3. Theory solvers, combinations and extensions

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Combinatorial Decision Making and Optimization

2nd cycle degree programme in Artificial Intelligence

University of Bologna, Academic Year 2024/25

Theory solvers

- So far, we introduced the basics of SMT solving without focusing much on background theories and their solvers
  - Eager vs lazy approaches
- In its simplest form, a $\mathcal{T}$-solver takes as input a conjunction of $\mathcal{T}$-literals μ and decides whether μ is $\mathcal{T}$-satisfiable
- We can see a SMT solver as a "collection" of theory solvers
- What are the crucial features for a $\mathcal{T}$ solver?

Theory solvers

- Early pruning: invoke $\mathcal{T}$-solver on partial Boolean assignments, especially during the early stages of the search
- Incrementality: when a new constraint is added, possibly avoid recomputing everything from scratch
- Backtrackability: support cheap (stack-based) removal of constraints when exploring the search tree without "resetting" the internal state

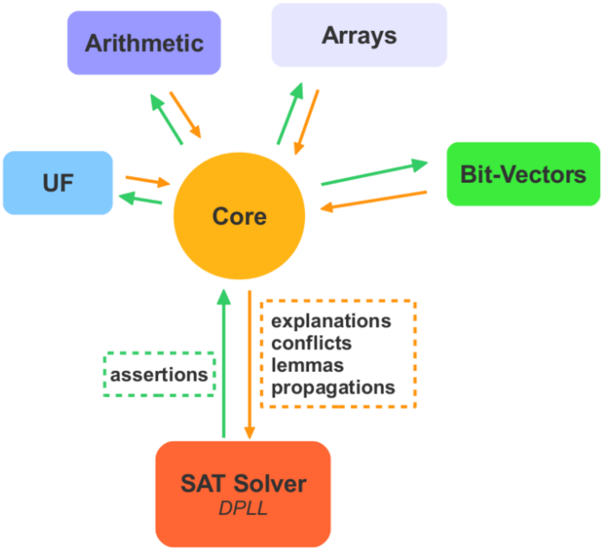
Theory solvers

- Literal deduction: $\mathcal{T}$-solver can perform deductions of literals not yet assigned in the input formula ($\mathcal{T}$-propagation)
- Explanation generation: when a conflict involving a literal ℓ is found, is necessary to get a (possibly short) explanation $\ell_1 \wedge \cdots \wedge \ell_n \rightarrow \ell$ to perform conflict analysis and backjumping

What theories?

- Uninterpreted functions (EUF)
- Arithmetic
  - LIA, LRA, LIRA, ...
- Arrays
- Bit-vectors
- Strings
- ...

What theories?



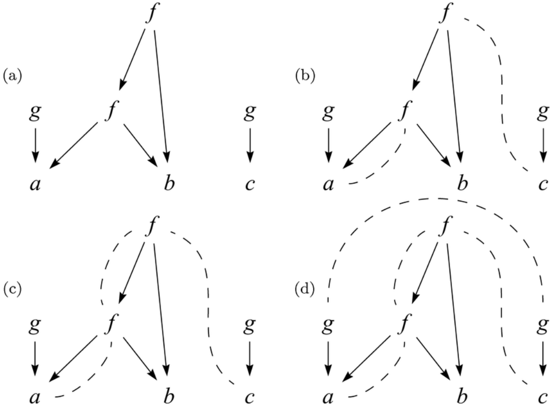
EUF theory

- A $\mathcal{T}_{EUF}$ formula can be decided in polynomial time with congruence closure procedure:
  - Add fresh c and replace each $p(t_1, \ldots, t_k)$ with $f_p(t_1, \ldots, t_k) = c$
  - Partition formula into equalities (E) and disequalities (D)
  - Compute the congruence closure $\equiv_E$ of E, i.e., the smallest equivalence relation over the terms of E such that:
    - $t_1 = t_2 \in E \implies t_1 \equiv_E t_2$
    - For each $f(s_1, \ldots, s_k), f(t_1, \ldots, t_k)$ occurring in E, if $s_i \equiv_E t_i$ for each $i \in \{1, \ldots, k\}$ then $f(s_1, \ldots, s_k) \equiv_E f(t_1, \ldots, t_k)$ (congruence property)
  - The formula is satisfiable iff for each $t_1 \neq t_2 \in D$ we have $t_1 \not\equiv_E t_2$

- Standard algorithms use a DAG to represent functions applications, and union-find (a.k.a. merge-find or disjoint-set) for the classes of $\equiv_E$

EUF theory

Example*: $\phi \equiv f(a,b) = a \wedge f(f(a,b),b) = c \wedge g(a) \neq g(c)$



(a) DAG for ϕ-terms. (b) E-graph: equivalences are the equalities in ϕ. (c) $f(f(a,b),b) \equiv_E f(a,b)$ because $f(a,b) \equiv_E a$. (d) $g(a) \equiv_E g(c)$ because $a \equiv_E c$. Since $g(a) \neq g(c)$ and $g(a) \equiv_E g(c)$, ϕ is unsatisfiable

*From: D. Detlefs, G. Nelson, J.B. Saxe. "Simplify: a theorem prover for program checking." JACM, 52.3 (2005): 365-473

LRA theory

- Consider LRA = Linear Real Arithmetic theory, having signature $\Sigma_{LRA} = (\mathbb{Q}, +, -, *, \leq)$ and linear multiplications only
- We could decide LRA-literals with Fourier-Motzkin elimination
  - Replace $t_1 \neq t_2$ with $t_1 < t_2 \vee t_2 < t_1$, and $t_1 \leq t_2$ with $t_1 < t_2 \vee t_1 = t_2$ (case splitting)
  - Eliminate equalities and apply Fourier-Motzkin elimination to all variables to determine its satisfiability
    - `https://en.wikipedia.org/wiki/Fourie-Motzkin_elimination`
- Not practical for large set of constraints, simplex method preferable

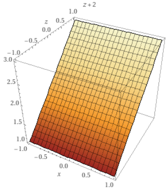
LIA theory

- Consider LIA = Linear Integer Arithmetic theory, having signature $\Sigma_{LIA} = (\mathbb{Z}, +, -, *, \leq)$ and linear multiplications only
  - if not linear, undecidable (*Peano arithmetic*)
  - if fully quantified, *Presburger arithmetic*
  - if quantifier-free, different decision procedures exist
- As for LRA, we can apply methods like Fourier-Motzkin, but Simplex + branch & bound/cut generally better
- Methods exist also for LIRA = integer + real arithmetic and NLA = nonlinear arithmetic
  - E.g., `https://microsoft.github.io/z3guide/docs/theories/Arithmetic/`

Difference logic

- Consider now DL = Difference Logic theory, having atomic formulas of the form $x - y \leq k$ with x, y variables and k constant
  - Constraints $x - y \bowtie k$ with $\bowtie \in \{=, \neq, <, \geq, >\}$ can be rewritten

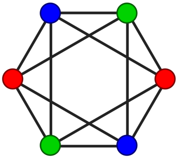
- E.g. if $x, y \in \mathbb{Z}$:
  $x - y > 5 \;\wedge\; x = z + 2 \implies$
  $x - y \geq 6 \;\wedge\; x - z \leq 2 \;\wedge\; x - z \geq 2 \implies$
  $y - x \leq -6 \;\wedge\; x - z \leq 2 \;\wedge\; z - x \leq -2$



- Unary constraints $x \leq k$ can be rewritten into $x + z_0 \leq k$ by enforcing $z_0 = 0$ in any satisfying assignment

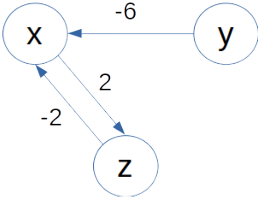
Difference logic and k-coloring

- If we allow $\neq$ and the domain is $\mathbb{Z}$, deciding satisfiability of DL formulas is NP-hard, e.g., it gets "as hard as" k-coloring problem
  - If we have k colors available, can we color a graph s.t. adjacent nodes have different colors? If $k \geq 3$ the problem is NP-hard
- Formally, given graph (V, E) and $k \in \mathbb{N}$, does it exist a function $c : V \to \{1, \ldots, k\}$ s.t. for each $(i, j) \in E$ we have $c(i) \neq c(j)$?
- Any k-coloring instance can be mapped to a DL formula with $|V|$ variables, $|E|$ disequalities $x_i \neq x_j$ for each $(i, j) \in E$ and $2|V|$ disequalities $1 \leq x_i \leq k$
  - If we can decide the DL formula in polynomial time, we can solve any problem of NP in polynomial time



Difference logic as graph problem

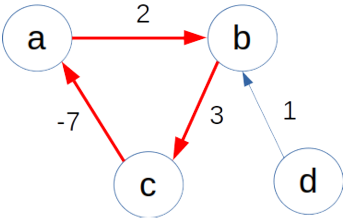
- From DL literals in Φ we can get a directed weighted graph $\mathcal{G}_\Phi$ with:
  - a node for each variable occuring in Φ
  - a weighted edge $x \xrightarrow{k} y$ for each $x - y \leq k \in \Phi$
- E.g., if $\Phi = \{y - x \leq -6,\ x - z \leq 2,\ z - x \leq -2\}$ then $\mathcal{G}_\Phi$ is:



- Theorem: Φ is inconsistent $\iff$ $\mathcal{G}_\Phi$ has a negative cycle

Difference logic as graph problem

- *Example*: let $\Phi = \{a - b \leq 2, b - c \leq 3, c - a \leq -7, d - b \leq 1\}$. The $\mathcal{G}_\Phi$ graph is:



- Negative loop $a \xrightarrow{2} b \xrightarrow{3} c \xrightarrow{-7} a$ (total weight -2): Φ inconsistent
  - $a \geq c + 7$ conflicts with $a \leq b + 2 \leq (c + 3) + 2 = c + 5$

Difference logic as graph problem

- Negative loops can be detected with Bellman-Ford in $O(|V||E|)$ by adding to V a source vertex x_0 and an edge $x_0 \xrightarrow{0} x$ for each $x \in E$
  - `https://en.wikipedia.org/wiki/Bellman%E2%80%93Ford_algorithm`
  - Other more efficient variant exists
- Negative loops denotes inconsistency explanations
  - Not minimal in general
- Theory propagations computed from consistent graphs: if there is a path between x and y with total weight k, we can deduce $x - y \leq k$
  - If $x \xrightarrow{k_1} x_1 \xrightarrow{k_2} x_2 \xrightarrow{k_3} \ldots \xrightarrow{k_n} y$ the total weight is $k = \sum_{i=1}^{n} k_i$ and $x - x_1 \leq k_1,\ x_1 - x_2 \leq k_2,\ \ldots,\ x_n - y \leq k_n$ hence $(x - x_1) + (x_1 - x_2) + \cdots + (x_n - y) \leq \sum_{i=1}^{n} k_i = k$ thus $x + (-x_1 + x_1) + \cdots + (-x_n + x_n) + y \leq k$, i.e., $x - y \leq k$

Other theories

- Bit-vectors: typically BV formulas are fist simplified, and then encoded into SAT formulas (bit-blasting)
- Arrays: typically flattening of terms + congruence closure + lazy axioms instantiation + optimizations
- (Multi)-sets
- Strings
- Floating points
- ...

Combining Theories

Need for combination

- So far we considered theories individually. But often SMT formulas contain atoms from (very) different theories
- In particular software verification applications can generate constraints over several data types
  - integers, floating points, bit-vectors, arrays, strings, ...
- E.g., formula $a = b + 2 \wedge A = write(B, a + 1, 4) \wedge (f(a) \vee g(b + 1))$ involves theory of linear arithmetic, arrays, and EUF
- Given $\mathcal{T}_i$-solvers for theories $\mathcal{T}_1, \ldots, \mathcal{T}_n$, can we combine them to get a solver for $\bigcup_i \mathcal{T}_i$?

Example

- Consider formula $f(f(x) - f(y)) = a \wedge f(0) = a + 2 \wedge x = y$
  - Two theories involved: EUF and linear arithmetic (LA)
- 1st step: purification. Each literal must belong to only one theory
  - Fresh constants needed: $e_1, e_2, \ldots, e_5$
- Purified formula: $\underbrace{e_1 = e_2 - e_3}_{LA}, \underbrace{f(e_1) = a, e_2 = f(x), e_3 = f(y)}_{EUF},$

$\underbrace{e_4 = 0, e_5 = a + 2}_{LA}, \underbrace{f(e_4) = e_5, x = y}_{EUF}$

- In this way EUF and LA solvers only share $a, e_1, \ldots, e_5$
- To merge the corresponding models, solvers must agree on equalities between shared constants, a.k.a. interface equalities

Example

- 2^{nd} step: satisfiability check and equalities exchange
- Purified formula: $\underbrace{e_1 = e_2 - e_3}_{LA}$, $\underbrace{f(e_1) = a, e_2 = f(x), e_3 = f(y)}_{EUF}$, $\underbrace{e_4 = 0, e_5 = a + 2}_{LA}$, $\underbrace{f(e_4) = e_5, x = y}_{EUF}$
- Both EUF-solver and LA-solver say SAT
- EUF solver deduces that $\{x = y, f(x) = e_2, f(y) = e_3\} \models e_2 = e_3$ and sends the literal to the LA solver

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$\underbrace{e_4 = 0, e_5 = a + 2, e_2 = e_3}_{LA}, \underbrace{f(e_4) = e_5, x = y}_{EUF}$
- Both EUF-solver and LA-solver say SAT
- LA solver deduces that $\{e_2 - e_3 = e_1, e_4 = 0, e_2 = e_3\} \models e_1 = e_4$ and sends the literal to the EUF solver

Example

- 2nd step: satisfiability check and equalities exchange

- Purified formula: $\underbrace{e_1 = e_2 - e_3}_{LA}, \underbrace{f(e_1) = a, e_2 = f(x), e_3 = f(y)}_{EUF},$
$\underbrace{e_4 = 0, e_5 = a + 2, e_2 = e_3}_{LA}, \underbrace{f(e_4) = e_5, x = y, e_1 = e_4}_{EUF}$

- Both EUF-solver and LA-solver say SAT

- LA solver deduces that $\{e_2 - e_3 = e_1, e_4 = 0, e_2 = e_3\} \models e_1 = e_4$ and sends the literal to the EUF solver

Example

- 2nd step: satisfiability check and equalities exchange
- Purified formula: $\underbrace{e_1 = e_2 - e_3}_{LA}, \underbrace{f(e_1) = a, e_2 = f(x), e_3 = f(y)}_{EUF},$ $\underbrace{e_4 = 0, e_5 = a + 2, e_2 = e_3}_{LA}, \underbrace{f(e_4) = e_5, x = y, e_1 = e_4}_{EUF}$
- Both EUF-solver and LA-solver say SAT
- EUF solver deduces that $\{f(e_1) = a, f(e_4) = e_5, e_1 = e_4\} \models a = e_5$ and sends the literal to the LA solver

Example

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- Purified formula: $\underbrace{e_1 = e_2 - e_3}_{LA}$, $\underbrace{f(e_1) = a, e_2 = f(x), e_3 = f(y)}_{EUF}$,
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- Both EUF-solver and LA-solver say SAT

- EUF solver deduces that $\{f(e_1) = a, f(e_4) = e_5, e_1 = e_4\} \models a = e_5$ and sends the literal to the LA solver

Example

- 2^{nd} step: satisfiability check and equalities exchange
- Purified formula: $\underbrace{e_1 = e_2 - e_3}_{LA}, \underbrace{f(e_1) = a, e_2 = f(x), e_3 = f(y)}_{EUF},$
  $\underbrace{e_4 = 0, e_5 = a + 2, e_2 = e_3, a = e_5}_{LA}, \underbrace{f(e_4) = e_5, x = y, e_1 = e_4}_{EUF}$
- EUF-solver say SAT...
- ...but LA-solver say UNSAT: $\{e_5 = a + 2, a = e_5\} \models \perp$
- Hence the original formula is UNSAT

Nelson-Oppen procedure (convex case)

- Let Σ_1, Σ_2 be signatures and $\mathcal{T}_1, \mathcal{T}_2$ their theories. If $\mathcal{T}_1$ and $\mathcal{T}_2$ are:
  - signature-disjoint
    - $\Sigma_1 \cap \Sigma_2 = \emptyset$
  - stably-infinite
    - Σ-theory $\mathcal{T}$ of sort σ is stably infinite if every T-satisfiable Σ-formula has a model interpreting σ as an infinite set
  - convex
    - For each set of $\mathcal{T}_i$-literals S, $S \models_{\mathcal{T}_i} (a_1 = b_1 \vee \cdots \vee a_n = b_n)$ must imply that $S \models a_k = b_k$ for some $k \in \{1, \ldots, n\}$

- then we can check the $(T_1 \cup T_2)$-satisfiability with the deterministic Nelson-Oppen algorithm

Nelson-Oppen procedure (convex case)

Let S be a $(T_1 \cup T_2)$-formula and E the set of interface equalities between S_1 and S_2. Deterministic Nelson-Oppen steps:

1. Purify S and split it into $\mathcal{S}_1$ and $\mathcal{S}_2$
   - $\mathcal{S}_i$ contains $\mathcal{T}_i$-literals only
2. If $\mathcal{S}_1 \models_{\mathcal{T}_1} \bot$, then return UNSAT
3. If $\mathcal{S}_2 \models_{\mathcal{T}_2} \bot$, then return UNSAT
4. If $\mathcal{S}_1 \models_{\mathcal{T}_1} (e = e')$ with $(e = e') \in E - S_2$, then $S_2 \leftarrow S_2 \cup \{(e = e')\}$ and go to 3
5. If $\mathcal{S}_2 \models_{\mathcal{T}_2} (e = e')$ with $(e = e') \in E - S_1$, then $S_1 \leftarrow S_1 \cup \{(e = e')\}$ and go to 2
6. return SAT

Nelson-Oppen procedure (non-convex case)

- Why we needed convex theories? Consider the following formula:

$$\begin{aligned} 1 \leq x &\leq 2 \\ f(1) &= a \\ f(x) &= b \\ a &= b + 2 \\ f(2) &= f(1) + 3 \end{aligned}$$

  involving linear integer arithmetic (LIA) and EUF theories

- Let's purify the formula by introducing the interface equalities:

$$e_1 = 1,\; e_2 = 2,\; e_3 = f(e_2),\; e_4 = f(e_1),\; e_3 = e_4 + 3$$

Nelson-Oppen procedure (non-convex case)

- Now let's check satisfiability and exchange entailed equalities:

LIA	EUF
$1 \leq x$	$f(e_1) = a$
$x \leq 2$	$f(x) = b$
$e_1 = 1$	$f(e_2) = e_3$
$a = b + 2$	$f(e_1) = e_4$
$e_2 = 2$	
$e_3 = e_4 + 3$	

- Both EUF-solver and LIA-solver say SAT
- EUF solver deduces that $\{f(e_1) = a, f(e_1) = e_4\} \models a = e_4$ and sends the literal to the LA solver

Nelson-Oppen procedure (non-convex case)

- Now let's check satisfiability and exchange entailed equalities:

LIA	*EUF*
$1 \leq x$	$f(e_1) = a$
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$e_1 = 1$	$f(e_2) = e_3$
$a = b + 2$	$f(e_1) = e_4$
$e_2 = 2$	
$e_3 = e_4 + 3$	
$a = e_4$	

- Both EUF-solver and LIA-solver say SAT
- EUF solver deduces that $\{f(e_1) = a, f(e_1) = e_4\} \models a = e_4$ and sends the literal to the LA solver

Nelson-Oppen procedure (non-convex case)

- Now let's check satisfiability and exchange entailed equalities:

LIA	*EUF*
$1 \leq x$	$f(e_1) = a$
$x \leq 2$	$f(x) = b$
$e_1 = 1$	$f(e_2) = e_3$
$a = b + 2$	$f(e_1) = e_4$
$e_2 = 2$	
$e_3 = e_4 + 3$	
$a = e_4$	

- Both EUF-solver and LIA-solver say SAT
- EUF and LIA theories cannot deduce any other interface equality
  - ...but LIA solver could deduce $x = e_1 \vee x = e_2$

Nelson-Oppen procedure (non-convex case)

- Now let's check satisfiability and exchange entailed equalities:

LIA	*EUF*
$1 \leq x$	$f(e_1) = a$
$x \leq 2$	$f(x) = b$
$e_1 = 1$	$f(e_2) = e_3$
$a = b + 2$	$f(e_1) = e_4$
$e_2 = 2$	
$e_3 = e_4 + 3$	
$a = e_4$	

- If $x = e_1$, EUF would deduce $a = b$: UNSAT
- If $x = e_2$, EUF would deduce $b = e_3$: UNSAT

Nelson-Oppen procedure (non-convex case)

- Now let's check satisfiability and exchange entailed equalities:

LIA	EUF
$1 \leq x$	$f(e_1) = a$
$x \leq 2$	$f(x) = b$
$e_1 = 1$	$f(e_2) = e_3$
$a = b + 2$	$f(e_1) = e_4$
$e_2 = 2$	
$e_3 = e_4 + 3$	
$a = e_4$	

- Hence, $x = e_1 \vee x = e_2$ is false and the original formula UNSAT
- ...But we can't infer this with deterministic Nelson-Oppen procedure!

Non-deterministic Nelson-Oppen

- Deterministic Nelson-Oppen procedure doesn't work in this example because $\mathcal{T}_{LIA}$ is not convex: $1 \leq x \leq 2 \models x = 1 \vee x = 2$ but in general neither $1 \leq x \leq 2 \not\models x = 1$ nor $1 \leq x \leq 2 \not\models x = 2$
- However, there is a non-deterministic Nelson-Oppen procedure that also works on non-convex theories
  - We still need disjoint and stably-infinite theories
- It works through arrangements of shared constants, basically doing case splitting $x = y \vee x \neq y$ between pair of shared constants x, y
  - Unsurprisingly, exponential worst-case time complexity

Optimization Modulo Theory

Extensions

- There are several extensions and enhancements to the SMT framework seen so far, e.g.
- Quantified formulas
- Layered solvers
- On-demand solvers
- Optimization Modulo Theory

Optimization Modulo Theory

- OMT is an extension of SMT where we need to find a model for an input formula φ that is optimal w.r.t. an objective function f_{obj}
- φ refers to a theory $\mathcal{T} = \mathcal{T}_{\preceq} \cup \mathcal{T}_1 \cup \cdots \cup \mathcal{T}_n$ where
  - $\mathcal{T}_{\preceq}$ contains a predicate $\preceq$ representing a total order
  - $\bigcup_{i=1}^{n} \mathcal{T}_i$ might be empty
- The goal is finding a model $\mathcal{M}$ s.t. $\varphi^{\mathcal{M}} = \mathit{true}$ and $f_{obj}^{\mathcal{M}}$ is minimal according to $\preceq$
  - Maximizing $f_{obj} \equiv$ minimizing $-f_{obj}$
- Typically, $\preceq$ is the $\leq$ predicate over integers or reals
  - E.g. $\mathcal{T}_{\mathcal{LIRA}}$ + Nelson-Oppen $\mathcal{T}_i$

Optimization Modulo Theory

- OMT is "much younger" than SMT: first proposal in 2006
  - R. Nieuwenhuis and A. Oliveras. *On SAT Modulo Theories and Optimization Problems*. In SAT, volume 4121 of LNCS. Springer, 2006
- Nowadays different OMT proposals (see Sebastiani et al. works)
  - Max-SMT
  - Bit-vectors
  - Floating points
  - ...
- Some state-of-the-art SMT solvers natively provide OMT capabilities (Z3, OptiMathSAT) but others still don't (e.g. CVC5)
- Let's see an example by R. Sebastiani of OMT($\mathcal{LRA}$) with linear search

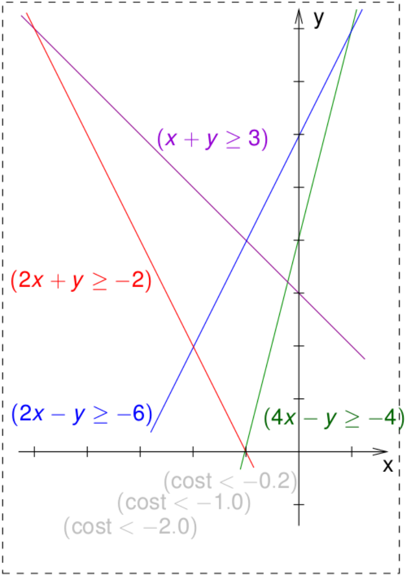
OMT($\mathcal{LRA}$) example

[w. pure-literal filt. $\Longrightarrow$ partial assignments]

- OMT($\mathcal{LRA}$) problem:

$$
\begin{array}{rl}
\varphi \stackrel{\text{def}}{=} & (\neg A_1 \vee (2x + y \geq -2)) \\
\wedge & (\;\; A_1 \vee (x + y \geq 3)) \\
\wedge & (\neg A_2 \vee (4x - y \geq -4)) \\
\wedge & (\;\; A_2 \vee (2x - y \geq -6)) \\
\wedge & (\mathrm{cost} < -0.2) \\
\wedge & (\mathrm{cost} < -1.0) \\
\wedge & (\mathrm{cost} < -2.0) \\
\mathrm{cost} \stackrel{\text{def}}{=} & x
\end{array}
$$

- $\mu = \left\{ \begin{array}{l} A_1, \neg A_1, \quad A_2, \neg A_2, \\ (4x - y \geq -4), \\ (x + y \geq 3), \\ (2x + y \geq -2), \\ (2x - y \geq -6) \\ (\mathrm{cost} < -0.2) \\ (\mathrm{cost} < -1.0) \\ (\mathrm{cost} < -2.0) \end{array} \right\}$



OMT($\mathcal{LRA}$) example

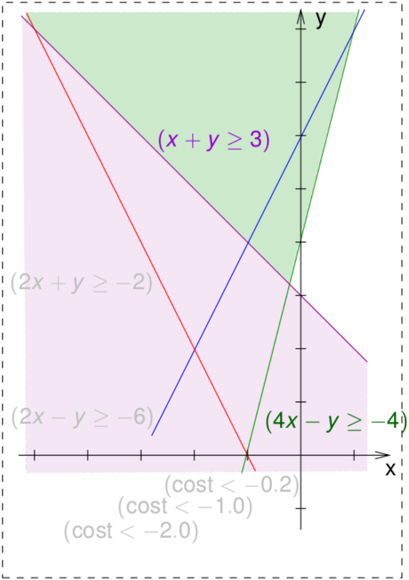
[w. pure-literal filt. $\Longrightarrow$ partial assignments]

- OMT($\mathcal{LRA}$) problem:

$$
\begin{array}{rcl}
\varphi & \stackrel{\text{def}}{=} & (\neg A_1 \vee (2x + y \geq -2)) \\
& \wedge & (\ A_1 \vee (x + y \geq 3)) \\
& \wedge & (\neg A_2 \vee (4x - y \geq -4)) \\
& \wedge & (\ A_2 \vee (2x - y \geq -6)) \\
& \wedge & (cost < -0.2) \\
& \wedge & (cost < -1.0) \\
& \wedge & (cost < -2.0) \\
cost & \stackrel{\text{def}}{=} & x
\end{array}
$$

- $\mu = \left\{ \begin{array}{l} A_1, \neg A_1, \quad A_2, \neg A_2, \\ (4x - y \geq -4), \\ (x + y \geq 3), \\ (2x + y \geq -2), \\ (2x - y \geq -6) \\ (cost < -0.2) \\ (cost < -1.0) \\ (cost < -2.0) \end{array} \right\}$

$\Longrightarrow$ SAT, $min = -0.2$



OMT($\mathcal{LRA}$) example

[w. pure-literal filt. $\Longrightarrow$ partial assignments]

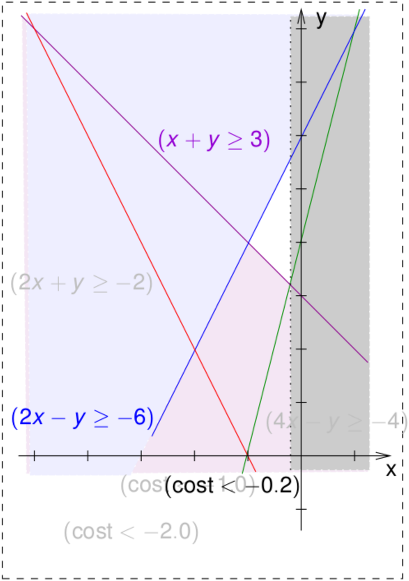
- OMT($\mathcal{LRA}$) problem:

$$
\begin{array}{rl}
\varphi \stackrel{\text{def}}{=} & (\neg A_1 \vee (2x + y \geq -2)) \\
\wedge & (\ \ A_1 \vee (x + y \geq 3)) \\
\wedge & (\neg A_2 \vee (4x - y \geq -4)) \\
\wedge & (\ \ A_2 \vee (2x - y \geq -6)) \\
\wedge & (cost < -0.2) \\
\wedge & (cost < -1.0) \\
\wedge & (cost < -2.0)
\end{array}
$$

$$cost \stackrel{\text{def}}{=} x$$

- $$\mu = \left\{ \begin{array}{l} A_1, \neg A_1, \quad A_2, \neg A_2, \\ (4x - y \geq -4), \\ (x + y \geq 3), \\ (2x + y \geq -2), \\ (2x - y \geq -6) \\ (cost < -0.2) \\ (cost < -1.0) \\ (cost < -2.0) \end{array} \right\}$$

$\Longrightarrow$ SAT, $min = -1.0$



OMT($\mathcal{LRA}$) example

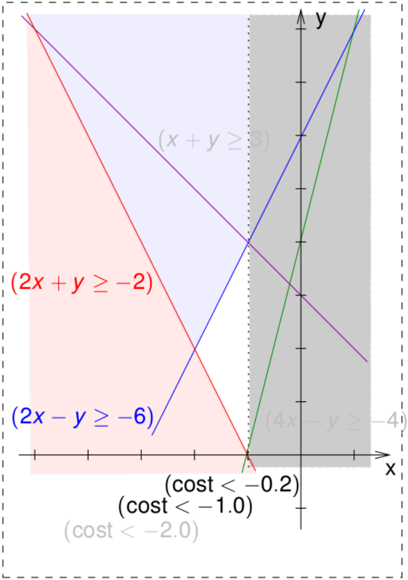
[w. pure-literal filt. $\Longrightarrow$ partial assignments]

- OMT($\mathcal{LRA}$) problem:

$$\begin{array}{rll} \varphi & \stackrel{\text{def}}{=} & (\neg A_1 \vee (2x + y \geq -2)) \\ & \wedge & (\ \ A_1 \vee (x + y \geq 3)) \\ & \wedge & (\neg A_2 \vee (4x - y \geq -4)) \\ & \wedge & (\ \ A_2 \vee (2x - y \geq -6)) \\ & \wedge & (\text{cost} < -0.2) \\ & \wedge & (\text{cost} < -1.0) \\ & \wedge & (\text{cost} < -2.0) \\ \text{cost} & \stackrel{\text{def}}{=} & x \end{array}$$

- $\mu = \left\{ \begin{array}{l} A_1, \neg A_1, \ \ A_2, \neg A_2, \\ (4x - y \geq -4), \\ (x + y \geq 3), \\ (2x + y \geq -2), \\ (2x - y \geq -6) \\ (\text{cost} < -0.2) \\ (\text{cost} < -1.0) \\ (\text{cost} < -2.0) \end{array} \right\}$

$\Longrightarrow$ SAT, $min = -2.0$



OMT($\mathcal{LRA}$) example

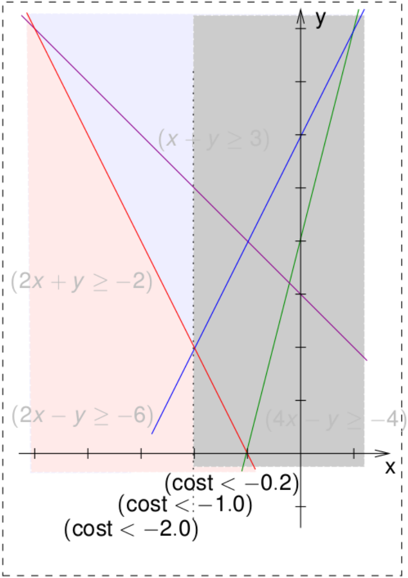
[w. pure-literal filt. $\Longrightarrow$ partial assignments]

- OMT($\mathcal{LRA}$) problem:

$$
\begin{array}{rcl}
\varphi & \stackrel{\text{def}}{=} & (\neg A_1 \vee (2x + y \geq -2)) \\
& \wedge & (\ \ A_1 \vee (x + y \geq 3)) \\
& \wedge & (\neg A_2 \vee (4x - y \geq -4)) \\
& \wedge & (\ \ A_2 \vee (2x - y \geq -6)) \\
& \wedge & (\text{cost} < -0.2) \\
& \wedge & (\text{cost} < -1.0) \\
& \wedge & (\text{cost} < -2.0) \\
\text{cost} & \stackrel{\text{def}}{=} & x
\end{array}
$$

- $\mu = \left\{ \begin{array}{l} A_1, \neg A_1, \quad A_2, \neg A_2, \\ (4x - y \geq -4), \\ (x + y \geq 3), \\ (2x + y \geq -2), \\ (2x - y \geq -6) \\ (\text{cost} < -0.2) \\ (\text{cost} < -1.0) \\ (\text{cost} < -2.0) \end{array} \right\}$

$\Longrightarrow$ UNSAT, $min = -2.0$



Offline OMT($\mathcal{LRA}$)

- Linear search repeatedly narrows the cost domain $[l_i, u_i)$ by adding $cost < c_i$ if a model with cost c_i is found at the i-th iteration
  - If no model is found, c_i is the minimum cost
- Binary search picks a pivot $p_i \in [l_i, u_i)$ and adds $cost < p_i$
  - $p_i \simeq (l_i + u_i)/2$
  - If no model is found, look into $[p_i, u_i)$
  - Can be more efficient, but we must know the cost bounds
- This approach is called offline because the SMT solvers used to find the models are black-boxes
  - No need to change their internals

Offline OMT($\mathcal{LRA}$)

Algorithm 1 Offline OMT($\mathcal{LA}(\mathbb{Q})$) Procedure based on Mixed Linear/Binary Search.

```
Require: ⟨φ, cost, lb, ub⟩ {ub can be +∞, lb can be −∞}
l ← lb; u ← ub; PIV ← ⊤; M ← ∅
φ ← φ ∪ {¬(cost < l), (cost < u)}
while (l < u ) do
    if (BinSearchMode()) then {Binary-search Mode}
        pivot ← ComputePivot(l, u)
        PIV ← (cost < pivot)
        φ ← φ ∪ {PIV}
        ⟨res, μ⟩ ← SMT.IncrementalSolve(φ)
        η ← SMT.ExtractUnsatCore(φ)
    else {Linear-search Mode}
        ⟨res, μ⟩ ← SMT.IncrementalSolve(φ)
        η ← ∅
    end if
    if (res = SAT) then
        ⟨M, u⟩ ← Minimize(cost, μ)
        φ ← φ ∪ {(cost < u)}                  u = current best bound
    else {res = UNSAT }
        if (PIV ∉ η) then
            l ← u                             Linear search completed
        else
            l ← pivot
            φ ← φ \ {PIV}
            φ ← φ ∪ {¬PIV}                    Updating binary search pivot
        end if
    end if
end while
return ⟨M, u⟩
```

From R. Sebastiani, S. Tomasi: *Optimization Modulo Theories with Linear Rational Costs*. ACM Trans. Comput. Log. 16(2): 12:1-12:43 (2015)

Inline OMT($\mathcal{LRA}$)

- The minimal cost is computed by a minimizer over linear rational inequalities
  - E.g., standard simplex techniques
- The offline approach can be improved by an inline schema
  - More efficient, but it requires modifying the internals of SMT solver
- In a nutshell, the inline approach integrates the optimization procedure into the SMT solver

Take-home messages

- Different theory solvers have been developed for different theories
  - E.g. EUF, DL, LRA, LIA, ...
- We often need to combine theories
  - Under certain conditions, Nelson-Oppen procedure can be used
- SMT solving can be optimized and extended
  - Optimization modulo theory

Resources

- Handbook of Satisfiability – Chapter 12 "*Satisfiability Modulo Theories*" by C. Barrett, R. Sebastiani, S.A. Seshia, C. Tinelli
  - Search "Satisfiability Modulo Theories - EECS at UC Berkeley"
- Barrett, Clark, and Cesare Tinelli. "Satisfiability modulo theories." Handbook of model checking. Springer, Cham, 2018. 305-343.
- SAT/SMT schools
  - `https://sat-smt.in/`
- ...