Local Consistency, Constraint Propagation & Global Constraints

Constraint Propagation

- Examination of the constraints to remove incompatible (inconsistent) values from the domains of the future (unexplored) variables.
  - Values that cannot be part of any solution.

Local Consistency

- A form of inference which detects inconsistent partial assignments.
  - Local, because we examine individual constraints.
- Popular local consistencies are domain-based.
  - They detect inconsistent partial assignments of the form $X_i = j$, hence:
    - j can be removed from $D(X_i)$ via propagation;
    - propagation can be implemented easily.

Generalised Arc Consistency (GAC)

- Also referred to as hyper-arc or domain consistency.
- A constraint C defined on k variables $C(X_1, \ldots, X_k)$ gives the set of allowed combinations of values (i.e. tuples).
  - $C \subseteq D(X_1) \times \ldots \times D(X_k)$
  - Each allowed tuple $(d_1, \ldots, d_k) \in C$ is a support for C.
- $C(X_1, \ldots, X_k)$ is GAC iff:
  - for all X_i in $\{X_1, \ldots, X_k\}$, for all $v \in D(X_i)$, v belongs to a support.
- Arc Consistency (AC) when $k = 2$.
- A CSP is GAC iff all its constraints are GAC.

Examples

- $D(X_1) = \{1,2,3\}$, $D(X_2) = \{2,3,4\}$, C: $X_1 = X_2$
  - AC(C)?
    - $1 \in D(X_1)$ and $4 \in D(X_2)$ do not have a support.
    - $X_1 = 1$ and $X_2 = 4$ are inconsistent partial assignments.

- $D(X_1) = \{1,2,3\}$, $D(X_2) = \{1,2\}$, $D(X_3) = \{1,2\}$, C: alldifferent($[X_1, X_2, X_3]$)
  - GAC(C)?
    - $1 \in D(X_1)$ and $2 \in D(X_1)$ do not have support.
    - $X_1 = 1$ and $X_1 = 2$ are inconsistent partial assignments.

Constraint Propagation

- A local consistency notion defines the properties that a constraint C must satisfy after constraint propagation.
  - The operational behaviour is completely left open (any algorithm could do).
  - The only requirement is to achieve the required property on C.
- A constraint propagation algorithm achieves a certain level of consistency by removing the inconsistent values from the domains of the variables in C.

Examples

- $D(X_1) = \{1,2,3\}$, $D(X_2) = \{2,3,4\}$, C: $X_1 = X_2$
  - AC(C)?
    - $1 \in D(X_1)$ and $4 \in D(X_2)$ do not have a support.
    - $X_1 = 1$ and $X_2 = 4$ are inconsistent partial assignments.
    - $D(X_1) = \{\cancel{1},2,3\}$, $D(X_2) = \{2,3,\cancel{4}\}$, C is now AC.

- $D(X_1) = \{1,2,3\}$, $D(X_2) = \{1,2\}$, $D(X_3) = \{1,2\}$, C: alldifferent($[X_1, X_2, X_3]$)
  - GAC(C)?
    - $1 \in D(X_1)$ and $2 \in D(X_1)$ do not have support.
    - $X_1 = 1$ and $X_1 = 2$ are inconsistent partial assignments.
    - $D(X_1) = \{\cancel{1},\cancel{2},3\}$, $D(X_2) = D(X_3) = \{1,2\}$, C is now GAC.

Constraint Propagation

- The targeted level of consistency depends on C.
  - GAC if an efficient propagation algorithm can be developed.
  - Otherwise BC or a lower level of consistency.

Bounds Consistency (BC)

- Defined for totally ordered (e.g. integer) domains.
- Relaxes the domain of X_i from $D(X_i)$ to $[\min(X_i)..\max(X_i)]$.
  - E.g., $D(X_i) = \{1,3,5\} \rightarrow [1..5]$
- A bound support is a tuple $(d_1,\ldots,d_k) \in C$ where $d_i \in [\min(X_i)..\max(Xi)]$.
- $C(X_1,\ldots, X_k)$ is BC iff:
  - For all X_i in $\{X_1,\ldots, X_k\}$, $\min(X_i)$ and $\max(X_i)$ belong to a bound support.

Bounds Consistency (BC)

- Disadvantage
  - BC might not detect all GAC inconsistencies in general.
    - We need to search more.
- Advantages
  - Might be easier to look for a support in a range than in a domain.
  - Achieving BC is often cheaper than achieving GAC.
    - Of interest in arithmetic constraints defined on integer variables with large domains.
  - Achieving BC is enough to achieve GAC for monotonic constraints.

(G)AC vs BC

- $D(X_1) = D(X_2) = D(X_3) = \{1,3\}$
  C: alldifferent($[X_1, X_2, X_3]$)
  - BC(C)
    - All $\min(X_i)$ and $\max(X_i)$ belong to a bound support.
    - No domain reduction with BC propagation.
  - Not GAC(C)
    - None of $\min(X_i)$ and $\max(X_i)$ belongs to a support.
    - C fails with GAC propagation.
- $D(X_1) = [2..6]$, $D(X_2) = [1..5]$
  C: $X_1 \leq X_2$
  - All values of $D(X_1) \leq \max(X_2)$ are AC.
  - All values of $D(X_2) \geq \min(X_1)$ are AC.
  - Enough to adjust $\max(X_1)$ and $\min(X_2)$.
    - $\max(X_1) \leq \max(X_2)$
    - $\min(X_1) \leq \min(X_2)$
    - $D(X_1) = [2..5]$, $D(X_2) = [2..5]$
    - AC(C), BC(C)

Propagation Algorithms

- When solving a CSP with multiple constraints:
  - propagation algorithms interact;
  - a propagation algorithm can wake up an already propagated constraint to be propagated again!
  - in the end, propagation reaches a fixed-point and all constraints reach a level of consistency;
  - the whole process is referred as constraint propagation.

Example

- $D(X_1) = D(X_2) = D(X_3) = \{1,2,3\}$
  C_1: alldifferent($[X_1, X_2, X_3]$) C_2: $X_2 < 3$ C_3: $X_3 < 3$
- Let's assume:
  - the order of propagation is C_1, C_2, C_3;
  - propagation algorithms maintain (G)AC.
- Propagation of C_1:
  - nothing happens, C_1 is GAC.
- Propagation of C_2:
  - $D(X_2) = \{1,2,\cancel{3}\}$, C_2 is now AC.
- Propagation of C_3:
  - $D(X_3) = \{1,2,\cancel{3}\}$, C_3 is now AC.
- C_1 is not GAC anymore, because the supports of $\{1,2\} \in D(X_1)$ in $D(X_2)$ and $D(X_3)$ are removed by the propagation of C_2 and C_3.
- Re-propagation of C_1:
  - $D(X_1) = \{\cancel{1},\cancel{2},3\}$, C_1 is now GAC.

Properties of Propagation Algorithms

- It may not be enough to remove inconsistent values from domains once.
- A propagation algorithm must wake up when necessary, otherwise may not achieve the desired local consistency property.
- Events that may trigger a constraint propagation:
  - when a variable is assigned
  - when the domain of a variable changes (for GAC);
  - when the domain bounds of a variable changes (for BC);
  - ...

Example

- $D(X_1) = D(X_2) = D(X_3) = \{1,2,3\}$
  C_1: alldifferent($[X_1, X_2, X_3]$) C_2: $X_2 \neq 2$ C_3: $X_3 \neq 2$
- Let's assume:
  - the order of propagation is C_1, C_2, C_3;
  - C_1 propagation algorithm maintains BC, the others AC.
- Propagation of C_1:
  - nothing happens, C_1 is BC.
- Propagation of C_2:
  - $D(X_2) = \{1,\cancel{2},3\}$, C_2 is now AC.
- Propagation of C_3:
  - $D(X_3) = \{1,\cancel{2},3\}$, C_3 is now AC.
- Does the propagator of C_1 wake up again?
- What happens if search assigns $X_1 = 1$?

Complexity of Propagation Algorithms

- Assume $|D(X_i)| = d$.
- Following the definition of the local consistency property:
  - one time AC propagation on a C(X_1,X_2) takes $O(d^2)$ time.
- We can do better!

Examples

- C: $X_1 = X_2$
  - $D(X_1) = D(X_2) = D(X_1) \cap D(X_2)$
    - Complexity: the cost of the set intersection operation
    - When should the propagation algorithm wake up?
- C: $X_1 \neq X_2$
  - When $D(X_i) = \{v\}$, remove v from $D(X_j)$.
    - Complexity: O(1)
    - When should the propagation algorithm wake up?
- C: $X_1 \leq X_2$
  - $\max(X_1) \leq \max(X_2)$, $\min(X_1) \leq \min(X_2)$
    - Complexity: O(1)
    - When should the propagation algorithm wake up?

Specialized Propagation

- Propagation specific to a given constraint.
- Advantages
  - Exploits the constraint semantics.
  - Potentially much more efficient than a general propagation approach.
- Disadvantages
  - Limited use.
  - Not always easy to develop one.
- Worth developing for recurring constraints.

Global Constraints

- Capture complex, non-binary and recurring combinatorial substructures arising in a variety of applications.
- Embed specialized propagation which exploits the substructure.

Benefits of Global Constraints

- Modelling benefits
  - Reduce the gap between the problem statement and the model.
  - May allow the expression of constraints that are otherwise not possible to state using primitive constraints (semantic).
- Solving benefits
  - Strong inference in propagation (operational).
  - Efficient propagation (algorithmic).

Some Groups of Global Constraints

- Counting
- Sequencing
- Scheduling
- Ordering
- Balancing
- Distance
- Packing
- Graph-based
- …

Counting Constraints

- Restrict the number of variables satisfying a condition or the number of times values are taken.

Alldifferent Constraint

- alldifferent($[X_1, X_2, \ldots, X_k]$) holds iff
  $X_i \neq X_j$ for $i < j \in \{1,\ldots,k\}$
  - permutation constraint with $|D(X_i)| = k$.
  - alldifferent([3,5,2,1,4])

- Useful in a variety of context, like:
  - puzzles and graph problems (e.g., sudoku and map colouring);
  - timetabling (e.g. allocation of activities to different slots);
  - tournament scheduling (e.g. a team can play at most once in a week);
  - configuration (e.g. a particular product cannot have repeating components).

Global Cardinality Constraint

- Constrains the number of times each value is taken by the variables.
- gcc($[X_1, X_2, \ldots, X_k], [v_1, \ldots, v_m], [O_1, \ldots, O_m]$) iff

  forall $j \in \{1,\ldots, m\}$ $O_j = |\{X_i \mid X_i = v_j, 1 \leq i \leq k\}|$
  - gcc([1, 1, 3, 2, 3], [1, 2, 3, 4], [2, 1, 2, 0])
  - alldifferent when $O_j \leq 1$.

- Useful e.g. in:
  - resource allocation (e.g. limit the usage of each resource).

Among Constraint

- Constrains the number of variables taking certain values.
- among($[X_1, X_2, \ldots, X_k], v, l, u$) iff

$$l \leq |\{i \mid X_i \in v, 1 \leq i \leq k\}| \leq u$$

  - among([1, 5, 3, 2, 5, 4], {1,2,3,4}, 3, 4)
- Useful in sequencing problems, as we see next.

Sequencing Constraints

- Ensure a sequence of variables obey certain patterns.

Sequence Constraint

- Constrains the number of values taken from a given set in any subsequence of q variables.
- sequence(l, u, q, [X_1, X_2, …, X_k], v) iff
  among([X_i, X_{i+1}, …, X_{i+q-1}], v, l, u) for $1 \leq i \leq k-q+1$
  - Known also as amongseq constraint.
  - sequence(1, 2, 3, [1,0,2,0,3], {0,1})
- Useful e.g. in:
  - rostering (e.g. every employee has 2 days off in any 7 day of period);
  - production line (e.g. at most 1 in 3 cars along the production line can have a sun-roof fitted).

Scheduling Constraints

- Help schedule tasks with respective release times, duration, and deadlines, using limited resources in a time interval D.

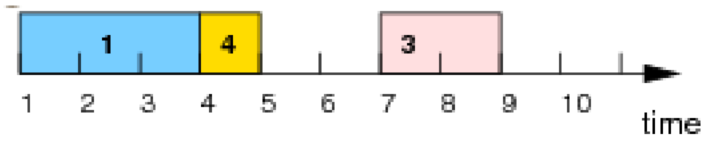
Disjunctive Resource Constraint

- Requires that tasks do not overlap in time.
  - Known also as noOverlap constraint.
- Given tasks $t_1, \ldots, t_k$, each associated with a start time S_i and duration D_i:

  disjunctive($[S_1, \ldots, S_k]$, $[D_1, \ldots, D_k]$) iff for all $i < j$

  $$(S_i + D_i \leq S_j) \vee (S_j + D_j \leq S_i)$$

- Useful when a resource can execute at most one task at a time.



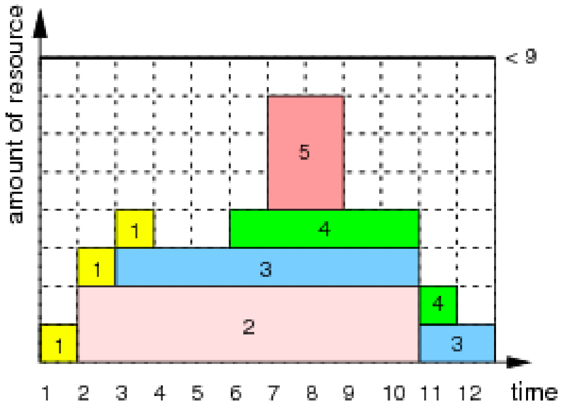
Cumulative Resource Constraint

- Constrains the usage of a shared resource.
- Given tasks $t_1, \ldots, t_k$, each associated with a start time S_i, duration D_i, resource requirement R_i, and a resource with a capacity C:

  cumulative($[S_1, \ldots, S_k]$, $[D_1, \ldots, D_k]$, $[R_1, \ldots, R_k]$, C) iff

  $\sum_{i \mid S_i \le u < S_i + D_i} R_i \le C$ forall u in D

- Useful when a fixed-capacity resource can execute multiple tasks at a time.



Ordering Constraints

- Enforce an ordering between the variables or the values.

Lexicographic Ordering Constraint

- Requires a sequence of variables to be lexicographically less than or equal to another sequence of variables.
- lex≤($[X_1, X_2, \ldots, X_k]$, $[Y_1, Y_2, \ldots, Y_k]$) holds iff:

  $X_1 \leq Y_1 \quad \wedge$

  $(X_1 = Y_1 \rightarrow X_2 \leq Y_2) \wedge$

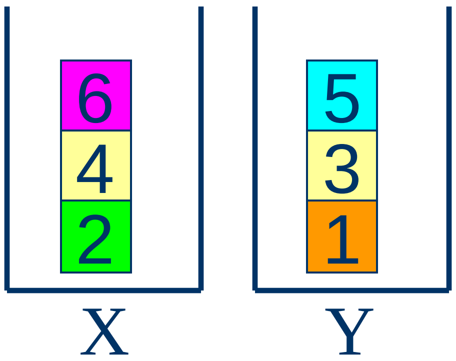
  $(X_1 = Y_1 \wedge X_2 = Y_2 \rightarrow X_3 \leq Y_3) \ldots$

  $(X_1 = Y_1 \wedge X_2 = Y_2 \ldots. \wedge X_{k-1} = Y_{k-1} \rightarrow X_k \leq Y_k)$

  - lex≤([1, 2, 4],[1, 3, 3])
- Useful for breaking symmetry.
  - lex≤(X , $\pi(X)$) for all π
  - lex≤(X , Y)

Lexicographic Ordering Constraint

- Consider the assignment of items to bins, which can be modelled by a matrix of Boolean variables:



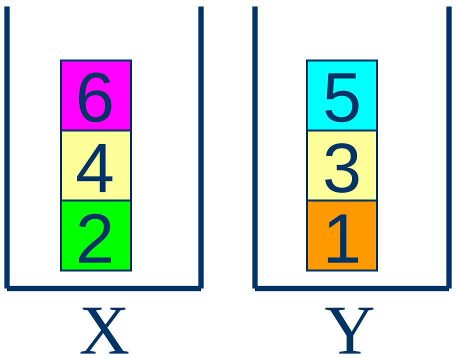
	i_1	i_2	i_3	i_4	i_5	i_6
X	0	1	0	1	0	1
Y	1	0	1	0	1	0

- What if the bins are symmetric?

Lexicographic Ordering Constraint

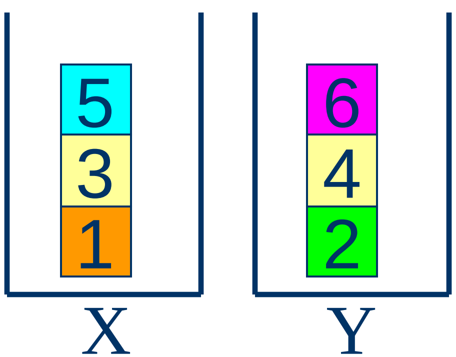
- Their item assignments can be permuted.

a)



	i_1	i_2	i_3	i_4	i_5	i_6
X	0	1	0	1	0	1
Y	1	0	1	0	1	0

b)

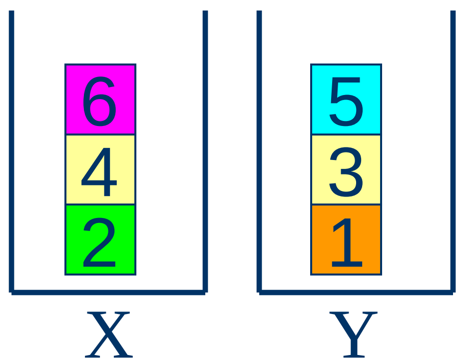


	i_1	i_2	i_3	i_4	i_5	i_6
X	1	0	1	0	1	0
Y	0	1	0	1	0	1

Lexicographic Ordering Constraint

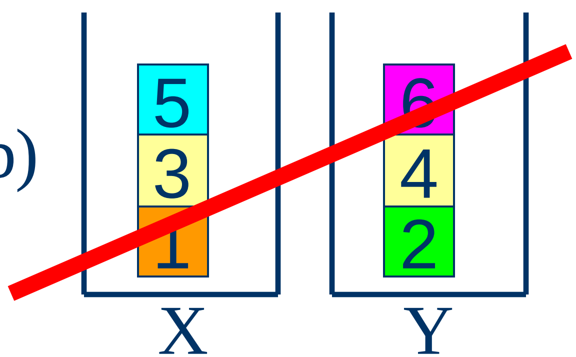
- lex≤(X , Y) avoids the symmetric assignments.

a)



	i_1	i_2	i_3	i_4	i_5	i_6
X	0	1	0	1	0	1
Y	1	0	1	0	1	0

b)

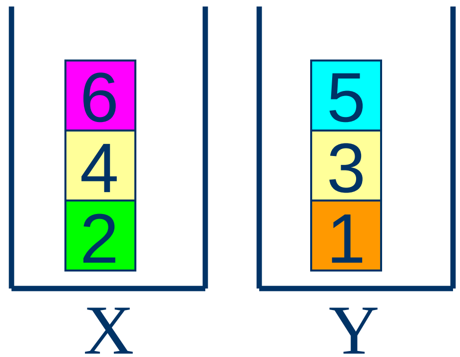


	i_1	i_2	i_3	i_4	i_5	i_6
X	1	0	1	0	1	0
Y	0	1	0	1	0	1

Lexicographic Ordering Constraint

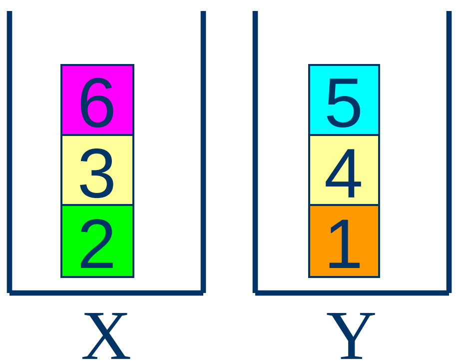
- What if items 3 and 4 are symmetric too?

a)



	i_1	i_2	i_3	i_4	i_5	i_6
X	0	1	0	1	0	1
Y	1	0	1	0	1	0

b)

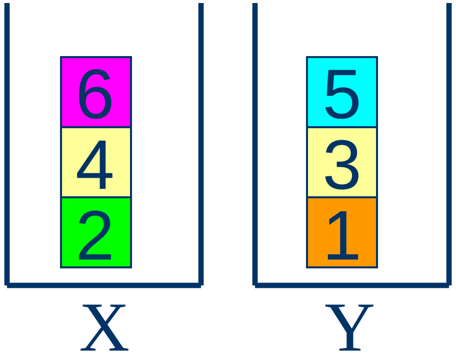


	i_1	i_2	i_3	i_4	i_5	i_6
X	0	1	1	0	0	1
Y	1	0	0	1	1	0

Lexicographic Ordering Constraint

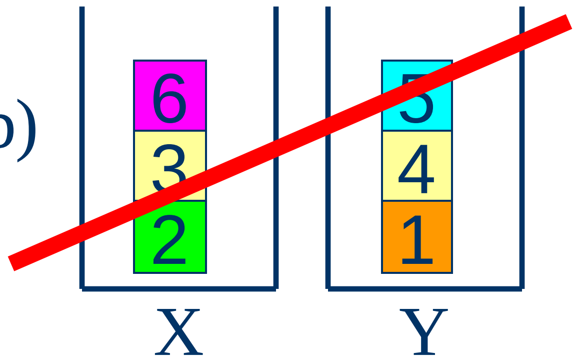
- lex≤(i_3 , i_4) avoids the symmetric assignments.

a)



	i_1	i_2	i_3	i_4	i_5	i_6
X	0	1	0	1	0	1
Y	1	0	1	0	1	0

b)



	i_1	i_2	i_3	i_4	i_5	i_6
X	0	1	1	0	0	1
Y	1	0	0	1	1	0

Specialized Propagation for Global Constraints

- How do we develop specialized propagation for global constraints?
- Two main approaches:
  - constraint decomposition;
  - dedicated propagation algorithm.

Constraint Decomposition

- A global constraint is decomposed into smaller and simpler constraints, each of which has a known propagation algorithm.
- Propagating each of the constraints gives a propagation algorithm for the original global constraint.
  - A very effective and efficient approach for some global constraints.

A Decomposition of Among

- among($[X_1, X_2, \ldots, X_k]$, v, N) iff
  $N = |\{i \mid X_i \in v, 1 \leq i \leq k\}|$
  - Decomposition as a conjunction of logical constraints and a sum constraint:

    B_i with $D(B_i) = \{0, 1\}$ for $1 \leq i \leq k$

    C_i: $B_i = 1 \leftrightarrow X_i \in v$ for $1 \leq i \leq k$

    C_{k+1}: $\sum_i B_i = N$
  - AC(C_i) for all i and BC(C_{k+1}) ensures GAC on among.

Constraint Decompositions

- May not always provide an effective propagation.
- Often GAC on the original constraint is stronger than (G)AC on the constraints in the decomposition.

A Decomposition of Alldifferent

- alldifferent($[X_1, X_2, \ldots, X_k]$)
  - Decomposition as a conjunction of difference constraints C_{ij}: $X_i \neq X_j$ for $i < j \in \{1,\ldots,k\}$
  - AC(C_{ij}) for all $i < j$ is weaker than GAC on alldifferent.
    - E.g., alldifferent($[X_1, X_2, X_3]$) with $D(X_1) = D(X_2) = D(X_3) = \{1,2\}$.
    - alldifferent is not GAC but the decomposition does not prune anything.

A Decomposition of Disjunctive

- disjunctive($[S_1, \ldots, S_k], [D_1, \ldots, D_k]$)
  - Decomposition as a conjunction of disjunctive constraints C_{ij}: $(S_i + D_i \leq S_j) \vee (S_j + D_j \leq S_i)$
    for $i < j \in \{1,\ldots,k\}$
  - Is GAC(C_{ij}) for all $i < j$ weaker than GAC on disjunctive?

Dedicated Propagation Algorithms

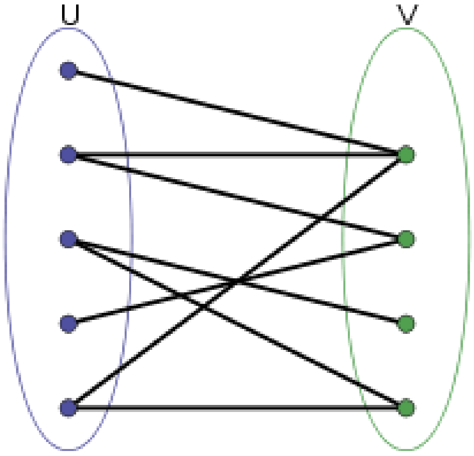
- Dedicated algorithms provide effective and efficient propagation.
- Often:
  - GAC is maintained in polynomial time;
  - many more inconsistent values are detected compared to the decompositions;
  - computation is done incrementally.

A Propagation Algorithm for alldifferent

- Jean-Charles Régin, “A Filtering Algorithm for Constraints of Difference in CSPs”, in the Proc. of AAAI’1994
  - Maintains GAC on alldifferent([X_1, X_2, …, X_k]) and runs in polynomial time.
  - Establishes a relation between the solutions of the constraint and the properties of a graph.
    - Maximal matching in a bipartite graph.
- A similar algorithm can be obtained with the use of flow theory.

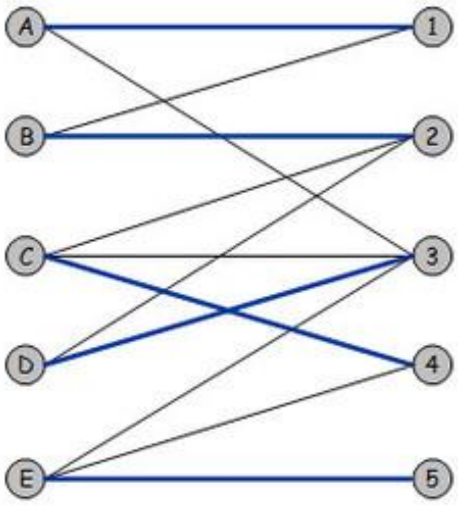
A Propagation Algorithm for alldifferent

- A bipartite graph is a graph whose vertices are divided into two disjoint sets U and V such that every edge connects a vertex in U to one in V.



A Propagation Algorithm for alldifferent

- A matching in a graph is a subset of its edges such that no two edges have a node in common.
  - Maximal matching is the largest possible matching.
- In a bipartite graph, maximal matching covers one set of nodes.

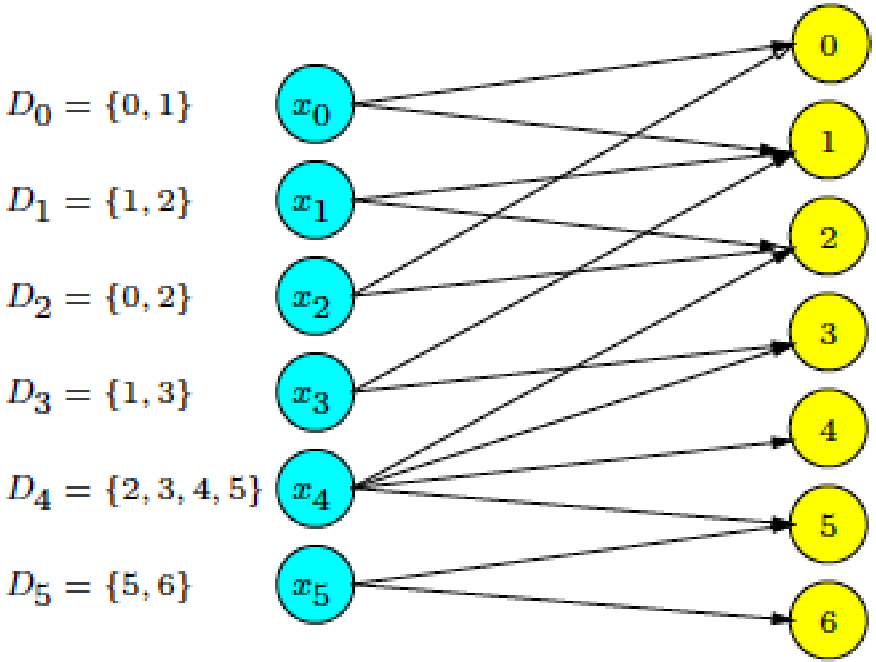


A Propagation Algorithm for alldifferent

- Observation
  - Given a bipartite graph G constructed between the variables [X_1, X_2, …, X_k] and their possible values (variable-value graph),
  - an assignment of values to the variables is a solution iff it corresponds to a maximal matching in G.
    - A maximal matching covers all the variables.
  - By computing all maximal matchings, we can find the consistent partial assignments.

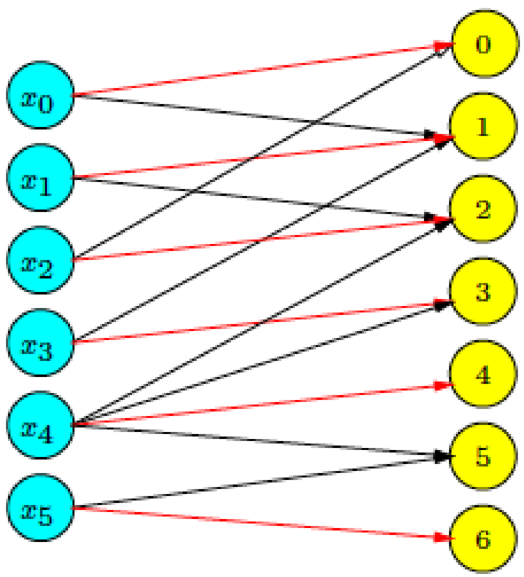
Example

Variable-value graph

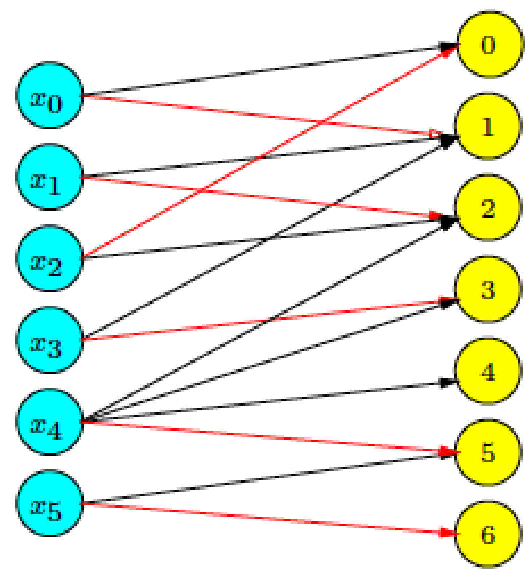


Some Maximal Matchings

$a(x_0) = 0$
$a(x_1) = 1$
$a(x_2) = 2$
$a(x_3) = 3$
$a(x_4) = 4$
$a(x_5) = 6$



$a(x_0) = 1$
$a(x_1) = 2$
$a(x_2) = 0$
$a(x_3) = 3$
$a(x_4) = 5$
$a(x_5) = 6$



All Maximal Matchings

- Inefficient to compute them naïvely.
- Theoretical results from matching theory to compute them efficiently.
  - One maximal matching can describe all maximal matchings!

Dedicated Propagation Algorithms

- GAC may as well be NP-hard!
  - E.g., gcc using variables for occurrences.
  - Algorithms which maintain weaker consistencies are of interest.
    - BC.
    - Between GAC and BC.
    - GAC on some variables, BC on others.
    - …

Dedicated Propagation Algorithms

- What if it is difficult to:
  - decompose a constraint;
  - build an efficient and effective dedicated algorithm?
- Consult global constraints for generic purposes!
  - E.g., table constraint.
    - Many solvers have efficient GAC algorithms.
    - Need to keep the table size small.

Crossword Puzzle Generation

- Valid words are defined in a table of compatible letters (i.e. dictionary).
  - table([X_1,X_2,X_3], dictionary)
  - table([X_1,X_{13},X_{16}], dictionary)
  - table([X_4,X_5,X_6,X_7], dictionary)
  - …
- No simple way to decide acceptable words other than to put them in a table.