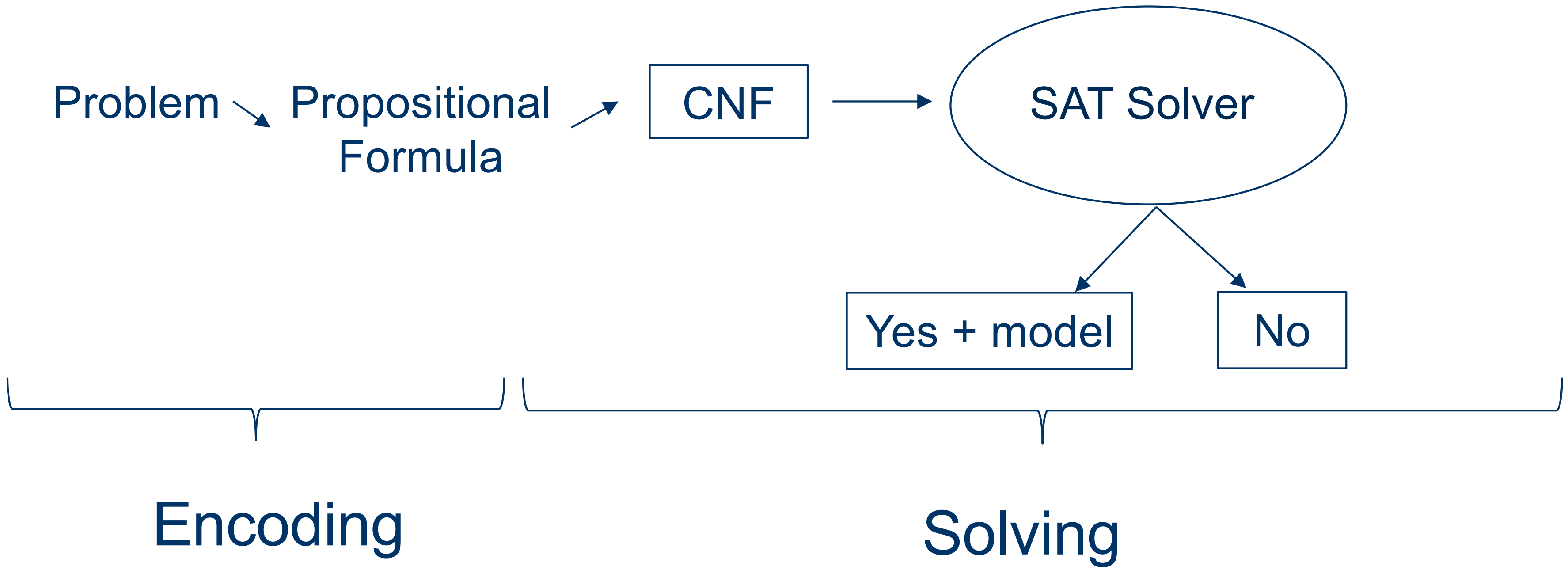
Problem Solving with SAT



Terminology

- Literal
  - Refers either to a Boolean variable p or to its negation $\neg p$.
- Clause
  - Disjunction of literals, e.g., $C = l_1 \vee l_2 \vee l_3$
  - Can be falsified with only one assignment to its literals, where all literals are assigned to F.
    - Satisfied with $2^k - 1$ assignments to its k literals.
  - The empty clause (denoted by $\bot$) is always falsified.

Terminology

- Propositional formula f in CNF
  - Conjunction of clauses, e.g., $f = C_1 \wedge C_2 \wedge C_3$
    - Conjunction of disjunction of literals.

$$\bigwedge_i \bigvee_j l_{ij}$$

  - Is satisfiable if there exists an assignment satisfying all clauses, otherwise unsatisfiable.
  - An arbitrary formula can be transformed into CNF preserving satisfiability.

Resolution

- Basic method of satisfiability of propositional formulas.
  - The basis of current SAT solvers.
- Applicable to formulas in CNF.
- Idea: from the given clauses, derive new clauses, with the aim of deriving the empty clause $\perp$ (contradiction).
  - Proves UNSAT.

Resolution Rule

- Given the clauses of the shape $p \vee V$ and $\neg p \vee W$, we can derive $V \vee W$.

$$\frac{p \vee V, \ \neg p \vee W}{V \vee W}$$

Unit Resolution

- If a clause consists of a single literal l (a unit clause), then the resolution rule allows to remove the literal $\neg l$ from a clause containing $\neg l$.

- When V or W in $\frac{p \vee V, \neg p \vee W}{V \vee W}$ is empty, we have:

$$\frac{p, \neg p \vee W}{W} \text{ or } \frac{p \vee V, \neg p}{V}$$

Example

- Prove that the CNF consisting of the following 5 clauses is UNSAT.

1. $p \vee q$
2. $\neg r \vee s$
3. $\neg q \vee r$
4. $\neg r \vee \neg s$
5. $\neg p \vee r$

Example

- Prove that the CNF consisting of the following 5 clauses is UNSAT.

1.	$p \vee q$	6.	$p \vee r$	$(1, 3, q)$
2.	$\neg r \vee s$	7.	r	$(5, 6, p)$
3.	$\neg q \vee r$	8.	s	$(2, 7, r)$
4.	$\neg r \vee \neg s$	9.	$\neg r$	$(4, 8, s)$
5.	$\neg p \vee r$	10.	$\perp$	$(7, 9, r)$

- Freedom in choice: several other sequences of resolution steps will lead to $\perp$ too.

DPLL

- Resolution
  - \+ Straightforward to give a refutation.
  - \+ Formula validation: f is a tautology iff $\neg f$ is UNSAT.
  - – Not direct to obtain a satisfying solution.
- DPLL is an algorithm to establish the SAT/UNSAT of a CNF.
  - Based on unit resolution.
  - Due to **D**avis, **P**utnam , **L**ogemann and **L**oveland in 1962.

DPLL

- Basic idea
  - First apply unit resolution as long as possible.
  - Then, choose a variable p.
  - Introduce the cases p and $\neg p$, and go on recursively.

DPLL Algorithm

DPLL(X):
$X :=$ unit-resol(X)

if $X = \emptyset$ then return(sat)

if $\perp \notin X$ then

choose variable p in X
DPLL($X \cup \{p\}$)
DPLL($X \cup \{\neg p\}$)

- unit-resol
  - While there exists a clause consisting of one literal l (a unit clause):
    - remove $\neg l$ from all clauses containing $\neg l$,
    - remove all clauses containing l (since they are now redundant).

DPLL

- Unit resolution and case analysis.
  - Similar to constraint propagation and search in CP.
- Complete method.
  - As CP.
- Efficiency strongly depends on the choice of the variable.
  - As in CP.

Example 1

1.	$\neg p \lor \neg s$	4.	$p \lor r$	7.	$\neg s \lor t$
2.	$\neg p \lor \neg r$	5.	$p \lor s$	8.	$q \lor s$
3.	$\neg q \lor \neg t$	6.	$r \lor t$	9.	$q \lor \neg r$

- No unit clause, choose a variable, say p.
- Add p + unit resolution

$$\neg s\ (1), \neg r\ (2)$$
$$q\ (\neg s, 8),\ \ t(\neg r, 6)$$
$$\neg t\ (q, 3)$$
$$\bot\ (t, \neg t)$$

- Add $\neg p$ + unit resolution

$$r(4), s\ (5)$$
$$q\ (r, 9),\ \ t(s, 7)$$
$$\neg t\ (q, 3)$$
$$\bot\ (t, \neg t)$$

Example 1

1.	$\neg p \vee \neg s$	*4.*	$p \vee r$	*7.*	$\neg s \vee t$
2.	$\neg p \vee \neg r$	*5.*	$p \vee s$	*8.*	$q \vee s$
3.	$\neg q \vee \neg t$	*6.*	$r \vee t$	*9.*	$q \vee \neg r$

UNSAT

- No unit clause, choose a variable, say p.
- Add p + unit resolution

$$\neg s\ (1), \neg r\ (2)$$
$$q\ (\neg s, 8),\ \ t(\neg r, 6)$$
$$\neg t\ (q, 3)$$
$$\bot\ (t, \neg t)$$

- Add $\neg p$ + unit resolution

$$r(4), s\ (5)$$
$$q\ (r, 9),\ \ t(s, 7)$$
$$\neg t\ (q, 3)$$
$$\bot\ (t, \neg t)$$

Example 2

1.	$\neg p \vee \neg s$	*4.*	$p \vee r$	*7.*	$\neg s \vee t$
2.	$\neg p \vee \neg r$	*5.*	$p \vee s$	*8.*	$q \vee s$
3.	$\neg q \vee \neg t$	*6.*	$r \vee t$		

- No unit clause, choose a variable, say p.
- Add p + unit resolution

$$\neg s\ (1), \neg r\ (2)$$
$$q\ (\neg s, 8),\ \ t(\neg r, 6)$$
$$\neg t\ (q, 3)$$
$$\bot\ (t, \neg t)$$

- Add $\neg p$ + unit resolution

$$r(4)$$
$$s\ (5)$$
$$t(s, 7)$$
$$\neg q\ (t, 3)$$

Example 2

1.	$\neg p \vee \neg s$	4.	$p \vee r$	7.	$\neg s \vee t$
2.	$\neg p \vee \neg r$	5.	$p \vee s$	8.	$q \vee s$
3.	$\neg q \vee \neg t$	6.	$r \vee t$		

SAT

- No unit clause, choose a variable, say p.
- Add p + unit resolution

$$\neg s\ (1), \neg r\ (2)$$
$$q\ (\neg s, 8),\ \ t(\neg r, 6)$$
$$\neg t\ (q, 3)$$
$$\bot\ (t, \neg t)$$

- Add $\neg p$ + unit resolution

$$r(4)$$
$$s\ (5)$$
$$t(s, 7)$$
$$\neg q\ (t, 3)$$

- Solution: $p = q = F, r = s = t = T$.

Example 3

1.	$\neg p \lor \neg s$	*4.*	$p \lor r$	*7.*	$\neg s \lor t$	*10.*	…
2.	$\neg p \lor \neg r$	*5.*	$p \lor s$	8.	$q \lor s$		
3.	$\neg q \lor \neg t$	*6.*	$r \lor t$	9.	$a \lor b \lor \neg c$		

- No unit clause, choose a variable, say p.
- Add p + unit resolution

$$\neg s\ (1), \neg r\ (2)$$
$$q\ (\neg s, 8),\ \ t(\neg r, 6)$$
$$\neg t\ (q, 3)$$
$$\perp\ (t, \neg t)$$

- Add $\neg p$ + unit resolution

$$r(4)$$
$$s\ (5)$$
$$t(s, 7)$$
$$\neg q\ (t, 3)$$

Example 3

1.	$\neg p \vee \neg s$	*4.*	$p \vee r$	*7.*	$\neg s \vee t$	*10.*	...
2.	$\neg p \vee \neg r$	*5.*	$p \vee s$	*8.*	$q \vee s$		
3.	$\neg q \vee \neg t$	*6.*	$r \vee t$	*9.*	$a \vee b \vee \neg c$		

- No unit clause, choose a variable, say p.
- Add p + unit resolution

$\neg s$ (1), $\neg r$ (2)

$q\ (\neg s, 8),\ t(\neg r, 6)$

$\neg t\ (q, 3)$

$\bot\ (t, \neg t)$

- Add $\neg p$ + unit resolution

$r(4)$

$s\ (5)$

$t(s, 7)$

$\neg q\ (t, 3)$

Continue with new decisions & unit propagation

Implementation of DPLL

- A direct implementation would make a copy of the CNF X at every recursive call.
  - Inefficient!
- Need to work on the original CNF X and mimic the DPLL algorithm which consists of a series of unit resolution, case analysis, backtrack and fail.

$\mathrm{DPLL}(X)$:
$X := \text{unit-resol}(X)$

if $X = \emptyset$ then return(sat)

if $\perp \notin X$ then

    choose variable p in X
    $\mathrm{DPLL}(X \cup \{p\})$
    $\mathrm{DPLL}(X \cup \{\neg p\})$

Efficient Implementation of DPLL

- Basic idea
  - Keep track of a list M of literals that have been **decided** and **derived** during the execution of DPLL.
- M is originally empty.
- M is extended when:
  - a literal is derived by unit resolution (UnitPropagate),
  - a case analysis starts (Decide).
- M is repaired when contradiction is found:
  - go back to the last decision, remove everything behind the last decision, negate the decision (Backtrack), and continue with a new decision,
  - otherwise (when it is not possible to backtrack), Fail.

Efficient Implementation of DPLL

- Notation
  - A literal l holds in M ($M \models l$) iff l occurs in M.
  - A clause C yields contradiction ($M \models \neg C$) iff for every literal l in C, we have, $M \models \neg l$.
  - l is undefined in M iff neither $M \models l$ nor $M \models \neg l$.
  - A decision literal l^d originates from a decision in the DPLL algorithm.

Efficient Implementation of DPLL

- The DPLL algorithm can be mimicked by starting with an empty M and applying four rules as long as possible.
  - At any moment, the current CNF of the DPLL algorithm corresponds to M + the original CNF from which all negations of literals from M have been stripped away.
- At the end, we have either:
  - fail, proving that the CNF is UNSAT, or
  - a list M containing p or $\neg p$ for every variable p, yielding a satisfying assignment.

UnitPropagate

- Mimics the generation of a new unit clause in DPLL.

$$M \Rightarrow Ml$$

if l is undefined in M and the CNF contains a clause $C \vee l$ satisfying $M \models \neg C$.

Decide

- Mimics the choice p in DPLL, when no UnitPropagate is possible.

$$M \Rightarrow M l^d$$

if l is undefined in M.

Backtrack

- Mimics backtracking to the negation of the last decision in case a branch is unsatisfiable.

$$M l^d N \Rightarrow M \neg l$$

if $M l^d N \models \neg C$ for a clause C in the CNF and N does not contain decision literals.

Fail

- Mimics the end of DPLL when every branch, and hence the CNF, is unsatisfiable.

$$M \Rightarrow \text{fail}$$

if $M \models \neg C$ for a clause C in the CNF and M does not contain decision literals.

Example

$\neg p \lor \neg s$ $\quad p \lor r$ $\quad \neg s \lor t$ $\quad \neg p \lor \neg r$ $\quad p \lor s$

$q \lor s$ $\quad \neg q \lor \neg t$ $\quad r \lor t$ $\quad q \lor \neg r$

Rule	M

Example

$\neg p \lor \neg s$ $\quad p \lor r$ $\quad \neg s \lor t$ $\quad \neg p \lor \neg r$ $\quad p \lor s$

$q \lor s$ $\quad \neg q \lor \neg t$ $\quad r \lor t$ $\quad q \lor \neg r$

Rule	M
Decide	p^d

Example

$\neg p \vee \neg s$ $\quad p \vee r$ $\quad \neg s \vee t$ $\quad \neg p \vee \neg r$ $\quad p \vee s$

$q \vee s$ $\quad \neg q \vee \neg t$ $\quad r \vee t$ $\quad q \vee \neg r$

Rule	M
Decide	p^d
UnitPropagate	$p^d \neg s$

Example

$\neg p \lor \neg s \quad p \lor r \quad \neg s \lor t \quad \neg p \lor \neg r \quad p \lor s$

$q \lor s \quad \neg q \lor \neg t \quad r \lor t \quad q \lor \neg r$

Rule	M
Decide	p^d
UnitPropagate	$p^d \neg s \neg r$

Example

$\neg p \vee \neg s \quad p \vee r \quad \neg s \vee t \quad \neg p \vee \neg r \quad p \vee s$

$q \vee s \quad \neg q \vee \neg t \quad r \vee t \quad q \vee \neg r$

Rule	M
Decide	p^d
UnitPropagate	$p^d \neg s \neg r\ t$

Example

$\neg p \vee \neg s$ $\quad p \vee r$ $\quad \neg s \vee t$ $\quad \neg p \vee \neg r$ $\quad p \vee s$

$q \vee s$ $\quad \neg q \vee \neg t$ $\quad r \vee t$ $\quad q \vee \neg r$

Rule	M
Decide	p^d
UnitPropagate	$p^d \neg s\ \neg r\ tq$

Example

$\neg p \vee \neg s$ $\quad p \vee r$ $\quad \neg s \vee t$ $\quad \neg p \vee \neg r$ $\quad p \vee s$

$q \vee s$ $\quad \neg q \vee \neg t$ $\quad r \vee t$ $\quad q \vee \neg r$

Rule	M
Decide	p^d
UnitPropagate	$p^d \neg s \ \neg r \ tq$ **CONTRADICTION**

Example

$\neg p \lor \neg s$ $p \lor r$ $\neg s \lor t$ $\neg p \lor \neg r$ $p \lor s$

$q \lor s$ $\neg q \lor \neg t$ $r \lor t$ $q \lor \neg r$

Rule	M
Decide	p^d
UnitPropagate	$p^d \neg s \ \neg r \ tq$
Backtrack	$\neg p$

Example

$\neg p \lor \neg s$ $\quad$ $p \lor r$ $\quad$ $\neg s \lor t$ $\quad$ $\neg p \lor \neg r$ $\quad$ $p \lor s$

$q \lor s$ $\quad$ $\neg q \lor \neg t$ $\quad$ $r \lor t$ $\quad$ $q \lor \neg r$

Rule	M
Decide	p^d
UnitPropagate	$p^d \neg s\ \neg r\ tq$
Backtrack	$\neg p$
UnitPropagate	$\neg pr$

Example

$\neg p \vee \neg s$ $\quad p \vee r$ $\quad \neg s \vee t$ $\quad \neg p \vee \neg r$ $\quad {\color{red} p \vee s}$

$q \vee s$ $\quad \neg q \vee \neg t$ $\quad r \vee t$ $\quad q \vee \neg r$

Rule	M
Decide	p^d
UnitPropagate	$p^d \neg s \ \neg r \ tq$
Backtrack	$\neg p$
UnitPropagate	$\neg p r {\color{red} s}$

Example

$\neg p \lor \neg s$ $\quad p \lor r$ $\quad \neg s \lor t$ $\quad \neg p \lor \neg r$ $\quad p \lor s$

$q \lor s$ $\quad \neg q \lor \neg t$ $\quad r \lor t$ $\quad q \lor \neg r$

Rule	M
Decide	p^d
UnitPropagate	$p^d \neg s \neg r\ tq$
Backtrack	$\neg p$
UnitPropagate	$\neg prsq$

Example

$\neg p \lor \neg s$ $\quad p \lor r$ $\quad \neg s \lor t$ $\quad \neg p \lor \neg r$ $\quad p \lor s$

$q \lor s$ $\quad \neg q \lor \neg t$ $\quad r \lor t$ $\quad q \lor \neg r$

Rule	M
Decide	p^d
UnitPropagate	$p^d \neg s \ \neg r \ tq$
Backtrack	$\neg p$
UnitPropagate	$\neg prsqt$

Example

$\neg p \vee \neg s \quad p \vee r \quad \neg s \vee t \quad \neg p \vee \neg r \quad p \vee s$

$q \vee s \quad \neg q \vee \neg t \quad r \vee t \quad q \vee \neg r$

Rule	M
Decide	p^d
UnitPropagate	$p^d \neg s\ \neg r\ tq$
Backtrack	$\neg p$
UnitPropagate	$\neg prsqt$ **CONTRADICTION**

Example

$\neg p \vee \neg s \quad p \vee r \quad \neg s \vee t \quad \neg p \vee \neg r \quad p \vee s$

$q \vee s \quad \neg q \vee \neg t \quad r \vee t \quad q \vee \neg r$

Rule	M
Decide	p^d
UnitPropagate	$p^d \neg s\ \neg r\ tq$
Backtrack	$\neg p$
UnitPropagate	$\neg prsqt$
Fail	