1. Satisfiability Modulo Theories

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From SAT to SMT

- Boolean satisfiability problem (SAT) well-studied
  - 1st problem proven NP-complete
- Several applications in AI and other fields
  - *The Silent (R)evolution of SAT*
    https://dl.acm.org/doi/10.1145/3560469
- Based on Propositional Logic
  - Atomic proposition or atoms (facts) can be either true or false
  - Atoms combined via connectives: $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$
  - Decidable, "efficient", but limited expressiveness
- Real-world applications often require a more expressive logic, e.g., First-Order Logic (FOL)

From SAT to SMT

- FOL formulas are *more general*, e.g., $(\forall x)(x < u \land (\nexists y)\ f(x, y) = u)$
- Many applications do not require "general-purpose" FOL satisfiability: only satisfiability w.r.t. a *fixed* background theory
- E.g., given formula $y = x + 1 \land x < z \land z < y$ one typically checks satisfiability w.r.t. an arithmetical theory, unless otherwise specified...
  - But one might interpret 1 as '1', $+$ as concatenation, $<$ as lex...
  - ...What arithmetic theory? Integers or reals?
- Focusing on a particular theory enables a more efficient solving via specialized decision procedures
  - Especially for quantifier-free formulas

From SAT to SMT

- A formula ϕ can be undecidable! We cannot prove neither ϕ nor $\neg\phi$
  - Undecidable $\neq$ unsatisfiable
- But we can restrict to decidable fragments of an undecidable theory
  - "subsets of the theory"
- **Satisfiability Modulo Theory** (SMT) concerns the study of the satisfiability of formulas w.r.t. some backgrounds theories
  - theory of integer arithmetic, real numbers, arrays, strings, (multi-)sets, trees, lists, bit-vectors, ...
- SMT solvers are used to determine the satisfiability of formulas through different procedures over different background theories
  - E.g., Z3 or CVC5 (formerly CVC4, CVC3, ...)

SMT history

- The roots of SMT date back to late 70s - early 80s
  - Nelson and Oppen, Shostak, Boyer and Moore, ...
- Modern SMT research started in the 90s
- From the 2000s up to now big development in SMT's foundational and practical aspects
  - SMT approaches integrated in various tools for theorem proving, program analysis and testing
  - Following big development in SAT solvers

SMT vs SAT/CP

- SMT extends SAT, and tackles combinatorial problems from an orthogonal perspective w.r.t. CP

- Like SAT solving, SMT historically specialised in theorem proving, software analysis and verification, (dynamic) symbolic execution
- CP more oriented to scheduling, resource allocation, optimization

- Like CP and unlike SAT, SMT employs domain-specific reasoning
- Unlike CP, SMT doesn't generally require finite domains

- SMT solving uses SAT abstractions instead of propagators and natively handles nogoods, which is not always true for CP solvers

Example: Dynamic Symbolic Execution of JavaScript

```
var x = symStr();    ▶ x is a symbolic variable having String type
var y = "length";
if (x[y] >= 2)
  console.log("PC1")
else if (y[x] === "g")
  console.log("PC2")
else
  console.log("PC3")
```

- Dynamic Symbolic Execution (DSE) runs a program with concrete inputs while tracking symbolic expressions (a.k.a. concolic execution)
  - DSE enables automated test generation and bug detection

- Consider the DSE of above JavaScript snippet starting with $\{x \leftarrow \texttt{""}\}$

Example: DSE of JavaScript

```
var x = "";
var y = "length";
if (""[y] >= 2)
  console.log("PC1")
else if (y[""] === "g")
  console.log("PC2")
else
  console.log("PC3")
```

- Property "length" of "" is 0

Example: DSE of JavaScript

```
var x = "";
var y = "length";
if (""[y] >= 2)
  console.log("PC1")
else if (y[""] === "g")
  console.log("PC2")
else
  console.log("PC3")
```

- Property `""` of `"length"` object is `undefined`

Example: DSE of JavaScript

```
1 var x = "";
2 var y = "length";
3 if (""[y] >= 2)
4   console.log("PC1")
5 else if (y[""] === "g")
6   console.log("PC2")
7 else
8   console.log("PC3")
```

▶ $PC = \{\neg(|x| \geq 2), \neg(\texttt{"length"}[x] = \texttt{"g"})\}$

- With input $x \leftarrow$ `""` we reached line 8 with associated path condition $PC = \{\neg(|x| \geq 2), \neg(\texttt{"length"}[x] = \texttt{"g"})\}$
- We negate one of the constraint of PC (e.g., $\neg(|x| \geq 2)$) and we solve $PC' = \{|x| \geq 2, \neg(\texttt{"length"}[x] = \texttt{"g"})\}$
  - A feasible solution is $x \leftarrow$ `"aa"`
  - We need a suitable string solver: PC' has (*reified*) constraints over string length, (in-)equality, indexing, ...

Example: DSE of JavaScript

```
1  var x = symVar();
2  var y = "length";
3  if (x[y] >= 2)          ▶ PC′ = {|x| ≥ 2, ¬("length"[x] = "g")}
4    console.log("PC1")
5  else if (y[x] === "g")
6    console.log("PC2")    ▶ PC″ = {¬(|x| ≥ 2), "length"[x] = "g"}
7  else
8    console.log("PC3")    ▶ PC = {¬(|x| ≥ 2), ¬("length"[x] = "g")}
```

- With $x \leftarrow$ `"aa"` we reach line 4 with associated PC'; we then iterate the procedure to solve $PC'' = \{\neg(|x| \geq 2), \texttt{"length"}[x] = \texttt{"g"}\}$
  - A feasible solution is $x \leftarrow$ `"3"`, because `"length"["3"] === "g"`
  - PC'' hard to solve if we don't know that $x \in \{\texttt{"0"}, \texttt{"1"}, \dots\}$
- With $x \leftarrow$ `"3"` we reach line 6: the set of inputs $\{\{x \leftarrow \texttt{""}\}, \{x \leftarrow \texttt{"aa"}\}, \{x \leftarrow \texttt{"3"}\}\}$ covers all the lines

Example: DSE of JavaScript

```
var x = symVar();
var y = "length";
if (x[y] >= 2)    ▶ PC′ = {|x| ≥ 2, ¬("length"[x] = "g")}
  console.log("PC1")
else if (y[x] === "g")
  console.log("PC2")    ▶ PC″ = {¬(|x| ≥ 2), "length"[x] = "g"}
else
  console.log("PC3")    ▶ PC = {¬(|x| ≥ 2), ¬("length"[x] = "g")}
```

- Example: ArathaJS
  - `https://github.com/ArathaJS/aratha`
  - `./run-analysis demo.js`
  - No longer developed / maintained :-(

SMT and Program Analysis

- Faithfully modelling JS, and in general programming languages semantics, is hard but not strictly necessary
- Difficult constructs and PCs can be ignored or approximated
  - This affects correctness (we might not achieve maximum coverage)
  - Acceptable if *"good enough"* coverage is reached in short time
- SMT solvers mostly used for software analysis: stronger reasoning capabilities and expressiveness w.r.t SAT, CP or MIP solvers
  - Achieved by leveraging FOL over multiple background theories

SMT Preliminaries

Formal preliminaries

- Let's recall (introduce?) the relevant FOL concepts and notation
  - We shall always assume FOL with equality
- The signature of a FOL is the set $\Sigma = \Sigma^F \cup \Sigma^P$ of its non-logical symbols, i.e., functions in Σ^F and predicates in Σ^P
  - We denote Σ_k^F (resp. Σ_k^P) the functions (predicates) with arity $k \geq 0$
  - So, $\Sigma^P = \bigcup_k \Sigma_k^P$ and $\Sigma^F = \bigcup_k \Sigma_k^F$
- The 0-arity functions of Σ_0^F are constant symbols
- The 0-arity predicates of Σ_0^P are propositional symbols
  - E.g., propositional logic has $\Sigma^F = \emptyset$ and $\Sigma^P = \Sigma_0^P \supseteq \{\bot, \top\}$
- We will consider only quantifier-free fragments (no $\exists, \forall$)
  - All variables are free variables

Terms and Formulas

- The set $\mathbb{T}^{\Sigma}$ of terms Σ is defined as:
  - $c \in \Sigma_0^F \implies c \in \mathbb{T}^{\Sigma}$
  - $f \in \Sigma_k^F$ and $t_1, \ldots, t_k \in \mathbb{T}^{\Sigma} \implies f(t_1, \ldots, t_k) \in \mathbb{T}^{\Sigma}$
  - $\varphi \in \mathbb{F}^{\Sigma}$ and $t_1, t_2 \in \mathbb{T}^{\Sigma} \implies ite(\varphi, t_1, t_2) \in \mathbb{T}^{\Sigma}$

- The set $\mathbb{F}^{\Sigma}$ of formulas of Σ is defined as:

$$\left.\begin{array}{l}
\bullet\ \bot, \top \in \mathbb{F}^{\Sigma} \\
\bullet\ t_1, t_2 \in \mathbb{T}^{\Sigma} \implies t_1 = t_2 \in \mathbb{F}^{\Sigma} \\
\bullet\ A \in \Sigma_0^P \implies A \in \mathbb{F}^{\Sigma} \\
\bullet\ p \in \Sigma_k^P \text{ and } t_1, \ldots, t_k \in \mathbb{T}^{\Sigma} \implies p(t_1, \ldots, t_k) \in \mathbb{F}^{\Sigma}
\end{array}\right\} \text{Atomic formulas}$$

  - $\varphi \in \mathbb{F}^{\Sigma} \implies \neg\varphi \in \mathbb{F}^{\Sigma}$
  - $\varphi_1, \varphi_2 \in \mathbb{F}^{\Sigma} \implies \varphi_1 \to \varphi_2,\ \varphi_1 \leftrightarrow \varphi_2,\ \varphi_1 \wedge \varphi_2,\ \varphi_1 \vee \varphi_2 \in \mathbb{F}^{\Sigma}$

Formal preliminaries

- An atomic formula is also called an atom
- A literal is either:
  - an atomic formula (positive literal), or
  - the negation of one (negative literal)
- A clause is a disjunction $\ell_1 \vee \cdots \vee \ell_k$ of literals
  - A unit clause is a clause consisting of a single literal $\neq \bot, \top$
- A formula is in Conjunctive Normal Form (CNF) if it is the conjunction $c_1 \wedge \cdots \wedge c_k$ of $k \geq 0$ clauses
  - Also denoted $\{c_1, \ldots, c_k\}$ or simply $c_1, \ldots, c_k$

Semantics

- The semantics of a formula denotes its *"meaning"*, i.e., a truth value in $\{true, false\}$, by means of a certain interpretation

- A model for Σ is a pair $\mathcal{M} = \langle M, (\cdot)^{\mathcal{M}} \rangle$ where set M is the universe of $\mathcal{M}$ and a mapping $(\cdot)^{\mathcal{M}}$ such that:
  - $f^{\mathcal{M}} \in \{\varphi \mid \varphi : M^k \to M\}$ for each function $f \in \Sigma_k^F$
    - In particular $c^{\mathcal{M}} \in M$ for each constant $c \in \Sigma_0^F$
  - $p^{\mathcal{M}} \in \{\varphi \mid \varphi : M^k \to \{true, false\}\}$ for each predicate $f \in \Sigma_k^P$
    - In particular $B^{\mathcal{M}} \in \{true, false\}$ for each proposition $B \in \Sigma_0^P$

- The $(\cdot)^{\mathcal{M}}$ extension to terms and formulas is called interpretation:
  - $\perp^{\mathcal{M}} = false$, $\top^{\mathcal{M}} = true$, $(t_1 = t_2)^{\mathcal{M}} = true \iff t_1^{\mathcal{M}} = t_2^{\mathcal{M}}$
  - $f(t_1, \dots, t_k)^{\mathcal{M}} = f^{\mathcal{M}}(t_1^{\mathcal{M}}, \dots, t_k^{\mathcal{M}})$
  - $p(t_1, \dots, t_k)^{\mathcal{M}} = p^{M}(t_1^{\mathcal{M}}, \dots, t_k^{\mathcal{M}})$
  - $ite(\varphi, t_1, t_2)^{\mathcal{M}} = \begin{cases} t_1^{\mathcal{M}} & \text{if } \varphi^{\mathcal{M}} = true \\ t_2^{\mathcal{M}} & \text{if } \varphi^{\mathcal{M}} = false \end{cases}$

Satisfiability

- $\mathcal{M}$ satisfies (resp. falsifies) $\varphi \in \mathbb{F}^\Sigma$ if $\varphi^\mathcal{M} = \mathit{true}$ (resp. $\varphi^\mathcal{M} = \mathit{false}$)
- A Σ-theory is a (*possibly infinite*) set $\mathcal{T}$ of Σ-models
- $\varphi \in \mathbb{F}^\Sigma$ is $\mathcal{T}$-satisfiable if there exists a model $\mathcal{M} \in \mathcal{T}$ satisfying φ
  - $\{\varphi_1, \ldots, \varphi_k\} \subseteq \mathbb{F}^\Sigma$ is $\mathcal{T}$-consistent iff $\varphi_1 \wedge \cdots \wedge \varphi_k$ is $\mathcal{T}$-satisfiable
- $\Gamma \subseteq \mathbb{F}^\Sigma$ $\mathcal{T}$-entails φ iff every $\mathcal{M} \in \mathcal{T}$ that satisfies Γ also satisfies φ
  - If Γ $\mathcal{T}$-entails φ, we write $\Gamma \models_\mathcal{T} \varphi$
  - Γ is $\mathcal{T}$-consistent iff $\Gamma \not\models_\mathcal{T} \bot$
- $\varphi \in \mathbb{F}^\Sigma$ is $\mathcal{T}$-valid iff $\emptyset \models_\mathcal{T} \varphi$, i.e., every $\mathcal{M} \in \mathcal{T}$ satisfies φ
  - A $\mathcal{T}$-valid clause $c = \ell_1 \vee \cdots \vee \ell_k$ is called theory lemma
  - φ is $\mathcal{T}$-consistent $\iff$ $\neg\varphi$ is not $\mathcal{T}$-valid

Example

- Suppose Σ defined by $\Sigma_0^F = \{a, b, c, d\}, \Sigma_2^F = \{f, g\}, \Sigma_1^P = \{p\}$
- Let $\mathcal{M}_1, \mathcal{M}_2$ be 2 models having universe $\mathcal{P}(\mathbb{Z})$ and such that:
  - $a^{\mathcal{M}_1} = \emptyset, b^{\mathcal{M}_1} = \{2x \mid x \in \mathbb{Z}\}, c^{\mathcal{M}_1} = \{2x+1 \mid x \in \mathbb{Z}\}, d^{\mathcal{M}_1} = \mathbb{Z}$
  - $a^{\mathcal{M}_2} = \{0\}, b^{\mathcal{M}_2} = \{x \in \mathbb{Z} \mid x > 0\}, c^{\mathcal{M}_2} = \{x \in \mathbb{Z} \mid x < 0\}, d^{\mathcal{M}_2} = \mathbb{Z}$
  - $f^{\mathcal{M}_1} = f^{\mathcal{M}_2} = \cup, \quad g^{\mathcal{M}_1} = g^{\mathcal{M}_2} = \cap, \quad p^{\mathcal{M}_1}(X) = p^{\mathcal{M}_2}(X) \Leftrightarrow X = \emptyset$
- Consider theory $\mathcal{T} = \{\mathcal{M}_1, \mathcal{M}_2\}$ and provide example(s) of:
  - A formula $\mathcal{T}$-satisfiable and not atomic
  - A set of ≥ 2 formulas not $\mathcal{T}$-consistent
  - A set of ≥ 2 formulas that $\mathcal{T}$-entails a not $\mathcal{T}$-valid formula
  - A $\mathcal{T}$-lemma of ≥ 3 clauses

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- Consider theory $\mathcal{T} = \{\mathcal{M}_1, \mathcal{M}_2\}$ and provide example(s) of:
  - A formula $\mathcal{T}$-satisfiable and not atomic: $p(a) \vee p(g(b, c))$
  - A set of ≥ 2 formulas not $\mathcal{T}$-consistent: $\{p(a), p(d)\}$
  - A set of ≥ 2 formulas that $\mathcal{T}$-entails a not $\mathcal{T}$-valid formula $\{p(a), \neg p(c)\} \models_{\mathcal{T}} (a = g(b, c))$
  - A $\mathcal{T}$-lemma of ≥ 3 clauses: $p(b) \vee p(c) \vee d = f(f(a, b), c)$

Expansion

- We check the $\mathcal{T}$-satisfiability of formulas with quantifier-free variables
  - In other terms, we find a consistent assignment values $\Rightarrow$ variables
- Because no quantification is involved, variables can be seen as "additional constants" not in Σ_0^F
- More generally, given signature Σ we can consider formulas with uninterpreted symbols, i.e., symbols not in Σ
  - Variable $\equiv$ uninterpreted constant $\equiv$ uninterpreted function
- Given Σ-model $\mathcal{M} = \langle M, (\cdot)^{\mathcal{M}} \rangle$ and $\Sigma' \supseteq \Sigma$, an expansion $\mathcal{M}'$ to Σ' of that model is any Σ'-model $\mathcal{M}' = \langle M', (\cdot)^{\mathcal{M}'} \rangle$ such that:
  - $M' = M$
  - $s^{\mathcal{M}'} = s^{\mathcal{M}}$ for each $s \in \Sigma$

Expansion

- Instead of a Σ-theory $\mathcal{T}$, we (sometimes implicitly) consider the theory $\mathcal{T}' = \{\mathcal{M}' \mid \mathcal{M}' \text{ is an expansion of a } \Sigma\text{-model } \mathcal{M}\ \}$
- The ground $\mathcal{T}$-satisfiability problem is determining, given Σ-theory $\mathcal{T}$, the $\mathcal{T}$-satisfiability of ground formulas over a Σ-expansion $\mathcal{T}'$
    - ground formula $\equiv$ formula with no variables: because uninterpreted constants play the role of variables, our formulas are always ground
- Because φ is $\mathcal{T}$-satisfiable $\iff$ $\neg\varphi$ is not $\mathcal{T}$-valid, the ground $\mathcal{T}$-satisfiability problem has a dual validity problem
    - E.g., $x > 5$ satisfiable iff $x \leq 5$ not valid

Example

- Let's take Σ as $\Sigma_0^F = \{a, b, c, d\}, \Sigma_1^F = \{f, g\}, \Sigma_2^P = \{p\}$ and a Σ-model $\mathcal{M} = \langle [0, 2\pi), (\cdot)^{\mathcal{M}} \rangle$ s.t.
  - $a^{\mathcal{M}} = 0, b^{\mathcal{M}} = \frac{\pi}{2}, c^{\mathcal{M}} = \pi, d^{\mathcal{M}} = \frac{3}{2}\pi,$
  - $f^{\mathcal{M}} = \sin$, $g^{\mathcal{M}} = \cos$, $p^{\mathcal{M}}(x, y) \Leftrightarrow x > y$
- By expanding Σ with uninterpreted constants $x, y, z, \ldots$ we can check the satisfiability of arbitrarily complex ground formulas
  - $p(f(y), g(g(d))) \vee p(a, f(b))$, $g(x) \iff g(c) \wedge f(g(z))$, $\ldots$
- E.g., is $p(g(x), f(d))$ is $\mathcal{M}$-satisfiable?
  - Let $\Sigma' = \Sigma \cup \{x\}$, and expansion $\mathcal{M}'$ of $\mathcal{M}$ s.t. $x^{\mathcal{M}'} = \frac{1}{2}\pi$
  - $p^{\mathcal{M}'}(g(x), f(d)) \equiv g^{\mathcal{M}}(x^{\mathcal{M}'}) > f^{\mathcal{M}}(d^{\mathcal{M}}) \equiv \cos(\frac{1}{2}\pi) > \sin(\frac{3}{2}\pi) \equiv 0 > -1 \equiv$ *true*

Axiomatic definition

- A theory can be defined axiomatically
- A (minimal) set of formulas $\Lambda \subseteq \mathbb{F}^{\Sigma}$ called axioms is given, and the corresponding theory is the set of all models of Λ
  - That is, $\mathcal{T}_{\Lambda} = \{\mathcal{M} \mid \forall \varphi \in \Lambda : \varphi^{M} = \mathit{true}\}$
- E.g., Peano axioms. Given Σ with constant 0 e unary function S:
  - $(\forall x)\ \neg(S(x) = 0)$
  - $(\forall x)(\forall y)\ S(x) = S(y) \rightarrow x = y$
  - $(\varphi(0) \wedge (\forall x)(\varphi(x) \rightarrow \varphi(S(x))) \rightarrow (\forall x)\ \varphi(x) \quad$ for any $\varphi \in \mathbb{F}^{\Sigma}$
- A theory satisfying Peano axioms is called arithmetic theory
- By adding $+$ and $*$, we can prove many arithmetic theorems
  - But not all of them! See Gödel incompleteness theorems

FOL Theories vs SMT Theories

- Naming alert!
- In FOL, a theory $\mathcal{T}$ is defined as a set of formulas that is closed under logical deduction: $\Gamma \models \varphi$ implies that $\varphi \in \mathcal{T}$ for each $\Gamma \subseteq \mathcal{T}$
  - E.g., the theory of arithmetic consists of all formulas derivable from the Peano axioms
- In SMT, a theory is defined as a set of models interpreting a given FOL signature in a domain-specific way.
  - SMT solvers check satisfiability of formulas w.r.t. a given theory $\mathcal{T}$: a formula ϕ is $\mathcal{T}$-satisfiable if there is a model $\mathcal{M} \in \mathcal{T}$ such that $\mathcal{M} \models \varphi$

Many-sorted logic

- Most of SMT applications involve different data types or sorts
- It may be convenient to formalize SMT problems with a many-sorted FOL having:
  - a set of sort symbols $\mathcal{S}$, representing different domains
  - a set of sorted variables uniquely associated with a sort $\sigma \in \mathcal{S}$
  - a sorted signature Σ including a set $\Sigma^{\mathcal{S}} \subseteq \mathcal{S}$ of sort symbols
  - corresponding semantics for variables and sorted signatures...
- ...Let's just assume that sort $\approx$ type without adding new formalism

Some theories of interest

Theories of interest

Let's have an overview of some SMT theories of interest:

- Uninterpreted functions
- Arithmetic
- Arrays
- Bit-vectors
- Strings

EUF Theory

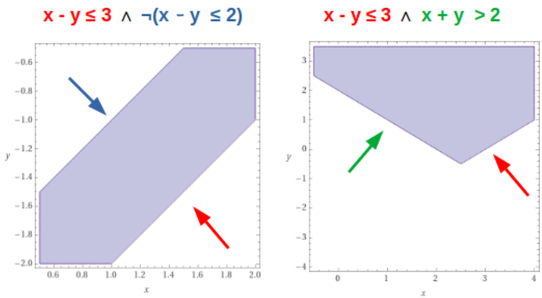
- EUF = Equality with Uninterpreted Functions theory ($\mathcal{T}_{EUF}$)
- No restrictions on how Σ-symbols interpretation, only the congruence closure property enforced: $\mathbf{x} = \mathbf{y} \Rightarrow f(\mathbf{x}) = f(\mathbf{y})$
  - for each $\mathbf{x} = (x_1, \ldots, x_k), \mathbf{y} = (y_1, \ldots, y_k)$ and $f \in \Sigma_k^F$
  - a.k.a. empty theory: its set of axioms is $\emptyset$
    - Why congruence closure cannot be expressed as a FOL axiom?
- Why $\mathcal{T}_{EUF}$? To abstract complex or "black-box" functions
- E.g., consider $a * (f(b) + f(c)) = d \wedge b * (f(a) + f(c)) \neq d \wedge a = b$
  - We don't need any arithmetic theory to prove it unsatisfiable!
- Just abstract $+$ and $*$ with fresh uninterpreted functions g and h: $h(a, g(f(b), f(c))) = d \wedge h(b, g(f(a), f(c))) \neq d \wedge a = b$

Arithmetic Theories

- Theory over numbers are clearly very used and useful
- Let $\Sigma \equiv (0, 1, +, -, \leq)$ and $\mathcal{T}_{\mathcal{Z}}$ interpreting Σ symbols in the usual way. $\mathcal{T}_{\mathcal{Z}}$ is a.k.a. Presburger arithmetic
  - We can define $\mathcal{T}_{\mathcal{R}}$ interpreting Σ symbols over reals
- The satisfiability of ground formulas for $\mathcal{T}_{\mathcal{Z}}$ and $\mathcal{T}_{\mathcal{R}}$ is decidable
  - There exist procedures to decide if a formula is true/false
- Ground satisfiability in $\mathcal{T}_{\mathcal{R}}$ is decidable in polynomial time
  - Simplex method is exponential but works fine in practice
- $\mathcal{T}_{\mathcal{Z}}$-satisfiability is harder: NP-complete in general

Arithmetic Theories

- $\mathcal{T}_{\mathbb{Z}}$ fragments have more efficient decision procedures, e.g.:
- Difference logic: every atom must be $x - y \bowtie k$ with $\bowtie \in \{=, \leq\}$, x, y variables and k integer
- UTPVI ("unit two variable per inequality") every atom $x \pm y \bowtie k$



Arithmetic Theories

- Things are much harder with multiplication:
  - The integer case becomes undecidable (*Peano arithmetic*)
  - The real case becomes doubly-exponential
- Another non-trivial case is the floating-point arithmetic, e.g.
  - For $\mathcal{T}_{\mathcal{Z}}$ and $\mathcal{T}_{\mathcal{R}}$, $(x+y)+z = x+(y+z)$ is valid (*associativity*)
  - For IEEE754 floating points, this is no longer true! E.g. for a sound floating-point model $\mathcal{M}$ s.t. $x^{\mathcal{M}} = 1, y^{\mathcal{M}} = 10^{100}, z^{\mathcal{M}} = -10^{100}$ we have $((x+y)+z)^{\mathcal{M}'} = \cdots = 0 \neq 1 = \cdots = x+(y+z)^{\mathcal{M}'}$
    - *"Catastrophic cancellation"*
  - *Why?*
- For floating points, $+$ and $\cdot$ are still commutative but not necessarily associative nor distributive

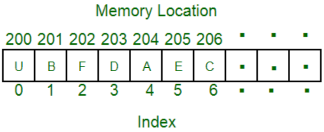
Arithmetic Theories

```python
x = 1
y = 1e100 # 10^100
z = -1e100 # 10^-100
print ("x =", x)
print ("y =", y)
print ("z =", z)
print ("x + y =", x + y)
print ("(x + y) + z =", (x + y) + z)
print ("x + (y + z) =",  x + (y + z))
```

```
x = 1
y = 1e+100
z = -1e+100
x + y = 1e+100
(x + y) + z = 0.0
x + (y + z) = 1.0
```

Theory of Arrays

- Arrays are homogeneous and indexed collections of elements



- Let $\Sigma_{\mathcal{A}}$ be a signature with 2 interpreted functions read and write:
    - $read(a, i)$ returns the value of $a[i]$
    - $write(a, i, v)$ returns the array obtained by replacing $a[i]$ with v

- The theory of array $\mathcal{T}_{\mathcal{A}}$ (with extensionality) is defined by:
    - (i) $(\forall a)(\forall i)(\forall v)\ read(write(a, i, v), i) = v$
    - (ii) $(\forall a)(\forall i)(\forall j)(\forall v)\ i \neq j \rightarrow read(write(a, i, v), j) = read(a, j)$
    - (iii) $(\forall a)(\forall a')\ ((\forall i)\ read(a, i) = read(a', i)) \rightarrow a = a' \qquad (extensionality)$

Theory of Arrays

- Exercise: is the following formula $\mathcal{T}_{\mathcal{A}}$-satisfiable?

$$
\begin{gathered}
write(a, i, x) \neq b \ \wedge \ a = b \ \wedge \ i = j \ \wedge \\
read(b, i) = y \ \wedge \ read(write(b, i, x), j) = y
\end{gathered}
$$

- Hint: remember the axioms:
  - (i) $(\forall a)(\forall i)(\forall v)\ read(write(a, i, v), i) = v$
  - (ii) $(\forall a)(\forall i)(\forall j)(\forall v)\ i \neq j \rightarrow read(write(a, i, v), j) = read(a, j)$
  - (iii) $(\forall a)(\forall a')\ ((\forall i)\ read(a, i) = read(a', i)) \rightarrow a = a'$

Theory of Arrays

- Exercise: is the following formula $\mathcal{T}_{\mathcal{A}}$-satisfiable?

$$
\begin{array}{c}
write(a, i, x) \neq b \ \wedge \ a = b \ \wedge \ i = j \ \wedge \\
read(b, i) = y \ \wedge \ read(write(b, i, x), j) = y
\end{array}
$$

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- Let $a' = write(a, i, x)$. Then, $a' \neq a \xrightarrow{(iii)} (\exists j) \ read(a', j) \neq read(a, j)$. We can prove the unsatisfiability by applying case splitting:
  - $i \neq j \xrightarrow{(ii)} read(a', j) = read(a, j)$ contradicting $read(a', j) \neq read(a, j)$
  - $i = j \longrightarrow read(a', i) \neq read(a, i) \xrightarrow{(i)} x \neq read(a, i)$ contradicting $read(a, i) = x$

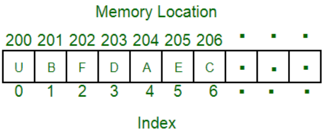
SMT-LIB Encoding

```
; Signature expansion.
(declare-fun a () (Array Int Real))
(declare-fun b () (Array Int Real))
(declare-fun i () Int)
(declare-fun j () Int)
(declare-fun x () Real)
(declare-fun y () Real)
; Formulas. select = read, store = write
(assert (not (= (store a i x) b)))
(assert (= (select b i) y))
(assert (= (select (store b i x) j) y))
(assert (= a b))
(assert (= i j))
; Checking satisfiability.
(check-sat)
```

arrays.smt2

Theory of Arrays

- The full $\mathcal{T}_{\mathcal{A}}$ theory is undecidable, but there are decidable fragments
  - Undecidability comes from universal quantifiers
- Useful theory for SW/HW verification
- In particular, arrays often used to abstract memory locations
  - Main advantage: the abstraction depends on the number of accesses to the memory rather than its actual size

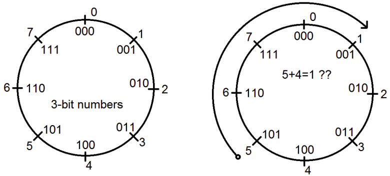


Theory of bit-vectors

- The theory of bit-vectors $\mathcal{T}_{\mathcal{BV}}$ naturally handles verification of programs and circuits
- $\mathcal{T}_{\mathcal{BV}} \neq \mathcal{T}_{\mathcal{A}}$: constants of $\mathcal{T}_{\mathcal{BV}}$ are vectors of bits with fixed length
- Typical operations: string-like (selection, slicing, concatenation, ...), logical (*bit-wise* NOT, OR, AND...), arithmetic $(+, -, \cdot, \ldots)$
- Straightforward reduction to SAT (bit-blasting)
- Exercise: if a, b, c are bit-vectors is $a[0:1] \neq b[0:1] \wedge (a \mid b) = c \wedge c[0] = 0 \wedge a[1] + b[1] = 0$ a $\mathcal{T}_{\mathcal{BV}}$-satisfiable formula?

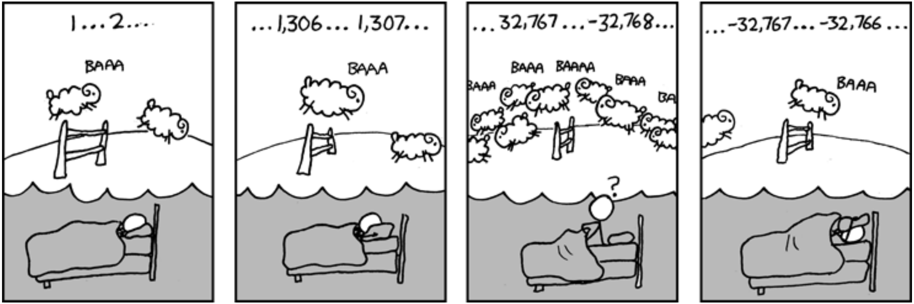
Theory of bit-vectors

- Bit-vectors better than integers or reals to model machine operations
- E.g., $x = 200 \wedge y = x + 100 \wedge y > x$ with x, y unsigned 8-bit integers
  - The formula is valid if we consider *"classical"* arithmetic theories
  - But machines operate differently! 8-bit unsigned integers are enclosed in $[0, 2^8 - 1] = [0, 255]$ so $y = x + 100 = 300$ is out of range
- In these cases typically $y = (x + 100) \mod 2^8 = 44$ so $y < x$
  - "wraparound"



Wraparound

- This also applies for signed integers, e.g., a 16-bit signed int can only be in $[-2^{15}, 2^{15} - 1] = [-32768, 32767]$ so e.g. $32767 + 3 = -32766$



`https://imgs.xkcd.com/comics/cant_sleep.png`

Theory of strings

- Over the last years the need for theory of strings emerged
  - Strings can enable vulnerabilities, especially in web programs
  - `https://mosca2023.github.io`
- Before, string solving typically handled with automata or bit-vectors
  - Automata limit the expressiveness and may be inefficient
  - Bit-vectors impose a fixed limit on string length
- Theory of strings can handle complex operations on unbounded-length strings natively, often in conjunction with other theories
  - E.g. arithmetic theory for string length, or regular expressions
- Some CP proposals too (over bounded-length strings)
  - `https://github.com/ArathaJS/aratha`

Theory of strings

- The theory of word equations is fundamental for string solving
- Fixed an alphabet $\mathcal{S}$, a word equation has form $L = R$ with L, R are concatenations of (uninterpreted) string constants
  - In other terms, L, R concatenate string variables and strings of $\mathcal{S}^*$
  - E.g. $X \cdot world \cdot Z = hello \cdot Y$
- The general theory of word equations is undecidable
  - Equivalent to arithmetic theory
- The quantifier-free theory of word equations is decidable
- Exercise: are the following formulas satisfiable?
  - $XY = YX \;\wedge\; X \neq Y$
  - $aX = Xb \;\wedge\; a \neq b$

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  - $XY = YX \ \wedge \ X \neq Y$ $\quad X = \epsilon, Y = a$
  - $aX = Xb \ \wedge \ a \neq b$ $\quad X$ starts with a and ends with b, so $X = aX_1b$, thus $aaX_1b = aX_1bb$, $aaaX_2bb = aaX_2bbb, \ldots, a^{k+1}X_kb^k = a^kX_kb^{k+1}$ with $|X_k| < |a| + |b|$: unsat because both a (as prefix) and b (suffix) should fit into X_k

SMT in practice

- In practice, theories not isolated: real-world applications often need a combination of arithmetic, strings, arrays, ... e.g.
    - $a = b + 2 \wedge A = write(B, a, 4) \wedge (read(A, b + 3) = 2 \vee f(a - 1) \neq f(b))$
    - $aX = Yb \ \wedge \ X \in \mathcal{L}(d \mid c^* ab) \ \wedge \ |Y| = 2 \cdot |X|$

- The goal is to efficiently combine decision procedures for each theory
    - Efficient procedures already exist for many theories of interest

- Decidability issues
    - ...But we can always restrict to fragments

Take-home messages

- SMT extends SAT to solve formulas in (quantifier-free) FOL
  - functions, (constants), predicates with arity > 1
- Similar/orthogonal to CP, it tackles combinatorial problems from a "more logical" perspective (formulas)
  - More oriented to problems derived from software analysis
  - Less optimization-oriented
- Eventually, SMT solving eagerly or lazily relies on SAT solving
  - SAT : machine language $\approx$ SMT : higher-level language
- Several theories of interest developed and studied over last decades
  - EUF, arithmetic, arrays, bit-vectors, strings, ...

Resources

- Handbook of Satisfiability – Chapter 12 "*Satisfiability Modulo Theories*" by C. Barrett, R. Sebastiani, S.A. Seshia, C. Tinelli
  - Search "Satisfiability Modulo Theories - EECS at UC Berkeley"
- Barrett, Clark, and Cesare Tinelli. "Satisfiability modulo theories." Handbook of model checking. Springer, Cham, 2018. 305-343.
- SAT/SMT schools
  - `https://sat-smt.in/`
- ...