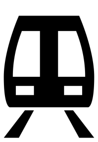
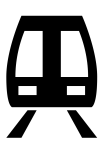
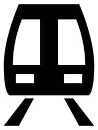
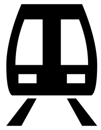
Planar Graphs

Alexander Golovnev

Outline

Subway Lines

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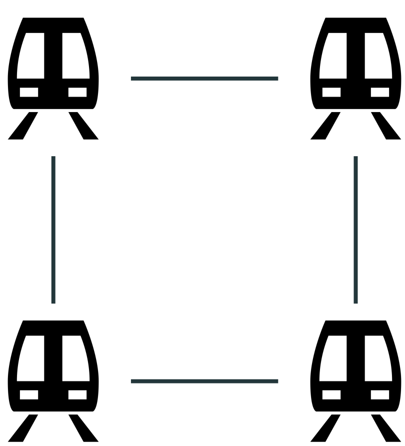

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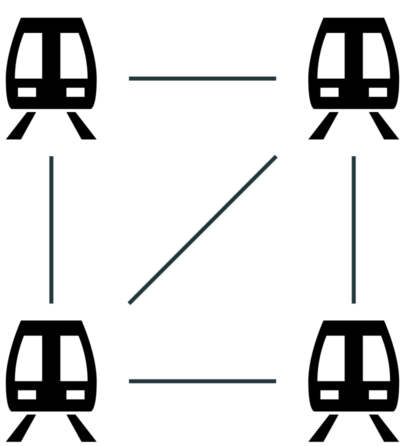

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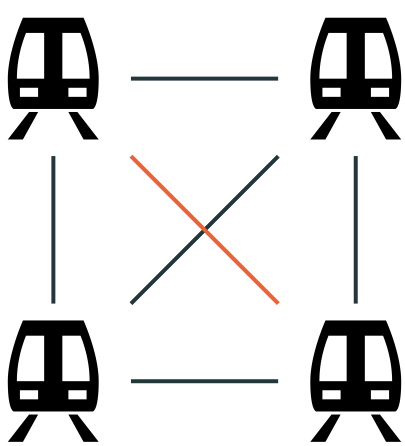

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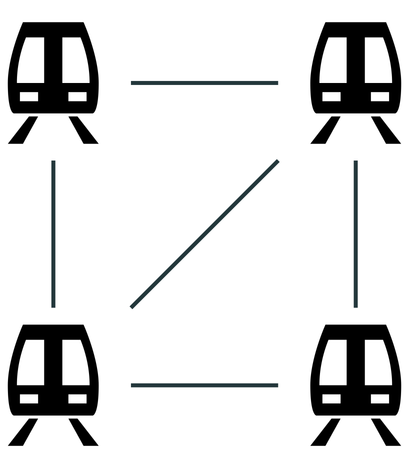

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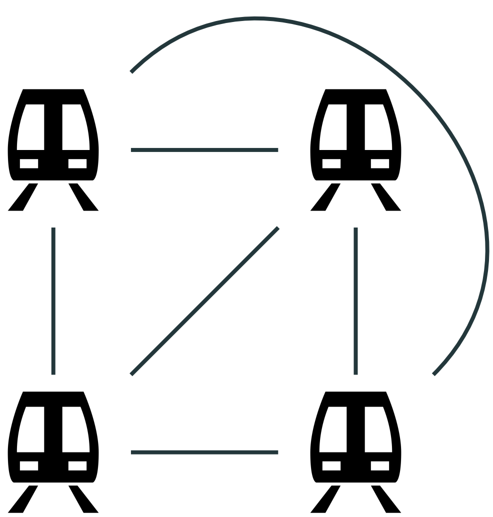

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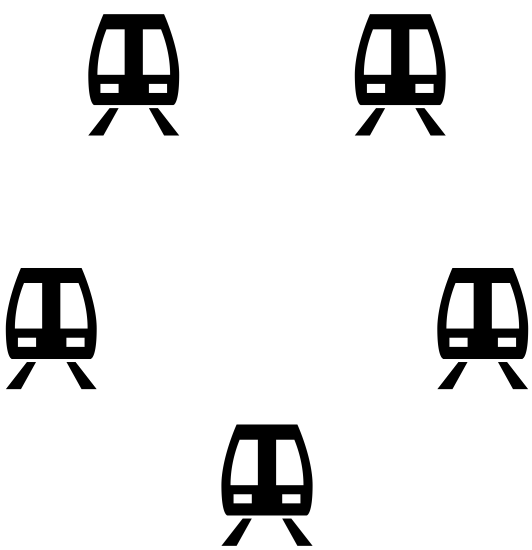

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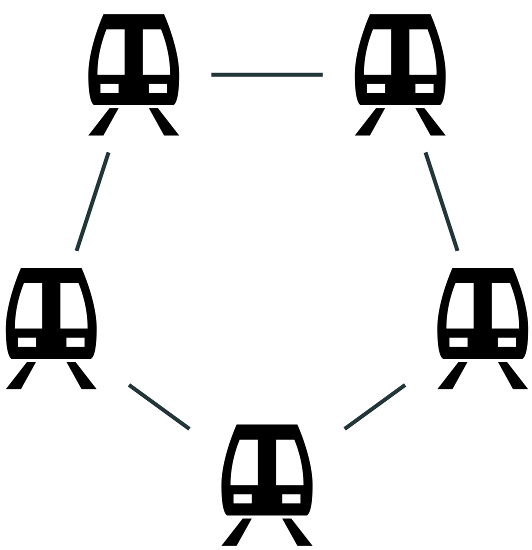

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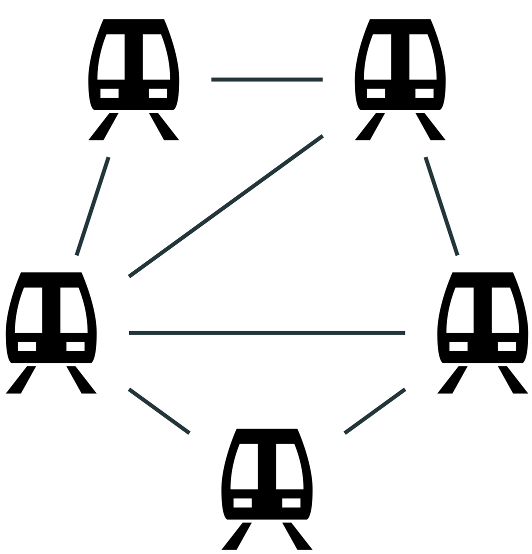

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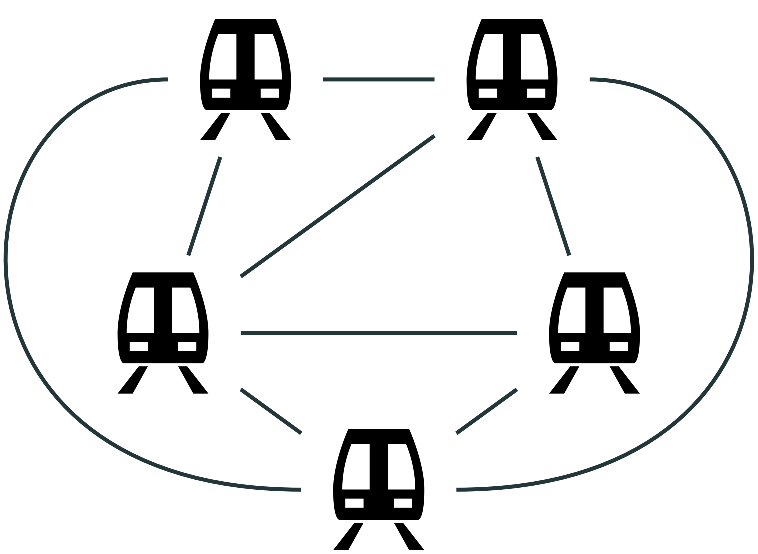

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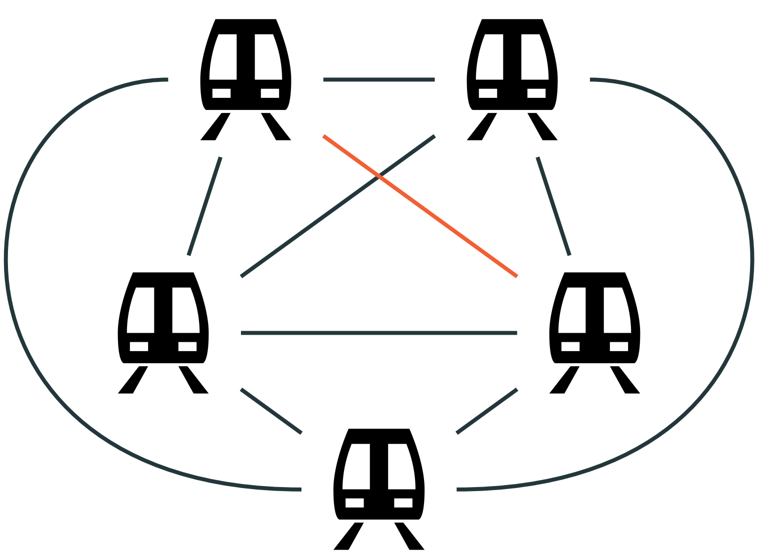

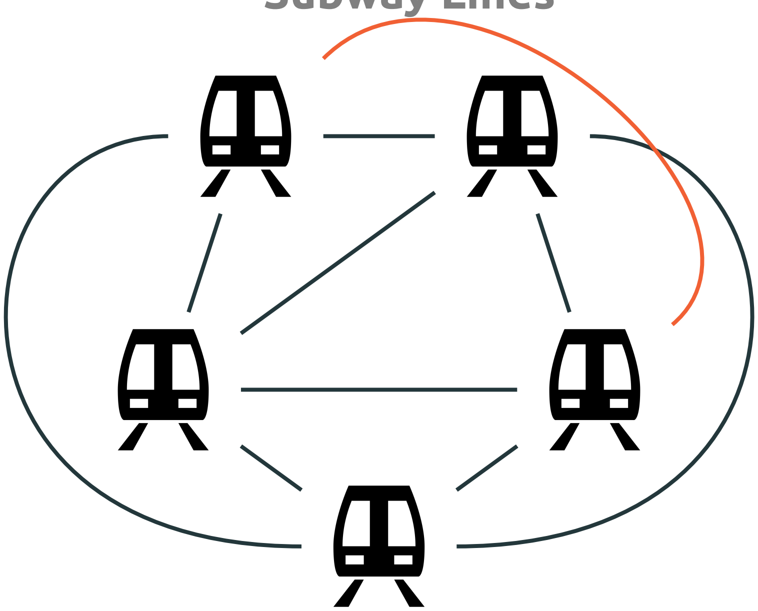

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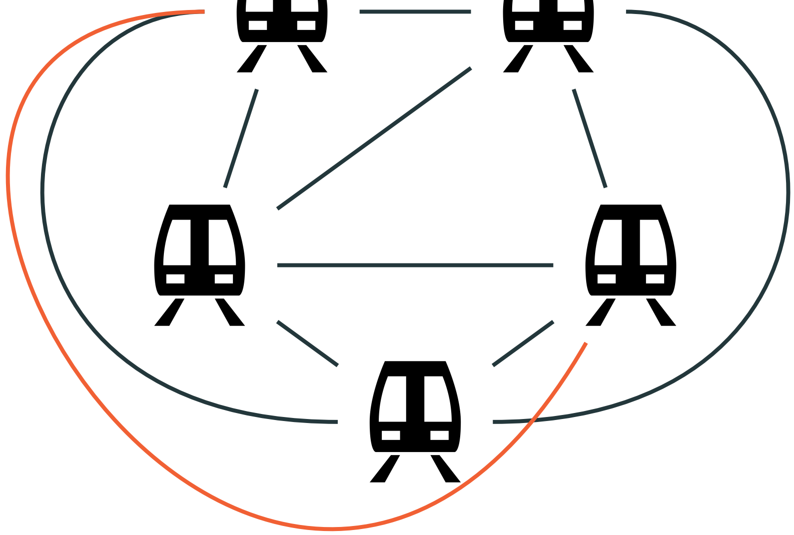

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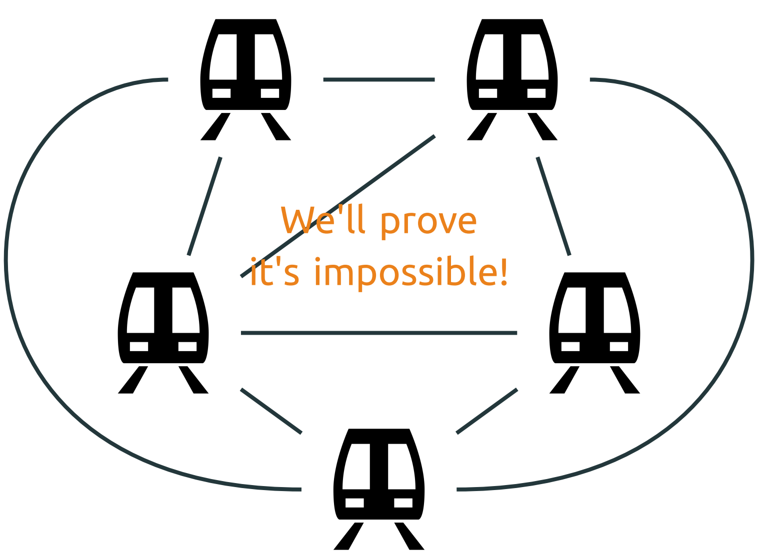

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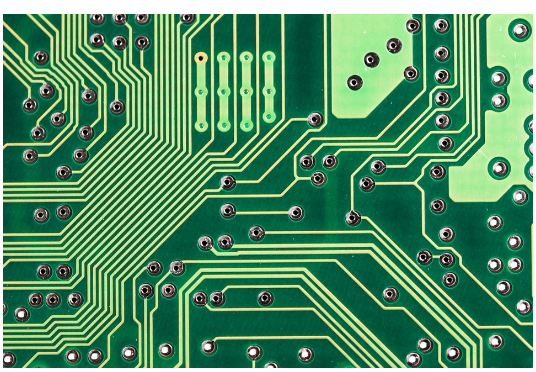
Planar Graphs

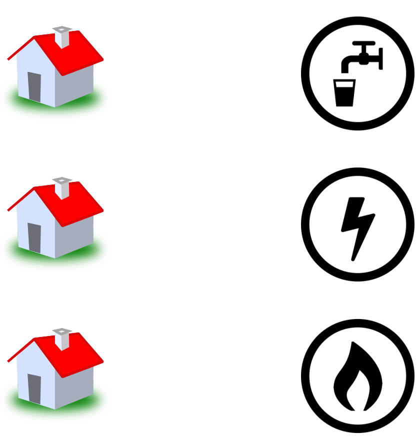
Euler's Formula

Applications of Euler's Formula

Design of Electronic Circuits



Three Utilities Problem



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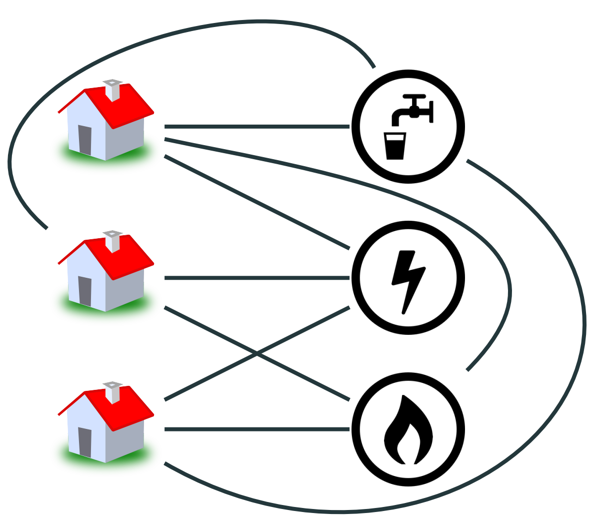

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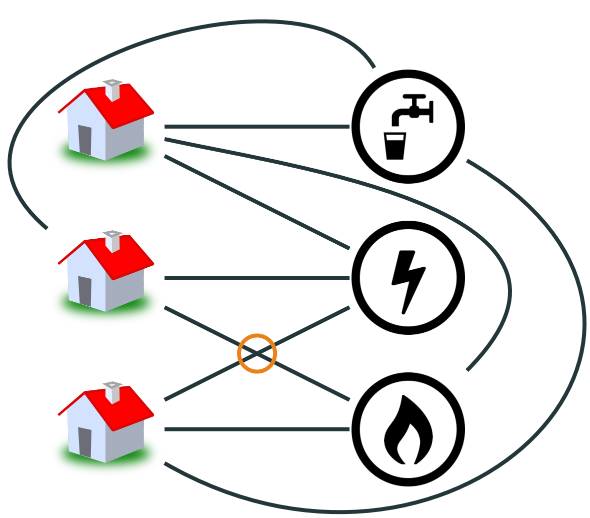

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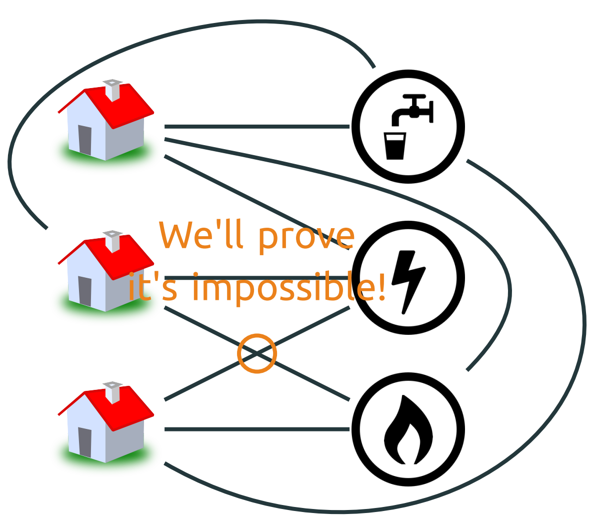

Planar Graphs

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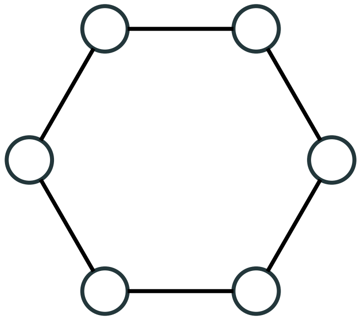
- Even if you usually draw a graph with intersecting edges, it is Planar if it can be drawn without crossing edges

Planar Graphs: Examples

This graph is planar because it can be drawn without crossing edges



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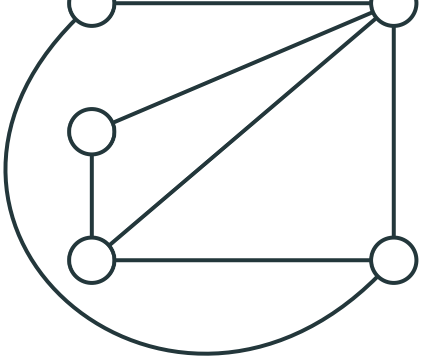

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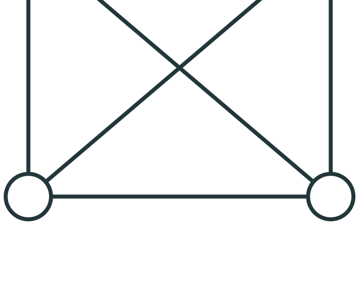

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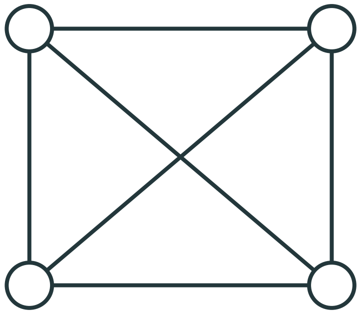

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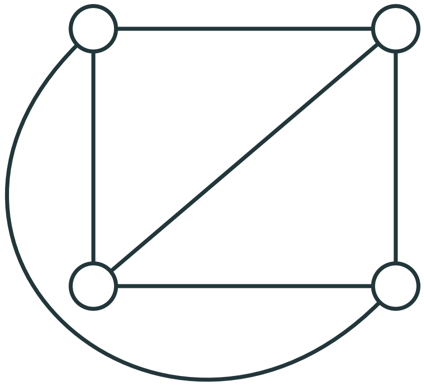

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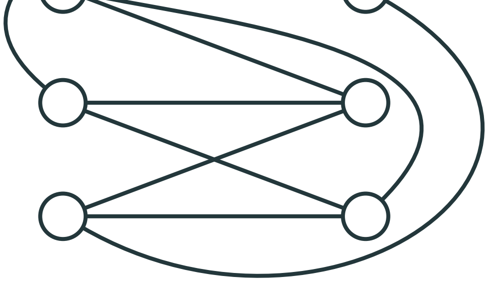
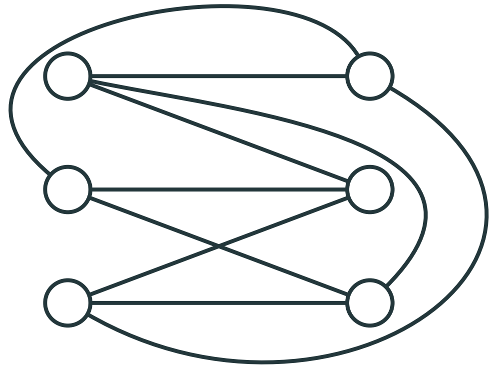


Planar Graphs: Examples

This graph is not planar because it cannot be drawn without crossing edges (we'll prove it later)



Maps and Planar Graphs

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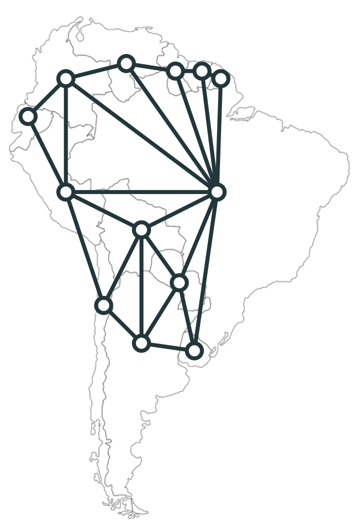

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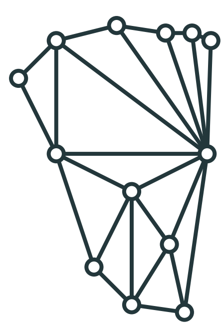

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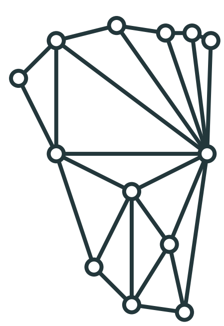

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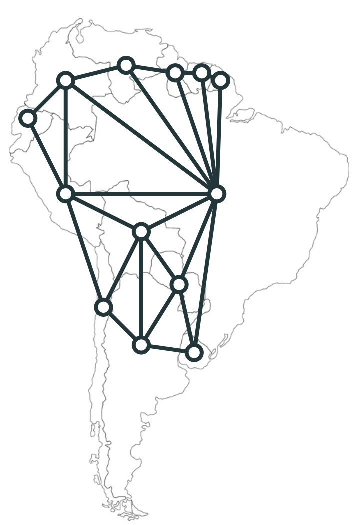

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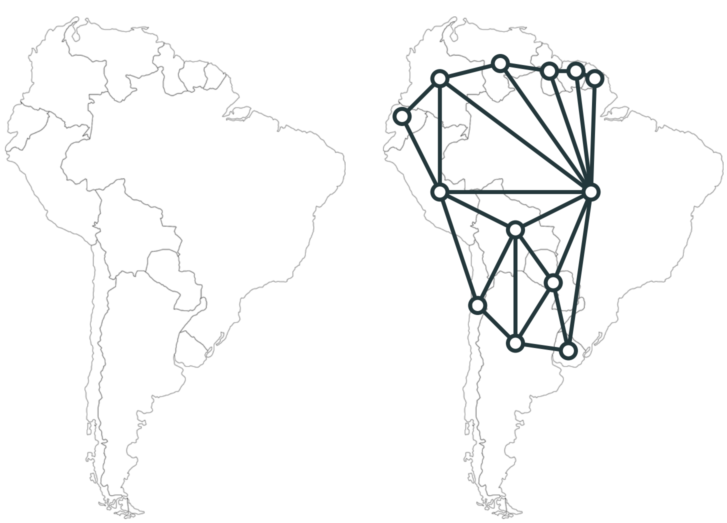

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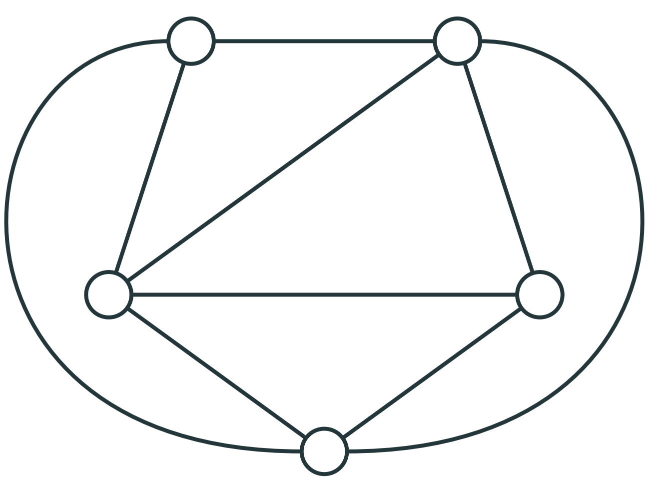

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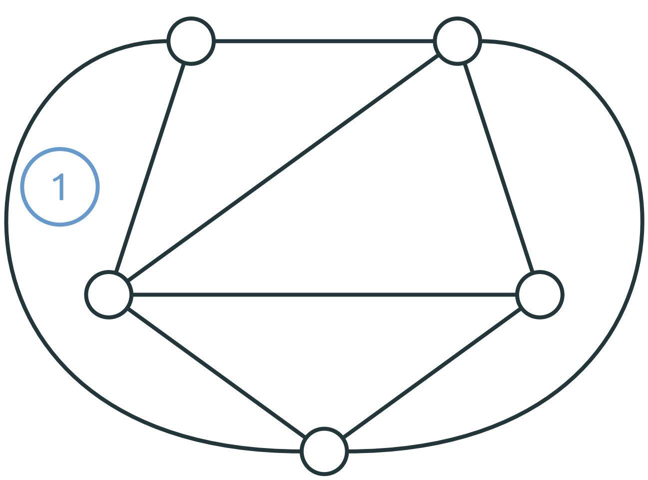

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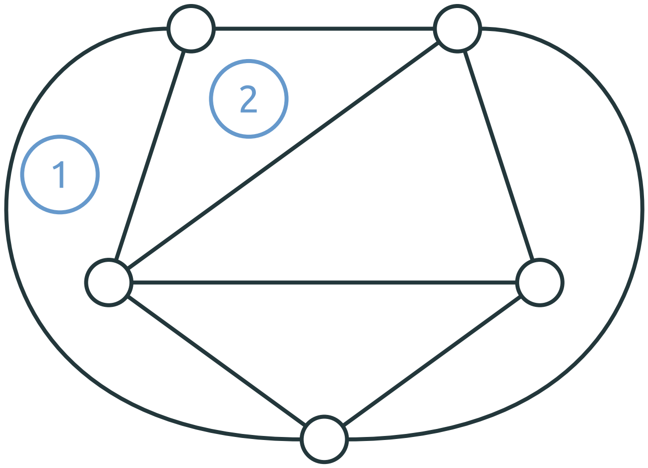

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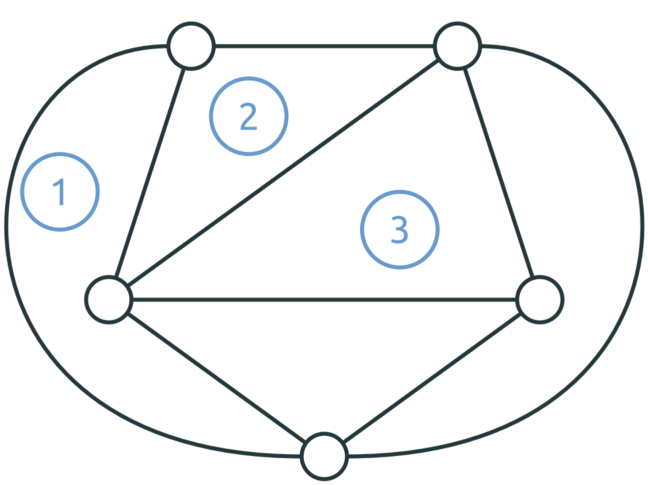

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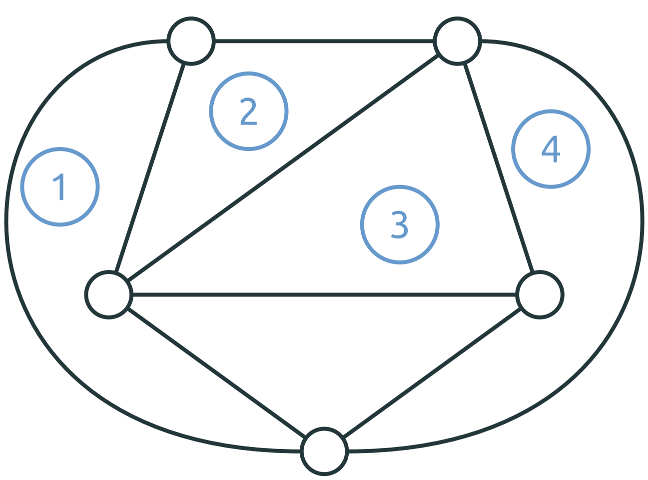

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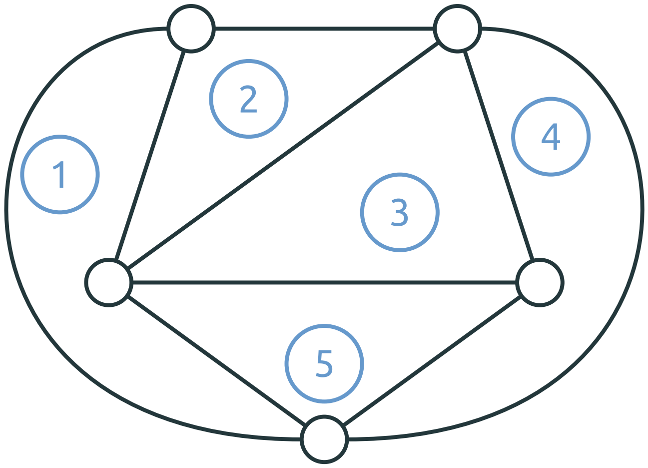

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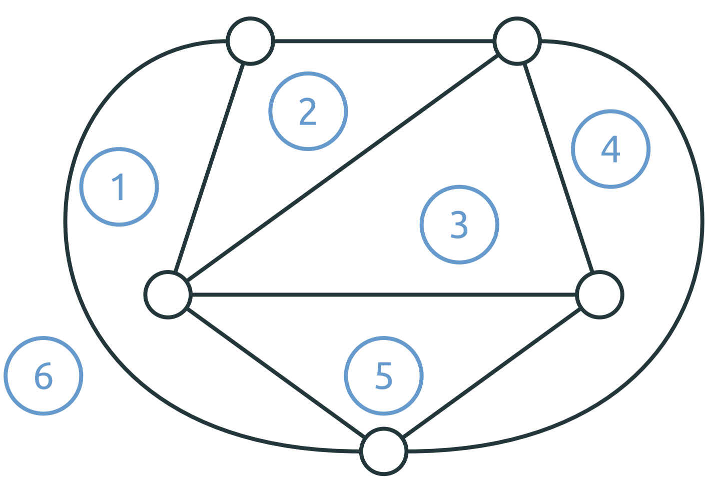


Euler's Formula

Theorem

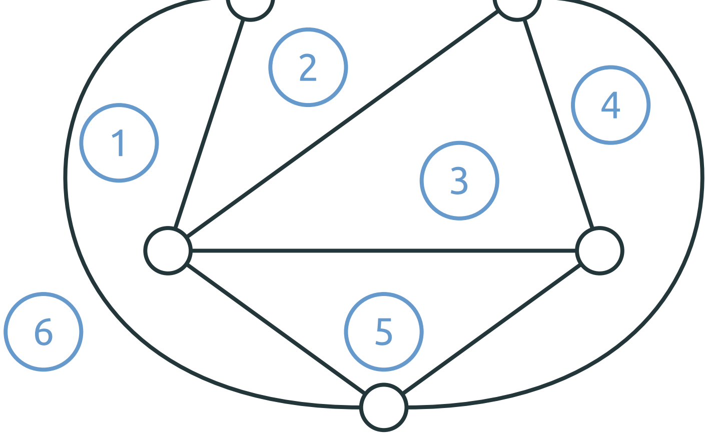
Let G be a connected planar graph drawn in the plane without edge intersections. Then

$$v - e + f = 2 \ ,$$

where v is the number of vertices, e is the number of edges, f is the number of faces in this drawing of G.

Euler's Formula

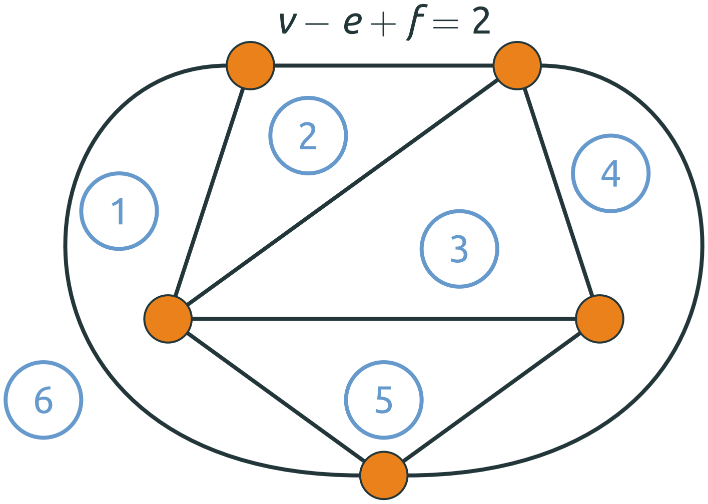
$$v - e + f = 2$$



Euler's Formula

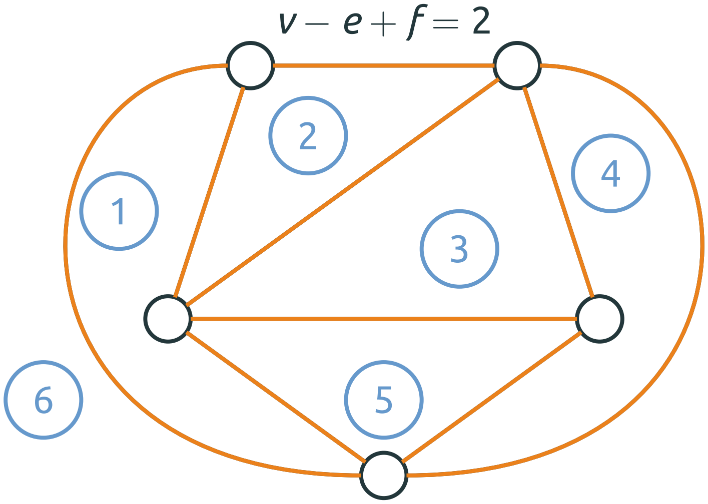
$5 -$

$v - e + f = 2$



Euler's Formula

$$\begin{array}{l} 5 - 9 + \\ v - e + f = 2 \end{array}$$



Euler's Formula

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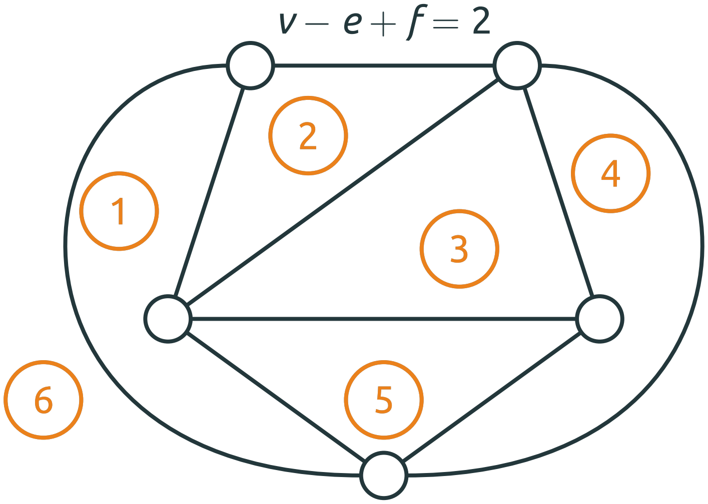

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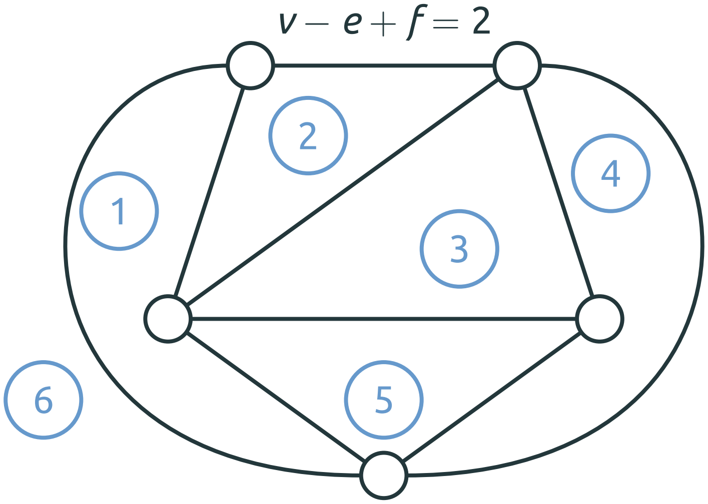

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- Induction Hypothesis. The formula holds for all graphs with $\leq c$ cycles
- Induction Step. We'll prove the formula for G with $c + 1$ cycles, v vertices, e edges, and f faces

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- Then $v - e + f = v_1 - (e_1 + 1) + (f_1 + 1) = v_1 - e_1 + f_1 = 2$

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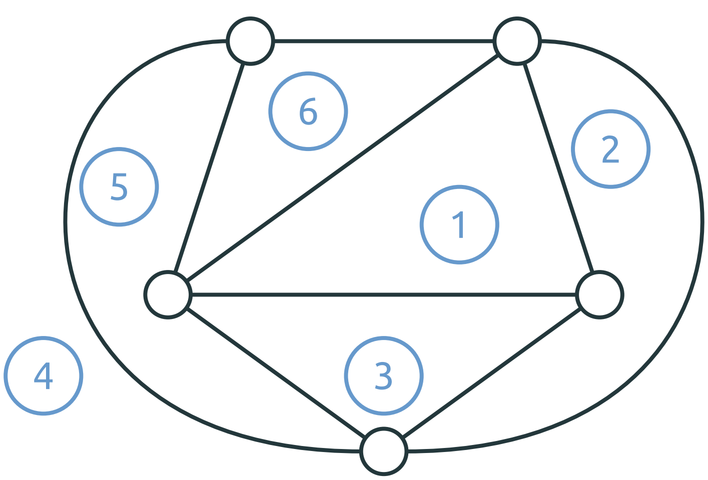

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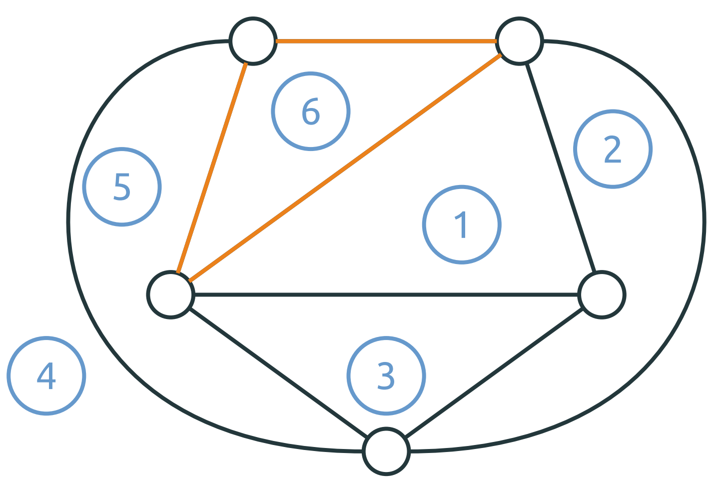

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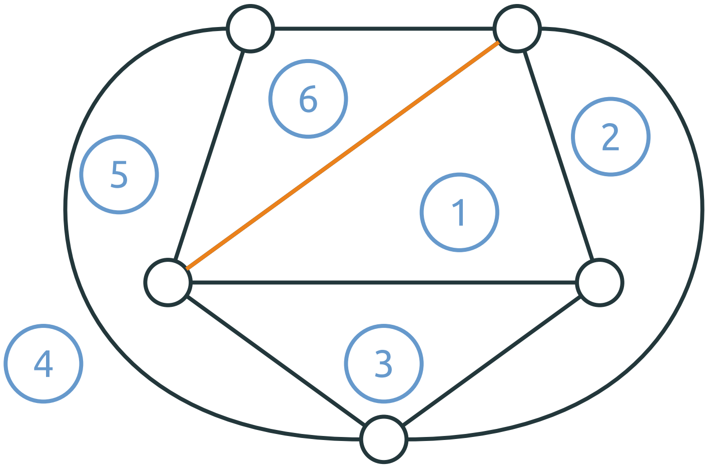

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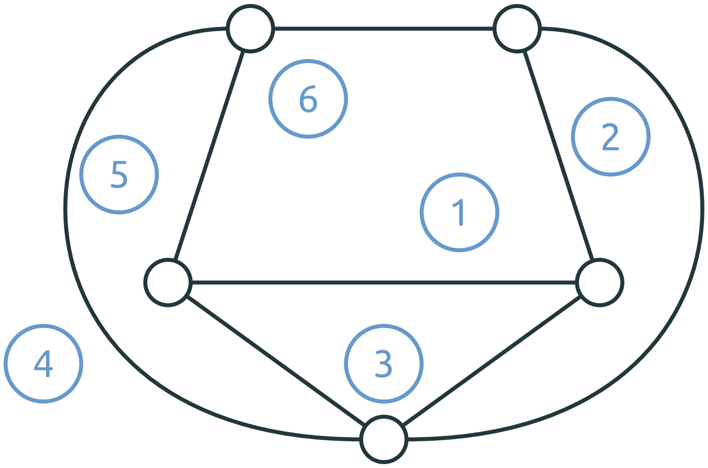

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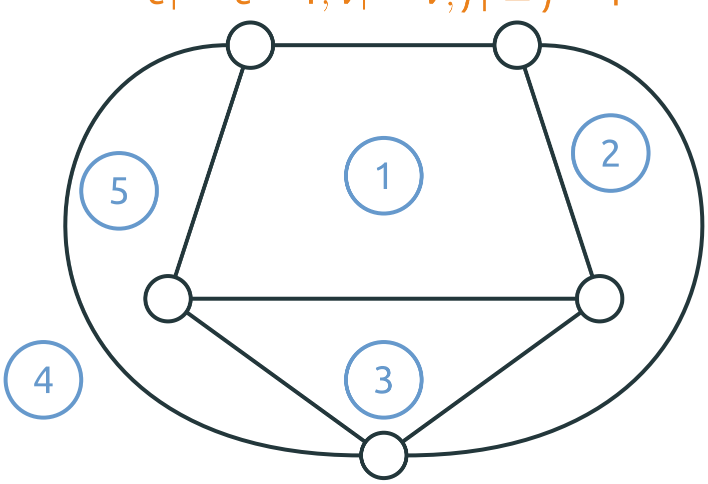

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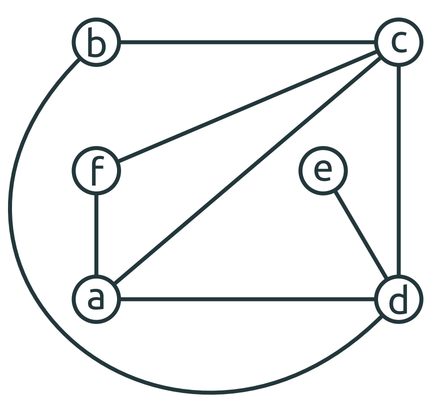
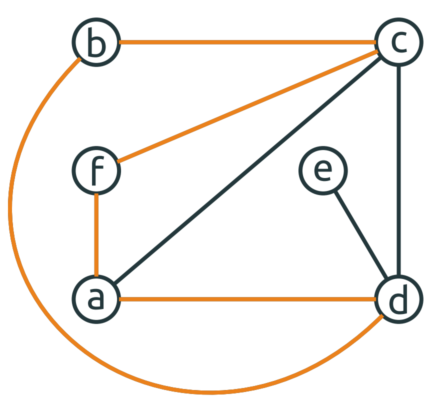
Applications of Euler's Formula

Faces and Edges



Faces and Edges

This face has 5 edges



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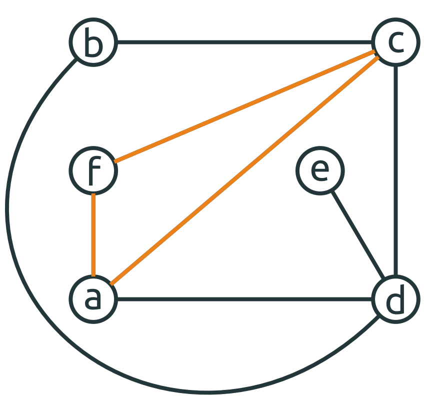
This face has 3 edges



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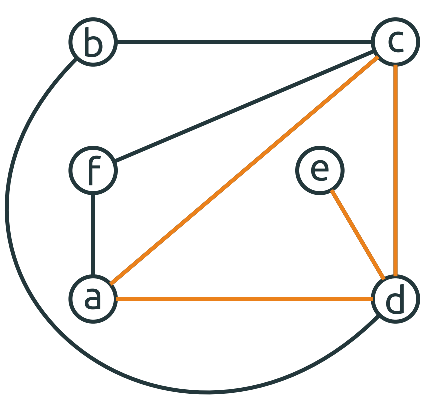
This face has 4 edges



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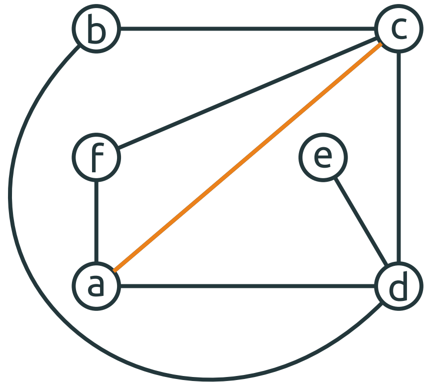

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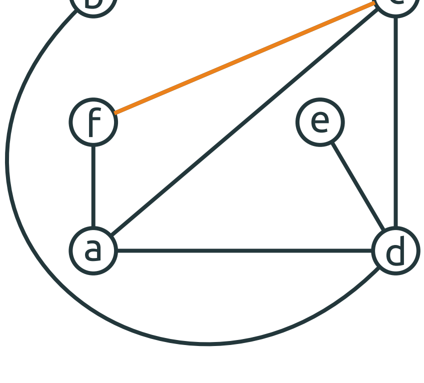
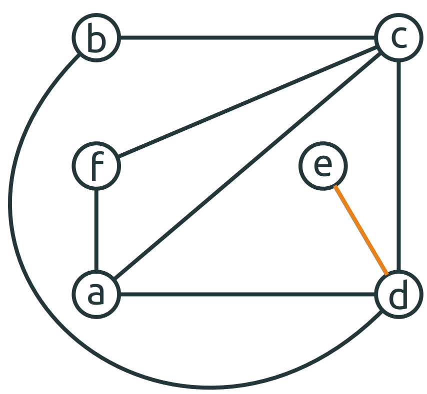


Faces and Edges

This edge belongs to 1 face



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The Number of Faces

- Consider a connected planar graph on ≥ 3 vertices
- Each face has at least 3 edges
- On the other hand, each edge belongs to at most 2 faces
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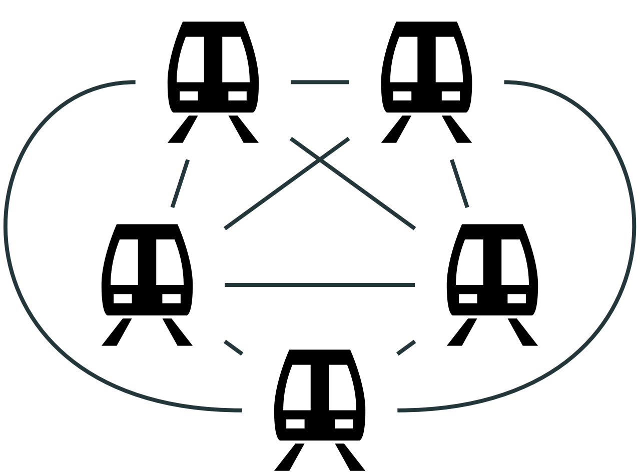
K_5 is Nonplanar

Why is K_5 nonplanar?



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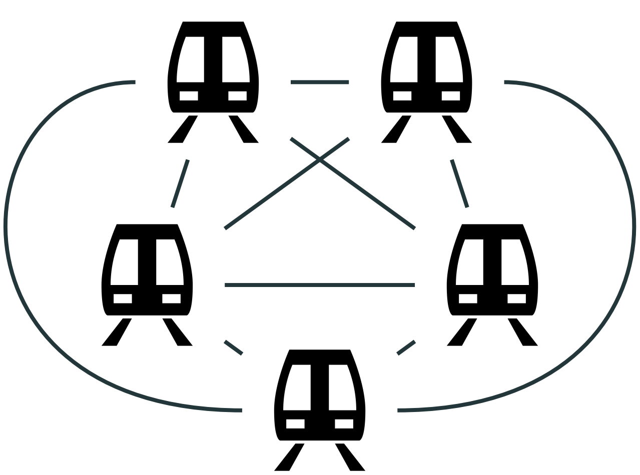
It has $v = 5$ vertices and $e = 10$ edges



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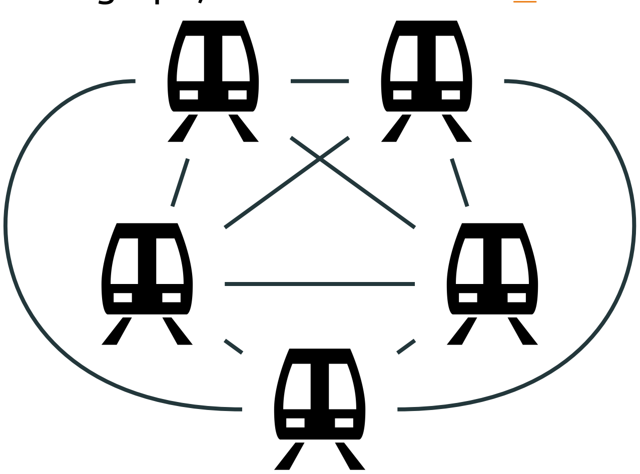
In a planar graph, $e = 10$ must be $\leq 3v - 6 = 9$



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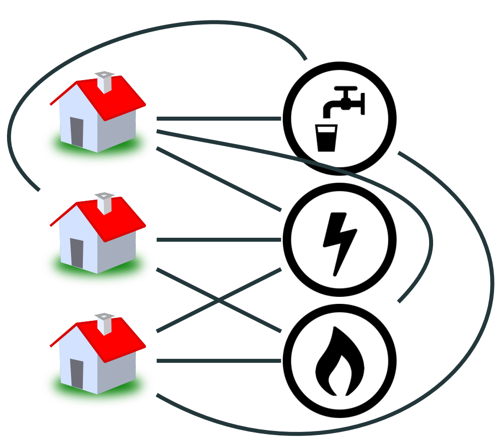

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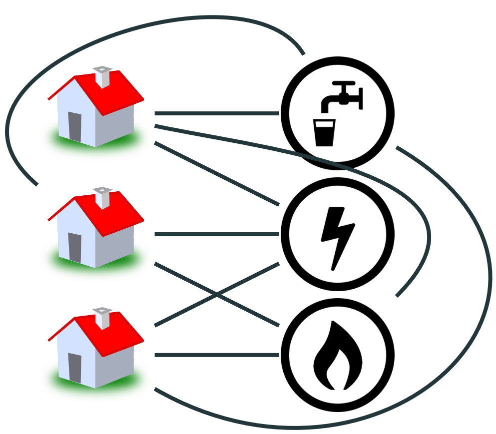

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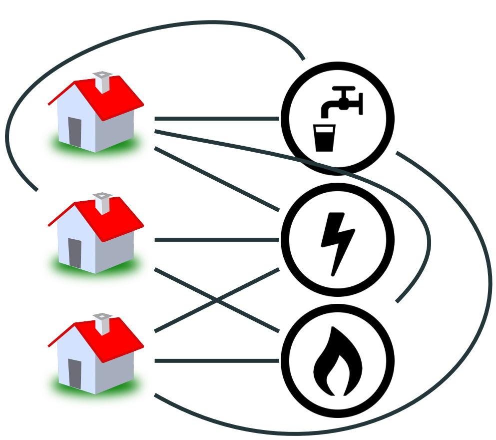
It does satisfy $e \leq 3v - 6$



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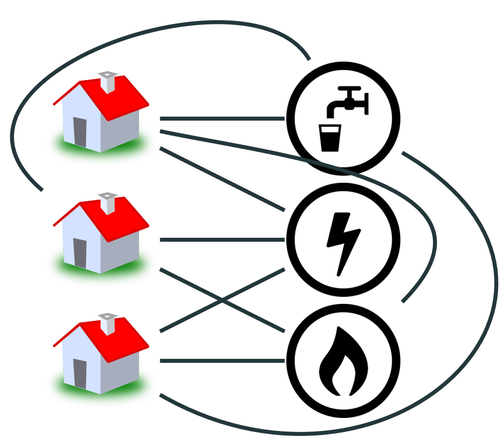

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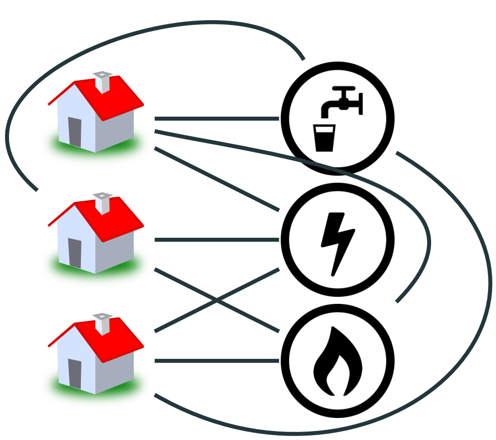

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In a planar bipartite graph,
$e = 9$ must be $\leq 2v - 4 = 8$