Languages and Algorithms for Artificial Intelligence (Third Module)

Between the Feasible and the Unfeasible

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The Border Between the Tractable and the Intractable

- Up to now, we have encountered (essentially speaking) two complexity classes, namely:
  - **P**, which contains those problems which can be solved in polynomial time, so the **tractable** ones.
  - **EXP**, which contains the whole of **P**, but also some problems which *cannot* be solved in polynomial time, intrinsically requiring exponential time (and as such, **intractable**).

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- It's now time to study the "border" between tractability and intractability:
  - Between **P** and **EXP**, one can define *many other* classes. i.e. there are many ways of defining a class **A** such that

$$\mathbf{P} \subseteq \mathbf{A} \subseteq \mathbf{EXP}$$

  - This is a formidable tool to classify those problems in **EXP** for which *we do not know* whether they are in **P** (and there are *so many* of them).

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- Very often, the language we would like to classify can be written as follows:

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- In other words, the elements of $\mathcal{L}$ are those strings for which we can find a *certificate* y (of polynomial length) such that the pair (x, y) passes the *test* $\mathcal{A} \subseteq \{0,1\}^* \times \{0,1\}^*$.

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  - *Not necessarily*: given x, we can check whether $x \in \mathcal{L}$ by checking whether $(x, y) \in \mathcal{A}$ for all possible possible y such that $|y| \leq p(|x|)$, of which however there are *exponentially* many.
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  - Of course this does *not* rule out other strategies to decide $\mathcal{L}$.
- The *take-away message* is thus the following: **crafting** a solution for the problem x (i.e., finding y) can potentially be more difficult than just **checking** y to be a solution to x.

The Complexity Class **NP**

- A language $\mathcal{L} \subseteq \{0,1\}^*$ is in the class **NP** iff there exist a polynomial $p : \mathbb{N} \to \mathbb{N}$ and a polynomial time TM $\mathcal{M}$ such that

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- Differently from **P** and **EXP**, the class **NP** does not have a natural counterpart as a class of *functions*.

Theorem

$\mathbf{P} \subseteq \mathbf{NP} \subseteq \mathbf{EXP}$

Examples Problems in **NP**

1. **Maximum Independent Set**
   - In its decision form, it asks whether a pair $(\mathbb{G}, k)$ of an undirected graph and a natural number $k \in \mathbb{N}$ is such that $\mathbb{G} = (V, E)$ admits an independent set $W \subseteq V$ of cardinality at least k.

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2. **Subset Sum**
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   - Given three numbers n, m, l, it asks whether n has a prime factor in the interval $[m, l]$.
   - Certificate: it can be taken to be the prime number p: checking that $p \in [m, l]$ and that p divides n is easy. Instead, checking that p is prime requires a lot of work, but can indeed be done in polynomial time.

Examples Problems in **NP**

3. **Decisional Linear Programming**
   - Given a sequence of m linear inequalities with rational coefficients over n variables (i.e. inequalities of the form $\sum_{i=1}^{n} a_i x_i \leq b$, where the coefficients $a_1, \ldots, a_n, b$ are in $\mathbb{Q}$), decide whether there is a rational assignment to the variables $x_1, \ldots, x_n$ which makes all the inequalities true.

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   - Certificate: again, the assignments suffices.

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- Some of the aforementioned problems are also in **P**:
  - *Decisional linear programming* can be proved to be in **P** thanks to, e.g., the Ellipsoid algorithm.
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- All the other problems are currently **not known** to be in **P**.
  - Are they all equivalent in terms of their inherent computational difficulty?
  - Is there any way isolate those problems in **NP** whose difficult is maximal, i.e. they are at least as hard as all other problems in **NP**?

Nondeterministic Turing Machines

- The class **NP** can also be defined using a variant of Turing machines, called the *nondeterministic* Turing machines (NDTM for short).
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- The only differences between a NDTM and an ordinary TM is that the former has:
  - Two transition functions δ_0 and δ_1 rather than just one. *At every step*, the machine chooses nondeterministically one between the two transition functions and proceed according to it
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  - A special state $q_{\mathtt{accept}}$.
- We say that a NDTM M:
  - **Accepts** the input $x \in \{0,1\}^*$ iff *there exists* one among the many possible evolutions of the machine M when fed with x which makes it reaching $q_{\mathtt{accept}}$.
  - **Rejects** the input $x \in \{0,1\}^*$ iff *none* of the aformentioned evolutions leads to $q_{\mathtt{accept}}$.

Time-Bounded NDTMs

- We say that a NDTM M *runs in time* $T : \mathbb{N} \rightarrow \mathbb{N}$ iff for every $x \in \{0,1\}^*$ *and for every possible nondeterministic evolution*, M reaches either the halting state or $q_{\mathtt{accept}}$ within $c \cdot T(|x|)$ steps, where $c > 0$.

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- For every function $T : \mathbb{N} \to \mathbb{N}$ and $\mathcal{L} \subseteq \{0,1\}^*$, we say that $\mathcal{L} \in \mathbf{NDTIME}(T(n))$ iff there is a NDTM M working in time T and such that $M(x) = 1$ iff $x \in \mathcal{L}$.

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- All this being said, NDTMs, contrarily to TMs, are *not* meant to model any form of physically realisable machine.

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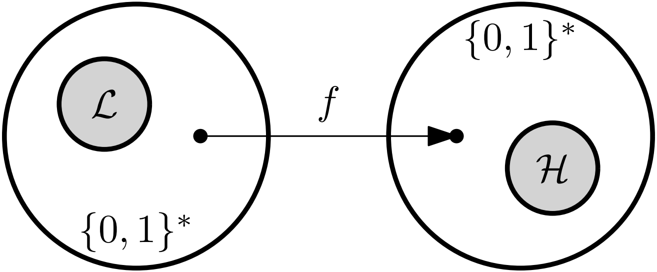
- We need something else, namely a (pre-order) relation between languages such that two languages being in relation tells us something precise about the **relative** difficulty of deciding them.

Reductions



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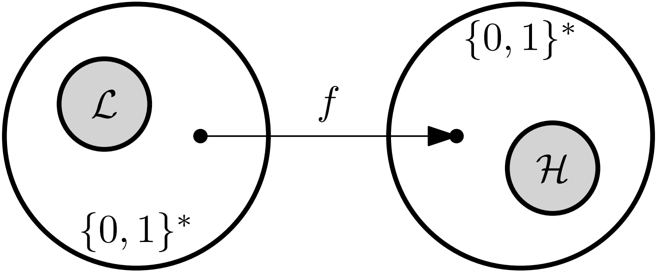

- The language $\mathcal{L}$ is said to be **polynomial-time reducible** to another language $\mathcal{H}$ iff there is a polytime computable function $f : \{0,1\}^* \to \{0,1\}^*$ such that $x \in \mathcal{L}$ iff $f(x) \in \mathcal{H}$.
- In this case, we write $\mathcal{L} \leq_p \mathcal{H}$.

Reductions and Complexity

- If $\mathcal{L} \leq_p \mathcal{H}$, then $\mathcal{H}$ is at least as difficult as $\mathcal{L}$, at least as far as classes like $\mathbf{P}$ (or above it) are concerned.
  - If, e.g., $\mathcal{L} \leq_p \mathcal{H}$ and $\mathcal{H} \in \mathbf{P}$, then also $\mathcal{L} \in \mathbf{P}$: a way to decide if $x \in \mathcal{L}$ consists in traslating it into $f(x)$ (which can be done in polynomial time), then checking whether $f(x) \in \mathcal{H}$.

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- A language $\mathcal{H} \subseteq \{0,1\}^*$ is said to be:
  - $\mathbf{NP}$-hard if $\mathcal{L} \leq_p \mathcal{H}$ for every $\mathcal{L} \in \mathbf{NP}$.
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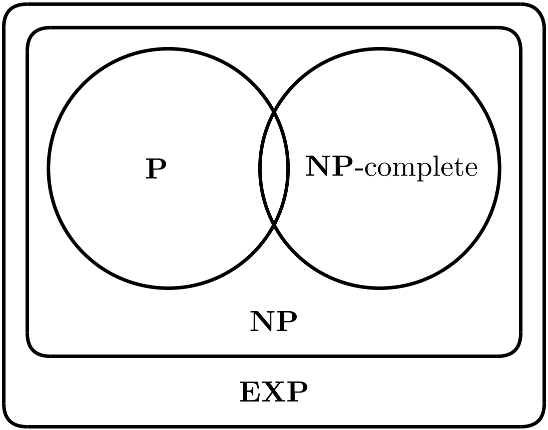
Theorem

1. *The relation $\leq_p$ is a pre-order (i.e. it is reflexive and transitive).*
2. *If $\mathcal{L}$ is $\mathbf{NP}$-hard and $\mathcal{L} \in \mathbf{P}$, then $\mathbf{P} = \mathbf{NP}$.*
3. *If $\mathcal{L}$ is $\mathbf{NP}$-complete, then $\mathcal{L} \in \mathbf{P}$ iff $\mathbf{P} = \mathbf{NP}$.*


P
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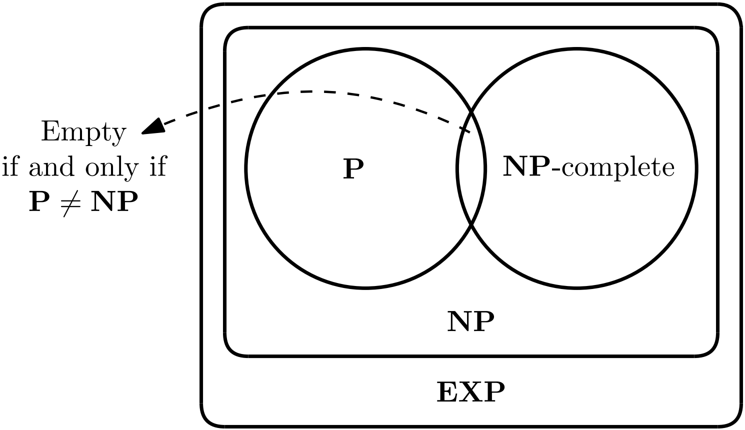
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- Let TMSAT be the following language:

$$\texttt{TMSAT} = \{(\alpha, x, 1^n, 1^t) \mid \exists u \in \{0,1\}^n . \mathcal{M}_\alpha \text{ outputs } 1 \\ \text{on input } (x, u) \text{ within } t \text{ steps}\}$$

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- Although interesting from a purely theoretical perspective, the language `TMSAT` is very specifically tied to Turing Machines, and thus of no practical importance.

A Quick Recap on Propositional Logic

- Formulas of **propositional logic** are either:
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- A formula F is **satisfiable** iff there is one ρ such that $[\![F]\!]_\rho = 1$.

The Cook-Levin Theorem

- A propositional formula F is said to be in **conjunctive normal form** (or a **CNF**) when it is a conjuction of disjunctions of *literals* (a literal being a variable or its negation).
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Theorem (Cook-Levin)

The following two languages are **NP***-complete:*

$$\mathtt{SAT} = \{\llcorner F \lrcorner \mid F \textit{ is a satisfiable CNF}\}$$

$$\mathtt{3SAT} = \{\llcorner F \lrcorner \mid F \textit{ is a satisfiable 3CNF}\}$$

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  - We define a polynomial-time transformation $x \mapsto \varphi_x$ from strings to CNFs such that

$$\varphi_x \in \mathtt{SAT} \Leftrightarrow \exists y \in \{0,1\}^{p(|x|)}.\mathcal{M}(x,y) = 1$$

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  - Finally, we need to show that $\texttt{SAT} \leq_p \texttt{3SAT}$.

Proving a Problem Hard

- Suppose you are studying the complexity of a language $\mathcal{L}$, and you are convinced that the underlying problem is **hard**. How should you back your claim?
- By proving that $\mathcal{L} \in \mathbf{P}$?
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- By proving that $\mathcal{L} \in \mathbf{NP}$?
  - Again, this **does not** mean much.
- By proving that $\mathcal{L}$ is $\mathbf{NP}$-complete?
  - **Yes**, this way you prove that the problem is not so hard (being in $\mathbf{NP}$), but not so easy either (unless $\mathbf{P} = \mathbf{NP}$).

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  2. That any other language $\mathcal{H} \in \mathbf{NP}$ is such that $\mathcal{H} \leq_p \mathcal{L}$.
     - We can of course prove the statement directly.
     - More often (e.g. when showing `3SAT` **NP**-complete), one rather proves that $\mathcal{J} \leq_p \mathcal{L}$ for a language $\mathcal{J}$ which is already known to be **NP**-complete.
     - This is correct, simply because $\leq_p$ is transitive:

$$\begin{array}{c}
\vdots \\
\mathcal{H} \xrightarrow{\leq_p} \mathcal{J} \xrightarrow{\leq_p} \mathcal{L} \\
\vdots
\end{array}$$

NP-complete Problems: Examples

Theorem

The Maximum Independent Set Problem `INDSET` *is* **NP***-complete.*

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Theorem

The Decisional 0/1 Linear Programming Problem `ILP` *is* **NP**-*complete*

- There is an easy polytime reduction from `SAT` to `ILP`.

The Zoo of **NP**-complete Problems

- For any pair $\mathcal{L}, \mathcal{H}$ of **NP**-complete problems, we have that

$$\mathcal{L} \leq_p \mathcal{H} \qquad \mathcal{H} \leq_p \mathcal{L}$$

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- This being said, defining a reduction from $\mathcal{L}$ to $\mathcal{H}$ is sometimes easy, and sometimes very difficult, and it is instructive to think at **NP**-complete problems as forming a graph, where edges are *natural* polytime reductions
  - A fragment, the one we have encountered so far is the following

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  - In fact, this graph is **huge**: thousands of different problems are known to be **NP**-complete

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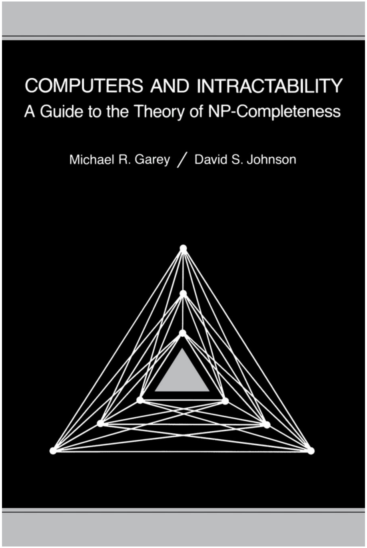

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- This is often **very useful**, because specialised tools for `SAT`, called SAT-solvers do exist.
  - They do not work in polynomial time.
  - Concretely, they work extremely well on a relatively large class of formulas.

Is This the End of The Story?

$$\mathbf{P} \subseteq \mathbf{NP} \subseteq \mathbf{EXP}$$

Is This the End of The Story?

$$\mathbf{L} \subseteq \mathbf{NL} \subseteq \mathbf{P} \subseteq \begin{array}{c}\mathbf{NP} \\ \mathbf{coNP}\end{array} \subseteq \cdots \subseteq \mathbf{PH} \subseteq \mathbf{PSPACE} \subseteq \mathbf{EXP} \subseteq \cdots$$

Thank You!

Questions?