Languages and Algorithms for Artificial Intelligence (Third Module)

A Glimpse into Computational Learning Theory

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University of Bologna, Academic Year 2020/2021

Learning Problems as Computational Problems

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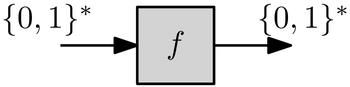

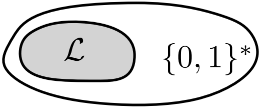

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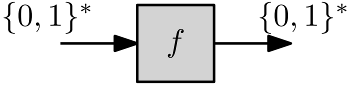

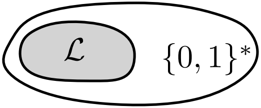

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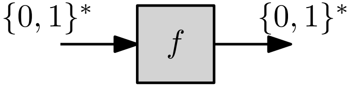

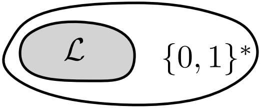

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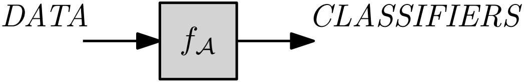

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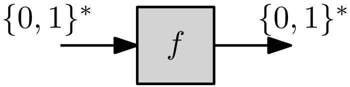

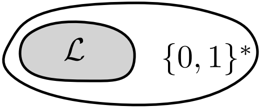

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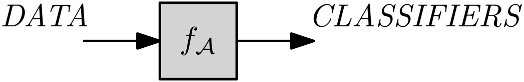

  - Could data and classifiers be encoded as strings, thus turning $f_{\mathcal{A}}$ as a function of the kind we know?
  - How could we formalize the fact that $\mathcal{A}$ correctly solves a given learning task?

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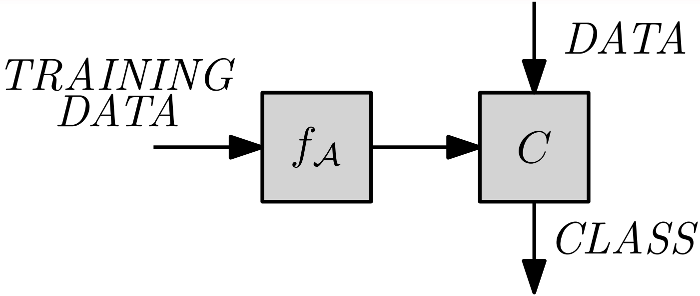

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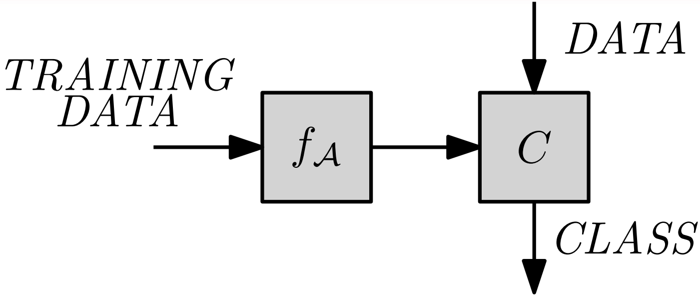
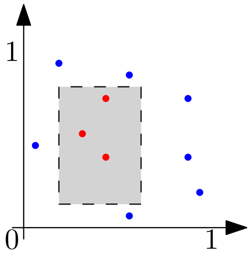


- Training data are **labelled**, so as to let $f_{\mathcal{A}}$ learn what the relationship exists between data and labels.
- The data C takes as input are **not labelled**, and the C's task is precisely the one of finding the appropriate label for any of them.
- Most often, C is drawn from a rather **restricted set of classifiers**, i.e. not all algorithms can be obtained in output from $f_{\mathcal{A}}$.

An Example Problem

Suppose that the data the algorithm $\mathcal{A}$ takes in input are points $(x, y) \in \mathbb{R}_{[0,1]} \times \mathbb{R}_{[0,1]}$ (where $\mathbb{R}_{[0,1]}$ is the set of real numbers between 0 and 1). These are labelled as positive or negative depending on they being inside a rectangle



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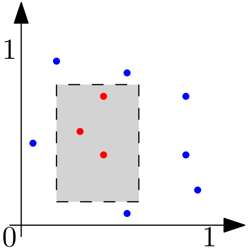

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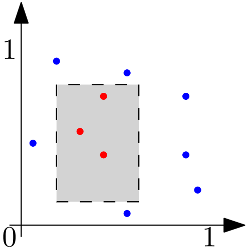

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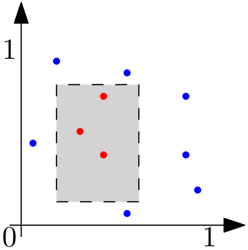

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- The algorithm $\mathcal{A}$ cannot guess the rectangle R with perfect accuracy if the data it receives in input are too few. As the data in D grow in number, we would expect the rectangle $f_{\mathcal{A}}(D)$ to converge to R, wouldn't we?

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  - It is supposed to "do the job" for each possible distribution $\mathbf{D}$.
- $\mathcal{A}$ is an ordinary algorithm.
  - Ultimately, it can be seen as a TM.
  - We thus assume that real numbers can be appropriately approximated as binary strings.
  - In some cases, it is useful to assume $\mathcal{A}$ to have the possibility to "flip a coin", i.e., to be a randomized algorithm.

The Algorithm $\mathcal{A}_{\mathsf{BFP}}$

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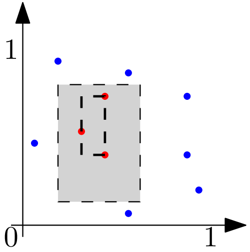

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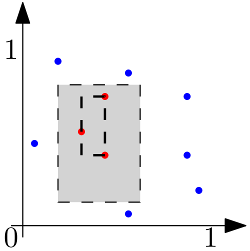

- The output classifier is a rectangle, which can be easily represented as a pair of coordinates.

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- As the number of samples in D grows, the result $\mathcal{A}_{\mathsf{BFP}}(D)$ does *not* necessarily approach the target rectangle, but its probability of error approaches zero.

Theorem

For every distribution $\mathbf{D}$*, for every* $0 < \varepsilon < \frac{1}{2}$ *and for every* $0 < \delta < \frac{1}{2}$*, if* $m \geq \frac{4}{\varepsilon}\ln\left(\frac{4}{\delta}\right)$*, then*

$$\Pr_{D\sim\mathbf{D}^m}[error_{\mathbf{D},T}(\mathcal{A}_{\mathsf{BFP}}(T(D)) < \varepsilon] > 1 - \delta$$

The General Model — Terminology

- We assume to work within an **instance space** X.
  - X is the set of (encodings) of instances of objects the learner wants to classify.
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- **Concepts** are subsets of X, i.e. collections of objects. These should be thought of as properties of objects.
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- A **concept class** $\mathcal{C}$ is a collection of concepts, namely a subset of $\mathcal{P}(X)$. These are the concepts which are sufficiently simple to describe, and that algorithms can handle.
  - The concept class $\mathcal{C}$ we work with in the example is the one of rectangles whose sides are parallel to the axes.
  - The **target concept** $c \in \mathcal{C}$ is the concept the learner wants to build a classifier for.

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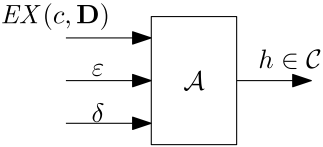

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  - ε is **error parameter**, while δ is the **confidence parameter**;
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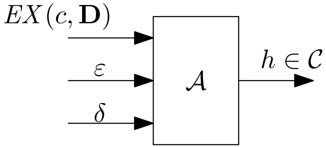

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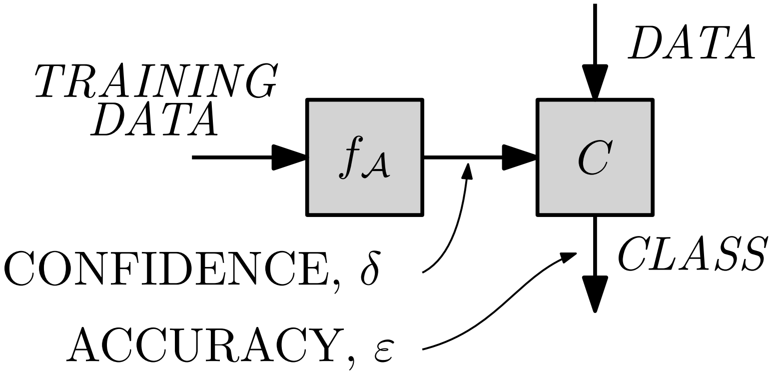
- The **error** of any $h \in \mathcal{C}$ is defined as $error_{\mathbf{D},c} = \Pr_{x \sim \mathbf{D}}[h(x) \neq c(x)]$.

The General Model — Two Kinds of Errors



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Corollary

The concept-class of axis-aligned rectangles over $\mathbb{R}^2_{[0,1]}$ is efficiently PAC-learnable.

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- Examples
  - X_n could be $\{0,1\}^n$, the set of **boolean vectors** of of (fixed!) length n, and $\mathcal{C}_n$ is the set of all subsets of $\{0,1\}^n$ *represented by CNFs*.
  - X_n could rather be $\mathbb{R}^n$, the set of **vectors of real numbers** of length n, while $\mathcal{C}_n$ are say, the subsets of $\mathbb{R}^n$ *represented by some form of neural network* with n inputs and 1 output.

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- In many cases (e.g. SGD), one has a *single* learning algorithm that work for every value of n. In that case, we allow (in the definition of efficient PAC learning) the algorithm $\mathcal{A}$ to take time polynomial in n, $\mathit{size}(c)$, $\frac{1}{\varepsilon}$ and $\frac{1}{\delta}$.

Boolean Functions as a Representation Class

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  - As an example, the conjunction

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  - *Not all* subsets of $\{0,1\}^n$ can be captured this way.
- A **second example** of a representation class for X_n is a class we know, namely the class $\mathbf{CNF}_n$ of CNFs over $x_1, \ldots, x_n$, which are conjunction *of disjunctions* of literals.
  - CNFs are normal forms of any boolean functions.
  - *All* subsets of $\{0,1\}^n$ can be captured this way.
  - We could even consider $k\mathbf{CNF}_n$ rather than arbitrary one, but this way we would lose universality.

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- A learning algorithm could proceed by keeping a conjunction of literals h as its state, initially set to

  $$x_1 \wedge \neg x_1 \wedge x_2 \wedge \neg x_2 \wedge \cdots \wedge x_n \wedge \neg x_n.$$

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Theorem

The representation class of boolean conjuctions of literals is efficiently PAC-learnable.

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  - A **3-term DNF formula** over n bits is a propositional formula in the form $T_1 \vee T_2 \vee T_3$, where each T_i is a conjunction of literals over $x_1, \ldots, x_n$.
  - In a sense, this class is the *dual* to 3CNFs!
  - As such, it is more expressive than conjunctions of literals, but still not universal.

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Theorem

If $\mathbf{NP} \neq \mathbf{RP}$, *then the representation class of* 3*-term DNF formulas is not efficiently PAC learnable.*

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- Models and results we did not have time to talk about include:
  - The VC Dimension.
  - The Fundamental Theorem of Learning.
  - The No-Free-Lunch Theorem.
  - Occam's Razor.
  - Positive and negative results about neural networks.
  - ...
- More information can be found in of the many excellent books on CLT, e.g.
  - Mehryar Mohri, Afshin Rostamizadeh, and Ameet Talwalkar *Foundations of Machine Learning* Second Edition. The MIT Press. 2018
  - Shai Shalev-Shwartz and Shai Ben-David. *Understanding Machine Learning: from Theory to Algorithms* Cambridge University Press. 2014.
  - Michael Kearns and Umesh Vazirani. *An Introduction to Computational Learning Theory* The MIT Press. 1994.

Example Results about Neural Networks (from Kearns and Vazirani's Book)

Theorem 3.7 *Let G be any directed acyclic graph, and let C_G be the class of neural networks on an architecture G with indegree r and s internal nodes. Then the number of examples required to learn C_G is*

$$O\left(\frac{1}{\epsilon}\log\frac{1}{\delta} + \frac{(rs+s)\log s}{\epsilon}\log\frac{1}{\epsilon}\right).$$

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Theorem 6.6 *Under the Discrete Cube Root Assumption, there is fixed polynomial $p(\cdot)$ and an infinite family of directed acyclic graphs (architectures) $G = \{G_{n^2}\}_{n\geq 1}$ such that each G_{n^2} has n^2 boolean inputs and at most $p(n)$ nodes, the depth of G_{n^2} is a fixed constant independent of n, but the representation class $C_G = \cup_{n\geq 1} C_{G_{n^2}}$ (where $C_{G_{n^2}}$ is the class of all neural networks over $\Re^{n^2}$ with underlying architecture G_{n^2}) is not efficiently PAC learnable (using any polynomially evaluatable hypothesis class). This holds even if we restrict the networks in $C_{G_{n^2}}$ to have only binary weights.*

Thank You!

Questions?