

# Handshaking Lemma

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and  
University of California, San Diego

## Outline

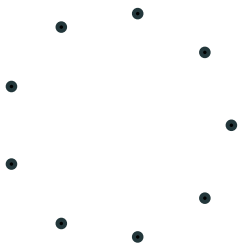
Handshaking Lemma

Total Degree

## Revisiting a Puzzle

### Puzzle

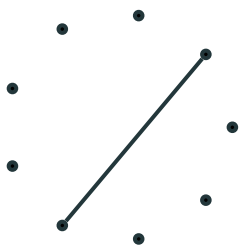
Is it possible to connect some pairs of nine points by segments so that each point is connected to five other points?



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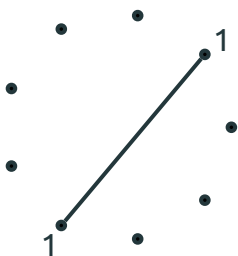
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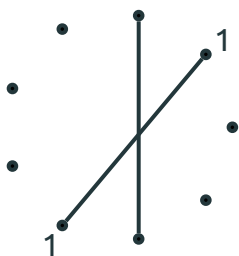
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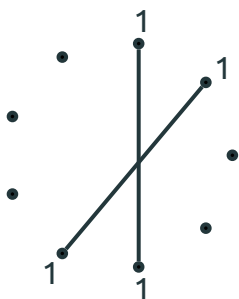
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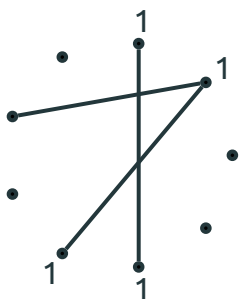
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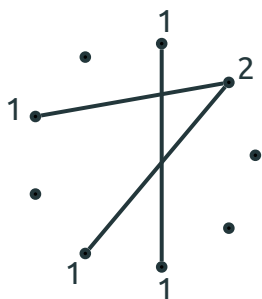




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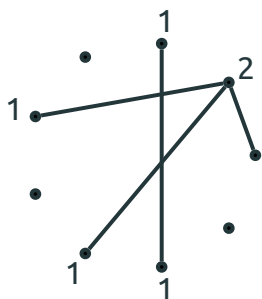
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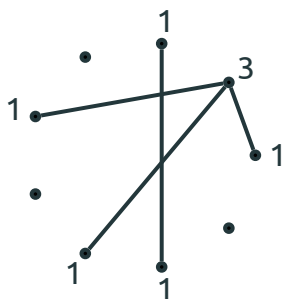
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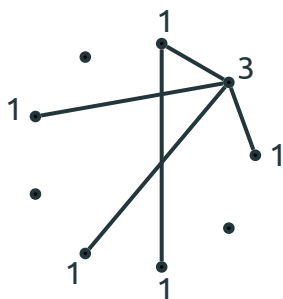
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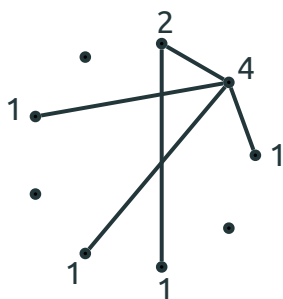
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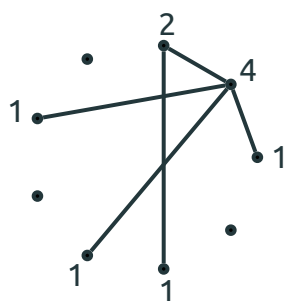
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## Revisiting a Puzzle

### Puzzle

Is it possible to connect some pairs of nine points by segments so that each point is connected to five other points?



the number of odd points is always even, hence we will never reach a situation when there are 9 points of degree 5

## Handshaking



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- Before a business meeting, some people shook hands. Then the number of people who made an odd number of handshakes is even
- In graph terms: A graph has an even number of odd nodes



## Degree Sum Formula

### Lemma

For any graph  $G(V, E)$ , the sum of degrees of all its nodes is twice the number of edges:

$$\sum_{v \in V} \text{degree}(v) = 2 \cdot |E|.$$

## Degree Sum Formula

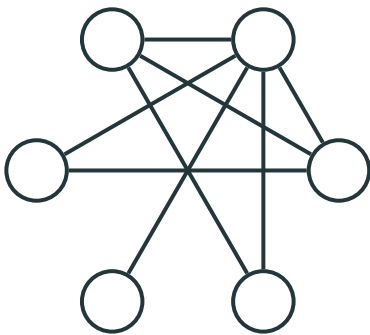
### Lemma

For any graph  $G(V, E)$ , the sum of degrees of all its nodes is twice the number of edges:

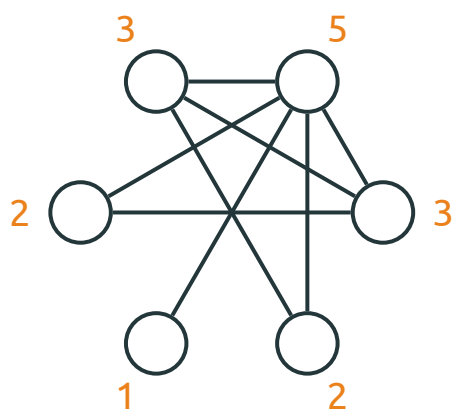
$$\sum_{v \in V} \text{degree}(v) = 2 \cdot |E|.$$

Implies the handshaking lemma: if a graph had an odd number of odd nodes, then the sum of degrees would be also odd.

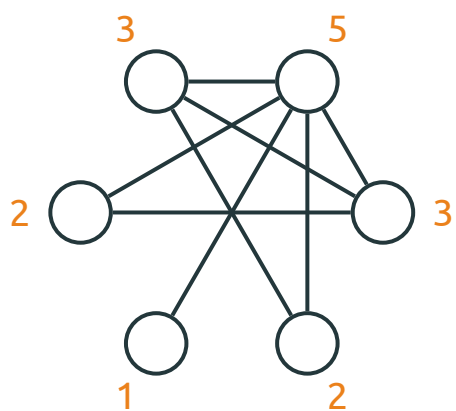
## Example



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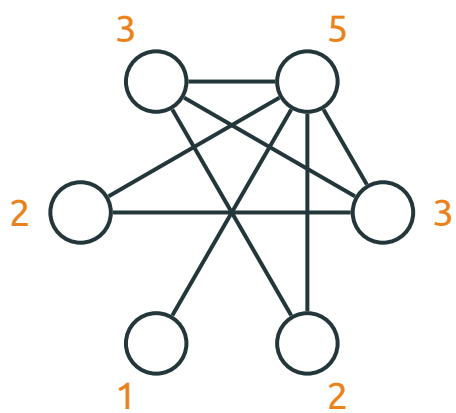


## Example

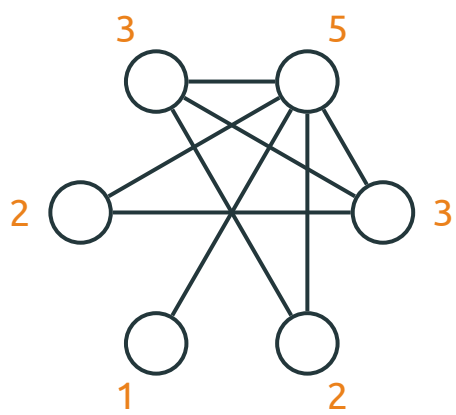


$$3 + 5 + 3 + 2 + 1 + 2 = 2 \cdot 8$$

## Proof

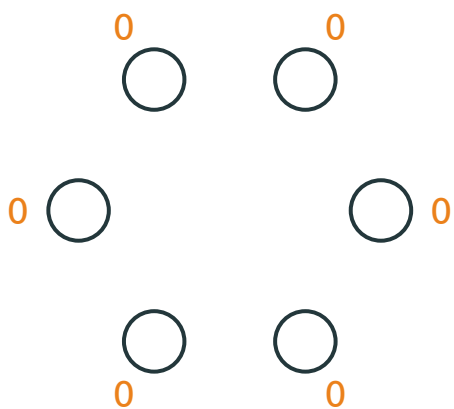


## Proof



remove all edges

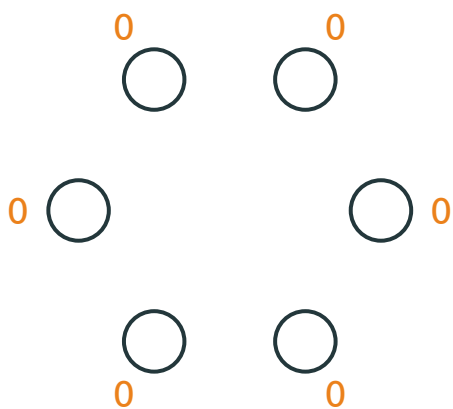
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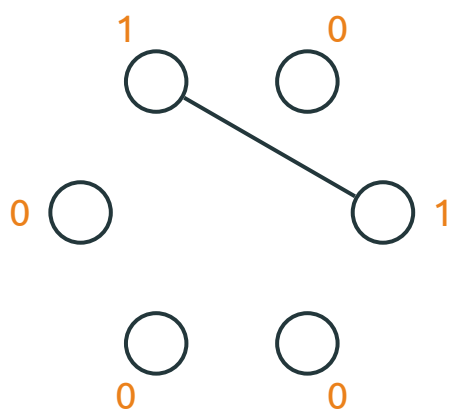


## Proof



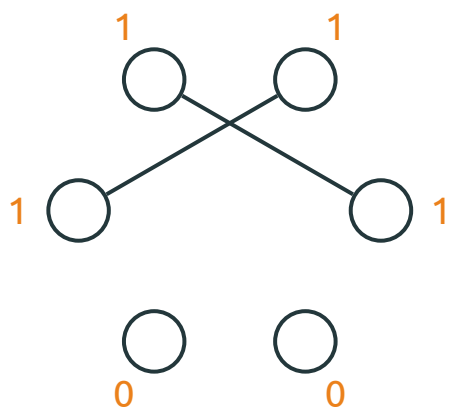
now, start adding them back, one by one

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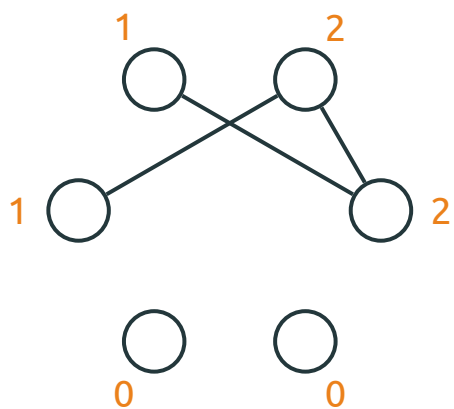
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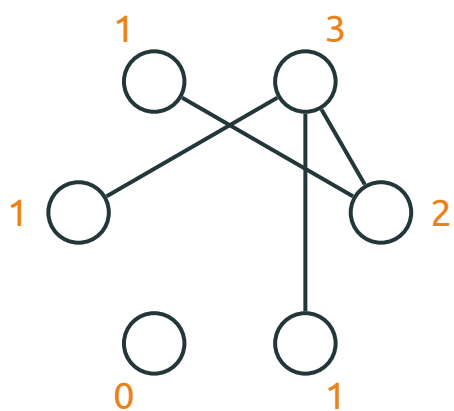
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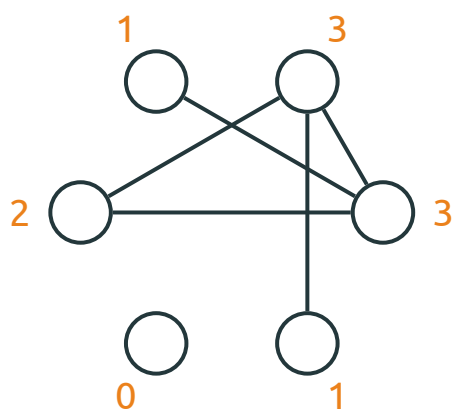
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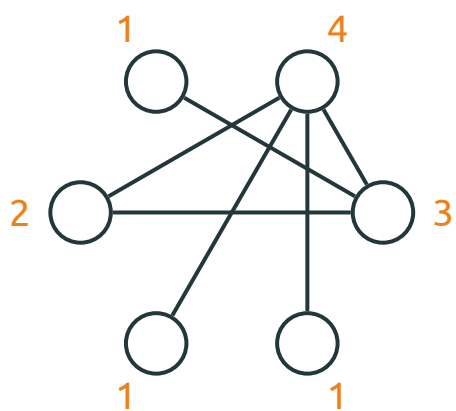
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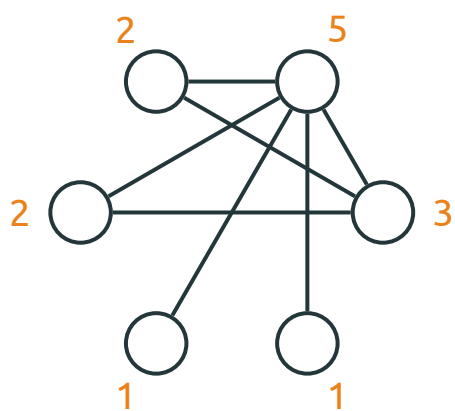
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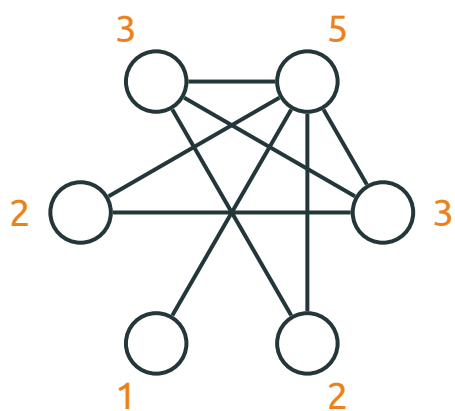
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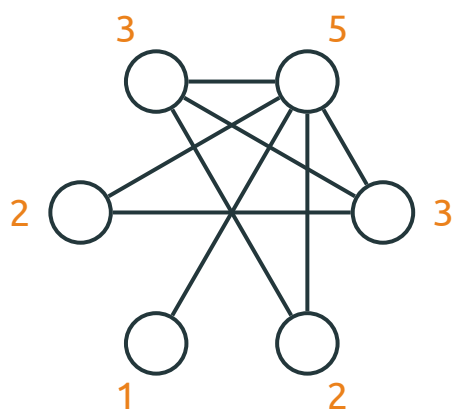


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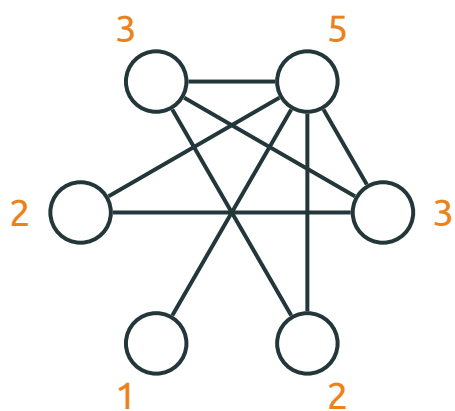
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## Proof



each edge contributes 2 to the sum of degrees

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as well as to twice the number of edges!

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- Base case: the formula holds if there are no edges (all degrees are equal to 0)
- Induction step: when we add an edge, the sum of degrees increases by 2

## Outline

Handshaking Lemma

Total Degree

## Exam

### Problem

At an exam, each of 20 students solved 3 problems. Each problem was solved by 5 students. What was the number of problems?





## **Solution**

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- Then, there are  $20 \times 3 = 60$  pieces of paper
- For each problem, let's stack together all its five solutions
- Thus, the number of problems is  $60/5 = 12$

**In Graph Terms**

20 students



In Graph Terms

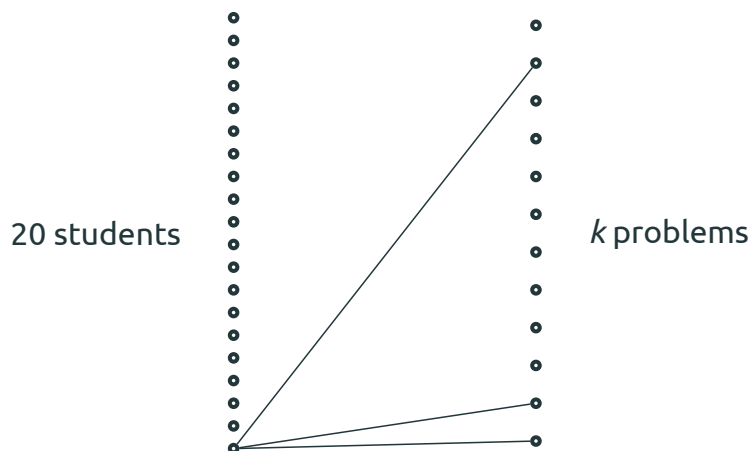
20 students



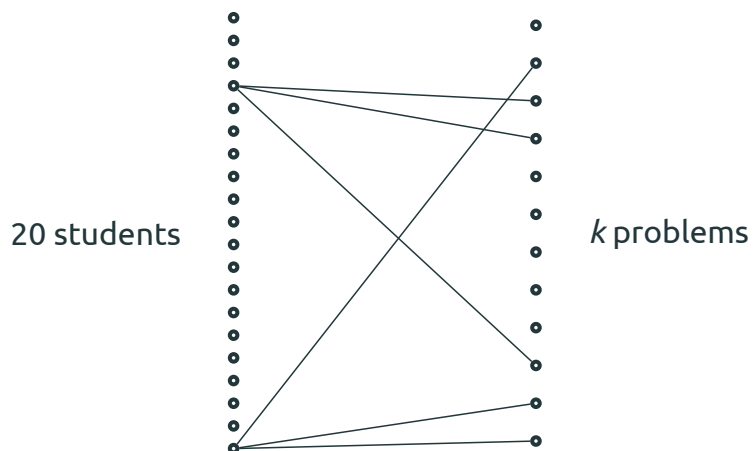
$k$  problems



## In Graph Terms

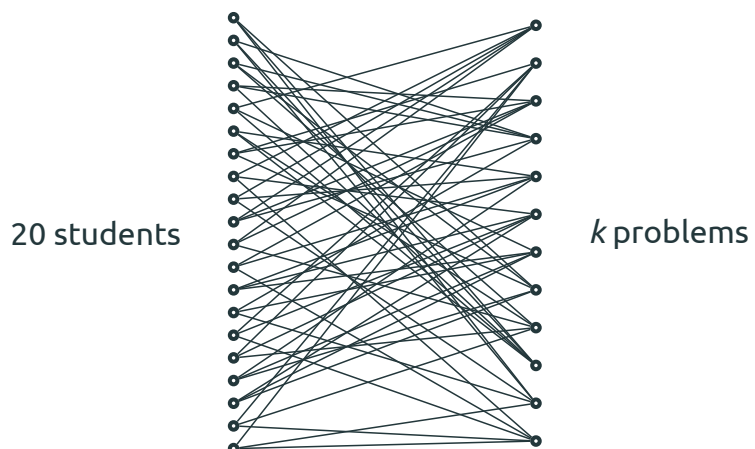


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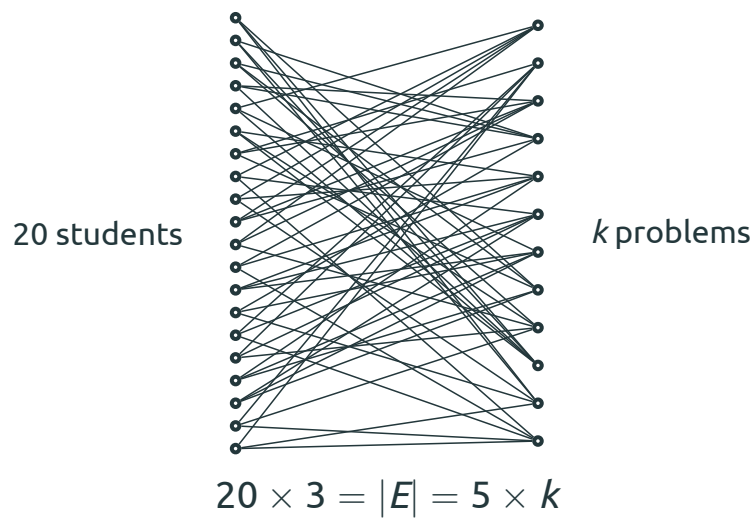




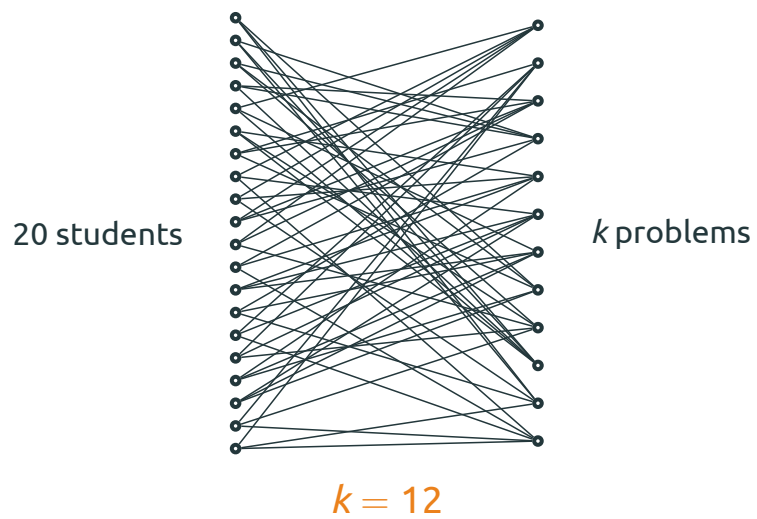
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## Summary

- This is a **bipartite** graph: each edge joins a student and a problem
- We used the **double counting** technique to solve the problem:
  - on one hand, the number of edges is equal to the **total degree** of the left part (i.e., the sum of all degree of the nodes from the left)
  - on the other hand, it is equal to the total degree of the right part