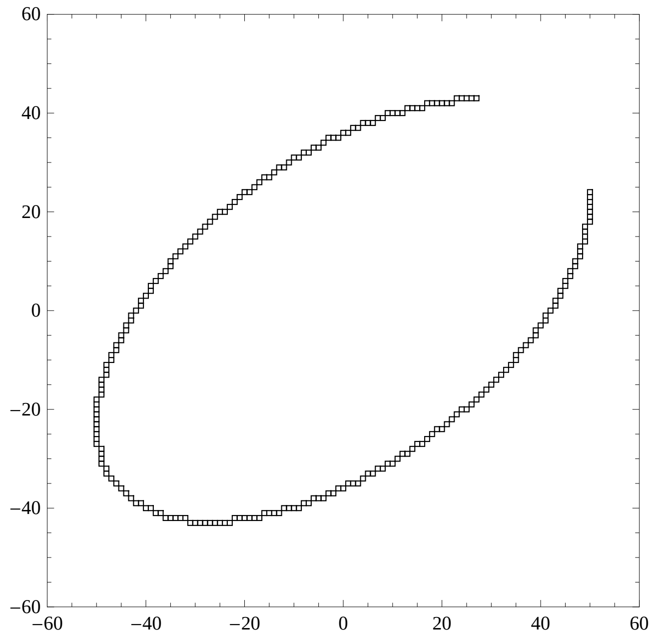
Now put everything together

```
EllipseDraw = Function[{τ1, τ2, Ξ, ab},
  NestWhileList[EllipseDiagonalIteration @@ # &,
   ellipsebegin[τ1, Ξ, ab], ! ((τ2.#1[[2, 2]] < 0) && (τ2.#2[[2, 2]] ≥ 0)) &, 2]
 ]
```

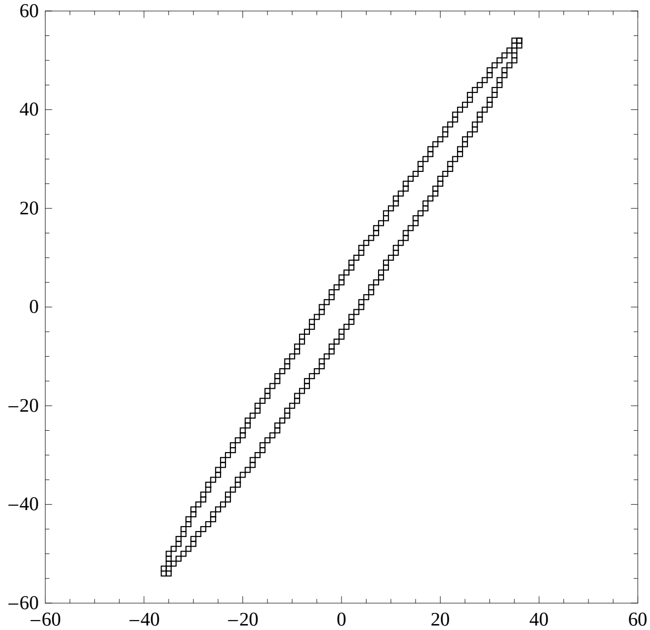
```
Function[{τ1, τ2, Ξ, ab}, NestWhileList[EllipseDiagonalIteration @@ #1 &,
  ellipsebegin[τ1, Ξ, ab], ! (τ2.#1[[2, 2]] < 0 && τ2.#2[[2, 2]] ≥ 0) &, 2]]
```

Examples

```
EllipseDraw[{-1, 0}, {0, 1}, {4, 3}, {58, 31}];
Show[Graphics[unitsquare[First[#]] & /@ %,
  Frame → True, AspectRatio → 1, PlotRange → 60 {{-1, 1}, {-1, 1}}]]
```

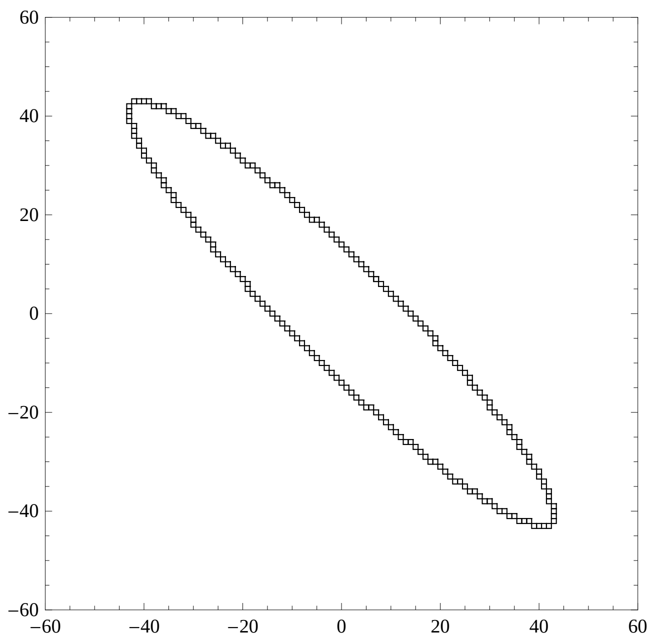


```
EllipseDraw[{-3, 2}, {-3, 2}, {2, 3}, {65, 3}];
Show[Graphics[unitsquare[First[#]] & /@ %,
  Frame → True, AspectRatio → 1, PlotRange → 60 {{-1, 1}, {-1, 1}}]]
```



■ Verification

```
ell60 = EllipseDraw @@ {{-1, 1}, {-1, 1}, {1, -1}, {60, 10}};
Show[Graphics[unitsquare[First[#]] & /@ ell60,
  Frame → True, AspectRatio → 1, PlotRange → 60 {{-1, 1}, {-1, 1}}]]
```



- **Data set of ellipse drawn**

$$\left\{\{x, y\}, \left\{\Delta, \left\{\frac{\partial\Delta}{\partial x}, \frac{\partial\Delta}{\partial y}\right\}\right\}, \{A, B, C\}\right\}$$

```
ell60
```

```
{{{7, 7}, {-14400, {100800, 100800}}, {3700, 3700, 3500}},
 {{6, 8}, {-14000, {100400, 101200}}, {3700, 3700, 3500}},
 {{5, 9}, {-12800, {100000, 101600}}, {3700, 3700, 3500}},
 {{4, 10}, {-10800, {99600, 102000}}, {3700, 3700, 3500}},
 {{3, 11}, {-8000, {99200, 102400}}, {3700, 3700, 3500}},
 {{2, 12}, {-4400, {98800, 102800}}, {3700, 3700, 3500}},
 {{1, 13}, {0, {98400, 103200}}, {3700, 3700, 3500}},
 {{0, 14}, {5200, {98000, 103600}}, {3700, 3700, 3500}},
 {{-1, 15}, {11200, {97600, 104000}}, {3700, 3700, 3500}},
 {{-2, 16}, {18000, {97200, 104400}}, {3700, 3700, 3500}},
 {{-3, 17}, {25600, {96800, 104800}}, {3700, 3700, 3500}},
 {{-4, 18}, {34000, {96400, 105200}}, {3700, 3700, 3500}},
 {{-5, 19}, {43200, {96000, 105600}}, {3700, 3700, 3500}},
 {{-6, 19}, {-49100, {88600, 98600}}, {3700, 3700, 3500}},
 {{-7, 20}, {-38700, {88200, 99000}}, {3700, 3700, 3500}},
 {{-8, 21}, {-27500, {87800, 99400}}, {3700, 3700, 3500}},
 {{-9, 22}, {-15500, {87400, 99800}}, {3700, 3700, 3500}},
 {{-10, 23}, {-2700, {87000, 100200}}, {3700, 3700, 3500}},
 {{-11, 24}, {10900, {86600, 100600}}, {3700, 3700, 3500}},
 {{-12, 25}, {25300, {86200, 101000}}, {3700, 3700, 3500}},
 {{-13, 26}, {40500, {85800, 101400}}, {3700, 3700, 3500}},
 {{-14, 26}, {-41600, {78400, 94400}}, {3700, 3700, 3500}},
 {{-15, 27}, {-25200, {78000, 94800}}, {3700, 3700, 3500}},
 {{-16, 28}, {-8000, {77600, 95200}}, {3700, 3700, 3500}},
 {{-17, 29}, {10000, {77200, 95600}}, {3700, 3700, 3500}},
 {{-18, 30}, {28800, {76800, 96000}}, {3700, 3700, 3500}},
 {{-19, 30}, {-44300, {69400, 89000}}, {3700, 3700, 3500}},
 {{-20, 31}, {-24300, {69000, 89400}}, {3700, 3700, 3500}},
 {{-21, 32}, {-3500, {68600, 89800}}, {3700, 3700, 3500}},
 {{-22, 33}, {18100, {68200, 90200}}, {3700, 3700, 3500}},
 {{-23, 34}, {40500, {67800, 90600}}, {3700, 3700, 3500}},
 {{-24, 34}, {-23600, {60400, 83600}}, {3700, 3700, 3500}},
 {{-25, 35}, {0, {60000, 84000}}, {3700, 3700, 3500}},
 {{-26, 36}, {24400, {59600, 84400}}, {3700, 3700, 3500}},
 {{-27, 36}, {-31500, {52200, 77400}}, {3700, 3700, 3500}},
 {{-28, 37}, {-5900, {51800, 77800}}, {3700, 3700, 3500}},
 {{-29, 38}, {20500, {51400, 78200}}, {3700, 3700, 3500}},
 {{-30, 38}, {-27200, {44000, 71200}}, {3700, 3700, 3500}},
 {{-31, 39}, {400, {43600, 71600}}, {3700, 3700, 3500}},
 {{-32, 40}, {28800, {43200, 72000}}, {3700, 3700, 3500}},
 {{-33, 40}, {-10700, {35800, 65000}}, {3700, 3700, 3500}},
 {{-34, 41}, {18900, {35400, 65400}}, {3700, 3700, 3500}},
 {{-35, 41}, {-12800, {28000, 58400}}, {3700, 3700, 3500}},
 {{-36, 42}, {18000, {27600, 58800}}, {3700, 3700, 3500}},
 {{-37, 42}, {-5900, {20200, 51800}}, {3700, 3700, 3500}},
 {{-38, 42}, {-22400, {12800, 44800}}, {3700, 3700, 3500}},
```

```
{{-39, 43}, {10 000, {12 400, 45 200}}, {3700, 3700, 3500}},
{{-40, 43}, {1300, {5000, 38 200}}, {3700, 3700, 3500}},
{{-41, 43}, {0, {-2400, 31 200}}, {3700, 3700, 3500}},
{{-42, 43}, {6100, {-9800, 24 200}}, {3700, 3700, 3500}},
{{-43, 42}, {6100, {-24 200, 9800}}, {3700, 3700, 3500}},
{{-43, 41}, {0, {-31 200, 2400}}, {3700, 3700, 3500}},
{{-43, 40}, {1300, {-38 200, -5000}}, {3700, 3700, 3500}},
{{-43, 39}, {10 000, {-45 200, -12 400}}, {3700, 3700, 3500}},
{{-42, 38}, {-22 400, {-44 800, -12 800}}, {3700, 3700, 3500}},
{{-42, 37}, {-5900, {-51 800, -20 200}}, {3700, 3700, 3500}},
{{-42, 36}, {18 000, {-58 800, -27 600}}, {3700, 3700, 3500}},
{{-41, 35}, {-12 800, {-58 400, -28 000}}, {3700, 3700, 3500}},
{{-41, 34}, {18 900, {-65 400, -35 400}}, {3700, 3700, 3500}},
{{-40, 33}, {-10 700, {-65 000, -35 800}}, {3700, 3700, 3500}},
{{-40, 32}, {28 800, {-72 000, -43 200}}, {3700, 3700, 3500}},
{{-39, 31}, {400, {-71 600, -43 600}}, {3700, 3700, 3500}},
{{-38, 30}, {-27 200, {-71 200, -44 000}}, {3700, 3700, 3500}},
{{-38, 29}, {20 500, {-78 200, -51 400}}, {3700, 3700, 3500}},
{{-37, 28}, {-5900, {-77 800, -51 800}}, {3700, 3700, 3500}},
{{-36, 27}, {-31 500, {-77 400, -52 200}}, {3700, 3700, 3500}},
{{-36, 26}, {24 400, {-84 400, -59 600}}, {3700, 3700, 3500}},
{{-35, 25}, {0, {-84 000, -60 000}}, {3700, 3700, 3500}},
{{-34, 24}, {-23 600, {-83 600, -60 400}}, {3700, 3700, 3500}},
{{-34, 23}, {40 500, {-90 600, -67 800}}, {3700, 3700, 3500}},
{{-33, 22}, {18 100, {-90 200, -68 200}}, {3700, 3700, 3500}},
{{-32, 21}, {-3500, {-89 800, -68 600}}, {3700, 3700, 3500}},
{{-31, 20}, {-24 300, {-89 400, -69 000}}, {3700, 3700, 3500}},
{{-30, 19}, {-44 300, {-89 000, -69 400}}, {3700, 3700, 3500}},
{{-30, 18}, {28 800, {-96 000, -76 800}}, {3700, 3700, 3500}},
{{-29, 17}, {10 000, {-95 600, -77 200}}, {3700, 3700, 3500}},
{{-28, 16}, {-8000, {-95 200, -77 600}}, {3700, 3700, 3500}},
{{-27, 15}, {-25 200, {-94 800, -78 000}}, {3700, 3700, 3500}},
{{-26, 14}, {-41 600, {-94 400, -78 400}}, {3700, 3700, 3500}},
{{-26, 13}, {40 500, {-101 400, -85 800}}, {3700, 3700, 3500}},
{{-25, 12}, {25 300, {-101 000, -86 200}}, {3700, 3700, 3500}},
{{-24, 11}, {10 900, {-100 600, -86 600}}, {3700, 3700, 3500}},
{{-23, 10}, {-2700, {-100 200, -87 000}}, {3700, 3700, 3500}},
{{-22, 9}, {-15 500, {-99 800, -87 400}}, {3700, 3700, 3500}},
{{-21, 8}, {-27 500, {-99 400, -87 800}}, {3700, 3700, 3500}},
{{-20, 7}, {-38 700, {-99 000, -88 200}}, {3700, 3700, 3500}},
{{-19, 6}, {-49 100, {-98 600, -88 600}}, {3700, 3700, 3500}},
{{-19, 5}, {43 200, {-105 600, -96 000}}, {3700, 3700, 3500}},
{{-18, 4}, {34 000, {-105 200, -96 400}}, {3700, 3700, 3500}},
{{-17, 3}, {25 600, {-104 800, -96 800}}, {3700, 3700, 3500}},
{{-16, 2}, {18 000, {-104 400, -97 200}}, {3700, 3700, 3500}},
{{-15, 1}, {11 200, {-104 000, -97 600}}, {3700, 3700, 3500}},
{{-14, 0}, {5200, {-103 600, -98 000}}, {3700, 3700, 3500}},
{{-13, -1}, {0, {-103 200, -98 400}}, {3700, 3700, 3500}},
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```

```
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{{43, -39}, {10 000, {45 200, 12 400}}, {3700, 3700, 3500}},
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```

```
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{{32, -21}, {-3500, {89 800, 68 600}}, {3700, 3700, 3500}},
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{{23, -10}, {-2700, {100 200, 87 000}}, {3700, 3700, 3500}},
{{22, -9}, {-15 500, {99 800, 87 400}}, {3700, 3700, 3500}},
{{21, -8}, {-27 500, {99 400, 87 800}}, {3700, 3700, 3500}},
{{20, -7}, {-38 700, {99 000, 88 200}}, {3700, 3700, 3500}},
{{19, -6}, {-49 100, {98 600, 88 600}}, {3700, 3700, 3500}},
{{19, -5}, {43 200, {105 600, 96 000}}, {3700, 3700, 3500}},
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{{8, 6}, {-14 000, {101 200, 100 400}}, {3700, 3700, 3500}},
{{7, 7}, {-14 400, {100 800, 100 800}}, {3700, 3700, 3500}}}
```

Verify that the iterated Δ and the Δ computed from the coordinates match:

```
ellipsediscriminant
disc60 = ellipsediscriminant /. {ξ → 1, υ → -1, a → 60, b → 10}
```

$$2\left(-a^2+b^2\right) x\, y\, \xi\, \upsilon - a^2 b^2 \left(\xi^2+\upsilon^2\right) + x^2 \left(b^2 \xi^2 + a^2 \upsilon^2\right) + y^2 \left(a^2 \xi^2 + b^2 \upsilon^2\right)$$

$$-720\,000 + 3700\, x^2 + 7000\, x\, y + 3700\, y^2$$

$\{x, y\}$ $\left\{\Delta_{\text{computed}}, \Delta_{\text{iterated}}\right\}$

```
TableForm[{#[[1]], {disc60 /. {x → #[[1, 1]], y → #[[1, 2]]}, First[#[[2]]]}} & /@ ell60, TableDepth → 2]
```

```
{7, 7}      {-14400, -14400}
{6, 8}      {-14000, -14000}
{5, 9}      {-12800, -12800}
{4, 10}     {-10800, -10800}
{3, 11}     {-8000, -8000}
{2, 12}     {-4400, -4400}
{1, 13}     {0, 0}
{0, 14}     {5200, 5200}
{-1, 15}    {11200, 11200}
{-2, 16}    {18000, 18000}
{-3, 17}    {25600, 25600}
{-4, 18}    {34000, 34000}
{-5, 19}    {43200, 43200}
{-6, 19}    {-49100, -49100}
{-7, 20}    {-38700, -38700}
{-8, 21}    {-27500, -27500}
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{-10, 23}   {-2700, -2700}
{-11, 24}   {10900, 10900}
{-12, 25}   {25300, 25300}
{-13, 26}   {40500, 40500}
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{-15, 27}   {-25200, -25200}
{-16, 28}   {-8000, -8000}
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{-19, 30}   {-44300, -44300}
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{-21, 32}   {-3500, -3500}
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{-25, 35}   {0, 0}
{-26, 36}   {24400, 24400}
{-27, 36}   {-31500, -31500}
{-28, 37}   {-5900, -5900}
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{-34, 41}   {18900, 18900}
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{-36, 42}   {18000, 18000}
{-37, 42}   {-5900, -5900}
```

```
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{-43, 41}  {0, 0}
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{-40, 33}  {-10 700, -10 700}
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{-39, 31}  {400, 400}
{-38, 30}  {-27 200, -27 200}
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{-35, 25}  {0, 0}
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{-33, 22}  {18 100, 18 100}
{-32, 21}  {-3500, -3500}
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{-23, 10}  {-2700, -2700}
{-22, 9}   {-15 500, -15 500}
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{-19, 6}   {-49 100, -49 100}
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```

```
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{-15, 1}   {11200, 11200}
{-14, 0}   {5200, 5200}
{-13, -1}  {0, 0}
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{-11, -3}  {-8000, -8000}
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{-9, -5}   {-12800, -12800}
{-8, -6}   {-14000, -14000}
{-7, -7}   {-14400, -14400}
{-6, -8}   {-14000, -14000}
{-5, -9}   {-12800, -12800}
{-4, -10}  {-10800, -10800}
{-3, -11}  {-8000, -8000}
{-2, -12}  {-4400, -4400}
{-1, -13}  {0, 0}
{0, -14}   {5200, 5200}
{1, -15}   {11200, 11200}
{2, -16}   {18000, 18000}
{3, -17}   {25600, 25600}
{4, -18}   {34000, 34000}
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{6, -19}   {-49100, -49100}
{7, -20}   {-38700, -38700}
{8, -21}   {-27500, -27500}
{9, -22}   {-15500, -15500}
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{11, -24}  {10900, 10900}
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{23, -34}  {40500, 40500}
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{27, -36}  {-31500, -31500}
{28, -37}  {-5900, -5900}
```

```
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{33, -40}  {-10 700, -10 700}
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{25, -12}  {25 300, 25 300}
```

```
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{22, -9}   {-15 500, -15 500}
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{19, -6}   {-49 100, -49 100}
{19, -5}   {43 200, 43 200}
{18, -4}   {34 000, 34 000}
{17, -3}   {25 600, 25 600}
{16, -2}   {18 000, 18 000}
{15, -1}   {11 200, 11 200}
{14, 0}    {5200, 5200}
{13, 1}    {0, 0}
{12, 2}    {-4400, -4400}
{11, 3}    {-8000, -8000}
{10, 4}    {-10 800, -10 800}
{9, 5}     {-12 800, -12 800}
{8, 6}     {-14 000, -14 000}
{7, 7}     {-14 400, -14 400}
```

Generalization to muralizer

Basic equation

To draw a straight line, the muralizer spoolers need to follow a trajectory defined by $\begin{pmatrix} r^2 & s^2 \end{pmatrix} \cdot \begin{pmatrix} A & C \\ C & B \end{pmatrix} \cdot \begin{pmatrix} r^2 \\ s^2 \end{pmatrix} + \begin{pmatrix} r^2 & s^2 \end{pmatrix} \cdot \begin{pmatrix} D \\ E \end{pmatrix} + F == 0$, where r and s are the lengths of thread spooled.

Iterated variables

The discriminant and its gradient need not be computed from scratch for each new point, but can be iterated, exploiting the fact that for continuous lines, each of the coordinate can only change by 0 or ± 1.

▪ Discriminant

```
a x^2 + c x y + b y^2 + d x + e y + f /. {x → r^2, y → s^2}
TableForm[% /. {{r → r + 1}, {r → r + 1, s → s + 1}, {s → s + 1}, {r → r - 1, s → s + 1},
   {r → r - 1}, {r → r - 1, s → s - 1}, {s → s - 1}, {r → r + 1, s → s - 1}}]
TableForm[Collect[#, {a, b, c, d, e, f}, FullSimplify] & /@ (% - %%)]
TableForm[FullSimplify[ListConvolve[{1, -1}, %, 1]]]
TableForm[Simplify[{-1, 1, -1}.%%[[#]]] & /@ {{1, 2, 3}, {3, 4, 5}, {5, 6, 7}, {7, 8, 1}}]
```

$f + d\,r^2 + a\,r^4 + e\,s^2 + c\,r^2\,s^2 + b\,s^4$

$f + d\,(1+r)^2 + a\,(1+r)^4 + e\,s^2 + c\,(1+r)^2\,s^2 + b\,s^4$
$f + d\,(1+r)^2 + a\,(1+r)^4 + e\,(1+s)^2 + c\,(1+r)^2\,(1+s)^2 + b\,(1+s)^4$
$f + d\,r^2 + a\,r^4 + e\,(1+s)^2 + c\,r^2\,(1+s)^2 + b\,(1+s)^4$
$f + d\,(-1+r)^2 + a\,(-1+r)^4 + e\,(1+s)^2 + c\,(-1+r)^2\,(1+s)^2 + b\,(1+s)^4$
$f + d\,(-1+r)^2 + a\,(-1+r)^4 + e\,s^2 + c\,(-1+r)^2\,s^2 + b\,s^4$
$f + d\,(-1+r)^2 + a\,(-1+r)^4 + e\,(-1+s)^2 + c\,(-1+r)^2\,(-1+s)^2 + b\,(-1+s)^4$
$f + d\,r^2 + a\,r^4 + e\,(-1+s)^2 + c\,r^2\,(-1+s)^2 + b\,(-1+s)^4$
$f + d\,(1+r)^2 + a\,(1+r)^4 + e\,(-1+s)^2 + c\,(1+r)^2\,(-1+s)^2 + b\,(-1+s)^4$

$d\,(1+2r) + a\left(-r^4 + (1+r)^4\right) + c\,(1+2r)\,s^2$
$d\,(1+2r) + a\left(-r^4 + (1+r)^4\right) + e\,(1+2s) + c\,(1+r+s)\,(1+r+s+2rs) + b\left(-s^4 + (1+s)^4\right)$
$e\,(1+2s) + c\,r^2\,(1+2s) + b\left(-s^4 + (1+s)^4\right)$
$d\,(1-2r) + a\left((-1+r)^4 - r^4\right) + e\,(1+2s) + c\,(-1+r-s)\,(-1+r-s+2rs) + b\left(-s^4 + (1+s)^4\right)$
$d\,(1-2r) + a\left((-1+r)^4 - r^4\right) + c\,(1-2r)\,s^2$
$d\,(1-2r) + a\left((-1+r)^4 - r^4\right) + e\,(1-2s) + b\left((-1+s)^4 - s^4\right) - c\,(-1+r+s)\,(1-s+r\,(-1+2s))$
$e\,(1-2s) + c\,r^2\,(1-2s) + b\left((-1+s)^4 - s^4\right)$
$d\,(1+2r) + a\left(-r^4 + (1+r)^4\right) + e\,(1-2s) + b\left((-1+s)^4 - s^4\right) - c\,(1+r-s)\,(-1+s+r\,(-1+2s))$

$(-1+2s)\left(e + c\,(1+r)^2 + b\,(1+2\,(-1+s)\,s)\right)$
$(1+2s)\left(e + c\,(1+r)^2 + b\,(1+2s\,(1+s))\right)$
$-(1+2r)\left(d + a\,(1+2r\,(1+r)) + c\,(1+s)^2\right)$
$-(-1+2r)\left(d + a\,(1+2\,(-1+r)\,r) + c\,(1+s)^2\right)$
$-(1+2s)\left(e + c\,(-1+r)^2 + b\,(1+2s\,(1+s))\right)$
$-(-1+2s)\left(e + c\,(-1+r)^2 + b\,(1+2\,(-1+s)\,s)\right)$
$(-1+2r)\left(d + a\,(1+2\,(-1+r)\,r) + c\,(-1+s)^2\right)$
$(1+2r)\left(d + a\,(1+2r\,(1+r)) + c\,(-1+s)^2\right)$

$c\,(1+2r)\,(1+2s)$
$-c\,(-1+2r)\,(1+2s)$
$c\,(-1+2r)\,(-1+2s)$
$-c\,(1+2r)\,(-1+2s)$

Step in r direction: $a\left((r \pm 1)^4 - r^4\right) + \left(c\,s^2 + d\right)\left((r \pm 1)^2 - r^2\right)$

Step in s direction: $b\left((s \pm 1)^4 - s^4\right) + \left(c\,r^2 + e\right)\left((s \pm 1)^2 - s^2\right)$

Diagonal step: sum of orthogonal steps, $+c\left(r^2 - (r \pm 1)^2\right)\left(s^2 - (s \pm 1)^2\right)$

- **Partial derivative of discriminant w.r.t. r^2**

```
D[a x^2 + c x y + b y^2 + d x + e y + f, x] /. {x → r^2, y → s^2}
TableForm[% /. {{r → r + 1}, {r → r + 1, s → s + 1}, {s → s + 1}, {r → r - 1, s → s + 1},
   {r → r - 1}, {r → r - 1, s → s - 1}, {s → s - 1}, {r → r + 1, s → s - 1}}]
TableForm[Collect[#, {a, b, c, d, e, f}, FullSimplify] & /@ (% - %%)]
TableForm[FullSimplify[ListConvolve[{1, -1}, %, 1]]]
TableForm[Simplify[{-1, 1, -1}.%%[[#]]] & /@ {{1, 2, 3}, {3, 4, 5}, {5, 6, 7}, {7, 8, 1}}]
```

$d + 2 a r^2 + c s^2$

$d + 2 a (1 + r)^2 + c s^2$
$d + 2 a (1 + r)^2 + c (1 + s)^2$
$d + 2 a r^2 + c (1 + s)^2$
$d + 2 a (-1 + r)^2 + c (1 + s)^2$
$d + 2 a (-1 + r)^2 + c s^2$
$d + 2 a (-1 + r)^2 + c (-1 + s)^2$
$d + 2 a r^2 + c (-1 + s)^2$
$d + 2 a (1 + r)^2 + c (-1 + s)^2$

$a (2 + 4 r)$
$a (2 + 4 r) + c (1 + 2 s)$
$c (1 + 2 s)$
$a (2 - 4 r) + c (1 + 2 s)$
$a (2 - 4 r)$
$a (2 - 4 r) + c (1 - 2 s)$
$c (1 - 2 s)$
$a (2 + 4 r) + c (1 - 2 s)$

$c (-1 + 2 s)$
$c + 2 c s$
$-2 a (1 + 2 r)$
$a (2 - 4 r)$
$-c (1 + 2 s)$
$c - 2 c s$
$2 a (-1 + 2 r)$
$a (2 + 4 r)$

0
0
0
0

Partial derivative of discriminant w.r.t. s^2

```
D[a x^2 + c x y + b y^2 + d x + e y + f, y] /. {x → r^2, y → s^2}
TableForm[% /. {{r → r + 1}, {r → r + 1, s → s + 1}, {s → s + 1}, {r → r - 1, s → s + 1},
   {r → r - 1}, {r → r - 1, s → s - 1}, {s → s - 1}, {r → r + 1, s → s - 1}}]
TableForm[Collect[#, {a, b, c, d, e, f}, FullSimplify] & /@ (% - %%)]
TableForm[FullSimplify[ListConvolve[{1, -1}, %, 1]]]
TableForm[Simplify[{-1, 1, -1}.%%[[#]]] & /@ {{1, 2, 3}, {3, 4, 5}, {5, 6, 7}, {7, 8, 1}}]
```

$e + c\, r^2 + 2\, b\, s^2$

$e + c\, (1 + r)^2 + 2\, b\, s^2$
$e + c\, (1 + r)^2 + 2\, b\, (1 + s)^2$
$e + c\, r^2 + 2\, b\, (1 + s)^2$
$e + c\, (-1 + r)^2 + 2\, b\, (1 + s)^2$
$e + c\, (-1 + r)^2 + 2\, b\, s^2$
$e + c\, (-1 + r)^2 + 2\, b\, (-1 + s)^2$
$e + c\, r^2 + 2\, b\, (-1 + s)^2$
$e + c\, (1 + r)^2 + 2\, b\, (-1 + s)^2$

$c\, (1 + 2\, r)$
$c\, (1 + 2\, r) + b\, (2 + 4\, s)$
$b\, (2 + 4\, s)$
$c\, (1 - 2\, r) + b\, (2 + 4\, s)$
$c\, (1 - 2\, r)$
$c\, (1 - 2\, r) + b\, (2 - 4\, s)$
$b\, (2 - 4\, s)$
$c\, (1 + 2\, r) + b\, (2 - 4\, s)$

$2\, b\, (-1 + 2\, s)$
$b\, (2 + 4\, s)$
$-c\, (1 + 2\, r)$
$c - 2\, c\, r$
$-2\, b\, (1 + 2\, s)$
$b\, (2 - 4\, s)$
$c\, (-1 + 2\, r)$
$c + 2\, c\, r$

0
0
0
0

Gradient of discriminant

```
(D[a x^2 + c x y + b y^2 + d x + e y + f, #] & /@ {x, y}) /. {x → r^2, y → s^2}
Outer[ReplaceAll, %, {{r → r + 1}, {r → r + 1, s → s + 1}, {s → s + 1}, {r → r - 1, s → s + 1},
   {r → r - 1}, {r → r - 1, s → s - 1}, {s → s - 1}, {r → r + 1, s → s - 1}}, 1];
TableForm[Transpose[%], TableDepth → 1]
TableForm[
 Transpose[Map[Collect[#, {a, b, c, d, e, f}, FullSimplify] &, %% - %%%, {2}]], TableDepth → 1]
TableForm[Simplify[{-1, 1, -1}.%[[#]]] & /@ {{1, 2, 3}, {3, 4, 5}, {5, 6, 7}, {7, 8, 1}},
 TableDepth → 1]
```

$\{d + 2 a r^2 + c s^2, e + c r^2 + 2 b s^2\}$

$\{d + 2 a (1 + r)^2 + c s^2, e + c (1 + r)^2 + 2 b s^2\}$
$\{d + 2 a (1 + r)^2 + c (1 + s)^2, e + c (1 + r)^2 + 2 b (1 + s)^2\}$
$\{d + 2 a r^2 + c (1 + s)^2, e + c r^2 + 2 b (1 + s)^2\}$
$\{d + 2 a (-1 + r)^2 + c (1 + s)^2, e + c (-1 + r)^2 + 2 b (1 + s)^2\}$
$\{d + 2 a (-1 + r)^2 + c s^2, e + c (-1 + r)^2 + 2 b s^2\}$
$\{d + 2 a (-1 + r)^2 + c (-1 + s)^2, e + c (-1 + r)^2 + 2 b (-1 + s)^2\}$
$\{d + 2 a r^2 + c (-1 + s)^2, e + c r^2 + 2 b (-1 + s)^2\}$
$\{d + 2 a (1 + r)^2 + c (-1 + s)^2, e + c (1 + r)^2 + 2 b (-1 + s)^2\}$

$\{a (2 + 4 r), c (1 + 2 r)\}$
$\{a (2 + 4 r) + c (1 + 2 s), c (1 + 2 r) + b (2 + 4 s)\}$
$\{c (1 + 2 s), b (2 + 4 s)\}$
$\{a (2 - 4 r) + c (1 + 2 s), c (1 - 2 r) + b (2 + 4 s)\}$
$\{a (2 - 4 r), c (1 - 2 r)\}$
$\{a (2 - 4 r) + c (1 - 2 s), c (1 - 2 r) + b (2 - 4 s)\}$
$\{c (1 - 2 s), b (2 - 4 s)\}$
$\{a (2 + 4 r) + c (1 - 2 s), c (1 + 2 r) + b (2 - 4 s)\}$

$\{0, 0\}$
$\{0, 0\}$
$\{0, 0\}$
$\{0, 0\}$

Step in r direction: $\begin{pmatrix} \pm\left(4\, a\left((r \pm 1)^3 - r^3\right) + 2\left(c\, s^2 + d\right)\right) \\ 2\, c\, s\left((r \pm 1)^2 - r^2\right) \end{pmatrix}$

Step in s direction: $\begin{pmatrix} 2\, c\, r\left((s \mp 1)^2 - s^2\right) \\ \mp\left(4\, b\left((s \mp 1)^3 - s^3\right) + 2\left(c\, r^2 + e\right)\right) \end{pmatrix}$

Diagonal step: sum of orthogonal steps, $+\begin{pmatrix} \pm 2\, c\left((s \mp 1)^2 - s^2\right) \\ \mp 2\, c\left((r \pm 1)^2 - r^2\right) \end{pmatrix}$

Terms that need to be updated/recalculated per iteration

$\delta\mathrm{r4} \equiv (r \pm 1)^4 - r^4 = \delta\mathrm{r4c} \pm \delta\mathrm{r4d} \equiv 1 + 6\, r^2 \pm r\left(1 + r^2\right)$

```
Apart[(r + #)^4 - r^4 & /@ {1, -1}]
Simplify[#@@%/2 & /@ {Plus, Subtract}]
```

$\{1 + 4\,r + 6\,r^2 + 4\,r^3,\ 1 - 4\,r + 6\,r^2 - 4\,r^3\}$

$\{1 + 6\,r^2,\ 4\,(r + r^3)\}$

$$\delta\text{r3} \equiv (r \pm 1)^3 - r^3 = \delta\text{r3c} \pm \delta\text{r3d} \equiv 3\,r \pm (1 + 3\,r^2)$$

```
Apart[(r + #)^3 - r^3 & /@ {1, -1}]
Simplify[#@@%/2 & /@ {Plus, Subtract}]
```

$\{1 + 3\,r + 3\,r^2,\ -1 + 3\,r - 3\,r^2\}$

$\{3\,r,\ 1 + 3\,r^2\}$

■ Setting up initial conditions for the iterations

```
coeffrules = FullSimplify[MapThread[Rule, {
    {a, b, c, d, e, f},
    {δx^2 + δy^2,
     δx^2 + δy^2,
     -2 (δx^2 + δy^2),
     -2 (δx ΔX + δy ΔY)^2 - 4 (ΔX δy - δx ΔY) (δy (x - X0) - δx (y - Y0)),
     -2 (δx ΔX + δy ΔY)^2 + 4 (ΔX δy - δx ΔY) (δy (x - X0) - δx (y - Y0)),
    (ΔX^2 + ΔY^2) ((δx ΔX + δy ΔY)^2 + 4 (δy (x - X0) - δx (y - Y0))^2)}}] /. {δx → x2 - x1, δy → y2 - y1,
   x → (x1 + x2)/2, y → (y1 + y2)/2, ΔX → Xb - Xa, ΔY → Yb - Ya, X0 → (Xa + Xb)/2, Y0 → (Ya + Yb)/2}]
```

$\{a \to (x1 - x2)^2 + (y1 - y2)^2,\ b \to (x1 - x2)^2 + (y1 - y2)^2,\ c \to -2\,((x1 - x2)^2 + (y1 - y2)^2),$
$d \to -2\,((x1 - x2)\,(Xa - Xb) + (y1 - y2)\,(Ya - Yb))^2 + 2\,(-(Xa - Xb)\,(y1 - y2) + (x1 - x2)\,(Ya - Yb))$
$((Xa + Xb)\,(y1 - y2) + x1\,(2\,y2 - Ya - Yb) + x2\,(-2\,y1 + Ya + Yb)),$
$e \to -2\,((x1 - x2)\,(Xa - Xb) + (y1 - y2)\,(Ya - Yb))^2 + 2\,((Xa - Xb)\,(y1 - y2) - (x1 - x2)\,(Ya - Yb))$
$((Xa + Xb)\,(y1 - y2) + x1\,(2\,y2 - Ya - Yb) + x2\,(-2\,y1 + Ya + Yb)),$
$f \to ((Xa - Xb)^2 + (Ya - Yb)^2)\,(((x1 - x2)\,(Xa - Xb) + (y1 - y2)\,(Ya - Yb))^2 +$
$((Xa + Xb)\,(y1 - y2) + x1\,(2\,y2 - Ya - Yb) + x2\,(-2\,y1 + Ya + Yb))^2)\}$

```
InitIteration[refpoints_] := Function[{points},
  Block[{
    Xa = refpoints[[1, 1]],
    Ya = refpoints[[1, 2]],
    Xb = refpoints[[2, 1]],
    Yb = refpoints[[2, 2]],
    x1 = points[[1, 1]],
    y1 = points[[1, 2]],
    x2 = points[[2, 1]],
    y2 = points[[2, 2]],
    signum = If[# ≥ 0, 1, -1] &,
    a, b, c, d, e, f,
    finalPoint,
    discriminant,
    gradient,
    sign
   },
   finalPoint = Round[{Sqrt[(x2 - Xa)^2 + (y2 - Ya)^2], Sqrt[(x2 - Xb)^2 + (y2 - Yb)^2]}];
   {a, b, c, d, e, f} = ({a, b, c, d, e, f} /. coeffrules);
   discriminant = f + d r^2 + a r^4 + e s^2 + c r^2 s^2 + b s^4;
   gradient = ({d + 2 a r^2 + c s^2, e + c r^2 + 2 b s^2});
   sign = signum[(x1 - x2) (Xa - Xb) + (y1 - y2) (Ya - Yb)];
   {{r, s}, {discriminant, gradient}, {a, b, c, d, e, f, sign, finalPoint}} /.
    {r -> Round[Sqrt[(x1 - Xa)^2 + (y1 - Ya)^2]], s -> Round[Sqrt[(x1 - Xb)^2 + (y1 - Yb)^2]]}
  ]
 ]

InitIteration[{{-100, 0}, {100, 0}}][{{-70, -120}, {80, -20}}]

{{124, 208}, {-373 992 320 000, {-4 492 720 000, 892 720 000}},
 {32 500, 32 500, -65 000, -2 680 000 000, -920 000 000, 55 360 000 000 000, 1, {181, 28}}}
```

■ Identifying quadrants and octants

Since $r \geq 0$ and $s \geq 0$, it doesn't make a difference if the sign of the derivatives of the discriminant w.r.t. $\{r, s\}$ or $\{x, y\} \equiv \{r^2, s^2\}$ is computed.

■ Diagonal iteration

Underlying general rules:
$\delta x = -\sigma \operatorname{Sign}\left[\frac{\partial \Delta}{\partial y}\right]$, $\delta y = \sigma \operatorname{Sign}\left[\frac{\partial \Delta}{\partial x}\right]$, with $\sigma \equiv \operatorname{Sign}[(x_1 - x_2)(X_a - X_b) + (y_1 - y_2)(Y_a - Y_b)]$; the straight direction is the direction for which the change in Δ is smaller.

"Cheating" iteration: compute discriminant and gradient from scratch each time

```
MuralizerIterationCheat = Function[{position, iterationvariables, iterationconstants},
  Block[{r, s,
    dfunc = Function[{r, s}, f + d r^2 + a r^4 + e s^2 + c r^2 s^2 + b s^4],
    dgradfunc = Function[{r, s}, {d + 2 a r^2 + c s^2, e + c r^2 + 2 b s^2}],
    a, b, c, d, e, f, sign, finalPoint,
    signum = If[# ≥ 0, 1, -1] &,
    σr, σs,
    discriminant,
    discriminantGradient,
    orthogonalStep,
    diagonalStep
   },
   {r, s} = position;
   {a, b, c, d, e, f, sign, finalPoint} = iterationconstants;
   discriminant = dfunc[r, s];
   discriminantGradient = dgradfunc[r, s];
   {σr, σs} = sign {{0, -1}, {1, 0}}.(signum /@ discriminantGradient);
   orthogonalStep = If[Abs[r discriminantGradient[[1]]] < Abs[s discriminantGradient[[2]]],
     {σr, 0}, {0, σs}
    ];
   diagonalStep = {σr, σs};
   Append[
    If[Abs[dfunc @@ (position + orthogonalStep)] < Abs[dfunc @@ (position + diagonalStep)],
     {position + orthogonalStep,
      {dfunc @@ (position + orthogonalStep), dgradfunc @@ (position + orthogonalStep)}},
     {position + diagonalStep, {dfunc @@ (position + diagonalStep),
       dgradfunc @@ (position + diagonalStep)}}
    ],
    iterationconstants]
  ]
 ]

Function[{position, iterationvariables, iterationconstants},
 Block[{r, s, dfunc = Function[{r, s}, f + d r^2 + a r^4 + e s^2 + c r^2 s^2 + b s^4],
   dgradfunc = Function[{r, s}, {d + 2 a r^2 + c s^2, e + c r^2 + 2 b s^2}], a, b,
   c, d, e, f, sign, finalPoint, signum = If[#1 ≥ 0, 1, -1] &, σr, σs,
   discriminant, discriminantGradient, orthogonalStep, diagonalStep},
  {r, s} = position; {a, b, c, d, e, f, sign, finalPoint} = iterationconstants;
  discriminant = dfunc[r, s]; discriminantGradient = dgradfunc[r, s];
  {σr, σs} = sign {{0, -1}, {1, 0}}.signum /@ discriminantGradient; orthogonalStep =
   If[Abs[r discriminantGradient[[1]]] < Abs[s discriminantGradient[[2]]], {σr, 0}, {0, σs}];
  diagonalStep = {σr, σs}; Append[If[Abs[dfunc @@ (position + orthogonalStep)] <
      Abs[dfunc @@ (position + diagonalStep)], {position + orthogonalStep,
      {dfunc @@ (position + orthogonalStep), dgradfunc @@ (position + orthogonalStep)}},
     {position + diagonalStep, {dfunc @@ (position + diagonalStep),
       dgradfunc @@ (position + diagonalStep)}}], iterationconstants]]]
```

Real iteration: Discriminant and gradients are iterated

```
MuralizerIteration = Function[{position, iterationvariables, iterationconstants},
  Block[{r, s,
    discriminant,
    discriminantGradient,
    a, b, c, d, e, f, sign, finalPoint,
    signum = If[# ≥ 0, 1, -1] &,
    σr, σs,
    discriminantSteps,
    gradientSteps,
    gradients,
    orthogonalStep,
    diagonalStep,
    discriminantValues
   },
   {r, s} = position;
   {discriminant, discriminantGradient} = iterationvariables;
   {a, b, c, d, e, f, sign, finalPoint} = iterationconstants;
   {σr, σs} = sign {{0, -1}, {1, 0}}.(signum /@ discriminantGradient);
   discriminantSteps = {
     a (6 r^2 + 1 + σr 4 r (r^2 + 1)) + (c s^2 + d) (1 + σr 2 r),
     b (6 s^2 + 1 + σs 4 s (s^2 + 1)) + (c r^2 + e) (1 + σs 2 s),
     c (2 r + σr) (2 s + σs)
    };
   gradientSteps = {
     (1 + σr 2 r) {2 a, c},
     (1 + σs 2 s) {c, 2 b}
    };
   orthogonalStep = (* {position step, discriminant step, discriminant gradient step} *)
    If[Subtract @@ Abs[discriminantGradient {r, s}] < 0,
     {{σr, 0}, {discriminantSteps⟦1⟧, gradientSteps⟦1⟧}},
     {{0, σs}, {discriminantSteps⟦2⟧, gradientSteps⟦2⟧}}];
   diagonalStep = {
     {σr, σs},
     {Plus @@ discriminantSteps, Plus @@ gradientSteps}};
   discriminantValues = discriminant + {orthogonalStep⟦2, 1⟧, diagonalStep⟦2, 1⟧};
   Append[{position, iterationvariables} +
     If[Subtract @@ Abs[discriminantValues] < 0,
      orthogonalStep,
      diagonalStep
     ],
    iterationconstants]
  ]
 ]
```

```
Function[{position, iterationvariables, iterationconstants},
 Block[{r, s, discriminant, discriminantGradient, a, b, c, d, e, f,
   sign, finalPoint, signum = If[#1 ≥ 0, 1, -1] &, σr, σs, discriminantSteps,
   gradientSteps, gradients, orthogonalStep, diagonalStep, discriminantValues},
  {r, s} = position; {discriminant, discriminantGradient} = iterationvariables;
  {a, b, c, d, e, f, sign, finalPoint} = iterationconstants;
  {σr, σs} = sign {{0, -1}, {1, 0}}.signum /@ discriminantGradient;
  discriminantSteps = {a (6 r^2 + 1 + σr 4 r (r^2 + 1)) + (c s^2 + d) (1 + σr 2 r),
    b (6 s^2 + 1 + σs 4 s (s^2 + 1)) + (c r^2 + e) (1 + σs 2 s), c (2 r + σr) (2 s + σs)};
  gradientSteps = {(1 + σr 2 r) {2 a, c}, (1 + σs 2 s) {c, 2 b}};
  orthogonalStep = If[Subtract @@ Abs[discriminantGradient {r, s}] < 0,
    {{σr, 0}, {discriminantSteps[[1]], gradientSteps[[1]]}},
    {{0, σs}, {discriminantSteps[[2]], gradientSteps[[2]]}}];
  diagonalStep = {{σr, σs}, {Plus @@ discriminantSteps, Plus @@ gradientSteps}};
  discriminantValues = discriminant + {orthogonalStep[[2, 1]], diagonalStep[[2, 1]]};
  Append[{position, iterationvariables} + If[Subtract @@ Abs[discriminantValues] < 0,
     orthogonalStep, diagonalStep], iterationconstants]]]
```

Termination criterion

```
continueDrawing = Function[{position, iterationvariables, iterationconstants},
  Block[{
    gradient = iterationvariables[[2]],
    finalpoint = Last[iterationconstants],
    distance
   },
   distance = (position - finalpoint).{{0, -1}, {1, 0}}.gradient;
   2 distance^2 > gradient.gradient
  ]
 ]
```

```
Function[{position, iterationvariables, iterationconstants},
 Block[{gradient = iterationvariables[[2]], finalpoint = Last[iterationconstants], distance},
  distance = (position - finalpoint).{{0, -1}, {1, 0}}.gradient;
  2 distance^2 > gradient.gradient]]
```

- **testrun**

- **Inverse transformation (to check results)**

```
inversexform[refpoints_] := Function[{radii},
  Block[{ra = radii[[1]], rb = radii[[2]], xa = refpoints[[1, 1]],
    ya = refpoints[[1, 2]], xb = refpoints[[2, 1]], yb = refpoints[[2, 2]]},
   {(xa + xb)/2 + ((ra - rb) (ra + rb) (-xa + xb))/(2 ((-xa + xb)^2 + (-ya + yb)^2)) +
     ((-ya + yb) Sqrt[-(-(ra - rb)^2 + (-xa + xb)^2 + (-ya + yb)^2) (-(ra + rb)^2 + (-xa + xb)^2 + (-ya + yb)^2)])/(2 ((-xa + xb)^2 + (-ya + yb)^2)),
    (ya + yb)/2 + ((ra - rb) (ra + rb) (-ya + yb))/(2 ((-xa + xb)^2 + (-ya + yb)^2)) -
     ((-xa + xb) Sqrt[-(-(ra - rb)^2 + (-xa + xb)^2 + (-ya + yb)^2) (-(ra + rb)^2 + (-xa + xb)^2 + (-ya + yb)^2)])/(2 ((-xa + xb)^2 + (-ya + yb)^2))}
  ]
 ]
```

- **"cheat" transform**

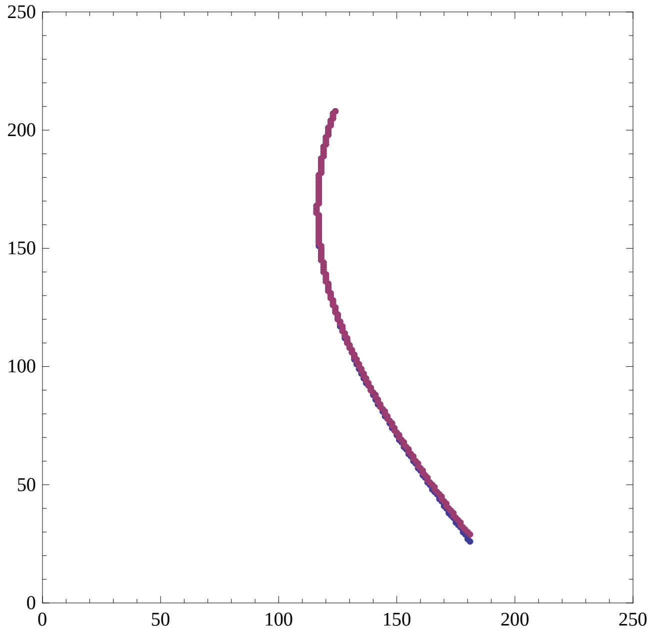
```
auxc = NestWhileList[MuralizerIterationCheat @@ # &, InitIteration[{{-100, 0}, {100, 0}}][
    {{-70, -120}, {80, -20}}], continueDrawing @@ # &, 1, 200];
```

- **real transform**

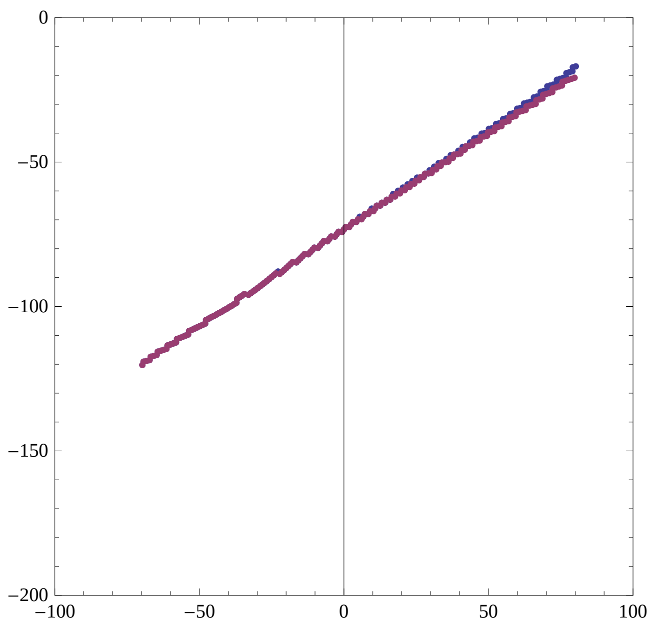
```
auxl = NestWhileList[MuralizerIteration @@ # &, InitIteration[{{-100, 0}, {100, 0}}][
    {{-70, -120}, {80, -20}}], continueDrawing @@ # &, 1, 200];
```

Evaluation

```
ListPlot[{First /@ auxl, First /@ auxc},
 Frame → True, PlotRange → {#, #} &[{0, 250}], AspectRatio → 1]
```



```
ListPlot[Map[inversexform[{{-100, 0}, {100, 0}}][First[#]] &, {auxl, auxc}, {2}],
 Frame → True, PlotRange → {{-100, 100}, {-200, 0}}, AspectRatio → 1]
```



```
Last[First[auxl]]
Last[First[auxc]]
```

{32 500, 32 500, -65 000, -2 680 000 000, -920 000 000, 55 360 000 000 000, 1, {181, 28}}

{32 500, 32 500, -65 000, -2 680 000 000, -920 000 000, 55 360 000 000 000, 1, {181, 28}}

```
Dimensions /@ {auxl, auxc}
```

```
{{183, 3}, {180, 3}}
```

■ several runs

```
coords = Table[{{Random[Real, {-100, 100}], Random[Real, {-200, 0}]},
    {Random[Real, {-100, 100}], Random[Real, {-200, 0}]}}, {n, 42}]
```

```
{{{-4.97228, -140.049}, {-79.0617, -24.8791}}, {{-55.363, -146.738}, {54.377, -64.9355}},
 {{-21.4175, -83.2669}, {-27.897, -197.281}}, {{87.927, -10.6215}, {13.7911, -156.958}},
 {{59.046, -4.45173}, {35.1657, -195.984}}, {{-71.9742, -6.67087}, {87.2992, -186.808}},
 {{32.9981, -66.6218}, {66.3609, -161.929}}, {{-11.6389, -119.883}, {-88.0161, -96.9935}},
 {{-90.2214, -36.6165}, {39.8809, -99.7124}}, {{-78.1484, -25.995}, {-73.9102, -142.754}},
 {{-37.1944, -21.5433}, {-9.07593, -146.77}}, {{-65.2202, -14.8724}, {3.62488, -159.962}},
 {{1.78176, -148.251}, {37.264, -198.033}}, {{-86.5793, -28.3672}, {25.2802, -101.04}},
 {{-96.3579, -191.751}, {85.3993, -1.32763}}, {{81.7904, -165.756}, {59.3095, -58.5735}},
 {{18.9848, -144.212}, {-31.6146, -111.803}}, {{-15.795, -129.34}, {64.7605, -151.841}},
 {{82.4232, -181.089}, {-72.5035, -153.807}}, {{69.0026, -152.722}, {2.21635, -52.7671}},
 {{65.3605, -160.971}, {16.8171, -51.4394}}, {{83.5701, -195.216}, {57.5076, -192.866}},
 {{-35.4147, -51.0035}, {-10.8778, -81.0629}}, {{80.3803, -121.664}, {24.3616, -129.222}},
 {{97.9571, -140.574}, {-3.1349, -175.415}}, {{-71.0455, -187.852}, {94.6487, -122.648}},
 {{-36.406, -26.8808}, {-22.1683, -71.2088}}, {{-19.976, -31.665}, {20.3241, -78.3429}},
 {{-84.5613, -180.662}, {-68.7981, -197.28}}, {{-64.9416, -58.9979}, {6.84032, -68.0576}},
 {{-62.8987, -118.424}, {-90.0248, -92.6423}}, {{-91.8532, -130.571}, {-84.6735, -169.994}},
 {{44.5528, -103.69}, {37.4948, -98.7853}}, {{-35.4712, -72.0254}, {-82.8293, -20.4425}},
 {{-50.9099, -91.3639}, {85.9688, -23.1625}}, {{-85.9683, -32.3661}, {-20.8715, -155.105}},
 {{76.9304, -113.943}, {-30.8467, -62.4626}}, {{68.7836, -183.371}, {-46.1732, -92.4685}},
 {{-75.7692, -79.6808}, {16.332, -193.683}}, {{59.702, -7.65538}, {-0.838781, -173.241}},
 {{10.6119, -116.291}, {13.1924, -150.078}}, {{-3.4198, -83.9254}, {-65.9361, -194.973}}}
```

```
cheats = NestWhileList[MuralizerIterationCheat @@ # &,
     InitIteration[{{-100, 0}, {100, 0}}][#], continueDrawing @@ # &, 1, 300] & /@ coords;
Dimensions /@
 cheats
```

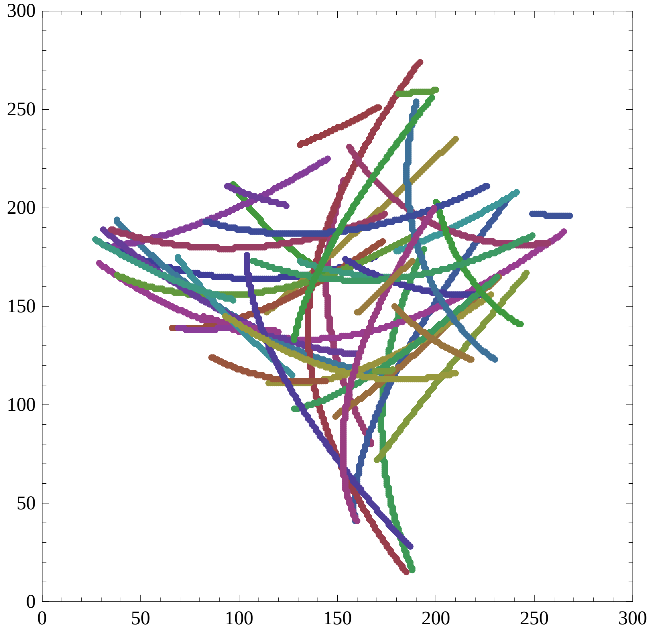
```
{{137, 3}, {135, 3}, {97, 3}, {164, 3}, {166, 3}, {237, 3}, {84, 3}, {58, 3},
 {136, 3}, {112, 3}, {108, 3}, {154, 3}, {62, 3}, {131, 3}, {260, 3}, {96, 3},
 {57, 3}, {71, 3}, {106, 3}, {114, 3}, {104, 3}, {20, 3}, {39, 3}, {40, 3}, {67, 3},
 {134, 3}, {37, 3}, {59, 3}, {20, 3}, {64, 3}, {31, 3}, {41, 3}, {10, 3}, {71, 3},
 {149, 3}, {140, 3}, {118, 3}, {143, 3}, {144, 3}, {160, 3}, {29, 3}, {124, 3}}
```

```
curves = NestWhileList[MuralizerIteration @@ # &,
     InitIteration[{{-100, 0}, {100, 0}}][#], continueDrawing @@ # &, 1, 300] & /@ coords;
Dimensions /@
 curves
```

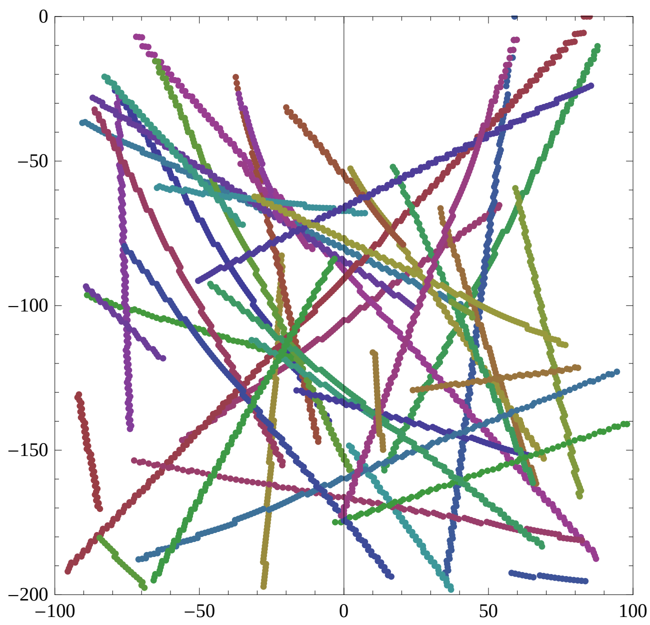
```
{{138, 3}, {135, 3}, {97, 3}, {164, 3}, {166, 3}, {237, 3}, {84, 3}, {58, 3},
 {136, 3}, {112, 3}, {108, 3}, {154, 3}, {62, 3}, {131, 3}, {263, 3}, {96, 3},
 {57, 3}, {71, 3}, {107, 3}, {114, 3}, {104, 3}, {20, 3}, {39, 3}, {40, 3}, {67, 3},
 {135, 3}, {37, 3}, {59, 3}, {20, 3}, {64, 3}, {31, 3}, {41, 3}, {10, 3}, {72, 3},
 {152, 3}, {140, 3}, {118, 3}, {143, 3}, {144, 3}, {160, 3}, {29, 3}, {124, 3}}
```

- **"cheat" function**

```
ListPlot[(First /@ #) & /@ cheats, Frame → True, PlotRange → {#, #} &[{0, 300}], AspectRatio → 1]
```

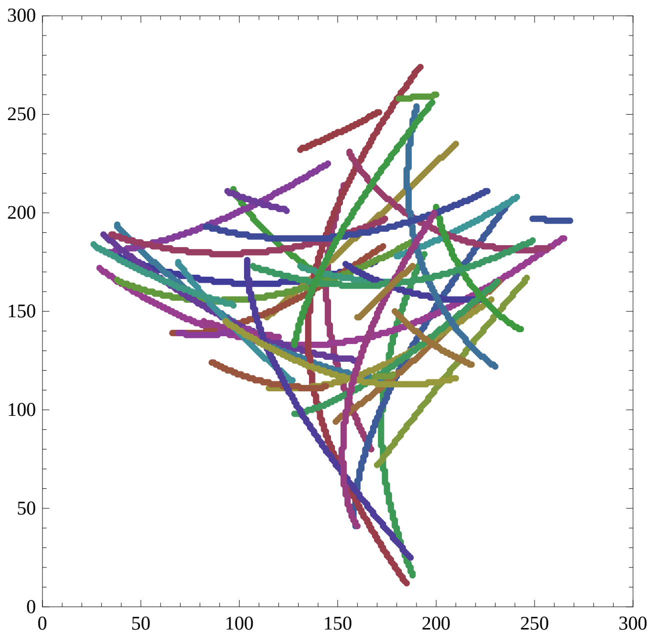



```
ListPlot[Map[inversexform[{{-100, 0}, {100, 0}}][First[#]] &, cheats, {2}],
 Frame → True, PlotRange → {{-100, 100}, {-200, 0}}, AspectRatio → 1]
```



- **function with iteration**

```
ListPlot[(First /@ #) & /@ curves, Frame → True, PlotRange → {#, #} &[{0, 300}], AspectRatio → 1]
```




```
ListPlot[Map[inversexform[{{-100, 0}, {100, 0}}][First[#]] &, curves, {2}],
 Frame → True, PlotRange → {{-100, 100}, {-200, 0}}, AspectRatio → 1]
```