Trees

Alexander Golovnev

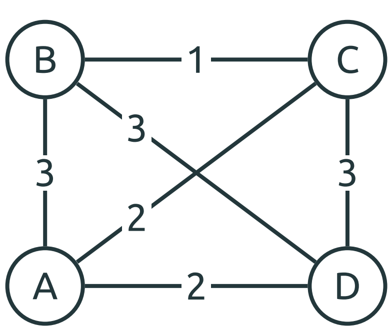
Outline

Road Repair

Trees

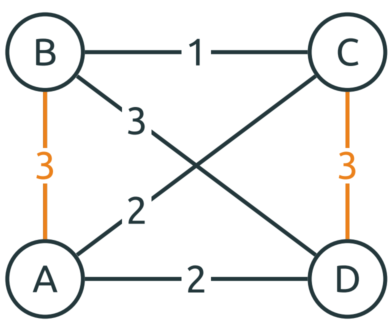
Minimum Spanning Tree

Road Repair



Road Repair

No pair of edges can connect all cities



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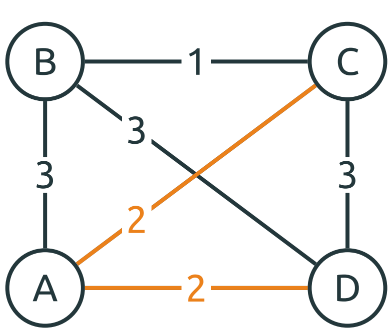
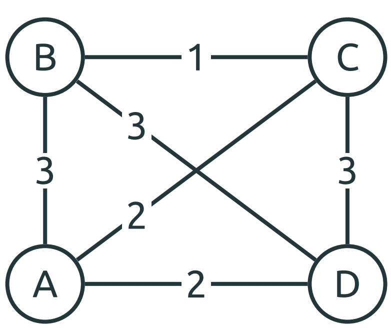


Road Repair

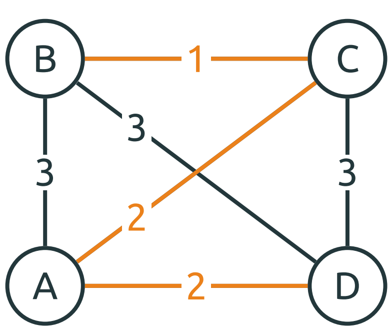
We need at least three edges



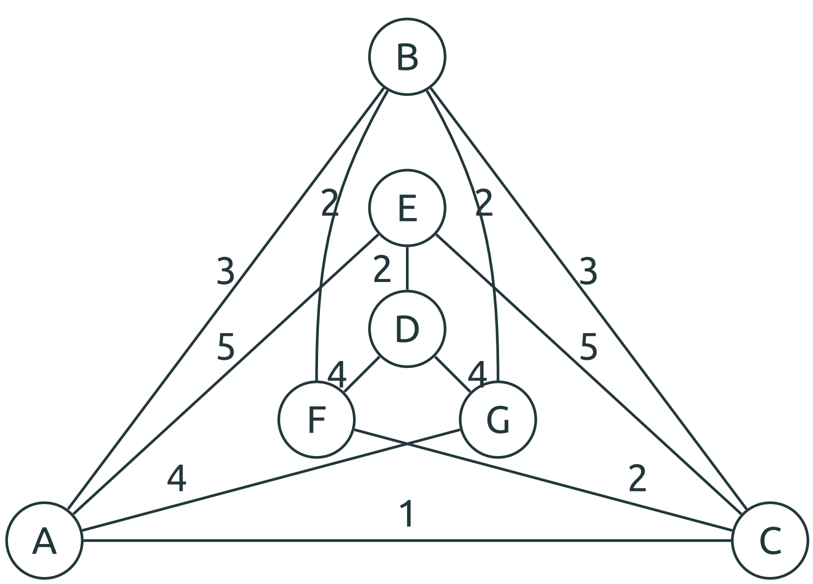
Road Repair

We need at least three edges

Three shortest edges work!



Road Repair



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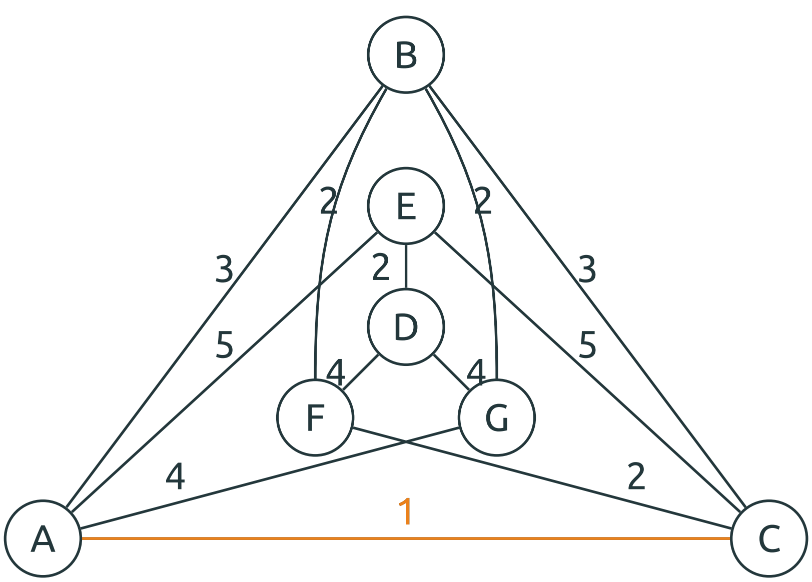

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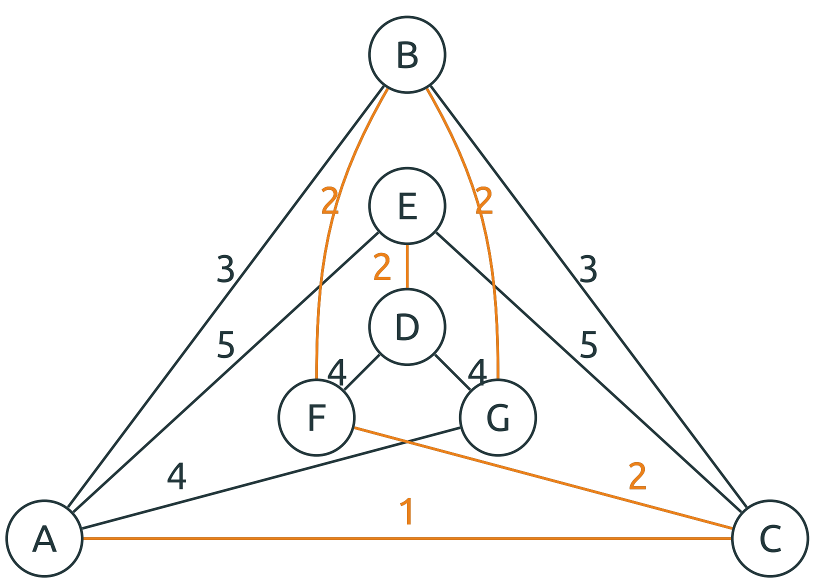

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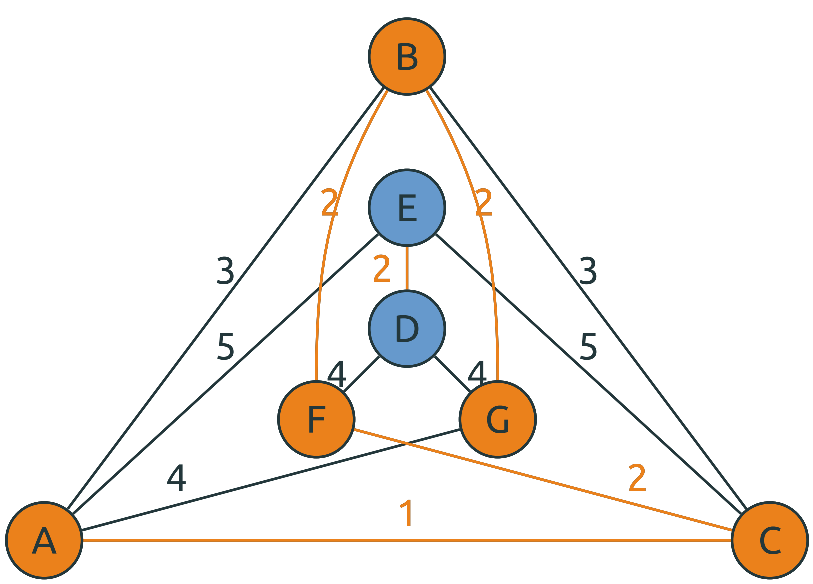

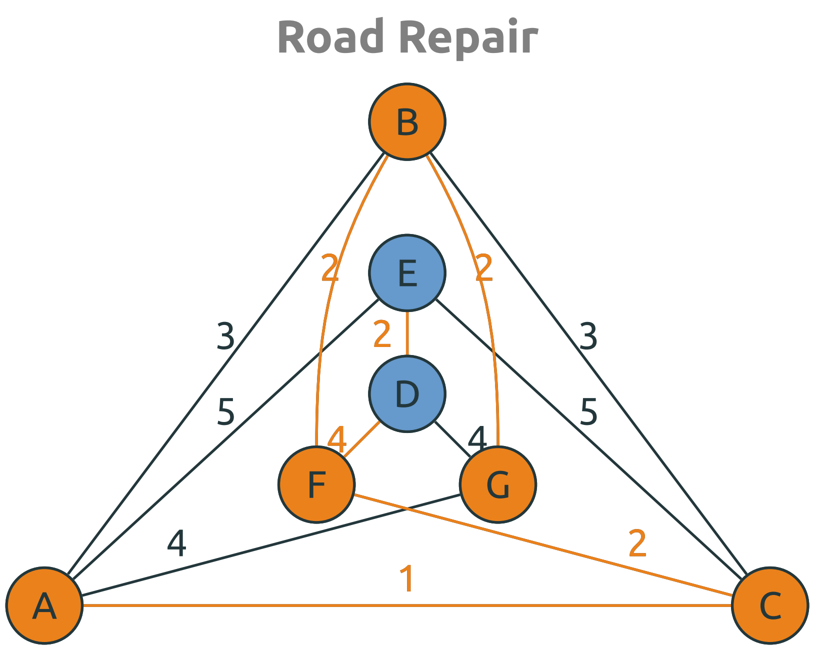
Road Repair
B
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D
F
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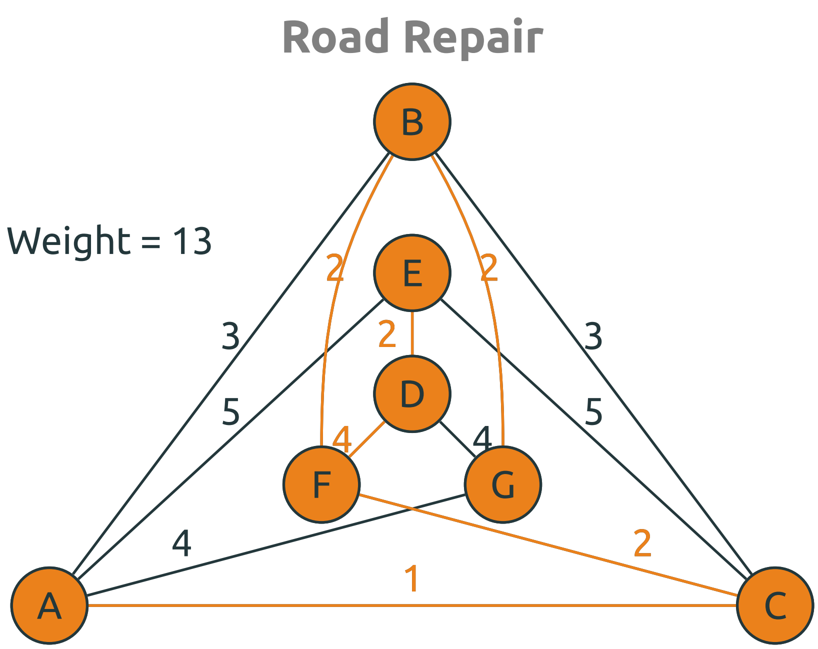
A
C
3
5
4
2
2
2
4
4
3
5
2
1


Road Repair
Weight = 13
B
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D
F
G
A
C
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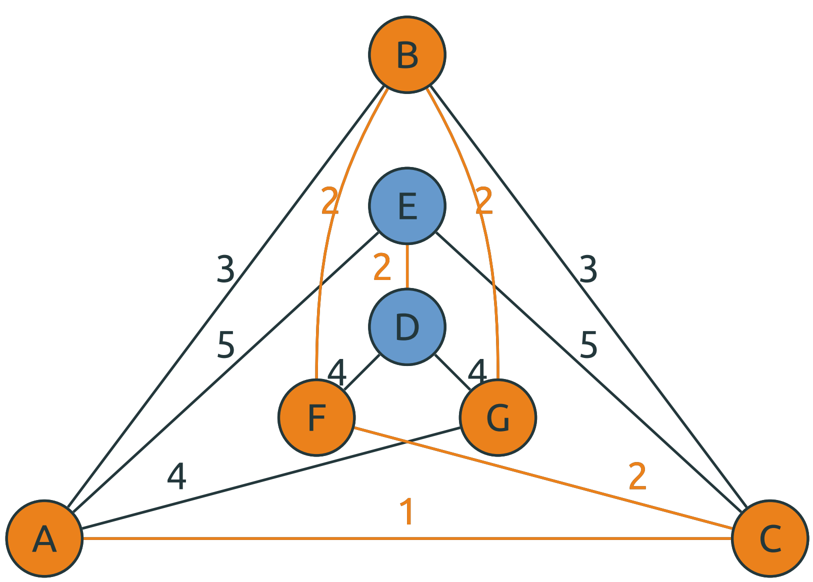
2
2
3
3
5
5
4
4
4
2
1

Road Repair



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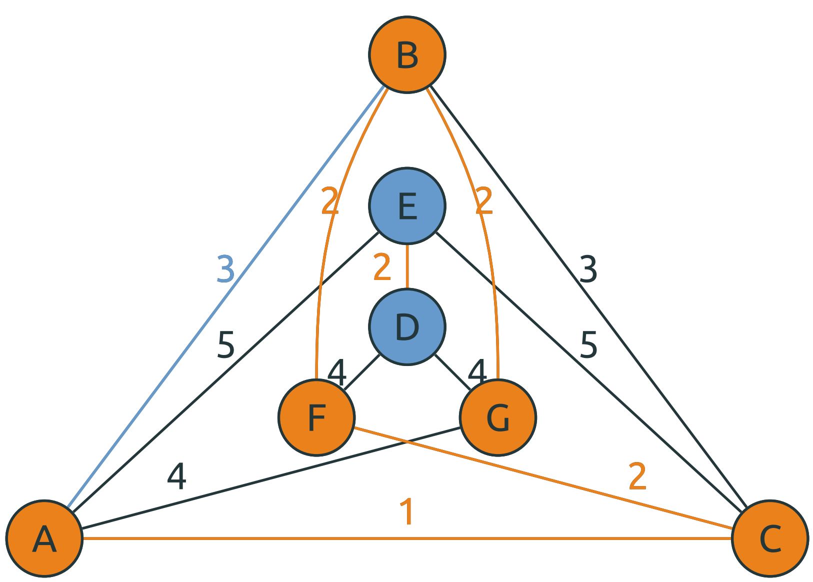

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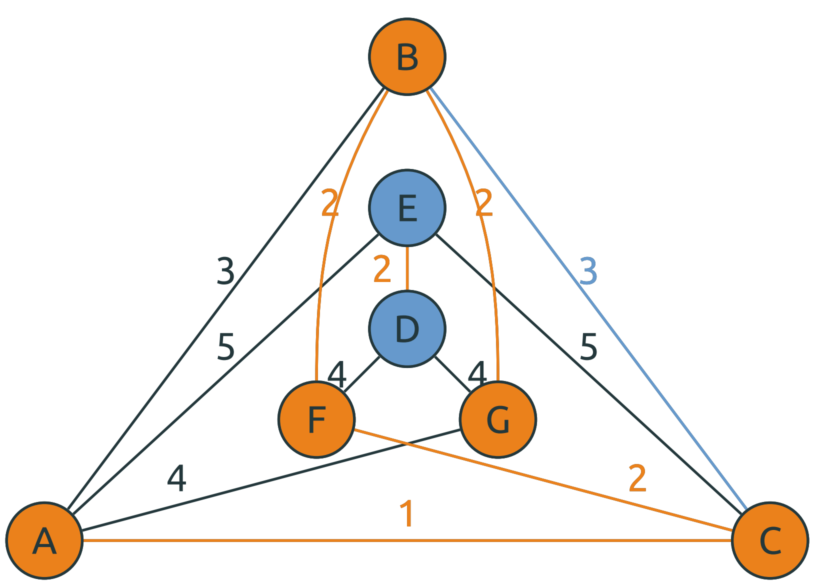

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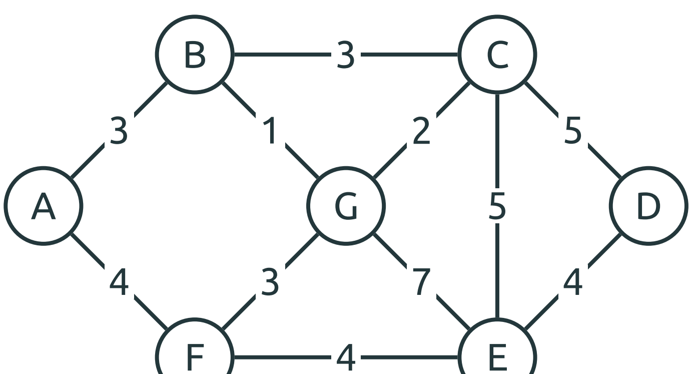

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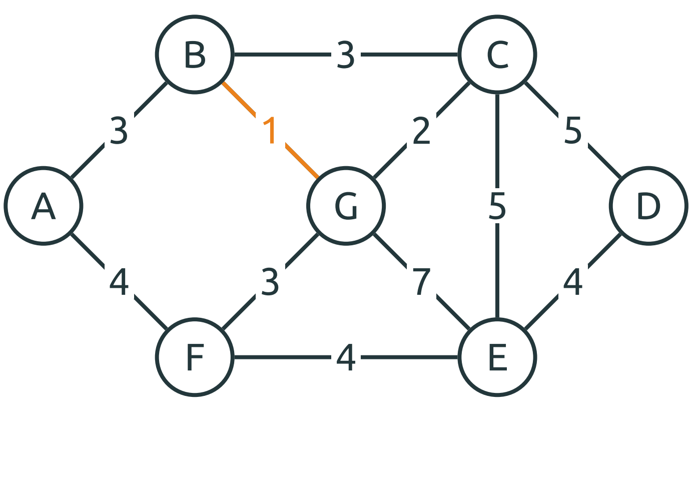

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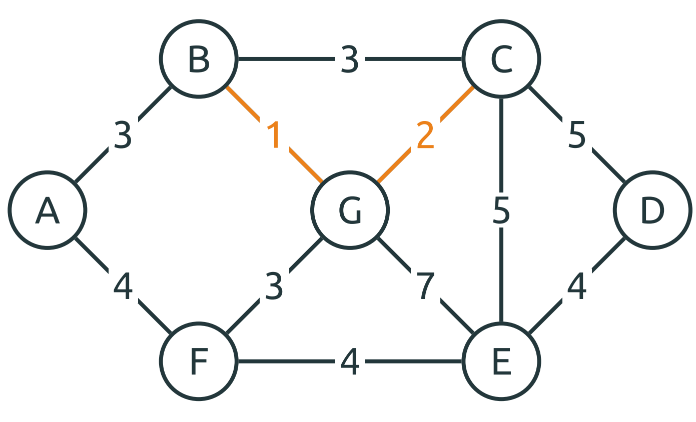

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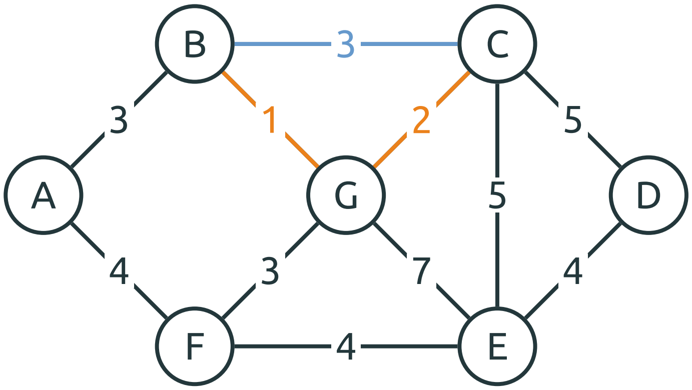

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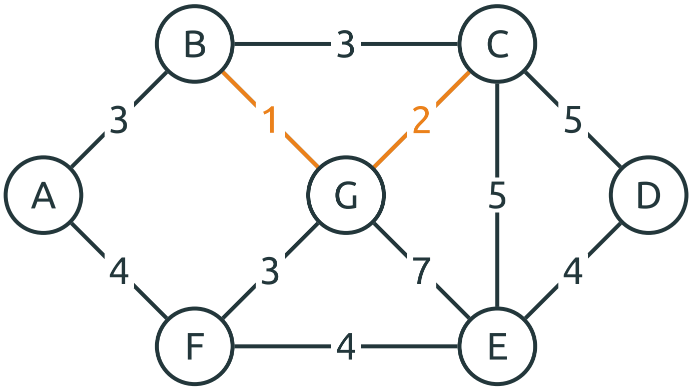

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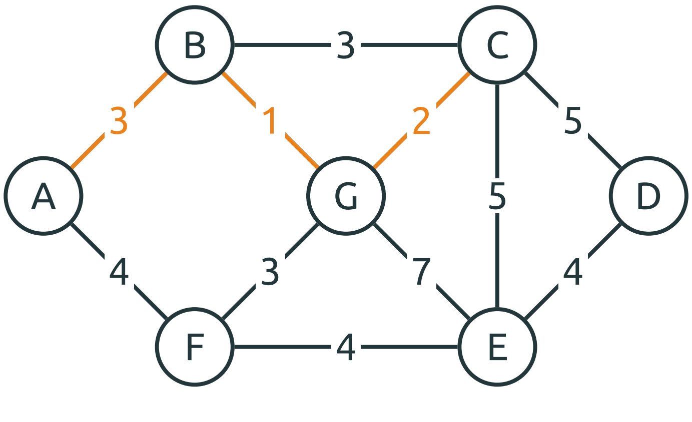

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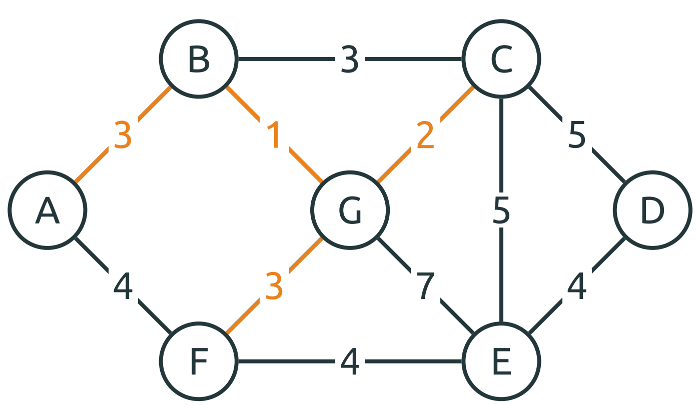

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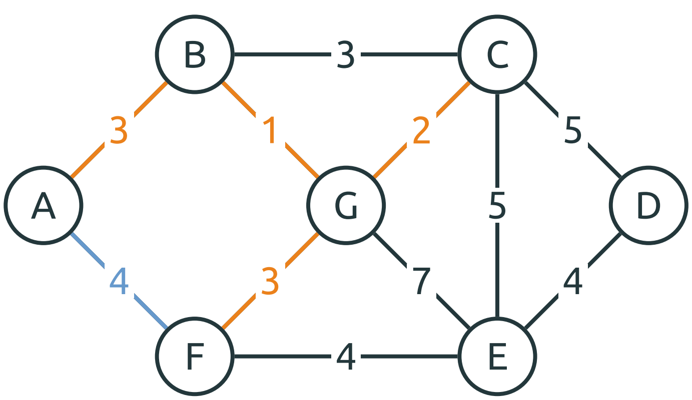

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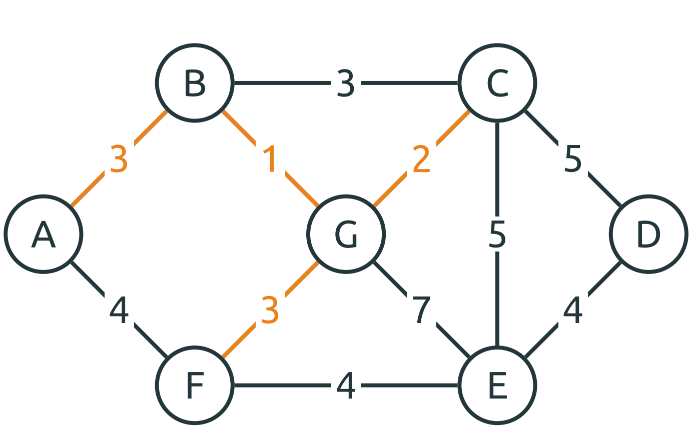

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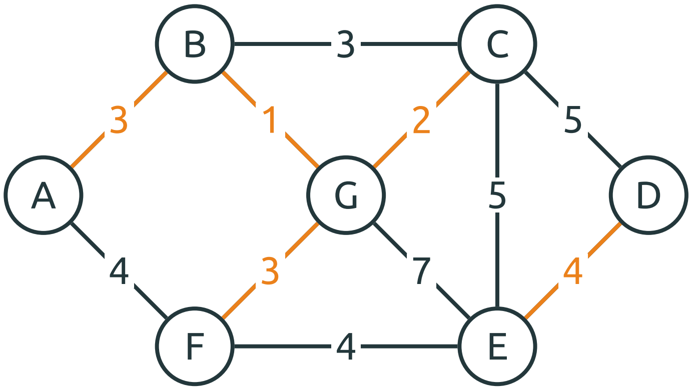

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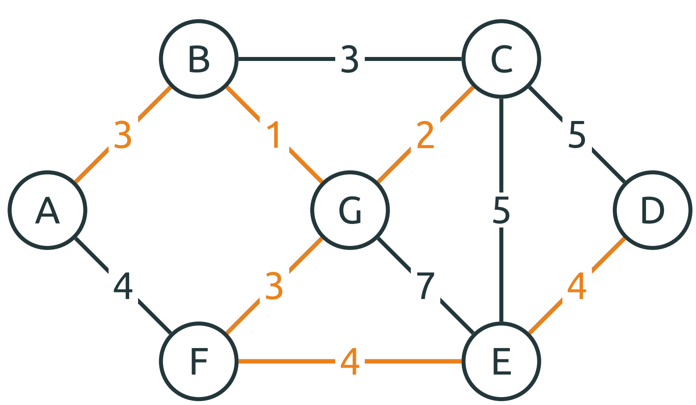

Outline

Road Repair

Trees

Minimum Spanning Tree

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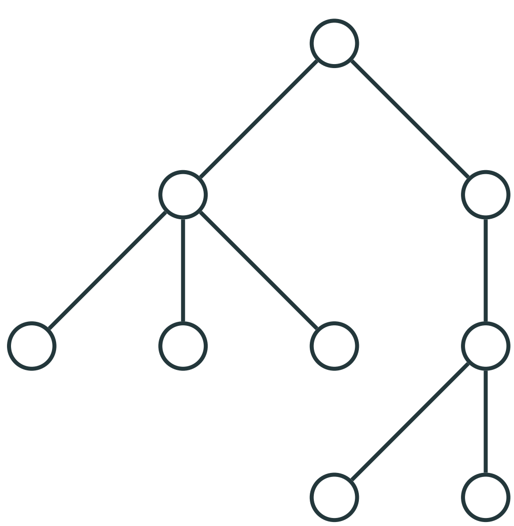
Trees: Examples



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(I) $\rightarrow$ (II)

- A connected graph on n vertices without cycles has $n - 1$ edges

(I) $\rightarrow$ (II)

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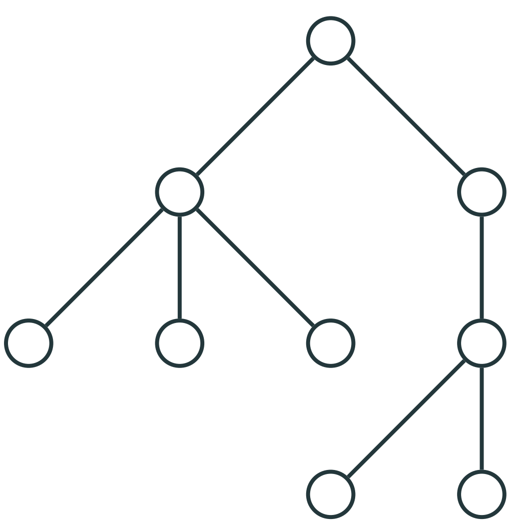
(I) $\rightarrow$ **(II)**



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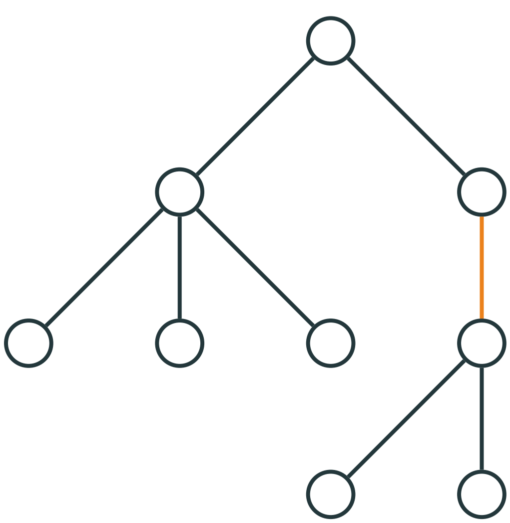

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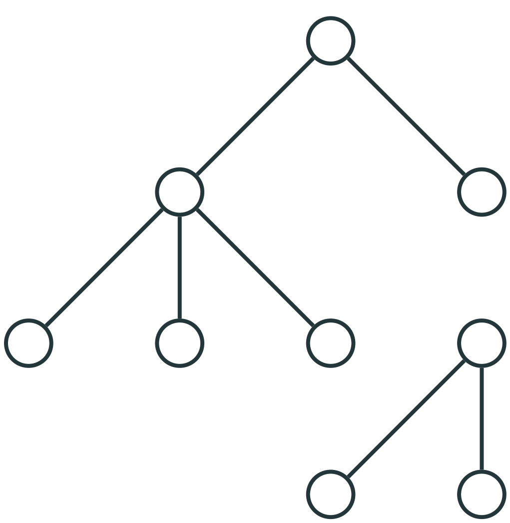

(I) $\rightarrow$ **(II)**

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- Thus, the original graph has $(n_1 - 1) + (n_2 - 1) + 1 = n - 1$ edges

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- Then the number of edges is n

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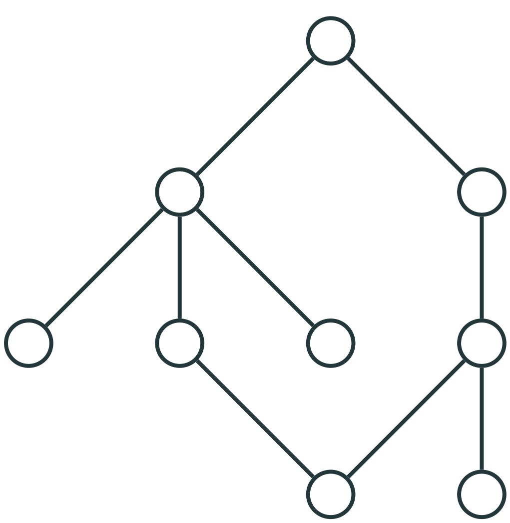

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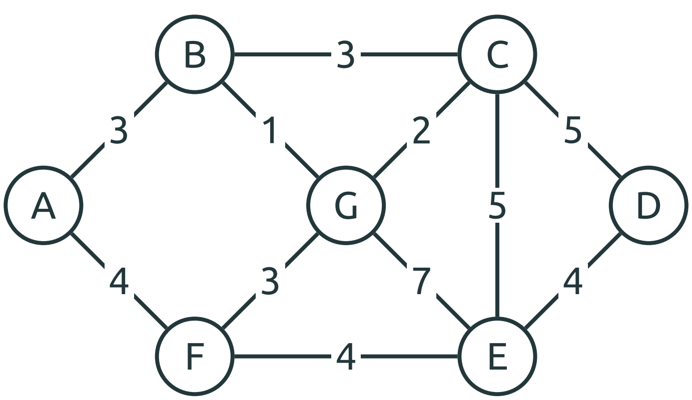
Minimum Spanning Tree: Examples



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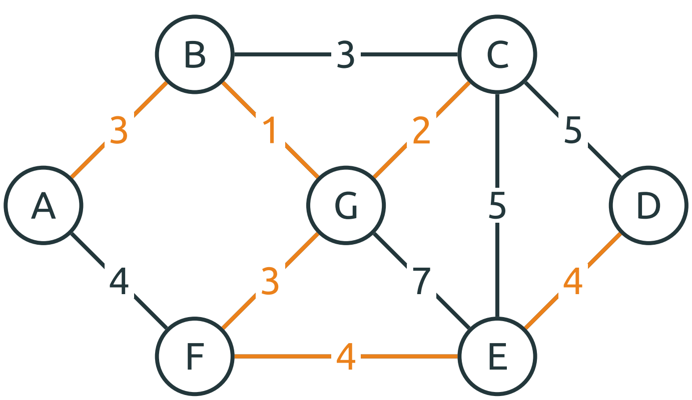

Kruskal's Algorithm

- Start with an empty graph T

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- Start with an empty graph T
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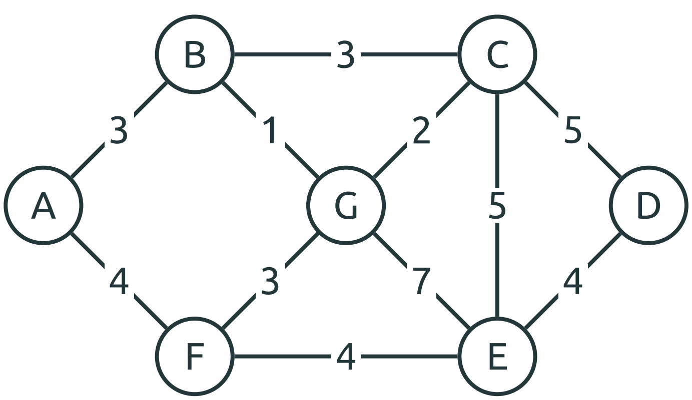
- Add to T an edge of the smallest weight which doesn't create a cycle in T

Kruskal's Algorithm: Examples



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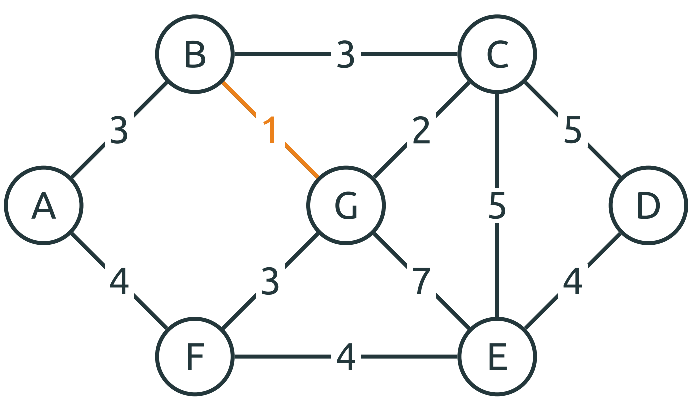

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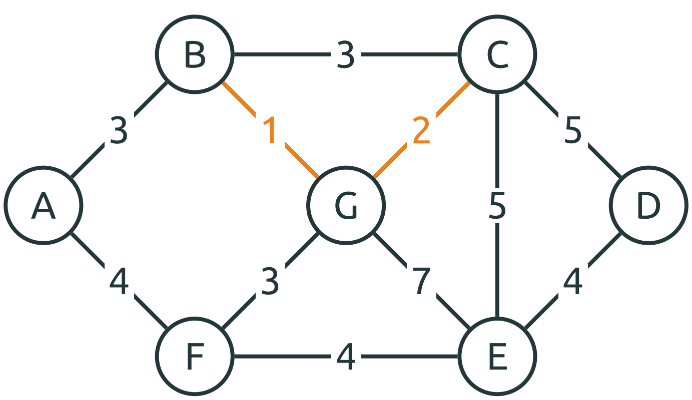

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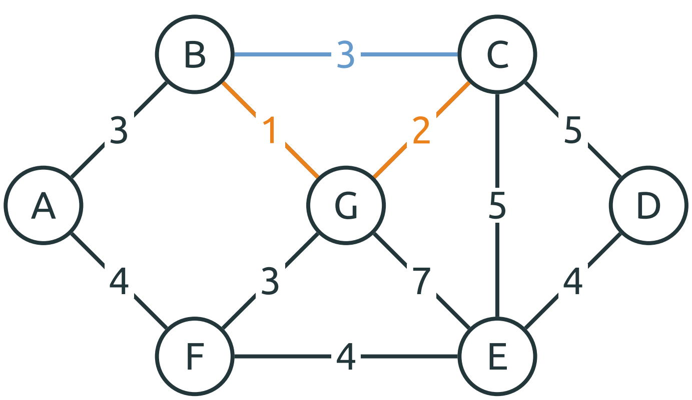

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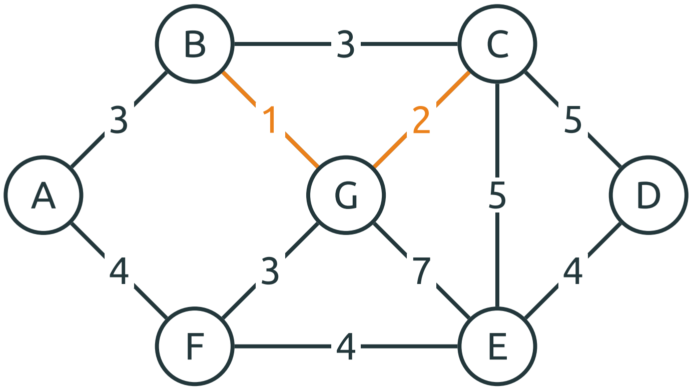

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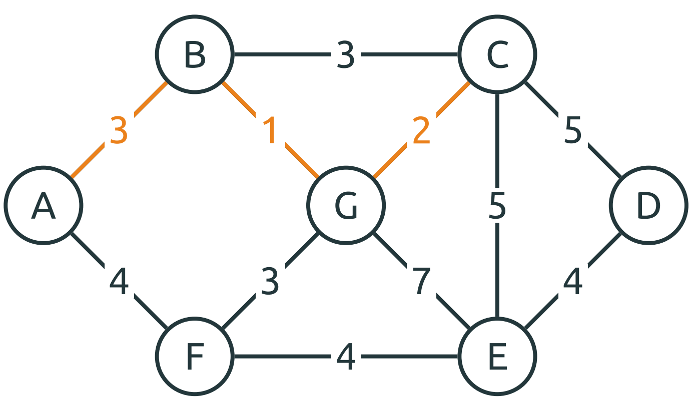

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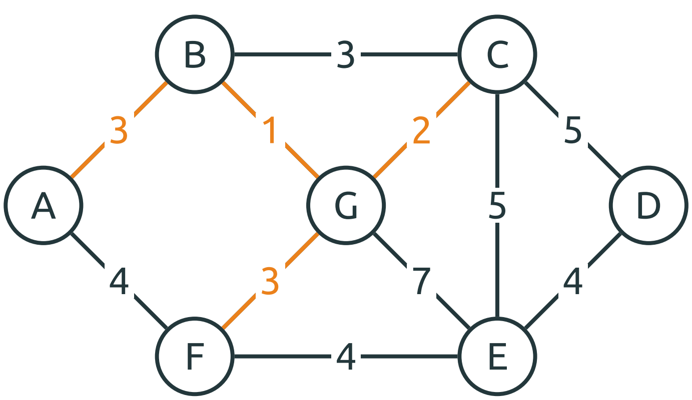

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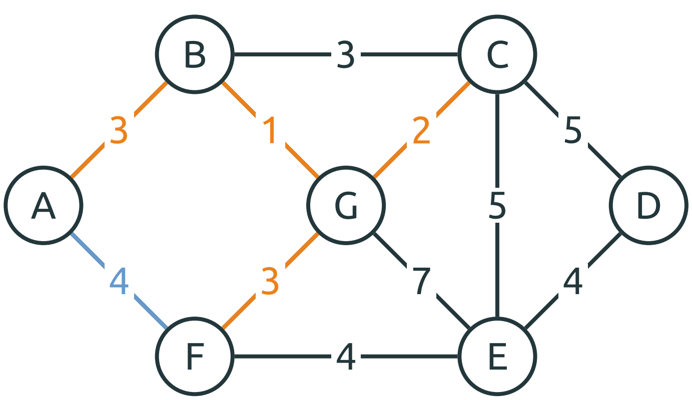

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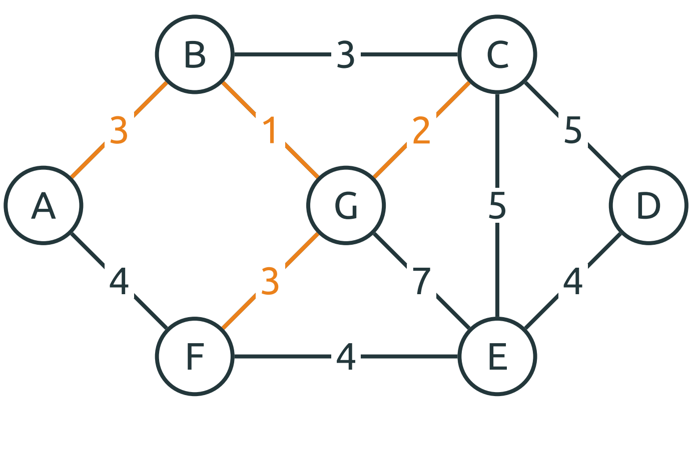

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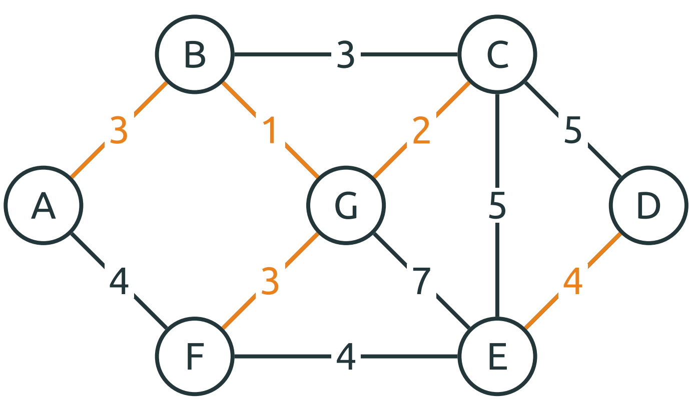

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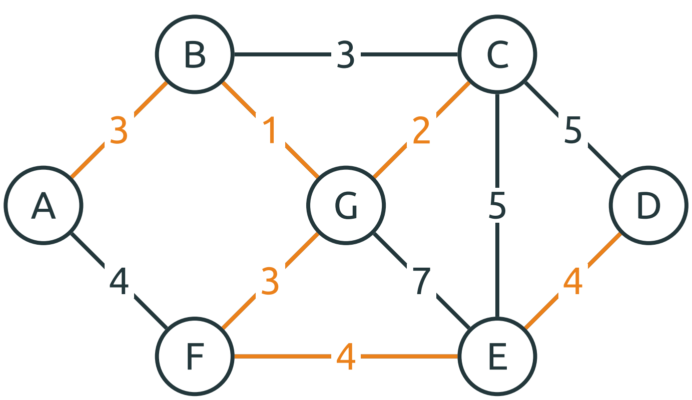

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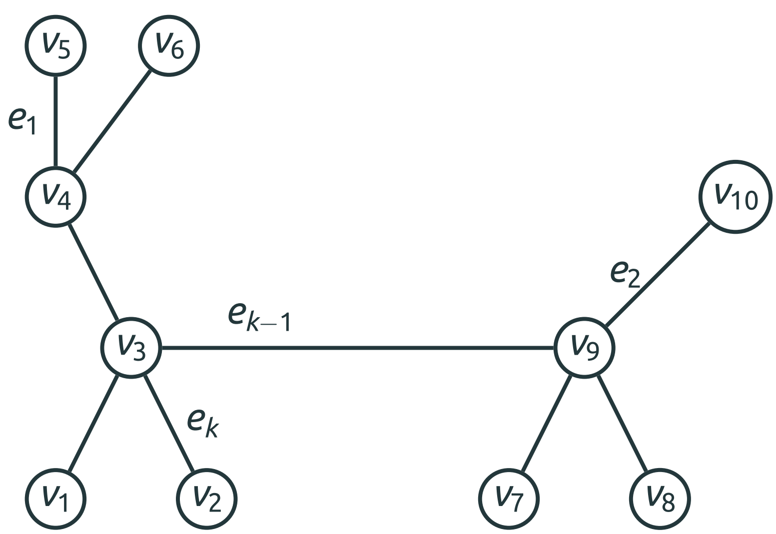

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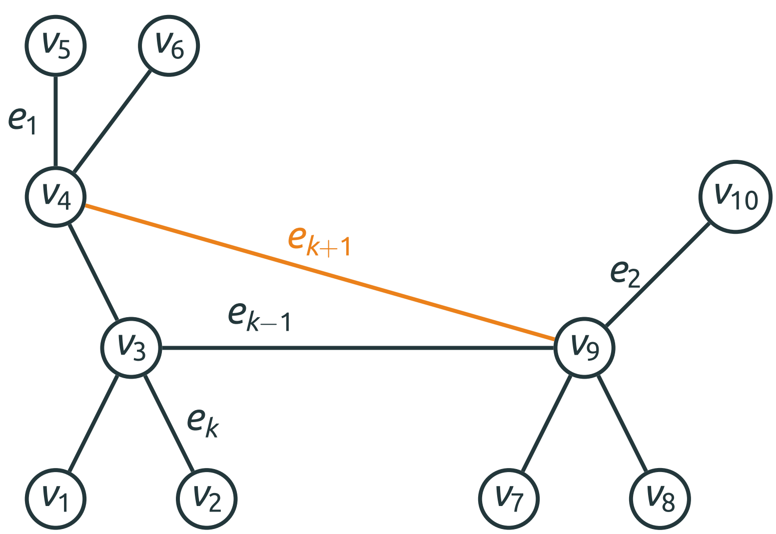

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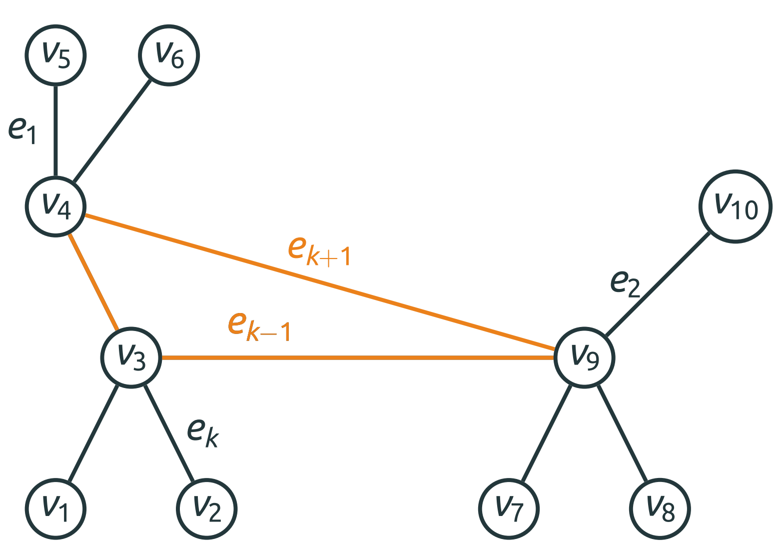

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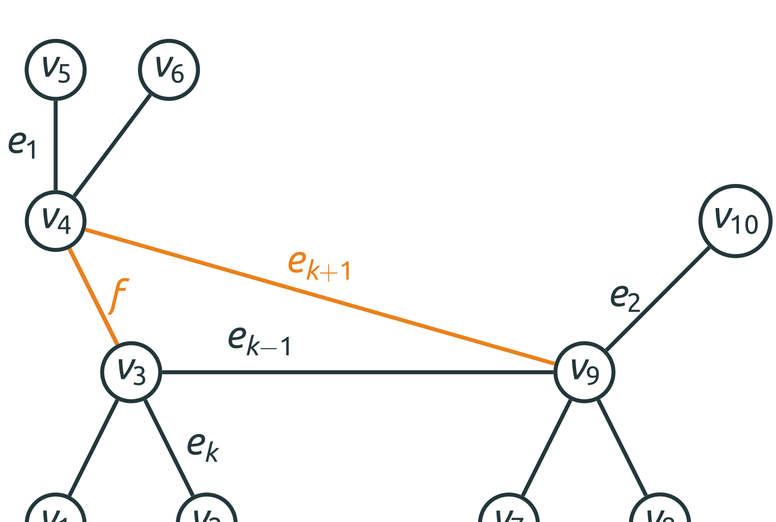

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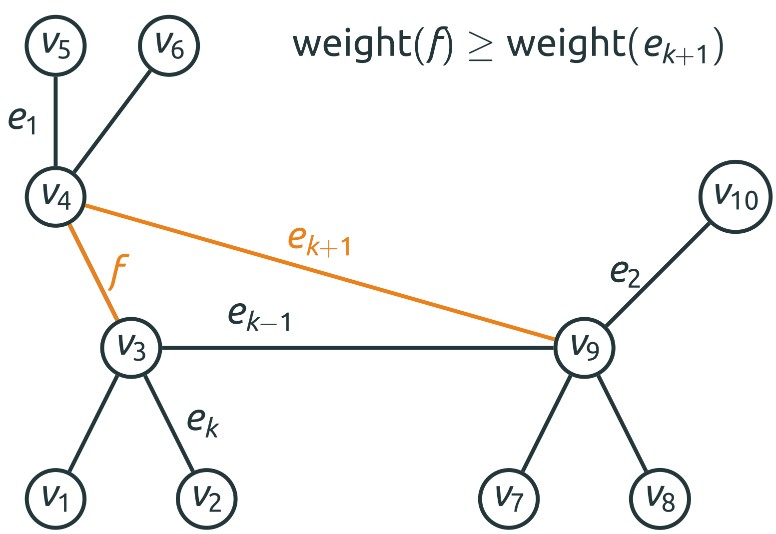
$$\text{weight}(f) \geq \text{weight}(e_{k+1})$$



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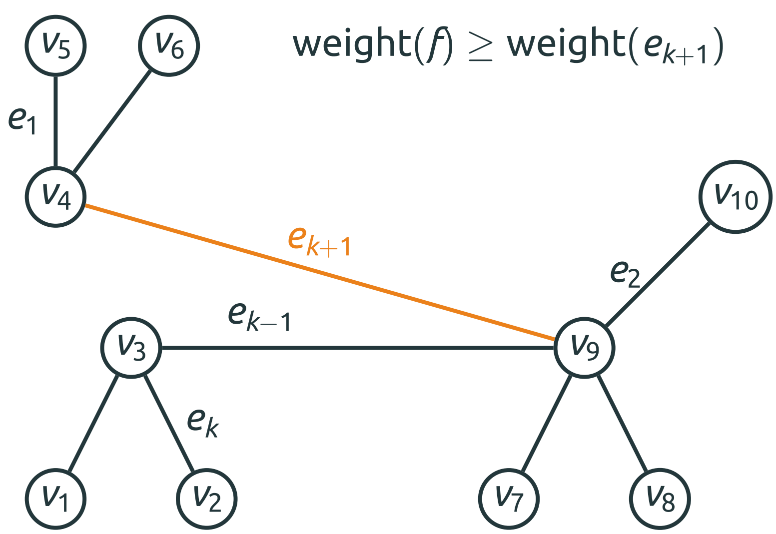

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