Homework #2 (NQe311, Spring 2020)

KAIST

(Due April 6)

1. To the Runge function

$$f(x) = \frac{1}{1+x^2},\ x \in [-5,5],$$

Construct (by the following algorithm on p. 12) Newton interpolation polynomials using n+1=11 nodes that are i) equally spaced, ii) Chebyshevian.

Also evaluate the interpolation polynomials for many points of x and plot them for comparison.

2. **Theorem**: If $p(x)$ is the polynomial of degree at most n that interpolate $f(x)$ at n+1 distinct nodes $x_0, x_1, \ldots, x_n$ belong to an interval [a,b] and if $f^{(n+1)}$ is continuous, then for each x in [a,b], there is a ξ in (a,b) for which

$$f(x) - p_n(x) = \frac{1}{(n+1)!} f^{(n+1)}(\xi) \prod_{i=0}^{n} (x - x_i)$$

Corollary: Let $f(x)$ be a function that $f^{(n+1)}$ is continuous on [a,b] and satisfies $|f^{(n+1)}(x)| \le M$. Let $p_n(x)$ be the polynomial of degree ≤n that interpolates $f(x)$ at n+1 equally-spaced nodes in [a,b], including the end points. Then on [a,b]

$$|f(x) - p_n(x)| \le \frac{1}{4(n+1)} M h^{n+1},$$

where $h = (b - a)/n$ is the spacing between nodes.

Show directly that the maximum interpolation error is bounded by the following expression and compare them to the bounds given by the Corollary.

a) $h^2 M/(8)$ for linear interpolation, where $h = x_1 - x_0$ and $M = \max_{x_0 \le x \le x_1} |f''(x)|$.

b) $h^3 M/(9\sqrt{3})$ for quadratic interpolation, $h = x_1 - x_0 = x_2 - x_1$ and $M = \max_{x_0 \le x \le x_2} |f^{(3)}(x)|$.

c) $h^4 M/24$ for cubic interpolation, $h = x_1 - x_0 = x_2 - x_1 = x_3 - x_2$ and $M = \max_{x_0 \le x \le x_3} |f^{(4)}(x)|$.