Exercice 1 :

Consider the following machine :

initial, accept, reject

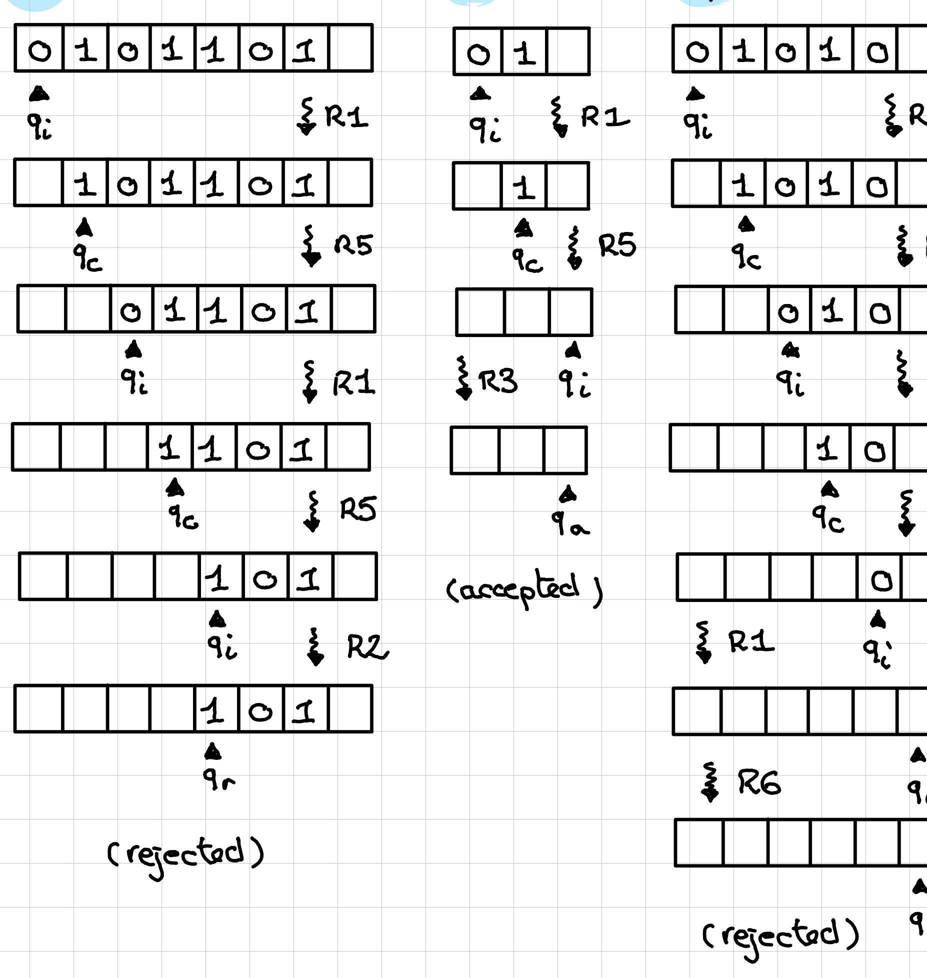
$\Sigma = \{0,1\}$ $\quad Q = \{q_i, q_c, q_a, q_r\}$

$\delta(q_i, 0) = (q_c, \square, R)$ (R1)
$\delta(q_i, 1) = (q_r, 1, S)$ (R2)
$\delta(q_i, \square) = (q_a, \square, S)$ (R3)

$\delta(q_c, 0) = (q_r, 0, S)$ (R4)
$\delta(q_c, 1) = (q_i, \square, R)$ (R5)
$\delta(q_c, \square) = (q_r, \square, S)$ (R6)

1) Are the following inputs accepted ?

a) b) c)



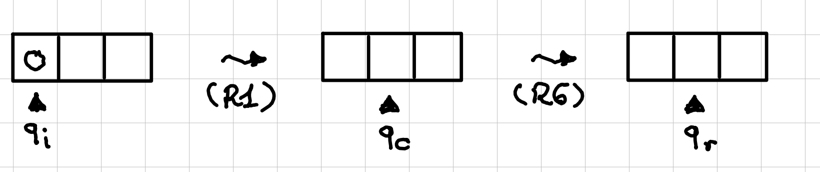
2) Does the machine always stops ?

<u>Lemma:</u> The machine stops in at most $|w| + 1$ steps.

<u>Proof:</u> By induction on the input word $w \in \{0,1\}^*$.

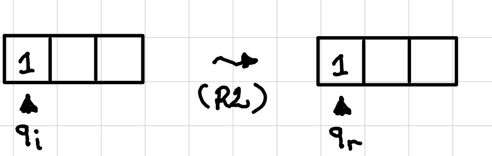
- $w = \varepsilon$: Then (R3) is applied, the state changes to q_a and the machine stops, which concludes this case since $|\varepsilon| + 1 = 0 + 1 = 1 \geqslant 1$.

- $w = 0$: Then the steps (R1) and (R6) are applied as follows :



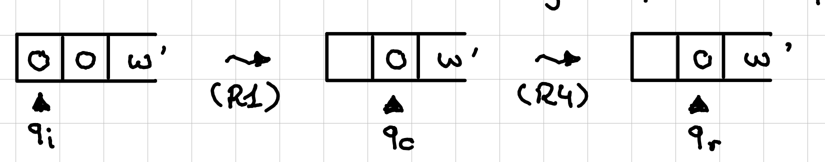
And the machine stops, which concludes this case since $|0| + 1 = 1 + 1 = 2 \geqslant 2$.

- $w = 1$: Then the step (R2) is applied :



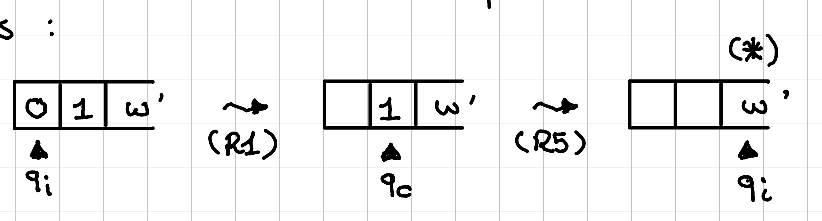
And the machine stops, which concludes this case since $|0| + 1 = 1 + 1 = 2 \geqslant 1$.

- $w = 00w'$: Then the following steps are applied:



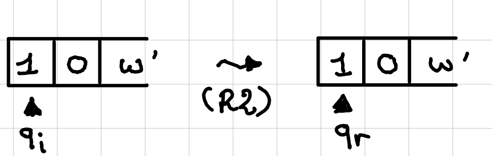
And the machine stops, which concludes this case since $|00w'| = 2 + |w'| \geqslant 2$.

- $w = 01w'$: The machine performs the following steps :



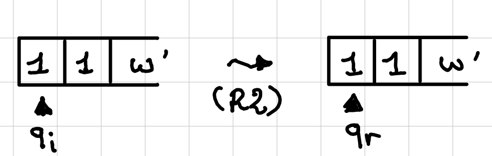
From (*), the induction hypothesis tells us that the machine stops in $|w'| + 1$ steps. We deduce that the machine on $|w|$ stops in at most $2 + |w'| + 1$ steps, which concludes this case since $2 + |w'| + 1 = |01w'| + 1 = |w| + 1$.

- $w = 10w'$: Then the following step is applied:



And the machine stops, which concludes this case since $|00w'| = 2 + |w'| \geqslant 1$.

- $w = 11w'$: Then the following step is applied:



And the machine stops, which concludes this case since $|00w'| = 2 + |w'| \geqslant 1$.

$\square$

3) What is the language accepted by the machine ?

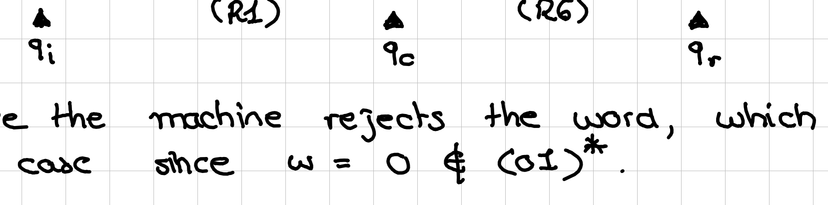
The machine accepts words which are repetitions of the word "01". Written otherwise, it accepts the language $(01)^*$.

<u>Lemma :</u> Let $w \in \{0,1\}^*$ be a word. Then w is accepted by the machine if and only if $w \in (01)^*$.

<u>Proof:</u> By induction on $w \in \{0,1\}^*$:

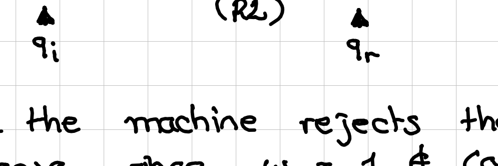
- $w = \varepsilon$: Then (R3) is applied and the machine accepts the word, which concludes this case since $\varepsilon \in (01)^*$.

- $w = 0$: Then the steps (R1) and (R6) are applied as follows :



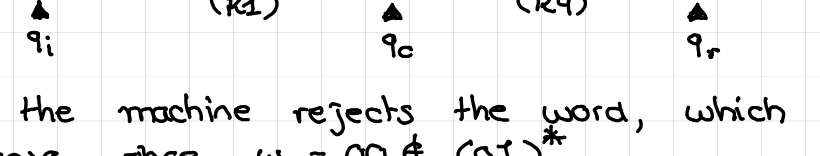
Hence the machine rejects the word, which concludes this case since $w = 0 \notin (01)^*$.

- $w = 1$: Then the step (R2) is applied :



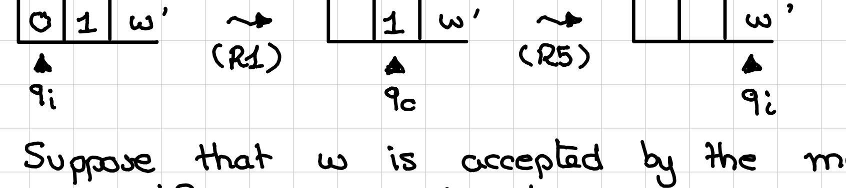
Hence the machine rejects the word, which concludes this case since $w = 1 \notin (01)^*$.

- $w = 00w'$: Then the following steps are applied:



Hence the machine rejects the word, which concludes this case since $w = 00 \notin (01)^*$.

- $w = 01w'$: The machine performs the following steps :

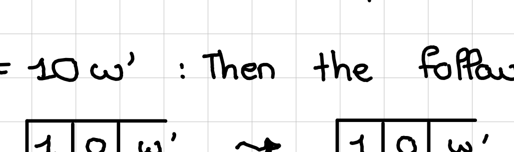


($\Rightarrow$) Suppose that w is accepted by the machine. Then necessarily (*) accepts, hence $w' \in (01)^*$ by the induction hypothesis. Therefore :

$$w = 01w' \in (01)(01)^* \subseteq (01)^*.$$

($\Leftarrow$) Suppose that $w \in (01)^*$, then necessarily $w' \in (01)^*$, hence (*) accepts and therefore the machine accepts w.

- $w = 10w'$: Then the following step is applied:



Hence the machine rejects the word, which concludes this case since $w = 10 \notin (01)^*$.

- $w = 11w'$: Then the following step is applied:

Hence the machine rejects the word, which concludes this case since $w = 11 \notin (01)^*$.

$\square$

4) Deduce the complexity of the membership problem for the set $(01)^*$.

We proved in question 2 that the machine is linear in time, and in question 3 that it is correct and complete, hence the problem is linear in time.

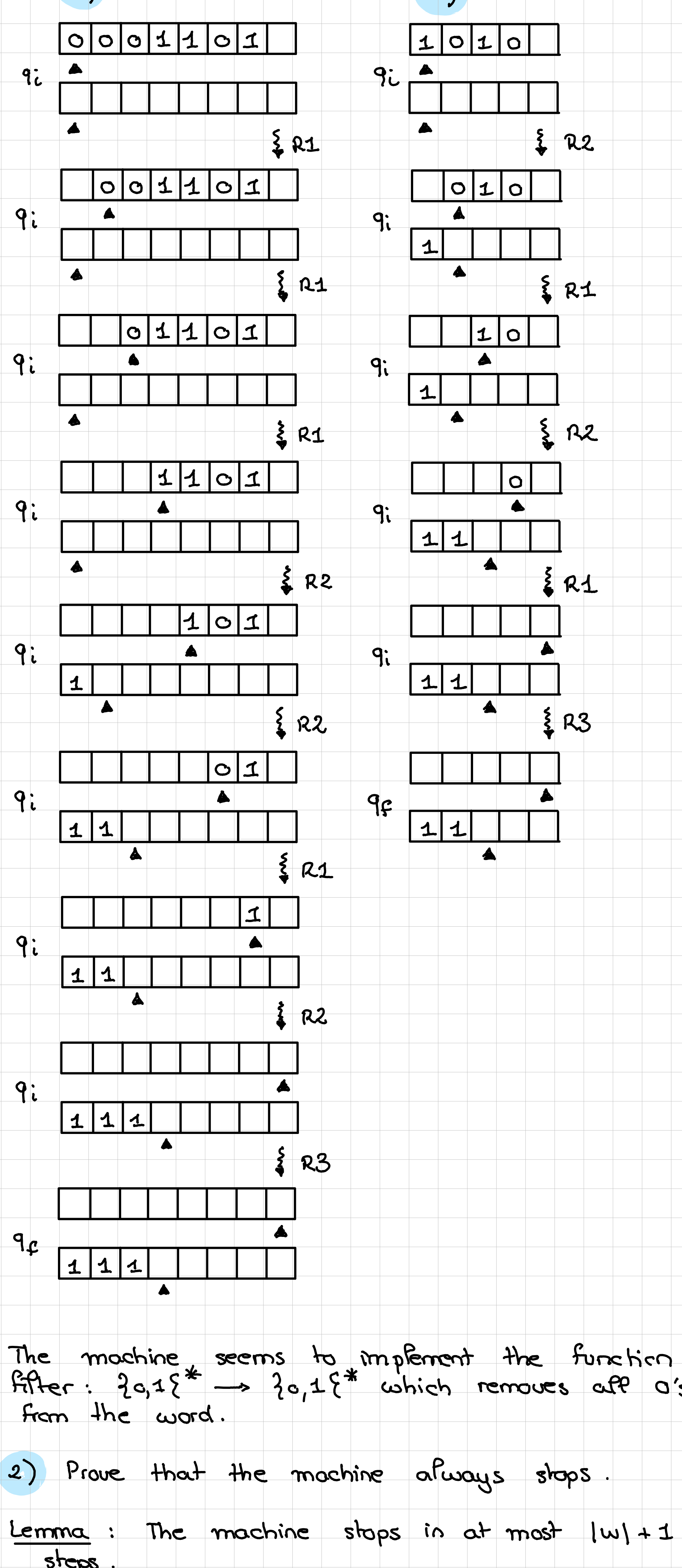
Exercice 2 :

$\Sigma = \{0, 1\}$ $\qquad$ $Q = \{ \underbrace{q_i}_{\text{initial}}, \underbrace{q_f}_{\text{final}} \}$

$$\delta(q_i, (0, \_)) = (q_i, (\square, \_), (R, S)) \qquad (R1)$$
$$\delta(q_i, (1, \_)) = (q_i, (\square, 1), (R, R)) \qquad (R2)$$
$$\delta(q_i, (\square, \_)) = (q_f, (\square, \_), (S, S)) \qquad (R3)$$

1) Run the machine on the following example and guess what it computes.

a) $\qquad$ b)



The machine seems to implement the function filter : $\{0,1\}^* \longrightarrow \{0,1\}^*$ which removes all 0's from the word.

2) Prove that the machine always stops.

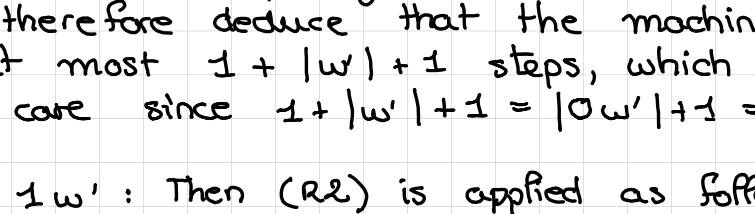
<u>Lemma</u> : The machine stops in at most $|w| + 1$ steps.

<u>Proof</u> : We slightly strengthen the property by stating that it hold watheuer is contained in the 2nd tape.

By induction on $w \in \{0,1\}^*$:

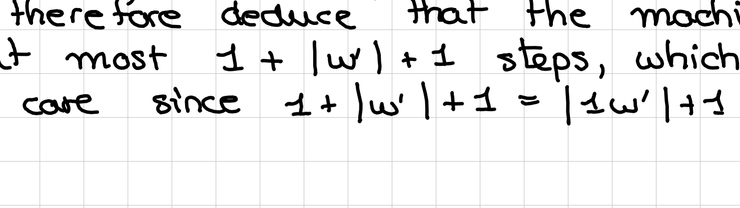
- $w = \varepsilon$: Then (R3) is applied, the states changes to q_f and the machine stops, which concludes this case since $|\varepsilon| + 1 = 1 \geq 1$.

- $w = 0w'$: Then (R1) is applied as follows :



If we disregards the 2nd tape, notice that (*) corresponds to the initial configuration of the machine on w'. Hence, it stops in at most $|w'|+1$ steps by the induction hypothesis. We therefore deduce that the machine stops in at most $1 + |w'| + 1$ steps, which concludes this case since $1 + |w'| + 1 = |0w'| + 1 = |w| + 1$.

- $w = 1w'$: Then (R2) is applied as follows :



If we disregards the 2nd tape, notice that (*) corresponds to the initial configuration of the machine on w'. Hence, it stops in at most $|w'|+1$ steps by the induction hypothesis. We therefore deduce that the machine stops in at most $1 + |w'| + 1$ steps, which concludes this case since $1 + |w'| + 1 = |1w'| + 1 = |w| + 1$.

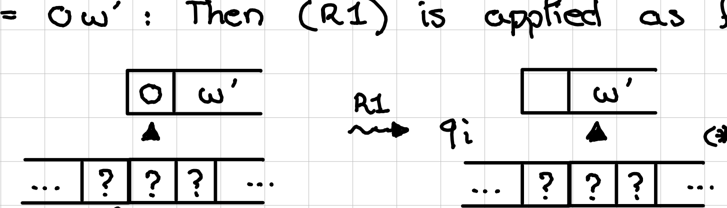
□

3) Prove the assertion you provided in question 1.

<u>Lemma</u> : The machine appends filter(w) on the 2nd tape without altering the left side of the 2nd tape.

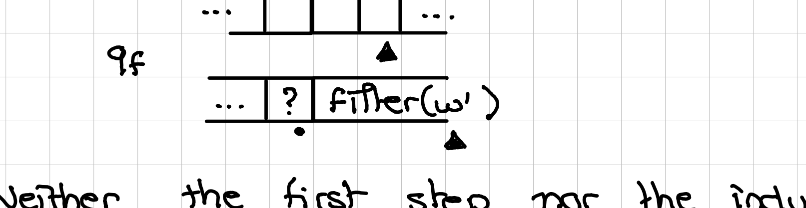
<u>Proof</u> : By induction on $w \in \{0,1\}^*$.

- $w = \varepsilon$: Then (R3) is performed as follows :

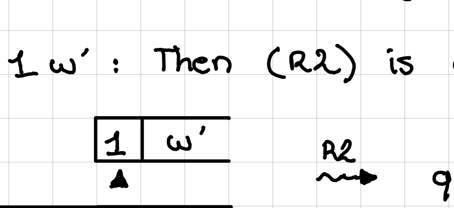


As expected, filter(ε) = ε was appened on the second tape and its left side was untouched.

- $w = 0w'$: Then (R1) is applied as follows :

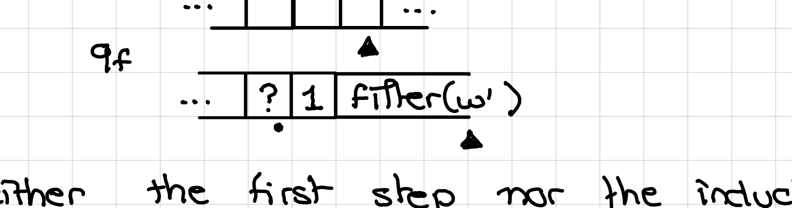


By the induction hypothesis, from (*) the machine appends filter(w) to the second tape without modifying its left side. Hence, we get :

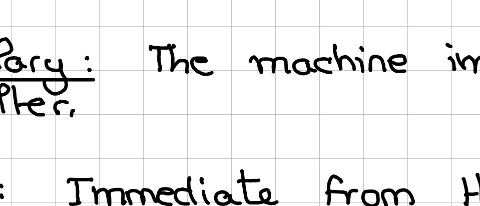


Neither the first step nor the inductive one affected the left side, hence it is left untouched, thus concluding this case since filter(w) = filter(0w') = filter(w')

- $w = 1w'$: Then (R2) is applied as follows :



By the induction hypothesis, from (*) the machine appends filter(w) to the second tape without modifying its left side. Hence, we get :



Neither the first step nor the inductive one affected the left side, hence it is left untouched, thus concluding this case since filter(w) = filter(1w') = 1 filter(w'). □

<u>Corollary</u> : The machine implements the function filter.

<u>Proof</u> : Immediate from the previous lemma and the empty 2nd tape on initial config.

Hence $\mathcal{L}(M) = \{ \text{filter}(w) \mid w \in \{0,1\}^* \}$,

4) Deduce the complexity of the problem of filtering 0's from a binary word.

We proved in question 2 that the machine is linear in time, and in question 3 that it is correct and complete, hence the problem is linear in time.

Exercice 3:

1) Write a 2-tape TM which accepts words with shape $0^n 1^n$.

$$\delta(q_{init}, (-,-)) = (q_{copy}, (-,-), (S,S))$$

Copy: (cut the first par to tape 2)

$$\delta(q_{copy}, (0,-)) = (q_{copy}, (\square, 0), (R,R))$$
$$\delta(q_{copy}, (1,-)) = (q_{comp}, (-,-), (S,L))$$
$$\delta(q_{copy}, (\square,-)) = (q_{comp}, (-,-), (S,L))$$

Compare: (compare tape 1 and 2).

$$\delta(q_{comp}, (1,0)) = (q_{comp}, (\square,\square), (R,L))$$
$$\delta(q_{comp}, (0,-)) = (q_{fail}, -)$$
$$\delta(q_{comp}, (-,1)) = (q_{fail}, -)$$
$$\delta(q_{comp}, (\square,\square)) = (q_{succ}, (\square,\square), (S,S)).$$

Bonus: Show that it is terminating, and that it only accepts words of the shape $0^n 1^n$ for $n \in \mathbb{N}$.

Exercise 4:

1) Write a 2-tape TM which recognizes palindromes of $\{0,1\}^*$.

We use the 2nd tape as a way to read the word backwards. Hence, we will first copy the word on the second tape, and then read it from the end to the start.

<u>Copy</u>: (copy tape 1 on tape 2)

(C1) $\delta(q_{init}, (-,-)) = (q_{copy}, (-,-), (S,S))$

(C2) $\delta(q_{copy}, (0,-)) = (q_{copy}, (0,0), (R,R))$

(C3) $\delta(q_{copy}, (1,-)) = (q_{copy}, (1,1), (R,R))$

(C4) $\delta(q_{copy}, (\square,-)) = (q_{move}, (-,-), (L,L))$

<u>Move</u>: (move cursor tape 1 to start)

(M1) $\delta(q_{move}, (0,-)) = (q_{move}, (0,0), (L,S))$

(M2) $\delta(q_{move}, (1,-)) = (q_{move}, (1,1), (L,S))$

(M3) $\delta(q_{move}, (\square,-)) = (q_{comp}, (-,-), (R,S))$

<u>Compare</u>: (compare tape 1 and tape 2)

(P1) $\delta(q_{comp}, (0,0)) = (q_{comp}, (\square,\square), (R,L))$

(P2) $\delta(q_{comp}, (1,1)) = (q_{comp}, (\square,\square), (R,L))$

(P3) $\delta(q_{comp}, (\square,\square)) = (q_{acc}, (\square,\square), (S,S))$

(P4) $\delta(q_{comp}, (0,1)) = (q_{fail}, (0,1), (S,S))$

(P5) $\delta(q_{comp}, (0,\square)) = (q_{fail}, (0,\square), (S,S))$ (*)

(P6) $\delta(q_{comp}, (1,0)) = (q_{fail}, (1,0), (S,S))$

(P7) $\delta(q_{comp}, (1,\square)) = (q_{fail}, (1,\square), (S,S))$ (*)

(P8) $\delta(q_{comp}, (\square,0)) = (q_{fail}, (\square,0), (S,S))$ (*)

(P9) $\delta(q_{comp}, (\square,1)) = (q_{fail}, (\square,1), (S,S))$ (*)

(*) cannot happen in practice.

2) Prove that the machine stops.

We prove this on each phase.

<u>Lemma 1</u>: Every run starting with q_{comp} stops in at most $|w|+1$ steps, where w is the word on the first tape.

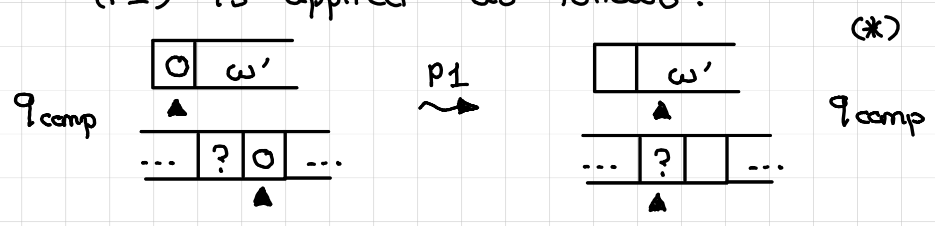
By induction on $w \in \{0,1\}^*$:

- $w = \varepsilon$: Then either (P3), (P8) or (P9) is applied and the machine stops, thus concluding this case since $|w|+1 = |\varepsilon|+1 = 1 \geq 1$.

- $w = 0w'$: We distinguish two cases:

  - (P4) or (P5) is applied: Then the machine stops thus concluding this case since $|w|+1 = |0w'|+1 \geq 1$.

  - (P1) is applied as follows:

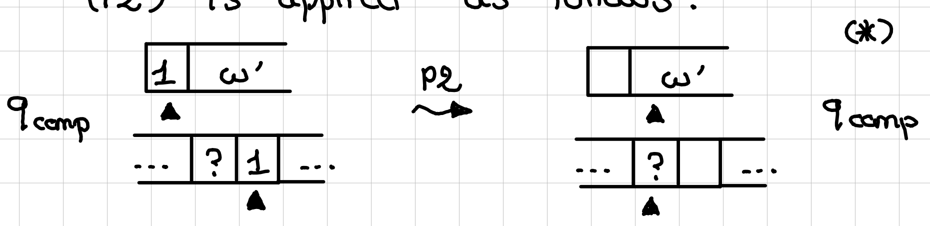


By ih, one has that from (*), the run stops in at most $|w'|+1$ steps. Hence, we deduce that the machine stops in at most $1+|w'|+1$ steps, therefore concluding this case since $|w|+1 = |0w'|+1 = 1+|w'|+1$

- $w = 1w'$: We distinguish two cases:

  - (P6) or (P7) is applied: Then the machine stops thus concluding this case since $|w|+1 = |1w'|+1 \geq 1$.

  - (P2) is applied as follows:



By ih, one has that from (*), the run stops in at most $|w'|+1$ steps. Hence, we deduce that the machine stops in at most $1+|w'|+1$ steps, therefore concluding this case since $|w|+1 = |1w'|+1 = 1+|w'|+1$ □

<u>Lemma 2</u>: Every run starting with q_{move} stops in at most $|w|+1$ steps, where w is the word on the first tape to the left of the current position. Moreover, the first tape is unchanged.

<u>Proof</u>: By induction on $w \in \{0,1\}^*$.

Same principle, using in particular Lemma 1 for the case (M3). □

<u>Lemma 3</u>: Every run starting with q_{copy} stops in at most $|w|+1$ steps, where w is the word on the first tape. Moreover, the first tape is unchanged.

<u>Proof</u>: By induction on $w \in \{0,1\}^*$.

Same principle, using in particular Lemma 2 for the case (C4). □

<u>Corollary</u>: The machine always stops in at most $3|w|+3$ steps.

2) Assuming the machine correctness and completeness, deduce the time complexity of the palindrome problem in such a model of computation.

From question 1, one has that the machine stops in at most $3|w|+3$ steps, hence the problem can be decided in a linear time in such a model.

! NO MORE TURING MACHINES !

Exercice 5 : For each language :

(a) $\mathcal{L} := \{ \lfloor \mathcal{M} \rfloor \mid \mathcal{M}$ squares the input $\}$

(b) $\mathcal{L} := \{ \lfloor \mathcal{M} \rfloor \mid \mathcal{M}$ decides a language in $\{0,1\}^* \}$

(c) $\mathcal{L}_n := \{ \lfloor \mathcal{M} \rfloor \mid \mathcal{M}$ accepts in at most n steps $\}$

(d) $\mathcal{L} := \{ \lfloor \mathcal{M} \rfloor \mid \mathcal{M}$ rejects every input $\}$

1) Is the language trivial ?

(a)

(b)

(c)

(d)

2) Is the language semantic ?

(a)

(b)

(c)

(d)

3) Is the language decidable ?
If so, explain how to build a TM deciding the language .

(a)

(b)

(c)

(d)

4) Pick one undecidable language and prove its undecidability by reduction to the halting problem .