Shortcomings of DPLL

DPLL(X):
$X :=$ unit-resol(X)

if $X = \emptyset$ then return(sat)

if $\bot \notin X$ then

choose variable p in X
DPLL($X \cup \{p\}$)
DPLL($X \cup \{\neg p\}$)

No learning

- Throws away all the work performed to conclude the current partial assignment (PA) is bad.
- Revisits bad PAs that lead to conflict due to the same root cause.

Chronological backtracking

Backtracks one level, even if it can be deduced that the current PA became doomed at another level.

Conflict-Driven Clause Learning (CDCL)

- Each time a contradiction occurs:
  - Clause Learning
    - Augments the original CNF X with a conflict clause that summarizes the root cause of the contradiction.
    - Can learn from its mistakes and ignore huge sections of the search space that will never satisfy the formula.
  - Backjumping
    - Based on the cause of the contradiction, skips the decisions (and derivations) that did not lead to conflict, and backtracks to an earlier decision literal in the search tree, instead of backtracking to the last decision literal.

Example

c_1: $\neg p \vee q$ c_2: $\neg r \vee s$ c_3: $\neg t \vee \neg u$ c_4: $\neg q \vee \neg t \vee u$

Rule	M

Example

c_1: $\neg p \lor q$ c_2: $\neg r \lor s$ c_3: $\neg t \lor \neg u$ c_4: $\neg q \lor \neg t \lor u$

Rule	M
Decide&UnitPropagate	$p^d q$

Example

c_1: $\neg p \vee q$ c_2: $\neg r \vee s$ c_3: $\neg t \vee \neg u$ c_4: $\neg q \vee \neg t \vee u$

Rule	M
Decide&UnitPropagate	$p^d q$
Decide&UnitPropagate	$p^d q r^d s$

Example

c_1: $\neg p \vee q$ c_2: $\neg r \vee s$ c_3: $\neg t \vee \neg u$ c_4: $\neg q \vee \neg t \vee u$

Rule	M
Decide&UnitPropagate	$p^d q$
Decide&UnitPropagate	$p^d q r^d s$
Decide&UnitPropagate	$p^d q r^d s t^d u$

Example

$c_1\colon \neg p \lor q \qquad c_2\colon \neg r \lor s \qquad c_3\colon \neg t \lor \neg u \qquad c_4\colon \neg q \lor \neg t \lor u$

Rule	M
Decide&UnitPropagate	$p^d q$
Decide&UnitPropagate	$p^d q r^d s$
Decide&UnitPropagate	$p^d q r^d s t^d u$ **CONTRADICTION**

Example

c_1: $\neg p \lor q$ c_2: $\neg r \lor s$ c_3: $\neg t \lor \neg u$ c_4: $\neg q \lor \neg t \lor u$

Rule	M
D&UP	$p^d q$
D&UP	$p^d q r^d s$
D&UP	$p^d q r^d s t^d u$ C

- For the contradiction between t, u, and c_3, $r^d s$ does not play role.
- u was derived using c_4 and contradiction was found using c_3.
- Resolution of c_3 and c_4 gives:

  $$c_5: \neg q \lor \neg t$$
- This newly learnt conflict clause gives a clue on where to backjump.

Example

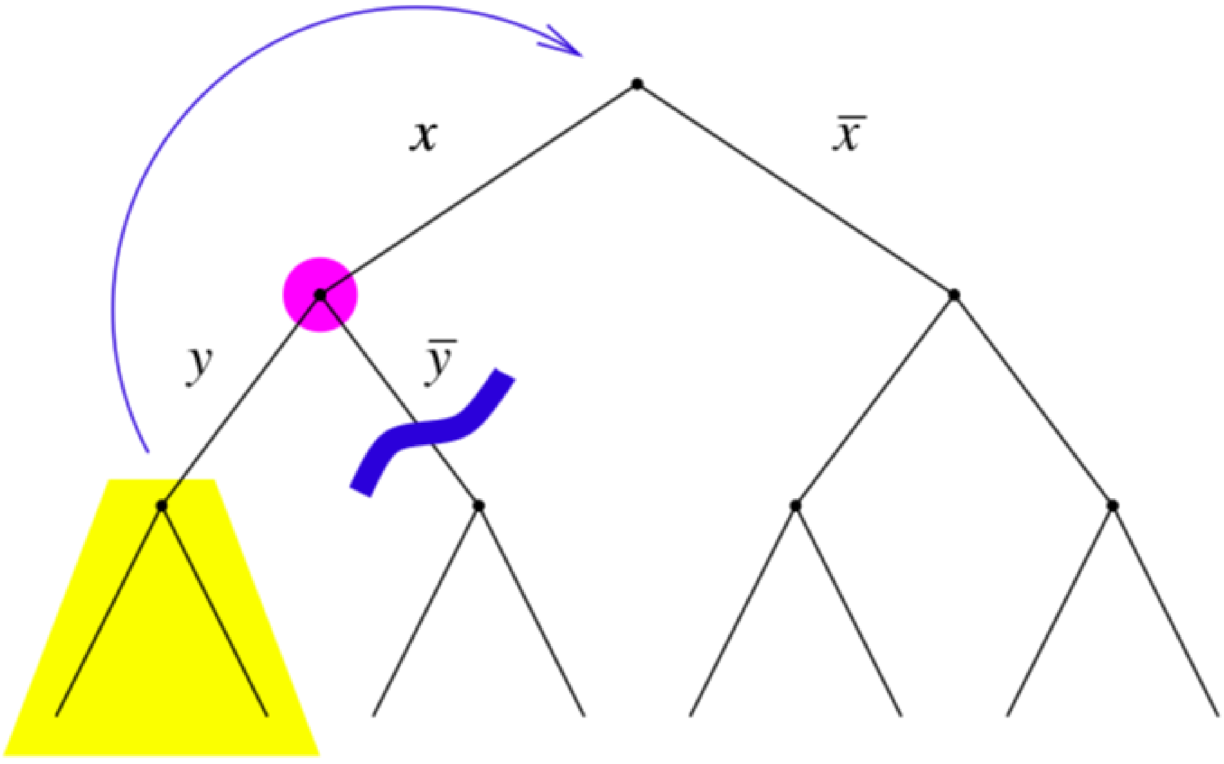
$c_1: \neg p \lor q$ $c_2: \neg r \lor s$ $c_3: \neg t \lor \neg u$ $c_4: \neg q \lor \neg t \lor u$

Rule	M
D&UP	$p^d q$
D&UP	$p^d q r^d s$
D&UP	$p^d q r^d s t^d u$ C
BJ	$p^d q \neg t$

- For the contradiction between t, u, and c_3, $r^d s$ does not play role.
- u was derived using c_4 and contradiction was found using c_3.
- Resolution of c_3 and c_4 gives:

  $c_5: \neg q \lor \neg t$
- This newly learnt conflict clause gives a clue on where to backjump.

Backjumping



Backjumping

- Mimics backtracking to an earlier decision literal than the last one.

$$M l^d N \Rightarrow M l'$$

if $M l^d N \models \neg C$ for a clause C in the CNF and there is a clause $C' \vee l'$ derivable from the CNF such that $M \models \neg C'$ and l' is not defined in M.

In Our Example

$$Ml^dN \Rightarrow Ml'$$

if $Ml^dN \vDash \neg C$ for a clause C in the CNF and there is a clause $C' \vee l'$ derivable from the CNF such that $M \vDash \neg C'$ and l' is not defined in M.

$$C' \vee l' = \neg q \vee \neg t$$

$$\underbrace{p^d q}_{M}\ \underbrace{r^d}_{l^d}\ \underbrace{st^d u}_{N} \quad \Rightarrow \quad \underbrace{p^d q}_{M}\ \underbrace{\neg t}_{l'}$$

Backjumping Clause

- Backjumping is unit propagation on a derivable conflict clause which is added to the original CNF.
- To derive such a conflict (backjumping) clause we store:
  - each decision;
  - at every UnitPropagate step, the corresponding clause and the derived literal.
- At every contradiction, we investigate which derived literals and corresponding clauses played a role.
- A backjumping clause can be obtained by resolution from the original CNF.

Implication Graph

- An implication graph $G = (V, E)$ is a DAG that records the history of decisions and the resulting deductions derived with UnitPropagate.
  - $v \in V$ represents a decision or a derived literal (l), or the conflict (κ) at a certain decision level (d).
    - v is labelled as $l@d$ or $\kappa@d$.
  - An edge $v \xrightarrow{c_i} w \in E$ indicates that w is derived with unit propagation from the clause c_i with one of its literals being v.

Example

$c_1: \neg x_1 \lor x_2 \lor \neg x_4$

$c_2: \neg x_1 \lor \neg x_2 \lor x_3$

$c_3: \neg x_3 \lor \neg x_4$

$c_4: x_4 \lor x_5 \lor x_6$

$c_5: \neg x_5 \lor x_7$

$c_6: \neg x_6 \lor x_7 \lor \neg x_8$

Implication Graph

$c_1 : \neg x_1 \vee x_2 \vee \neg x_4$

$c_2 : \neg x_1 \vee \neg x_2 \vee x_3$

$c_3 : \neg x_3 \vee \neg x_4$

$c_4 : x_4 \vee x_5 \vee x_6$

$c_5 : \neg x_5 \vee x_7$

$c_6 : \neg x_6 \vee x_7 \vee \neg x_8$

True literals highlighted in green; false literals highlighted in pink

Decision literal

Implication Graph

$c_1 : \neg x_1 \vee x_2 \vee \neg x_4$

$c_2 : \neg x_1 \vee \neg x_2 \vee x_3$

$c_3 : \neg x_3 \vee \neg x_4$

$c_4 : x_4 \vee x_5 \vee x_6$

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Implication Graph

$c_1 : \neg x_1 \vee x_2 \vee \neg x_4$

$c_2 : \neg x_1 \vee \neg x_2 \vee x_3$

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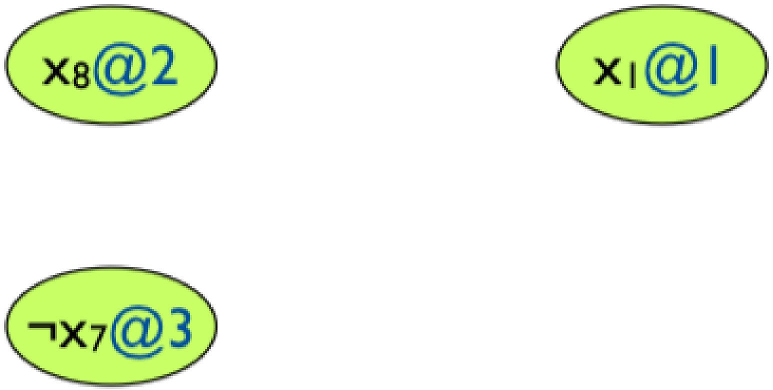

Implication Graph

$c_1 : \neg x_1 \vee x_2 \vee \neg x_4$

$c_2 : \neg x_1 \vee \neg x_2 \vee x_3$

$c_3 : \neg x_3 \vee \neg x_4$

$c_4 : x_4 \vee x_5 \vee x_6$

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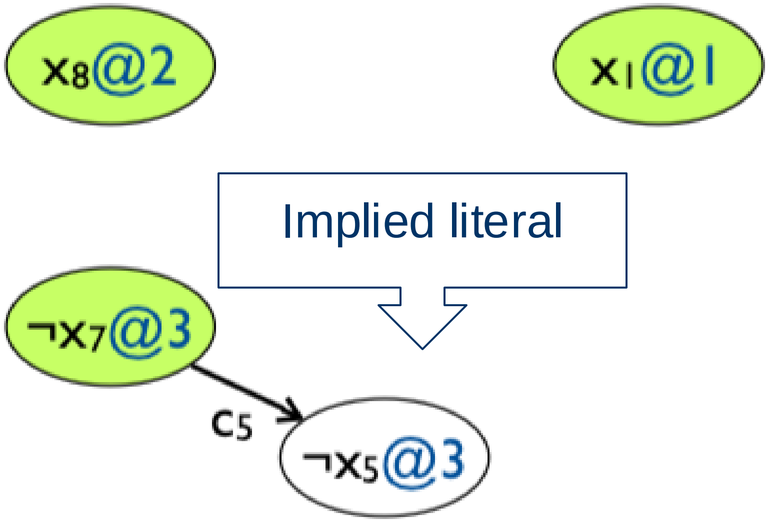

Implication Graph

$c_1: \neg x_1 \vee x_2 \vee \neg x_4$

$c_2: \neg x_1 \vee \neg x_2 \vee x_3$

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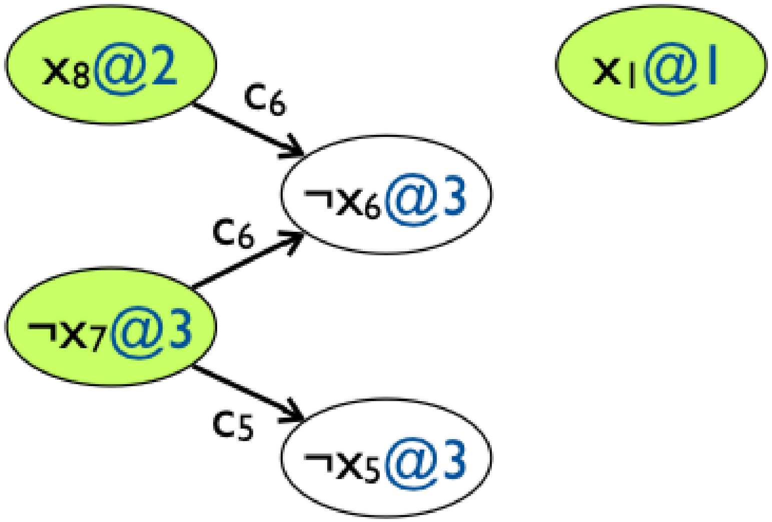

Implication Graph

$c_1 : \neg x_1 \lor x_2 \lor \neg x_4$

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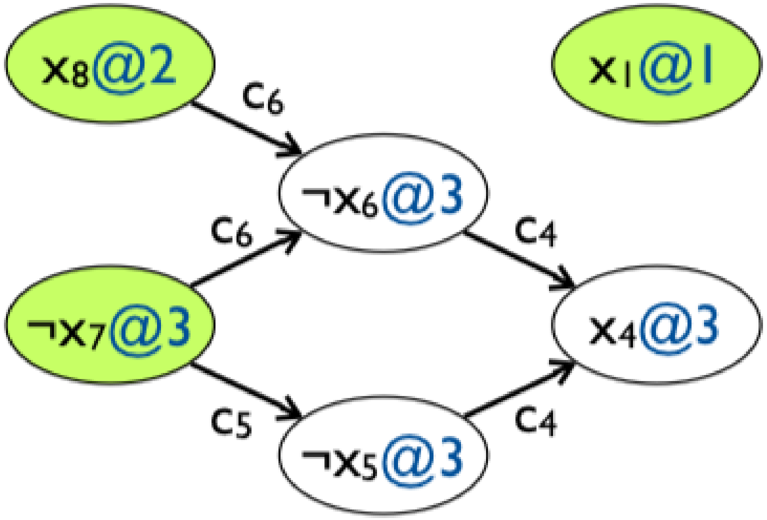

Implication Graph

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$c_2 : \neg x_1 \lor \neg x_2 \lor x_3$

$c_3 : \neg x_3 \lor \neg x_4$

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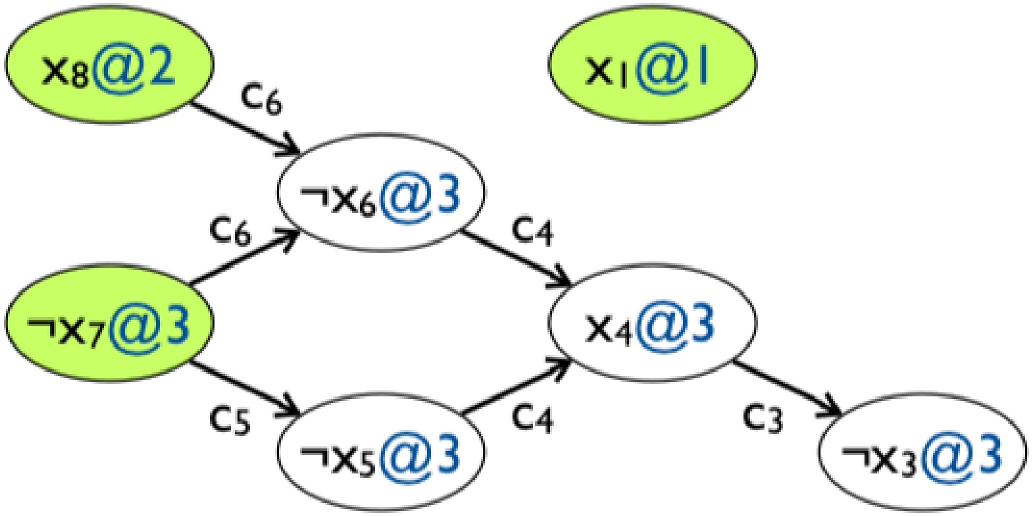

Implication Graph

$c_1 : \neg x_1 \vee x_2 \vee \neg x_4$

$c_2 : \neg x_1 \vee \neg x_2 \vee x_3$

$c_3 : \neg x_3 \vee \neg x_4$

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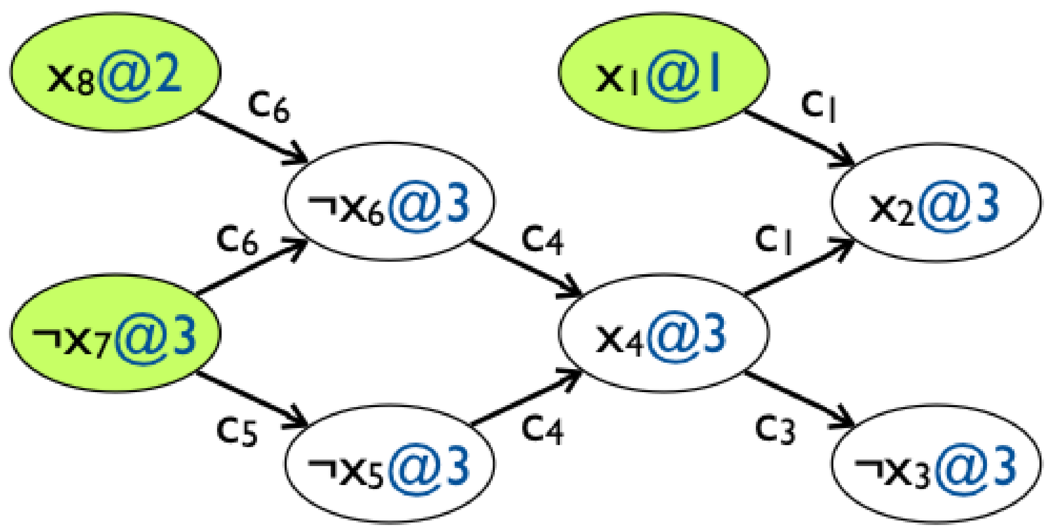

Implication Graph

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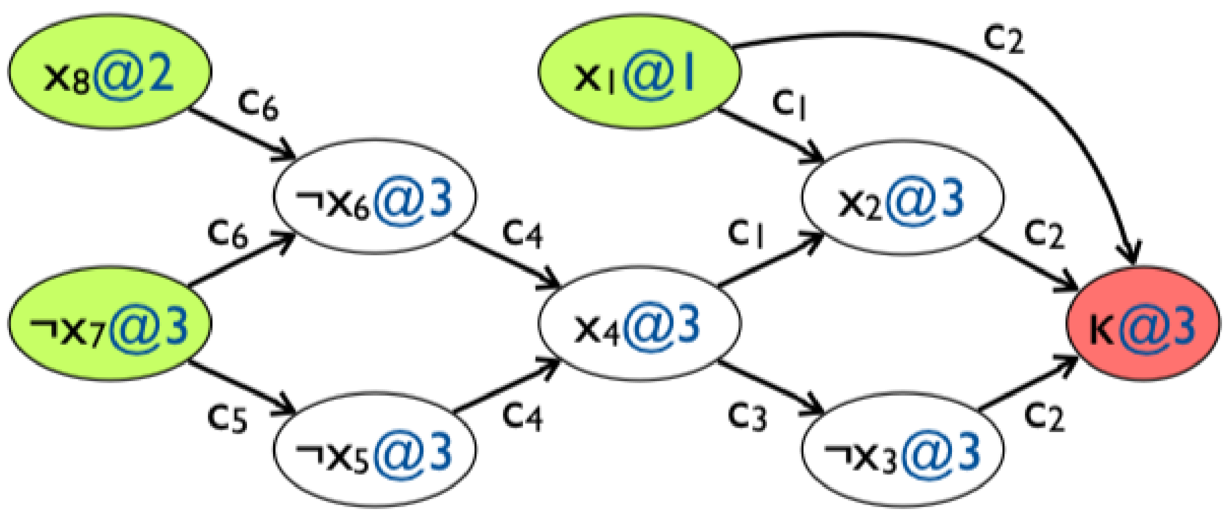

Clause Learning

$c_1: \neg x_1 \lor x_2 \lor \neg x_4$

$c_2: \neg x_1 \lor \neg x_2 \lor x_3$

$c_3: \neg x_3 \lor \neg x_4$

$c_4: x_4 \lor x_5 \lor x_6$

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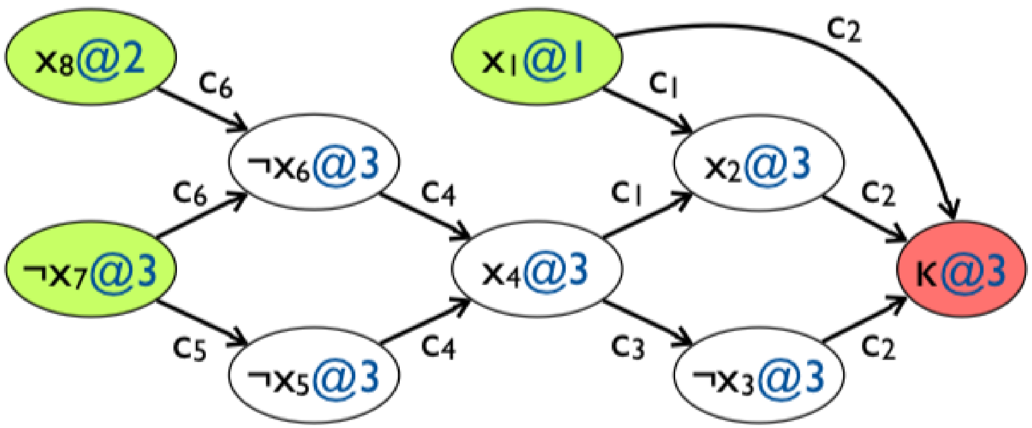
$c_6: \neg x_6 \lor x_7 \lor \neg x_8$



c_2: $\neg x_1 \lor \neg x_2 \lor x_3$ (clause causing the conflict)

Clause Learning

$c_1: \neg x_1 \vee x_2 \vee \neg x_4$

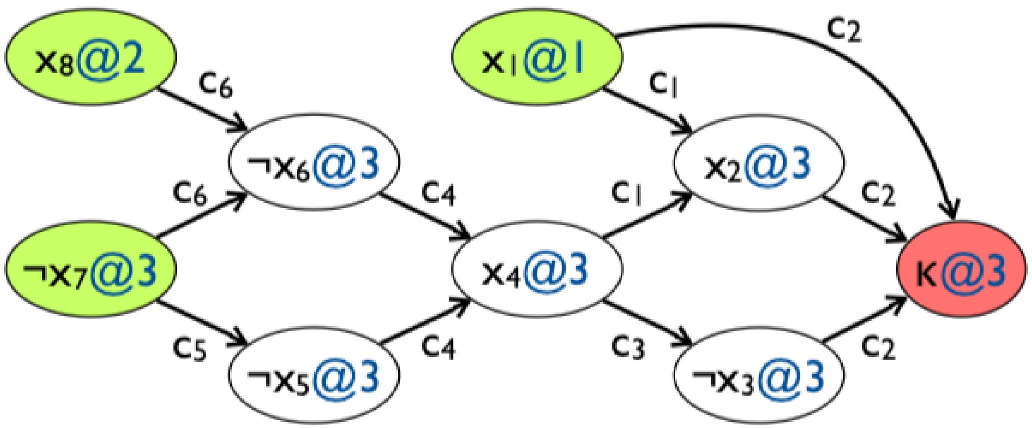
$c_2: \neg x_1 \vee \neg x_2 \vee x_3$

$c_3: \neg x_3 \vee \neg x_4$

$c_4: x_4 \vee x_5 \vee x_6$

$c_5: \neg x_5 \vee x_7$

$c_6: \neg x_6 \vee x_7 \vee \neg x_8$



c_2: $\neg x_1 \vee \neg x_2 \vee x_3$ (clause causing the conflict)

c: $\neg x_1 \vee x_3 \vee \neg x_4$ (resolution with c_1)

Clause Learning

$c_1: \neg x_1 \vee x_2 \vee \neg x_4$

$c_2: \neg x_1 \vee \neg x_2 \vee x_3$

$c_3: \neg x_3 \vee \neg x_4$

$c_4: x_4 \vee x_5 \vee x_6$

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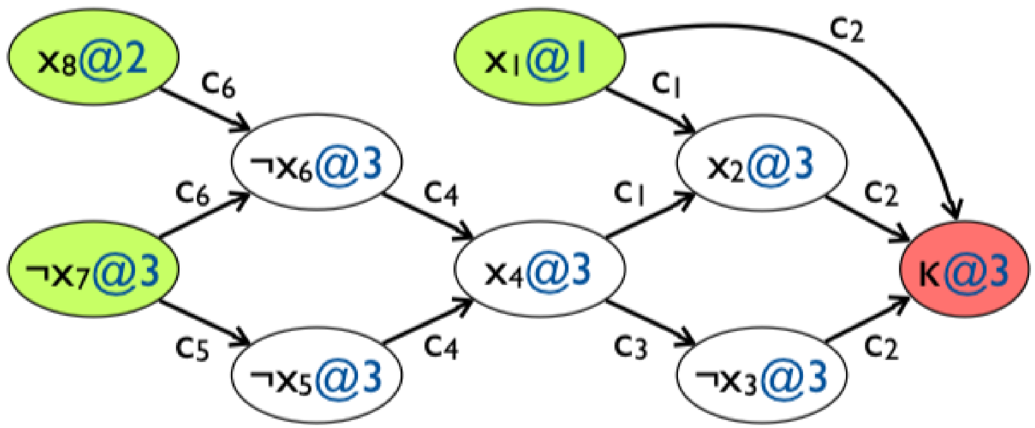


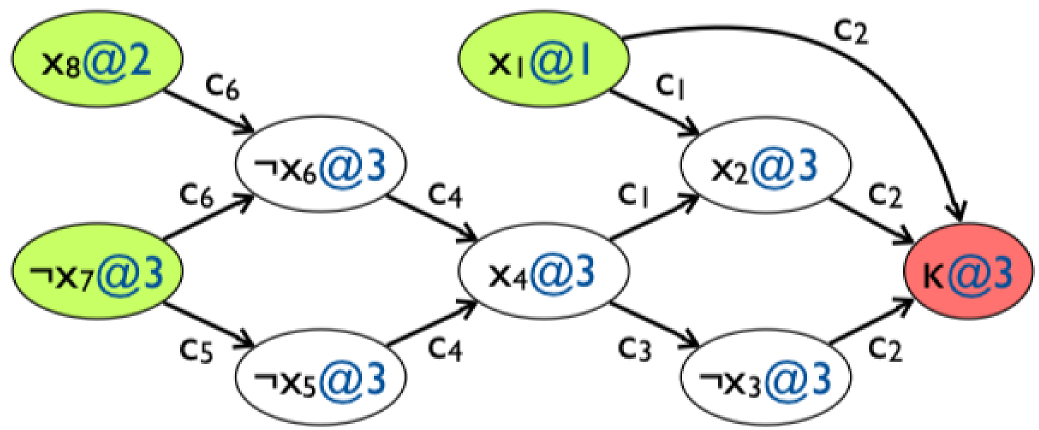
c_2: $\neg x_1 \vee \neg x_2 \vee x_3$ (clause causing the conflict)

c: $\neg x_1 \vee x_3 \vee \neg x_4$ (resolution with c_1)

c: $\neg x_1 \vee \neg x_4$ (resolution with c_3)

Clause Learning

c_1 : $\neg x_1 \lor x_2 \lor \neg x_4$
c_2 : $\neg x_1 \lor \neg x_2 \lor x_3$
c_3 : $\neg x_3 \lor \neg x_4$
c_4 : $x_4 \lor x_5 \lor x_6$
c_5 : $\neg x_5 \lor x_7$
c_6 : $\neg x_6 \lor x_7 \lor \neg x_8$



c_2: $\neg x_1 \lor \neg x_2 \lor x_3$ (clause causing the conflict)

c: $\neg x_1 \lor x_3 \lor \neg x_4$ (resolution with c_1)

c: $\neg x_1 \lor \neg x_4$ (resolution with c_3)

c: $\neg x_1 \lor x_5 \lor x_6$ (resolution with c_4)

Clause Learning

$c_1: \neg x_1 \lor x_2 \lor \neg x_4$

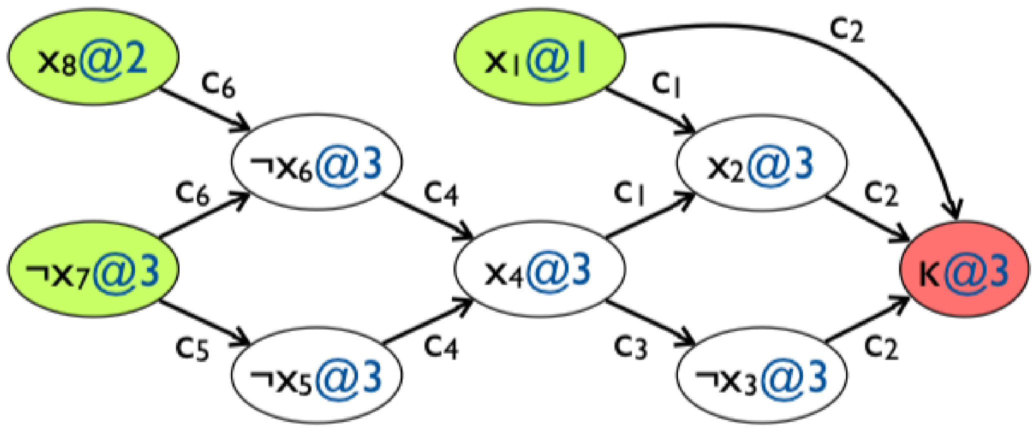
$c_2: \neg x_1 \lor \neg x_2 \lor x_3$

$c_3: \neg x_3 \lor \neg x_4$

$c_4: x_4 \lor x_5 \lor x_6$

$c_5: \neg x_5 \lor x_7$

$c_6: \neg x_6 \lor x_7 \lor \neg x_8$



c_2: $\neg x_1 \lor \neg x_2 \lor x_3$ (clause causing the conflict)

c: $\neg x_1 \lor x_3 \lor \neg x_4$ (resolution with c_1)

c: $\neg x_1 \lor \neg x_4$ (resolution with c_3)

c: $\neg x_1 \lor x_5 \lor x_6$ (resolution with c_4)

c: $\neg x_1 \lor x_6 \lor x_7$ (resolution with c_5)

Clause Learning

$c_1: \neg x_1 \vee x_2 \vee \neg x_4$

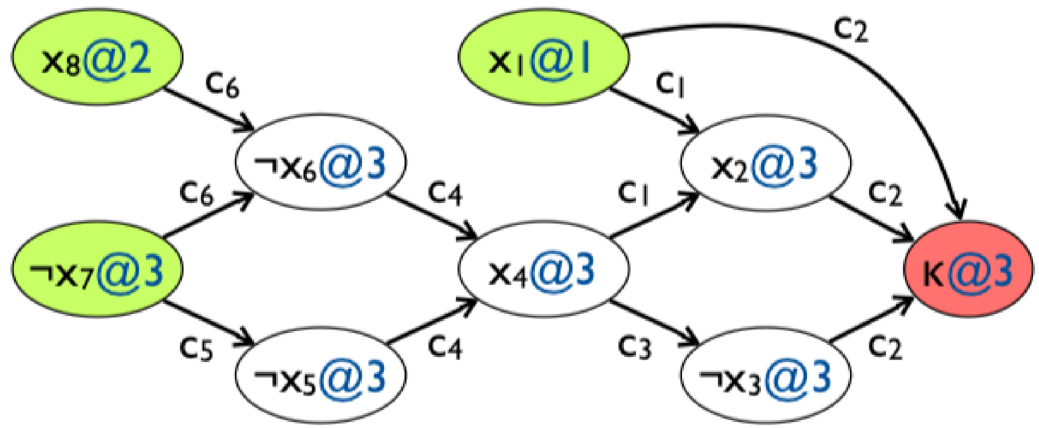
$c_2: \neg x_1 \vee \neg x_2 \vee x_3$

$c_3: \neg x_3 \vee \neg x_4$

$c_4: x_4 \vee x_5 \vee x_6$

$c_5: \neg x_5 \vee x_7$

$c_6: \neg x_6 \vee x_7 \vee \neg x_8$



c_2: $\neg x_1 \vee \neg x_2 \vee x_3$ (clause causing the conflict)

c: $\neg x_1 \vee x_3 \vee \neg x_4$ (resolution with c_1)

c: $\neg x_1 \vee \neg x_4$ (resolution with c_3)

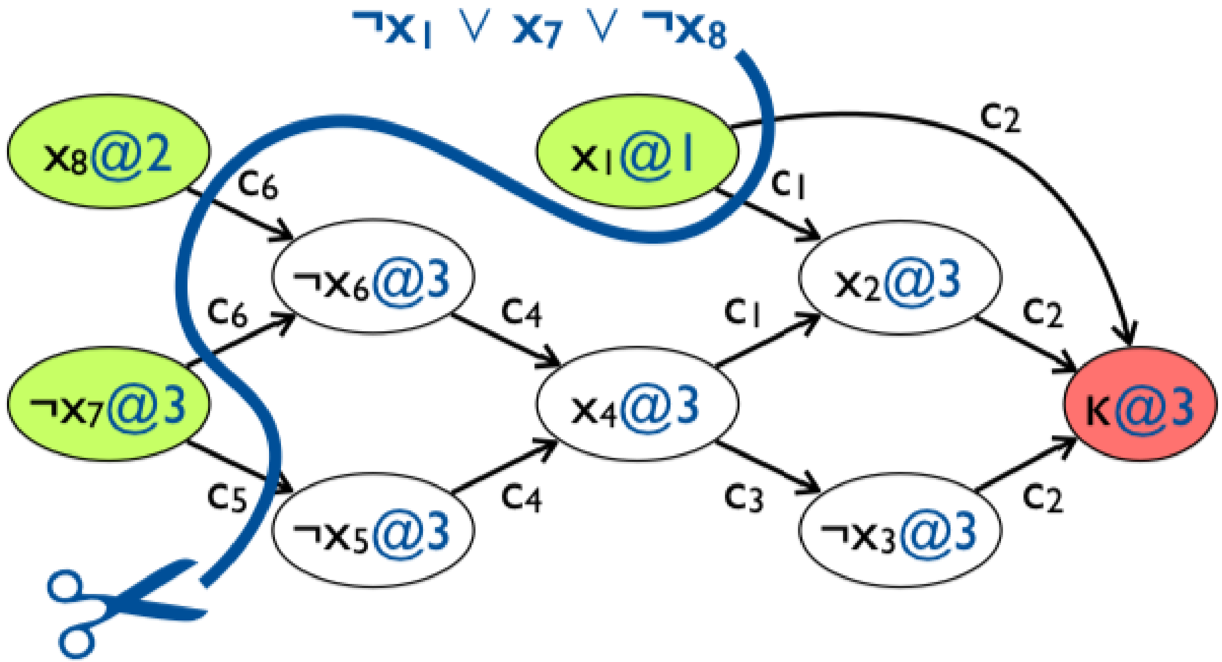
c: $\neg x_1 \vee x_5 \vee x_6$ (resolution with c_4)

c: $\neg x_1 \vee x_6 \vee x_7$ (resolution with c_5)

c: $\neg x_1 \vee x_7 \vee \neg x_8$ (resolution with c_6) Backjumping clause

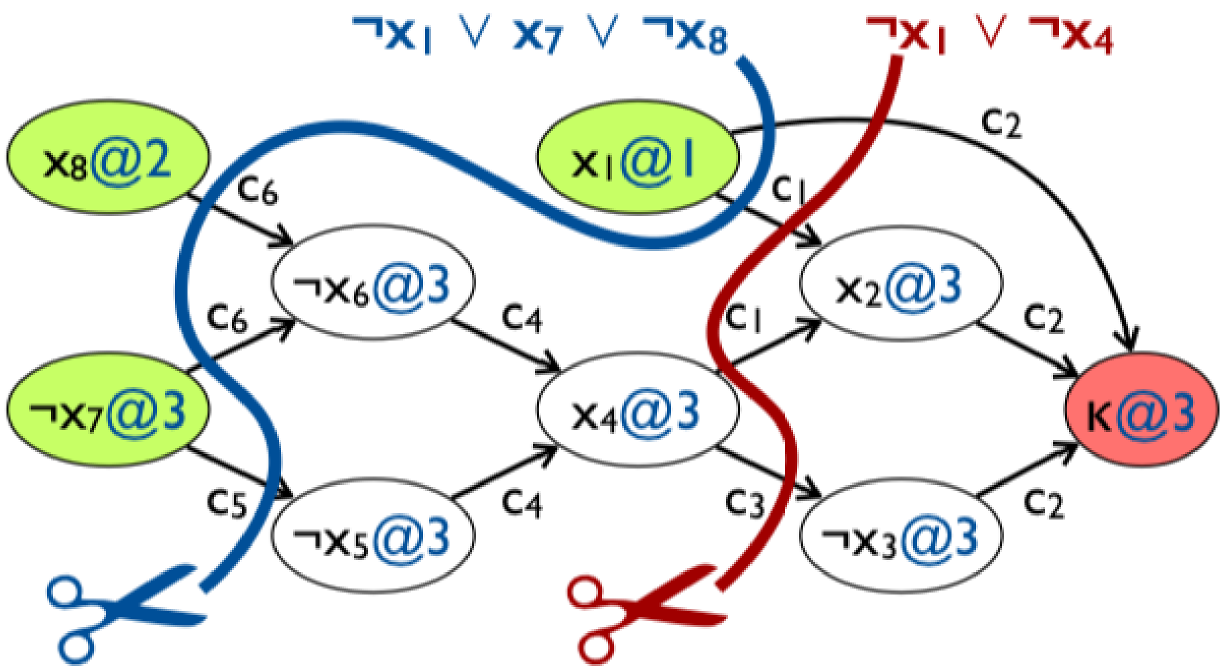
Clause Learning

- The same clause could have been obtained by the decision assignments.



Clause Learning

- Every cut that separates sources from the sink defines a valid conflict clause.



Clause Learning

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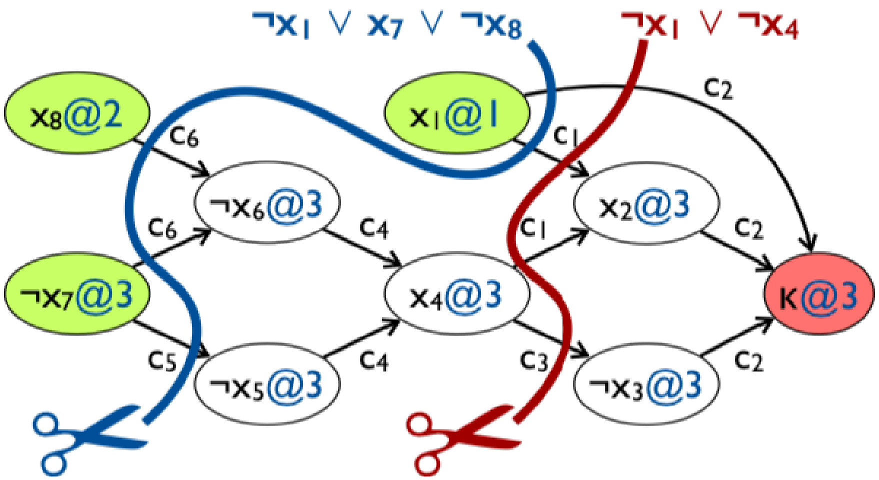


c_2: $\neg x_1 \vee \neg x_2 \vee x_3$

c: $\neg x_1 \vee x_3 \vee \neg x_4$

c: $\neg x_1 \vee \neg x_4$ Backjumping clause

c: $\neg x_1 \vee x_5 \vee x_6$

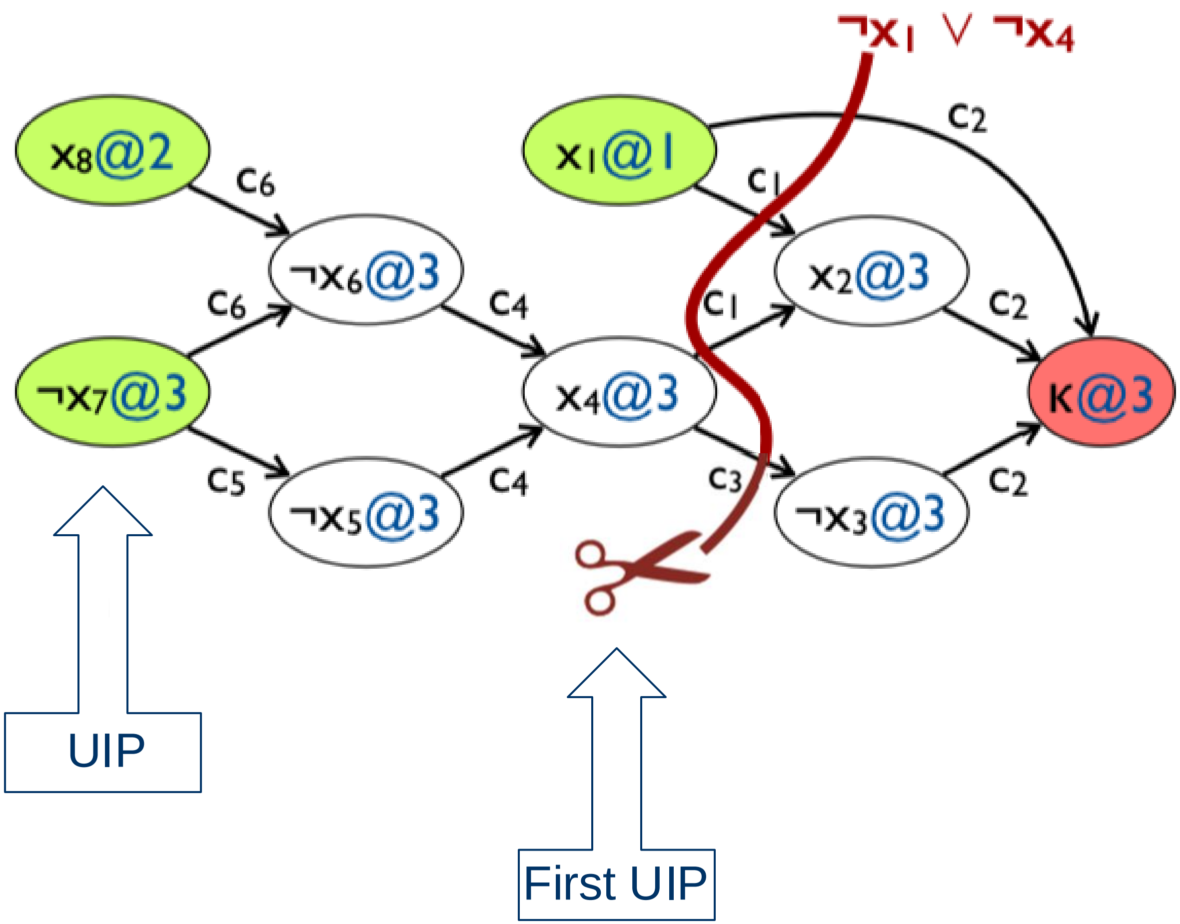
c: $\neg x_1 \vee x_6 \vee x_7$

c: $\neg x_1 \vee x_7 \vee \neg x_8$

Unit Implication Point (UIP)

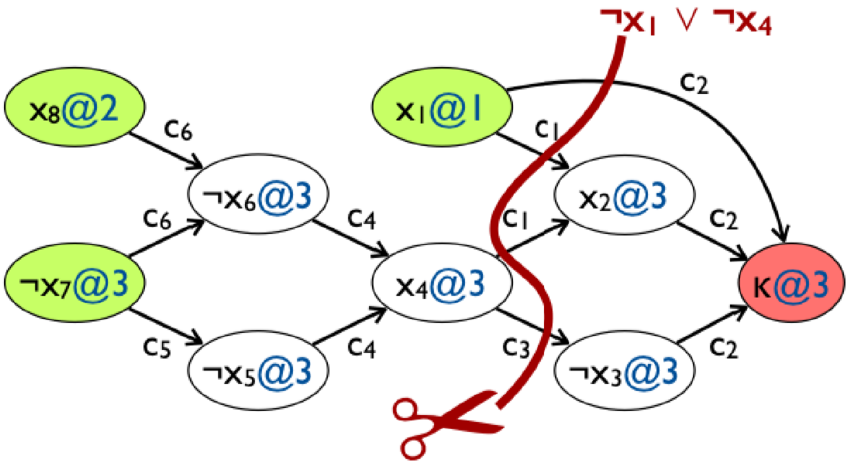
- Any node in the implication graph, other than the conflict, that is on all paths from the current decision literal ($l@d$) to the conflict ($\kappa@d$).
  - A first UIP (**1UIP**) is the UIP that is closest to the conflict.
- Motivation
  - Reduces the size of the learnt clause.
  - Guarantees the highest backtrack jump in the search tree.

First Unit Implication Point (1UIP)



First Unit Implication Point (1UIP)

- There is a UIP at the decision level d, when the number of literals assigned at a decision level d in an intermediate learnt clause is 1.
  - Starting from the conflict clause, the first found UIP is 1UIP.



c_2: $\neg x_1 \vee \neg x_2 \vee x_3$ (clause causing the conflict)

c: $\neg x_1 \vee x_3 \vee \neg x_4$ (resolution with c_1)

c: $\neg x_1 \vee \neg x_4$ (resolution with c_3)

1UIP-based Backjumping

$c_1: \neg x_1 \vee x_2 \vee \neg x_4$

$c_2: \neg x_1 \vee \neg x_2 \vee x_3$

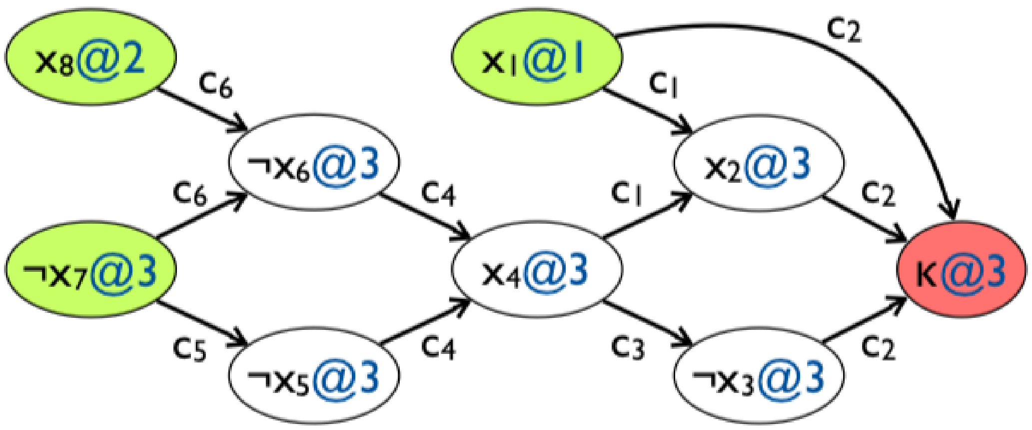
$c_3: \neg x_3 \vee \neg x_4$

$c_4: x_4 \vee x_5 \vee x_6$

$c_5: \neg x_5 \vee x_7$

$c_6: \neg x_6 \vee x_7 \vee \neg x_8$

$c\ : \neg x_1 \vee \neg x_4$



1UIP-based Backjumping

$c_1 : \neg x_1 \lor x_2 \lor \neg x_4$

$c_2 : \neg x_1 \lor \neg x_2 \lor x_3$

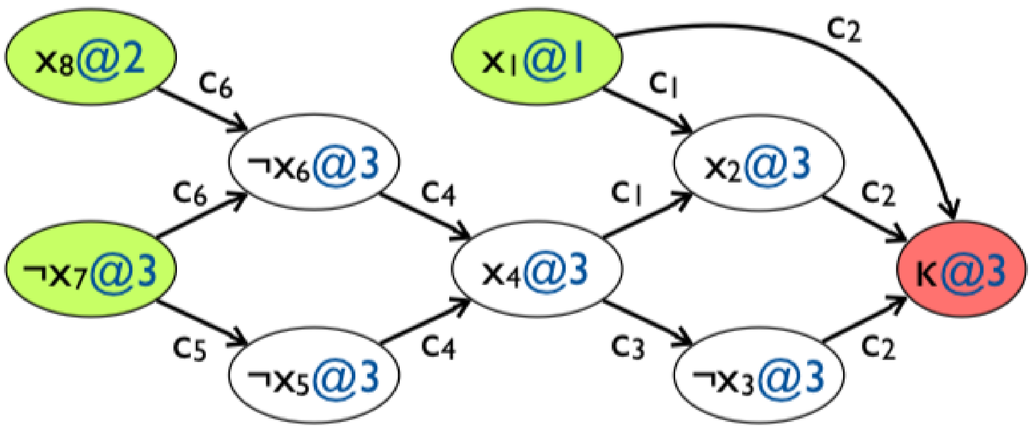
$c_3 : \neg x_3 \lor \neg x_4$

$c_4 : x_4 \lor x_5 \lor x_6$

$c_5 : \neg x_5 \lor x_7$

$c_6 : \neg x_6 \lor x_7 \lor \neg x_8$

$c : \neg x_1 \lor \neg x_4$



- $x_1^d x_8^d x_7^d \neg x_6 \neg x_5 x_4 x_2 \neg x_3 \rightarrow$

1UIP-based Backjumping

$c_1 : \neg x_1 \vee x_2 \vee \neg x_4$

$c_2 : \neg x_1 \vee \neg x_2 \vee x_3$

$c_3 : \neg x_3 \vee \neg x_4$

$c_4 : x_4 \vee x_5 \vee x_6$

$c_5 : \neg x_5 \vee x_7$

$c_6 : \neg x_6 \vee x_7 \vee \neg x_8$

...

$c \;: \neg x_1 \vee \neg x_4$

Conflict clause is unit after backtracking!

$x_1@1$

- $x_1^d x_8^d x_7^d \neg x_6 \neg x_5 x_4 x_2 \neg x_3 \Rightarrow x_1^d \neg x_4$

Characteristics of CDCL Solvers

- Activity-based search concentrates on choosing the variables involved in clause learning.
- Clause learning restriction and deletion.
- Preprocessing and inprocessing.
- Restarting with the learnt clauses, activity-based search and randomization.
- Optimized unit propagation to avoid inspecting all clauses.

Recent Advances

- Parallel solving.
- Unsatisfiability proofs.
- Pseudo-Boolean Solving.
  - SAT with linear constraints.
- Maximum Satisfiability (MaxSAT).
  - SAT for optimization problems.

Why is Chuffed Fast

- Read its description at github!
- Watch the invited talk of CDMO 2021/2022
  - Lazy clause generation: a hybrid CP and SAT approach to combinatorial optimization
  - by Peter Stuckey, co-leader of the MiniZinc team