Propositional Logic:

Syntax, Semantics, Natural Deduction

Summary

- Syntax
- Semantics
- Normal forms
- Deduction and refutation
- Natural Deduction

For any formal system (programming language, logic etc.)

- Syntax: the rules which tell us how the well formed sentences are constructed
  - Informal definitions
  - BNF
  - Inductive definitions
  - ….
- Semantics: the rules which tell us the meaning of the well formed sentences
  - Operational semantics
  - Denotational semantics
  - Logical semantics
  - ….

For logic(s)

- Syntax:
  - which are the symbols used to represent propositions, variables, functions, predicates and
  - which are the symbols used to represent connectives …
  - In order to obtain well formed formulas

- Semantics, two main realms:
  - Model theory: what is true
  - Proof theory: what is provable
  - Hopefully the two above coincide

Propositional Logic

Propositional logic is the simplest logic- illustrates basic ideas using **propositions**

- P1 Snow is white
- P2 Today it is raining
- P3 This course is boring

Pi is an atom or atomic formula
Each Pi can be either **true** or **false** but never both
The values true or false assigned to each proposition is called **truth value** of the proposition

Connectives

Many connectives used in natural language: and, or, not, but, since, if ... then ... etc. Why in logic we use a few of them?

- Max is a Marxist, but he is not humourless
- π irrational, but it is not algebraic

"But" in the first statement carries some emotional meaning (surprise) while in the second statement such a surprise is less immediate (one has to know that almost all irrationals are algebraic) and we can use "and".

We stick to some principle of economy and we select connectives which arde useful for expressing proofs. such as "and" and "or" . However also in this case we have some ambiguities ...

Connectives problems 1

- John drove on and hit a pedestrian.
- John hit a pedestrian and drove on.

Problems:

- It seems that "and" 'and' may have an ordering function in time, while this is not the case in mathematics (and logic)

Need to create an artificial language with precise meanings of connectives.

Connectives problems 2

1. If I open the window then we'll have fresh air.
2. If I open the window then $1 + 3 = 4$.
3. If $1 + 2 = 4$, then we'll have fresh air.
4. John is working or he is at home.
5. Euclid was a Greek or a mathematician.

Problems:

- From 1: there is a relation between the premise and conclusion
- From 2 and 3: no relation between the premise and conclusion. Perfectly adequate in mathematics though, often called *material implication*
- We tend to accept 4 and to reject 5.

Need to create an artificial language with precise meanings of

Alphabet for Propositional Logic

Definition 1.1.1 *The language of propositional logic has an alphabet consisting of*

(i) *proposition symbols :* $p_0,\ p_1,\ p_2,\ \ldots,$

(ii) *connectives :* $\wedge$, $\vee$, $\rightarrow$, $\neg$, $\leftrightarrow$, $\perp$,

(iii) *auxiliary symbols :* (,).

The connectives carry traditional names:

$\wedge$	- *and*	- *conjunction*
$\vee$	- *or*	- *disjunction*
$\rightarrow$	- *if ..., then ...*	- *implication*
$\neg$	- *not*	- *negation*
$\leftrightarrow$	- *iff*	- *equivalence, bi-implication*
$\perp$	- *falsity*	- *falsum, absurdum*

The proposition symbols and $\perp$ stand for the indecomposable propositions, which we call *atoms*, or *atomic propositions*.

Syntax of propositional logic I

Recursive definition of **well-formed formulas**

1. An atom is a formula
2. If S is a formula, $\neg S$ is a formula (**negation**)
3. If S_1 and S_2 are formulas, $S_1 \wedge S_2$ is a formula (**conjunction**)
4. If S_1 and S_2 are formulas, $S_1 \vee S_2$ is a formula (**disjunction**)
5. All well-formed formulas are generated by applying above rules

Shortcuts:

- $S_1 \rightarrow S_2$ can be written as $\neg S1 \vee S2$

Syntax of propositional logic II (using BNF)

Backus-Naur Form (BNF) :

$$
\begin{array}{lcl}
< \text{formula} > & ::= & \text{Atomic Proposition} \\
& | & \neg < \text{formula} > \\
& | & < \text{formula} > \wedge < \text{formula} > \\
& | & < \text{formula} > \vee < \text{formula} > \\
& | & < \text{formula} > \rightarrow < \text{formula} > \\
& | & < \text{formula} > \Leftrightarrow < \text{formula} > \\
& | & (< \text{formula} >)
\end{array}
$$

Syntax of propositional logic III (inductive definition)

Definition 1.1.2 *The set PROP of propositions is the smallest set X with the properties*

$(i)\quad p_i \in X (i \in N),\ \bot \in X,$

$(ii)\quad \varphi, \psi \in X \Rightarrow\ (\varphi \wedge \psi),\ (\varphi \vee \psi),\ (\varphi \to \psi),\ (\varphi \leftrightarrow \psi) \in X,$

$(iii)\ \varphi \in X \Rightarrow (\neg \varphi) \in X.$

Useful for proving (by induction) properties of formulae

How to use induction to prove properties of formulas*

Theorem 1.1.3 (Induction Principle) *Let A be a property, then $A(\varphi)$ holds for all $\varphi \in PROP$ if*

(i) $A(p_i)$, *for all* i,*and* $A(\perp)$,

(ii) $A(\varphi),\ A(\psi) \Rightarrow A(\,(\varphi \Box \psi)),$

(iii) $A(\varphi) \Rightarrow A(\,(\neg\varphi)).$

Proof. Let $X = \{\varphi \in PROP \mid A(\varphi)\}$, then X satisfies (i), (ii) and (iii) of definition 1.1.2. So $PROP \subseteq X$, i.e. for all $\varphi \in PROP$ $A(\varphi)$ holds. $\Box$

Where $\Box$ indicate any binary connective

Some example of well formed formulas

Semantics I for propositional logic: models, i.e. truth tables

Semantics

- In order to define the semantics, a.k.a. the meaning, of propositional logic (PL) we need to define the meaning of the propostional symbols: if we write the proposition “It is raining” is it true or false? Of course it depends on the “world” we are considering, so we need the notion of interpretation (see next slide)

Interpretation

Definition

Interpretation: Given a propositional formula G, let $\{A_1, \cdots, A_n\}$ be the set of atoms which occur in the formula, an **Interpretation** I of G is an assignment of truth values to $\{A_1, \cdots, A_n\}$.

Example

Consider the formula: $G \triangleq (P \vee Q) \wedge \neg(P \wedge Q)$

Set of atoms: $\{P, Q\}$

Interpretation for G: $I = \{P = \mathbf{T}, Q = \mathbf{F}\}$

Interpretation (ctnd)

- Each atom A_i can be assigned either **True** or **False** but never both.
- Given an interpretation I a formula G is said to be true in I iff G is evaluated to **True** in the interpretation
- Given a formula G with n distinct atoms there will be 2^n distinct interpretations for the atoms in G.
- Convention: $\{P, \neg Q, \neg R, S\}$ represents an interpretation $I : \{P = T, Q = F, R = F, S = T\}$.
- Given a formula G and an interpretation I, if G is true under I we say that I is a model for G.and we can write $I \models G$

Semantics of connectives

- The semantics, or meaning, of the connectives is defined by using “truth tables”: these are tables that given the truth values (True or False) of the arguments, provide the truth value of the formula obtained by applying a connective to those argiment, as shown in the following slides

Semantics

Relationships between truth values of atoms and truth values of formulas

$\neg S$	is true iff	S	is false		
$S_1 \wedge S_2$	is true iff	S_1	is true **and**	S_2	is true
$S_1 \vee S_2$	is true iff	S_1	is true **or**	S_2	is true
$S_1 \rightarrow S_2$	is true iff	S_1	is false **or**	S_2	is true
i.e.,	is false iff	S_1	is true **and**	S_2	is false
$S_1 \leftrightarrow S_2$	is true iff	$S_1 \rightarrow S_2$	is true **and**	$S_2 \rightarrow S_1$	is true

Truth Tables

Example (Truth Tables for main logical connectives)

P_1	P_2	$\neg P_1$	$P_1 \wedge P_2$	$P_1 \vee P_2$	$P_1 \rightarrow P_2$	$P_1 \leftrightarrow P_2$
T	T	F	T	T	T	T
T	F	F	F	T	F	F
F	T	T	F	T	T	F
F	F	T	F	F	T	T

Evaluation of a formula

- Once we have a meaning for propositional symbols we can used truth tables for evaluating the truth value of a formula:

Evaluation of a formula

Recursive Evaluation

Consider the formula $G \triangleq \neg P_1 \wedge (P_2 \vee P_3)$
Suppose we know that $P_1 = F,\ P_2 = F,\ P_3 = T$
Then we have

$$\neg P_1 \wedge (P_2 \vee P_3) = \mathit{true} \wedge (\mathit{false} \vee \mathit{true}) = \mathit{true} \wedge \mathit{true} = \mathit{true}$$

Note

We evaluate $\neg P_1$ before $P_1 \wedge P_2$, this is because the following decreasing rank for connectives operator holds:

$$\leftrightarrow \quad \rightarrow \quad \vee \quad \wedge \quad \neg$$

Exercise

Example (XOR)

Write the truth table for the formula:

$$G \triangleq (P \vee Q) \wedge \neg(P \wedge Q)$$

Exercise

Example (XOR)

Write the truth table for the formula:

$$G \triangleq (P \vee Q) \wedge \neg(P \wedge Q)$$

Example (XOR)

Write the truth table for the formula:

$$G \triangleq (P \vee Q) \wedge \neg(P \wedge Q)$$

Sol.

P	Q	$P \vee Q$	$P \wedge Q$	$\neg(P \wedge Q)$	G
T	T	T	T	F	F
T	F	T	F	T	T
F	T	T	F	T	T
F	F	F	F	T	F

Model

- The notion of model is very important: we say that an interpretation INT is a model of a formula F, written

  INT |= F

  if F is true when the truth value of propositional symbols is defined according to INT.

  Note that we use the same symbol (|=) that we use for logical consequence.

Validity

Definition

Valid Formula: A formula F is **valid** iff it is true in all its interpretation

- A valid formula can be also called a **Tautology**
- A formula which is not valid is **invalid**
- If F is valid we can write $\models F$

Example (de Morgan's Law)

$(\neg(P \wedge Q) \leftrightarrow (\neg P \vee \neg Q))$ is a valid formula

P	Q	$\neg(P \wedge Q)$	$\neg P \vee \neg Q$	$(\neg(P \wedge Q) \leftrightarrow (\neg P \vee \neg Q))$
T	T	F	F	T
T	F	T	T	T
F	T	T	T	T
F	F	T	T	T

Inconsistency

Definition

Inconsistent Formula: A formula F is **inconsistent** iff it is false in all its interpretation

- An inconsistent formula is said to be **unsatisfiable**
- A formula which is not inconsistent is **consistent** or **satisfiable**
- Invalid is different from Inconsistent

Example

$\neg((\neg(P \wedge Q) \leftrightarrow (\neg P \vee \neg Q)))$ is inconsistent

P	Q	$(\neg(P \wedge Q) \leftrightarrow (\neg P \vee \neg Q))$	$\neg(\neg(P \wedge Q) \leftrightarrow (\neg P \vee \neg Q))$
T	T	T	F
T	F	T	F
F	T	T	F
F	F	T	F

Inconsistency vs validity

- A formula is valid iff its negation is inconsistent (and vice versa)
- A formula is invalid (consistent) iff there is at least an interpretation in which the formula is false (true)
- An inconsistent formula is invalid but the opposite does not hold
- A valid formula is consistent but the opposite does not hold

Example

The formula $G \triangleq P \vee Q$ is invalid (e.g., it is false when P and Q are false) but is not inconsistent because it is true in all other cases. Moreover, G is consistent (e.g., it is true whenever P or Q are false) but is not valid because it is false when both P and Q are false.

AX is set of axioms which describe some “world”

Consider a formula F: we are interested in the following questions:

1) AX |= F ? I.e., is F a logical consequence of AX?

2) | = F ? I.e., is F valid?

3) Is F satisfiable? Means: there exists an I such that

I|= F ?

For example, F is a formula describing a fault in an electric circuit involving the variables X1, X2, …. Xn, and knowing whether it is satisfiable in importnat in order to know whether there is a fault.

Knowing whether we can answer yes or no to previous questions is important. In other terms, it is important to know whether these problems are decidable.

Decidability

Property

Propositional Logic is decidable: there is a terminating method to decide whether a formula is ***valid****.*

- To decide whether a formula is valid:
  1. we can enumerate all possible interpretations
  2. for each interpretation evaluate the formula
- Number of interpretations for a formula are finite (2^n)
- Decidability is a very strong and desirable property for a Logical System
- Trade off between representational power and decidability

Decidability

The fastest known algorithms for deciding propositional satisfiability are based on the Davis-Putnam Algorithm.

A *unit clause* is a clause that consists of a single literal.

function Satisfiable (clause list S) **returns** boolean;
    /* unit propagation */
    repeat
        for each unit clause $L \in S$ **do**
            delete from S every clause containing L
            delete $\neg L$ from every clause of S in which it occurs
        end for
        if S is empty **then return** TRUE
        else if null clause is in S then **return** FALSE **end if**
    until no further changes result **end repeat**
    /* splitting */
    choose a literal L occurring in S
    if Satisfiable ($S \cup \{L\}$) **then return** TRUE
    else if Satisfiable ($S \cup \{\neg L\}$ **then return** TRUE
    else return FALSE **end if**
end function

Logical equivalence

Definition

Logical Equivalence: Two formulas F and G are logically equivalent $F \equiv G$ iff the truth values of F and G are the same under every interpretation of F and G.

Useful equivalence rules

$$
\begin{array}{rcll}
(P \wedge Q) & \equiv & (Q \wedge P) & \text{commutativity of } \wedge \\
(P \vee Q) & \equiv & (Q \vee P) & \text{commutativity of } \vee \\
((P \wedge Q) \wedge R) & \equiv & (P \wedge (Q \wedge R)) & \text{associativity of } \wedge \\
((P \vee Q) \vee R) & \equiv & (P \vee (Q \vee R)) & \text{associativity of } \vee \\
\neg(\neg P) & \equiv & P & \text{double-negation elimination} \\
(P \to Q) & \equiv & (\neg Q \to \neg P) & \text{contraposition} \\
(P \to Q) & \equiv & (\neg P \vee Q) & \text{implication elimination} \\
(P \leftrightarrow Q) & \equiv & ((P \to Q) \wedge (Q \to P)) & \text{biconditional elimination} \\
\neg(P \wedge Q) & \equiv & (\neg P \vee \neg Q) & \text{de Morgan} \\
\neg(P \vee Q) & \equiv & (\neg P \wedge \neg Q) & \text{de Morgan} \\
(P \wedge (Q \vee R)) & \equiv & ((P \wedge Q) \vee (P \wedge R)) & \text{distributivity of } \wedge \text{ over } \vee \\
(P \vee (Q \wedge R)) & \equiv & ((P \vee Q) \wedge (P \vee R)) & \text{distributivity of } \vee \text{ over } \wedge
\end{array}
$$

Normal forms

Standard ways of writing formulas
Two main normal forms:

- Conjunctive Normal Form (CNF)
- Disjunctive Normal Form (DNF)

Definition

Literal: a literal is an atom or the negation of an atom

Definition

Negation Normal Form: A formula is in Negation Normal Form (NNF) iff negations appears only in front of atoms

CNF

Definition

Conjunctive Normal Form: A formula F is in Conjunctive Normal Form (CNF) iff it is in Negation Normal Form and it has the form $F \triangleq F_1 \wedge F_2 \wedge \cdots \wedge F_n$, where each F_i is a disjunction of literals.

- If F is in CNF Each F_i is called a **clause**
- CNF is also refered to as Clausal Form

Example

The formula $G \triangleq (\neg P \vee Q) \wedge (\neg P \vee R)$ is in CNF. We can write G as a set of clauses $\{C_1, C_2\}$ where $C_1 = \neg P \vee Q$ and $C_2 = \neg P \vee R$.
The formula $G \triangleq \neg(P \vee Q) \wedge (\neg P \vee R)$ is not in CNF because negation appears in front of a formula and not only in front of atoms.

DNF

Definition

Disjunctive Normal Form: A formula F is in Disjunctive Normal Form (DNF) iff it is in Negation Normal Form and it has the form $F \triangleq F_1 \vee F_2 \vee \cdots \vee F_n$, where each F_i is a conjunction of literals.

Example

The formula $G \triangleq (\neg P \wedge R) \vee (Q \wedge \neg P) \vee (Q \wedge P)$ is in DNF.

Any formula can be transformed into a normal form by using the equivalence rules given above.

A clause is a formula L1 $\vee$ ..... $\vee$ Ln

where each Li is a literal

Example of tranformation in CNF

(P-> Q) $\wedge$ (R $\vee$ (B $\wedge$ A))

(logically) equivalent to

(neg P $\vee$ Q) $\wedge$ (R $\vee$ (B $\wedge$ A))

equivalent to

(neg P $\vee$ Q) $\wedge$ ((R $\vee$ B) $\wedge$ (R $\vee$ A))

equivalent to

(neg P $\vee$ Q) $\wedge$ (R $\vee$ B) $\wedge$ (R $\vee$ A)

Other example

$(\text{neg } P \lor Q) \land (R \lor (A \to \text{not } A)) \land (R \lor (A \lor \text{not } A))$

equivalent to

$(\text{neg } P \lor Q) \land (R \lor \text{False}) \land (R \lor \text{true})$

equivalent to

$(\text{neg } P \lor Q) \land (R \lor \text{False}) \land (\text{true})$

equivalent to

$(\text{neg } P \lor Q) \land (R \lor \text{False})$

equivalent to

$(\text{neg } P \lor Q) \land (R)$

Example of transformation

Example (Formula transformations)

Prove that the following logical equivalences hold by transforming formulas:
$P \vee Q \wedge \neg(P \wedge Q) \leftrightarrow (P \vee Q) \wedge (\neg P \vee \neg Q) \leftrightarrow (\neg P \wedge Q) \vee (P \wedge \neg Q)$

Sol.

Given $P \vee Q \wedge \neg(P \wedge Q)$ apply de Morgan's law on the second part and directly obtain $(P \vee Q) \wedge (\neg P \vee \neg Q)$
For more examples see Examples 2.8, 2.9 [Chang and Lee Ch. 2]
Try to prove the other equivalence

Logical consequence

Definition

Given a set of formulas $\{F_1, \cdots, F_n\}$ and a formula G, G is said to be a logical consequence of $F_1, \cdots, F_n$ iff for any interpretation I in which $F_1 \wedge \cdots \wedge F_n$ is true G is also true.

- If G is a logical consequence of $\{F_1, \cdots, F_n\}$ we write $F_1 \wedge \cdots \wedge F_n \models G$.
- $F_1, \cdots, F_n$ are called axioms or premises for G.
- $F \equiv Q$ iff $F \models Q$ and $Q \models F$

Example

$S \to C, C \to F, S$ are premises for F

Deduction theorem

Theorem

Given a set of formulas $\{F_1, \cdots, F_n\}$ *and a formula* G*,* $(F_1 \wedge \cdots \wedge F_n) \models G$ *if and only if* $\models (F_1 \wedge \cdots \wedge F_n) \rightarrow G$*.*

Sketch of proof.

- $\Rightarrow$ For each interpretation I in which $F_1 \wedge \cdots \wedge F_n$ is true G is true, $I \models (F_1 \wedge \cdots \wedge F_n) \rightarrow G$, however for every interpretation I' in which $F_1 \wedge \cdots \wedge F_n$ is false then $(F_1 \wedge \cdots \wedge F_n \rightarrow G)$ is true, thus $I' \models (F_1 \wedge \cdots \wedge F_n) \rightarrow G$. Therefore, $\models (F_1 \wedge \cdots \wedge F_n) \rightarrow G$.
- $\Leftarrow$ for every interpretation we have that when $F_1 \wedge \cdots \wedge F_n$ is true G is true therefore $(F_1 \wedge \cdots \wedge F_n) \models G$.

Proof by refutation

Theorem

Given a set of formulas $\{F_1, \cdots, F_n\}$ and a formula G, $(F_1 \wedge \cdots \wedge F_n) \models G$ if and only if $F_1 \wedge \cdots \wedge F_n \wedge \neg G$ is inconsistent.

Sketch of proof.

$(F_1 \wedge \cdots \wedge F_n) \models G$ holds iff for every interpretation under which $F_1 \wedge \cdots \wedge F_n$ is true also G is true. This holds iff there is no interpretation for which $F_1 \wedge \cdots \wedge F_n$ is true and G is false, but this happens precisely when $F_1 \wedge \cdots \wedge F_n \wedge \neg G$ is false for every interpretation, i.e. when $F_1 \wedge \cdots \wedge F_n \wedge \neg G$ is inconsistent. □

Discussion

Previous theorems show that:

- We can prove logical consequence by proving validity of a formula
- We can prove logical consequence by refuting a given formula, i.e. by proving a given formula is inconsistent

Notice that we did not use any specific property of propositional logic

Example

Example

We want to show that $(P \to Q) \wedge P \models Q$

Using definition

We show that for each interpretation in which $(P \to Q) \wedge P$ is true, also Q is true. We can do that by writing the truth table of the formulas.

Example

Using deduction theorem

We know from the deduction theorem that $(P \to Q) \land P \models Q$ iff $\models ((P \to Q) \land P) \to Q$. Therefore we need to show that $((P \to Q) \land P) \to Q$ is valid, we can do that by writing the truth table of the formula and verifying that the formula is evaluated true for all its possible interpretation.

Using Refutation

We know that $(P \to Q) \land P \models Q$ iff $(P \to Q) \land P \land \neg Q$ is inconsistent. Therefore we need to show that $(P \to Q) \land P \land \neg Q$ is inconsistent, we can do that by writing the truth table of the formula and verifying that the formula is evaluated false for all its possible interpretation.

Exercise

Exercise

- Consider the following formulas: $F_1 \triangleq (P \rightarrow Q)$, $F_2 \triangleq \neg Q$, $G \triangleq \neg P$. Show that $F_1 \wedge F_2 \models G$ using all three approaches [Chang-Lee example 2.11]
- Given that if the congress refuses to enact new laws, then the strike will not be over unless it lasts for more than a year or the president of the firm resigns, will the strike be over if the congress refuses to act and the strike just started ? [Chang-Lee example 2.12]

Exercise

Which of the following arguments are valid?

1. If I am wealthy, then I am happy. I am happy. Therefore, I am wealthy.
2. If John drinks beer, he is at least 18 years old. John does not drink beer. Therefore, John is not yet 18 years old.
3. If girls are blonde, they are popular with boys. Ugly girls are unpopular with boys. Intellectual girls are ugly. Therefore, blonde girls are not intellectual.
4. If I study, then I will not fail basket weaving 101. If I do not play cards to often, then I will study. I failed basket weaving 101. Therefore, I played cards too often.

Exercise

The following example is due to Lewis Carroll. Prove that it is a valid argument.

1. All the dated letters in this room are written on blue paper.
2. None of them are in black ink, except those that are written in the third person.
3. I have not filed any of those that I can read.
4. None of those that are written on one sheet are undated.
5. All of those that are not crossed out are in black ink.
6. All of those that are written by Brown begin with “Dear Sir.”
7. All of those that are written on blue paper are filed.
8. None of those that are written on more than one sheet are crossed out.
9. None of those that begin with “Dear sir” are written in the third person.

Therefore, I cannot read any of Brown’s letters.

Exercise (ctnd)

Let

- p be "the letter is dated,"
- q be "the letter is written on blue paper,"
- r be "the letter is written in black ink,"
- s be "the letter is written in the third person,"
- t be "the letter is filed,"
- u be "I can read the letter,"
- v be "the letter is written on one sheet,"
- w be "the letter is crossed out,"
- x be "the letter is written by Brown,"
- y be "the letter begins with 'Dear Sir' "

Exercise (ctnd)

Now, we can write the argument in propositional logic.

1. $p \to q$
2. $\neg s \to \neg r$
3. $u \to \neg t$
4. $v \to p$
5. $\neg w \to r$
6. $x \to y$
7. $q \to t$
8. $\neg v \to \neg w$
9. $y \to \neg s$

Therefore $x \to \neg u$

Semantics II for propositional logic: Proof theory (natural deduction)

Proof theory

- As mentioned earlier in the course we are interested in logical consequences, since these allow us to derive conclusions about truth in specific worlds described ny specific axioms.
- We have seen a way to prove logical consequences based on the (semantic) notion of model, namely by using truth tables
- Is there a way to prove logical consequences by using syntax, i.e. by syntactic manipulations of the symbols which appear in our formulas? This would allow to use computation, as we know it, to perform (automatically) deduction (or inference).
- The answer is yeas and it is provided by proof theory.

Proof theory

- There are many systems, for many different logic, consisting of set of rules which allow to deduce, or derive, conclusions from premises.

- Of course we would like that our rules are correct w.r.t. the semantics based on truth tables … more on this later

- We will see one specific case: natural deduction for propositional logic

Natural deduction (for PL)

- A set of rules which allow to deduce, or derive, conclusions from premises.

- Rules can introduce or eliminate connectives

- Rules can be composed to form derivations

- A sound and complete system (see later)

Natural deduction

- We consider only the connectives $\wedge, \rightarrow$ and $\perp$.
- The reason is that V has a quite different meaning in constructive and non constructive approaches.
- This is not a limitation, since the above set of connectives is functionally complete.

Natural deduction

INTRODUCTION RULES   ELIMINATION RULES

$$(\wedge I) \quad \frac{\varphi \quad \psi}{\varphi \wedge \psi} \wedge I \qquad\qquad (\wedge E) \quad \frac{\varphi \wedge \psi}{\varphi} \wedge E \quad \frac{\varphi \wedge \psi}{\psi} \wedge E$$

$$(\rightarrow I) \quad \begin{array}{c} [\varphi] \\ \vdots \\ \psi \\ \hline \varphi \rightarrow \psi \end{array} \rightarrow I \qquad\qquad (\rightarrow E) \quad \frac{\varphi \quad \varphi \rightarrow \psi}{\psi} \rightarrow E$$

Note the discard of the hypothesis indicated with [ ]

Natural deduction

Ex falso sequitur quodlibet

$$(\bot)\ \frac{\bot}{\varphi}\,\bot$$

Reduction Ad Absurdum

$$(\mathrm{RAA})\ \begin{array}{c}[\neg\varphi] \\ \vdots \\ \dfrac{\bot}{\varphi}\,\mathrm{RAA}\end{array}$$

Some examples of derivations (proofs)

I

$$\dfrac{\dfrac{[\varphi \wedge \psi]^1}{\psi}\wedge E \qquad \dfrac{[\varphi \wedge \psi]^1}{\varphi}\wedge E}{\dfrac{\psi \wedge \varphi}{\varphi \wedge \psi \rightarrow \psi \wedge \varphi}\rightarrow I_1}\wedge I$$

Some examples

II

$$\dfrac{\dfrac{\dfrac{[\varphi]^2 \quad [\varphi \to \bot]^1}{\bot} \to E}{(\varphi \to \bot) \to \bot} \to I_1}{\varphi \to ((\varphi \to \bot) \to \bot)} \to I_2$$

Some examples

III

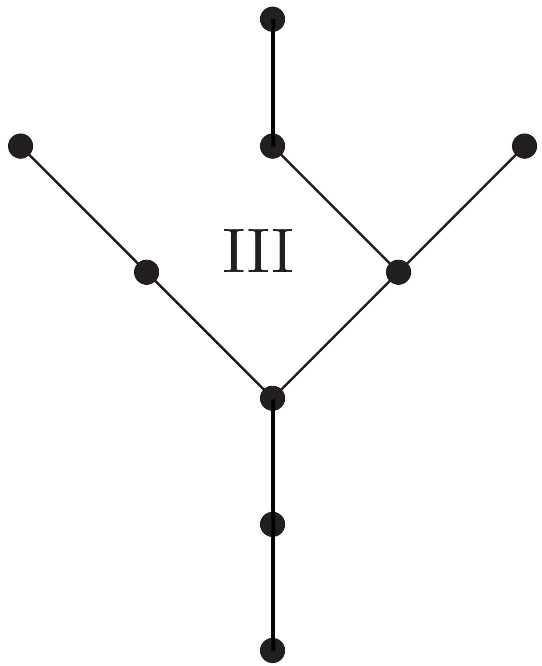
$$\dfrac{\dfrac{\dfrac{\dfrac{[\varphi \wedge \psi]^1}{\psi}\wedge E \qquad \dfrac{\dfrac{[\varphi \wedge \psi]^1}{\varphi}\wedge E \qquad [\varphi \to (\psi \to \sigma)]^2}{\psi \to \sigma}\to E}{\sigma}\to E}{\varphi \wedge \psi \to \sigma}\to I_1}{(\varphi \to (\psi \to \sigma)) \to (\varphi \wedge \psi \to \sigma)}\to I_2$$

Some examples

$$\mathbf{II'} \qquad \cfrac{\cfrac{\cfrac{[\varphi]^2 \quad [\neg\varphi]^1}{\bot}\rightarrow E}{\neg\neg\varphi}\rightarrow I_1}{\varphi \rightarrow \neg\neg\varphi}\rightarrow I_2$$

Derivations are trees

Each derivation is a tree: the leafs contain the premises while the root is the conclusion.



Previous example III

Derivations

Definition 1.4.1 *The set of derivations is the smallest set X such that*

(1) The one element tree φ belongs to X for all $\varphi \in PROP$.

(2$\wedge$) If $\begin{array}{c}\mathcal{D}\\ \varphi\end{array}, \begin{array}{c}\mathcal{D}'\\ \varphi'\end{array} \in X$, *then* $\dfrac{\begin{array}{cc}\mathcal{D} & \mathcal{D}'\\ \varphi & \varphi'\end{array}}{\varphi \wedge \varphi'} \in X$.

If $\begin{array}{c}\mathcal{D}\\ \varphi \wedge \psi\end{array} \in X$, *then* $\dfrac{\begin{array}{c}\mathcal{D}\\ \varphi \wedge \psi\end{array}}{\varphi}, \dfrac{\begin{array}{c}\mathcal{D}\\ \varphi \wedge \psi\end{array}}{\psi} \in X$.

Definition of Derivations (ctnd)

$$(2{\rightarrow})\ \mathit{If}\ \begin{array}{c}\varphi\\ \mathcal{D}\\ \psi\end{array} \in X,\ \mathit{then}\ \begin{array}{c}[\varphi]\\ \mathcal{D}\\ \psi\\ \hline \varphi \rightarrow \psi\end{array} \in X.$$

$$\mathit{If}\ \begin{array}{c}\mathcal{D}\\ \varphi\end{array},\ \begin{array}{c}\mathcal{D}'\\ \varphi \rightarrow \psi\end{array} \in X,\ \mathit{then}\ \begin{array}{cc}\mathcal{D} & \mathcal{D}'\\ \varphi & \varphi \rightarrow \psi\\ \hline \multicolumn{2}{c}{\psi}\end{array} \in X.$$

Definition of Derivations (ctnd)

$$(2\bot)\ \textit{If}\ \begin{array}{c}\mathcal{D}\\ \bot\end{array} \in X,\ \textit{then}\ \dfrac{\begin{array}{c}\mathcal{D}\\ \bot\end{array}}{\varphi} \in X.$$

$$\textit{If}\ \begin{array}{c}\neg\varphi\\ \mathcal{D}\\ \bot\end{array} \in X,\ \textit{then}\ \dfrac{\begin{array}{c}[\neg\varphi]\\ \mathcal{D}\\ \bot\end{array}}{\varphi} \in X.$$

Derivation and theorems

We write

$$\Gamma \vdash \varphi$$

If there exists a derivation with$^{\emptyset}$ (uncancelled) hypothesis (premises) Γ and conclusion φ

When $\Gamma = \emptyset$ we say that φ is a theorem

Completeness theorem

One can problem that the notion of derivability and the notion of truth coincide, that is the following holds:

$$\Gamma \vdash \varphi \Leftrightarrow \Gamma \models \varphi.$$

Corollary: the set of theorems coincide with the set of tautologies

Soundness

$$\textbf{(Soundness)} \quad \Gamma \vdash \varphi \Rightarrow \Gamma \models \varphi.$$

Proof: Induction on the derivations (see book)

Completeness

$$\Gamma \models \varphi \Rightarrow \Gamma \vdash \varphi$$

Proof: One proves that $\Gamma \nvdash \varphi \Rightarrow \Gamma \not\models \varphi$ by showing that:

$\Gamma \nvdash \varphi$ iff

$\Gamma \cup \{\neg\varphi\}$ is consistent iff

there is a valuation v such that $[\![\psi]\!] = 1$

for all $\psi \in \Gamma \cup \{\neg\varphi\}$

Completeness

$$\Gamma \models \varphi \Rightarrow \Gamma \vdash \varphi$$

Proof: One proves that $\Gamma \nvdash \varphi \Rightarrow \Gamma \not\models \varphi$ by showing that:

$\Gamma \nvdash \varphi$ iff

$\Gamma \cup \{\neg\varphi\}$ is consistent iff

there is a valuation v such that $[\![\psi]\!] = 1$

for all $\psi \in \Gamma \cup \{\neg\varphi\}$.

The proof uses next lemma (and some other things, see book).

A useful Lemma

Definition .

Γ is consistent if $\Gamma \nvdash \bot$

Γ is inconsistent if $\Gamma \vdash \bot$

Lemma 1.5.5 (a) $\Gamma \cup \{\neg\varphi\}$ *is inconsistent* $\Rightarrow \Gamma \vdash \varphi$,
(b) $\Gamma \cup \{\varphi\}$ *is inconsistent* $\Rightarrow \Gamma \vdash \neg\varphi$.

Or, if $\Gamma \nvdash \varphi$ then $\Gamma \cup \{\neg\varphi\}$ is consistent