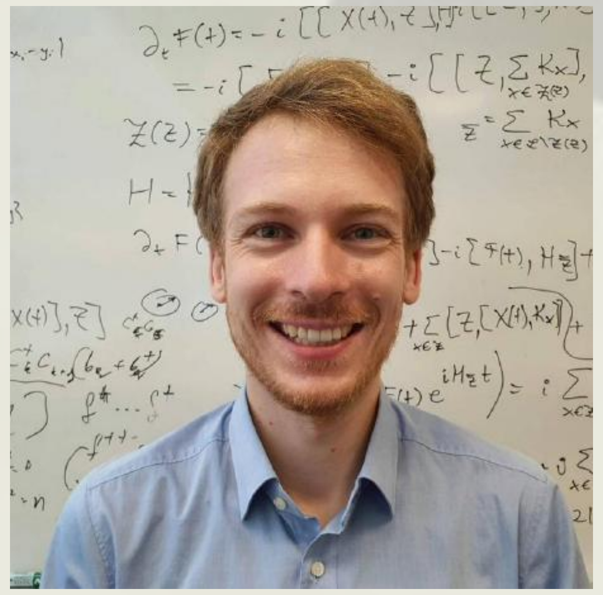
Illustration Credit: Atomchip group, TU Wien

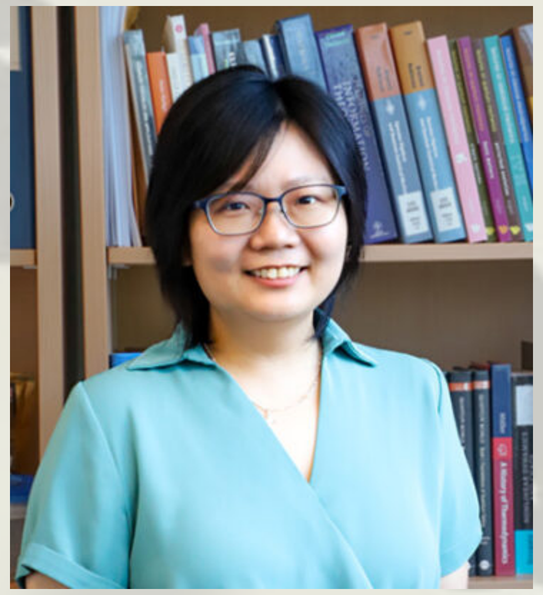
arXiv 2508.20533

Extensive entanglement between coupled quantum fields in and out of equilibrium

Taufiq Murtadho*

Collaborator: Marek Gluza, Nelly H. Y. Ng





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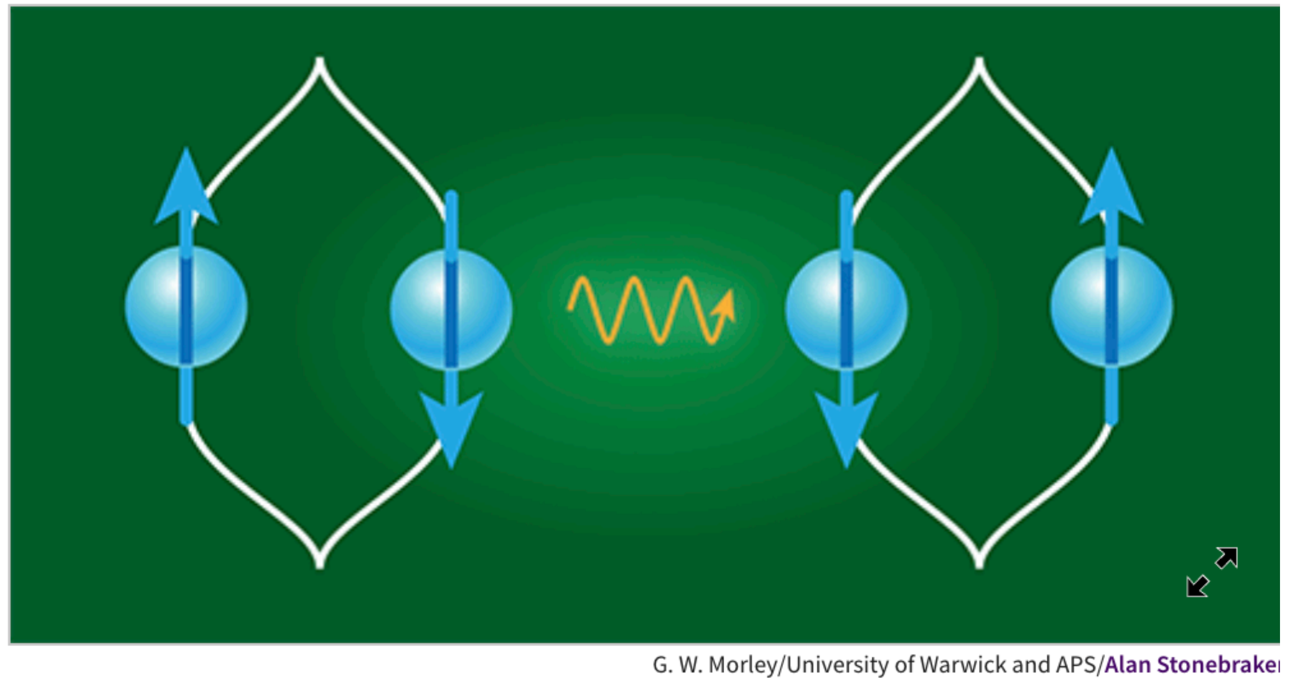
Quantum physics goes "big"

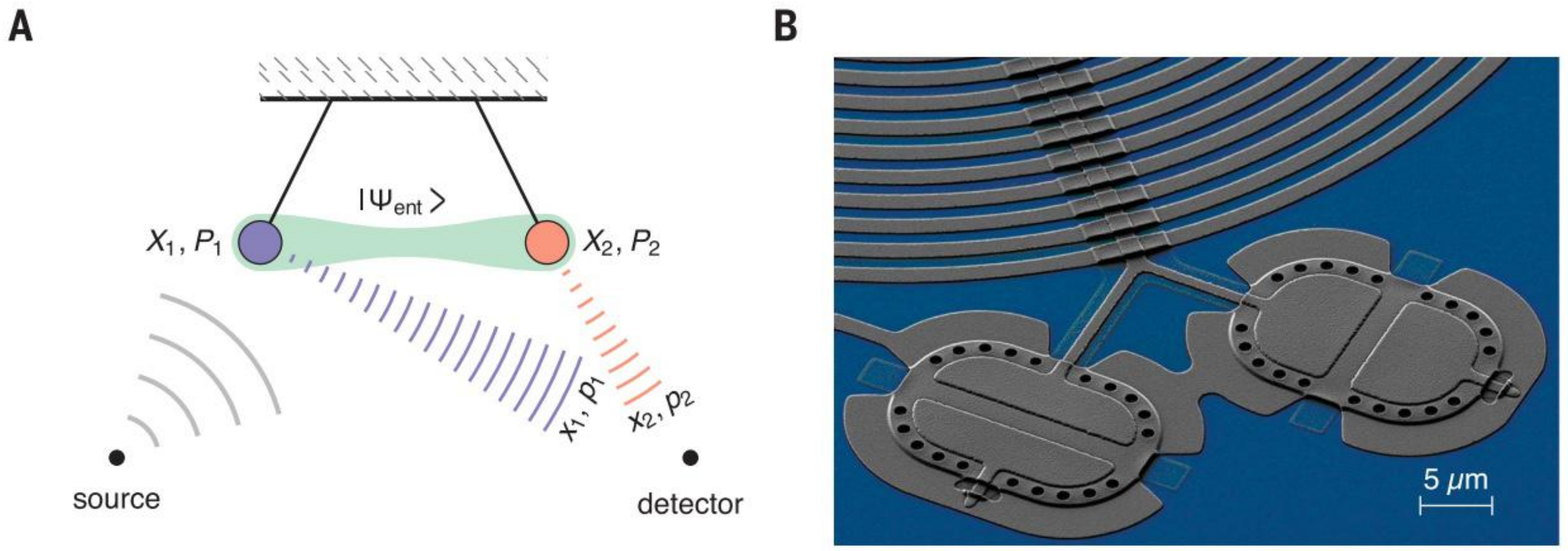
- There has been intensive efforts to **observe quantum effects (e.g. superposition, entanglement) in mesoscopic and macroscopic systems.**

- Aim to better understand fundamental physics: **decoherence, quantum gravity, many-body physics etc.**

- Developing **quantum computers and quantum information technologies** (sensor, memory, communication, etc.)

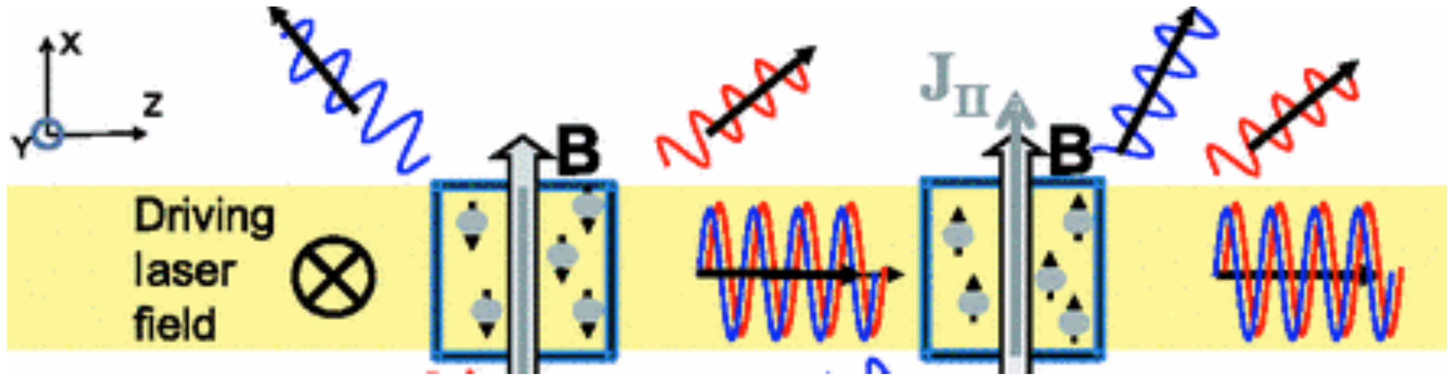


Bose, Sougato, et al. "Spin entanglement witness for quantum gravity." *PRL* 19.24: 240401 (2017).



Kotler, S., et al. "Direct observation of deterministic macroscopic entanglement." *Science* 372.6542: 622-625 (2021).

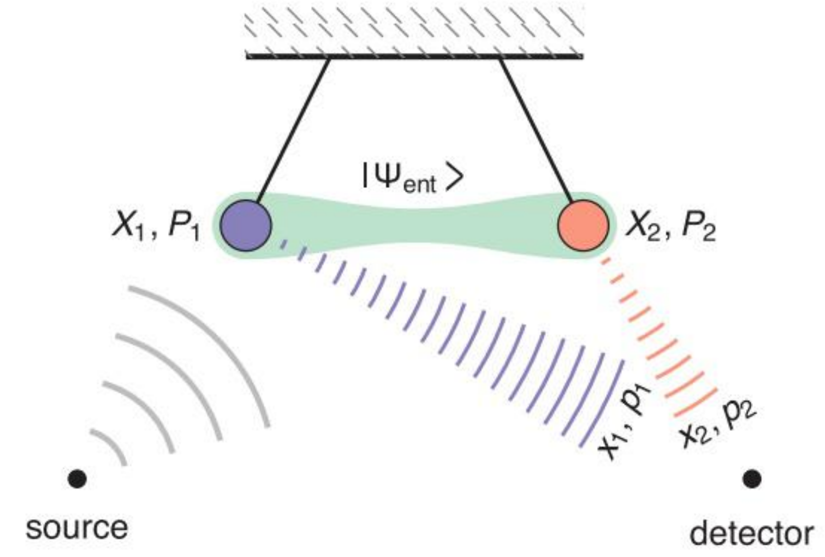
Quantum entanglement between two “large” objects



Dissipative entanglement generation between Cs atoms ensembles

Krauter, H., et al. *PRL* 107.8: 080503 (2011).

A



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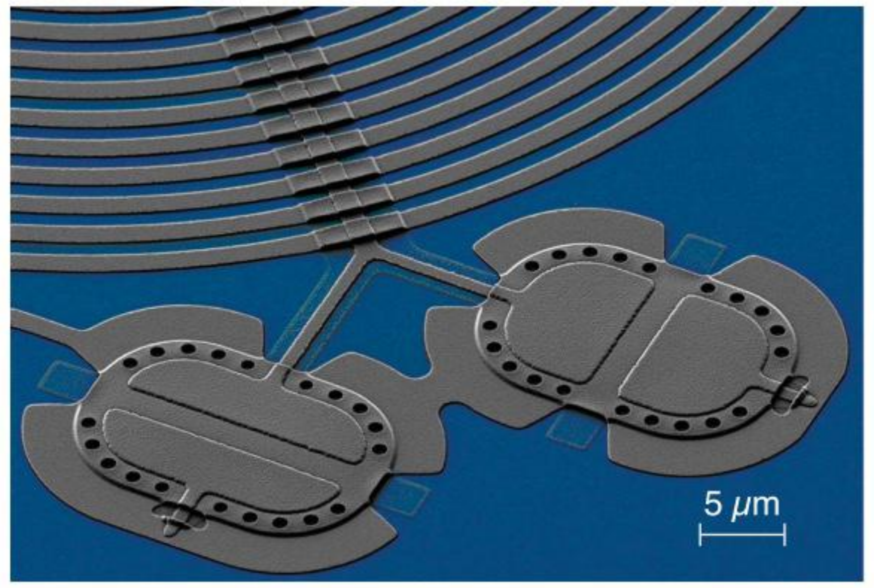
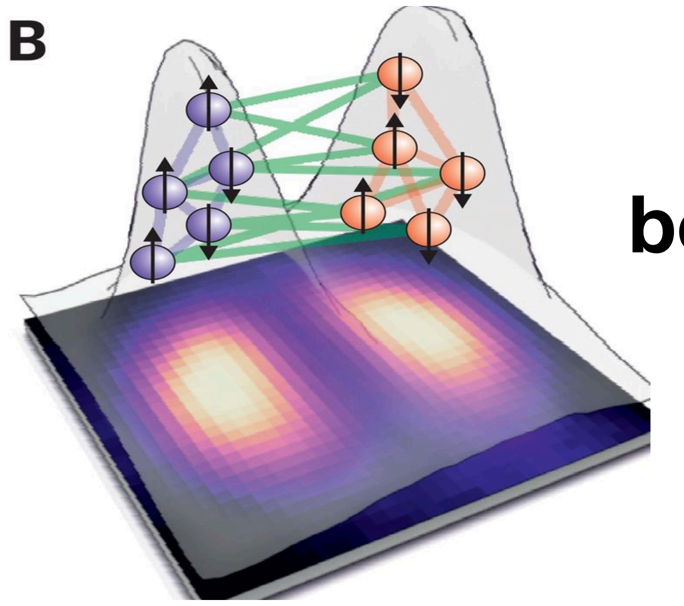


Macroscopic entanglement between mechanical oscillators

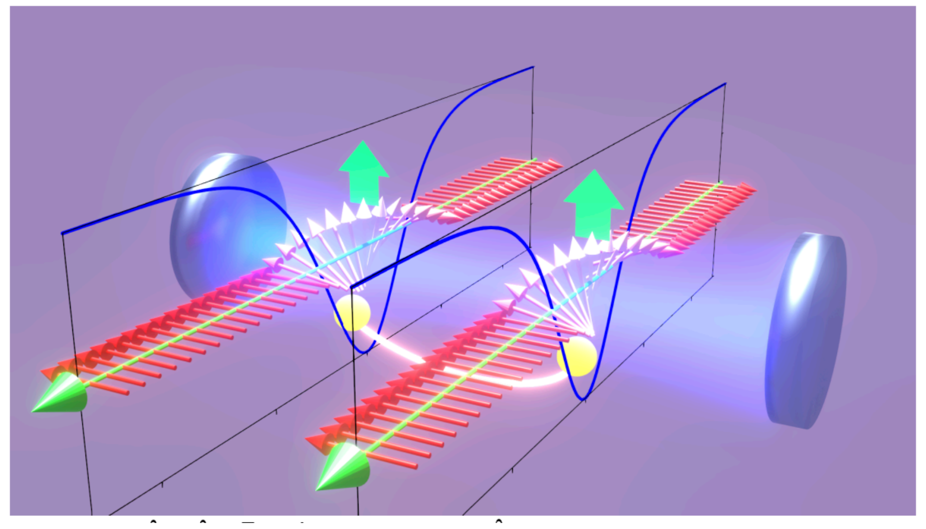
Kotler, S., et al. *Science* 372.6542: 622-625 (2021).

B



Macroscopic entanglement between spatially-separated BECs

Lange, Karsten, et al. *Science* 360.6387: 416-418 (2018).



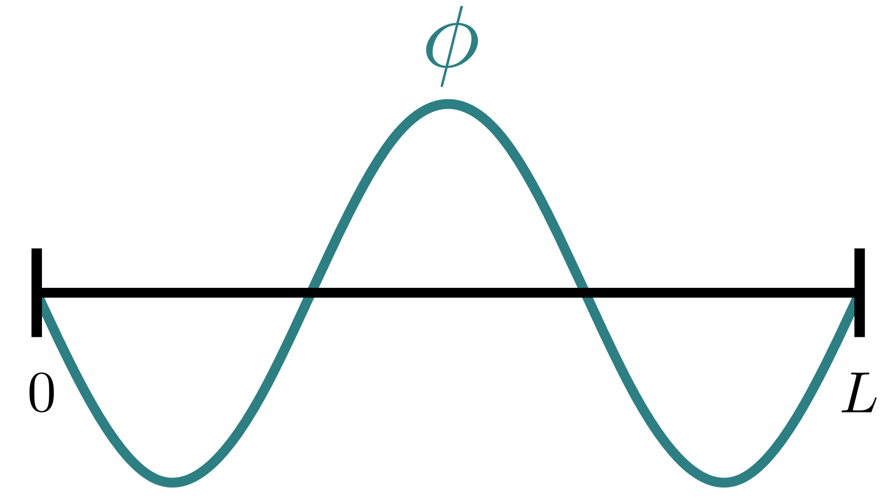
Macroscopic entanglement between magnetic domain walls

Gupta, Rahul, Huaiyang Yuan, and Himadri Shekhar Dhar. *arXiv:2508.03450* (2025).

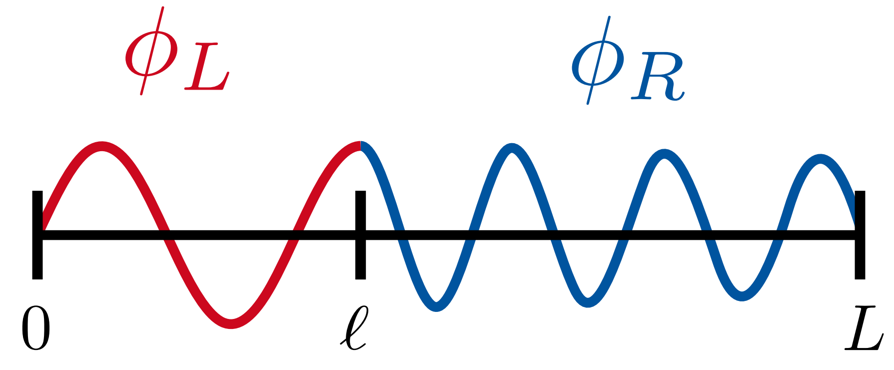
- Typically probed in the limit of **two-mode** approximations.
- Can we show **entanglement between ‘macroscopic’ objects with each possibly occupying multiple modes, e.g. quantum fields?**

Entanglement in quantum field or many-body systems

Real Space (Geometric) Entanglement



Emonts, P., & Kukuljan, I. *PRR*, *4*(3), 033039 (2022).



Entanglement entropy

$$S(\rho) = -\,\mathrm{Tr}\left(\rho_L \log \rho_L\right)$$

$$\rho_L = \mathrm{Tr}_R \rho$$

'Area Law' of Entanglement

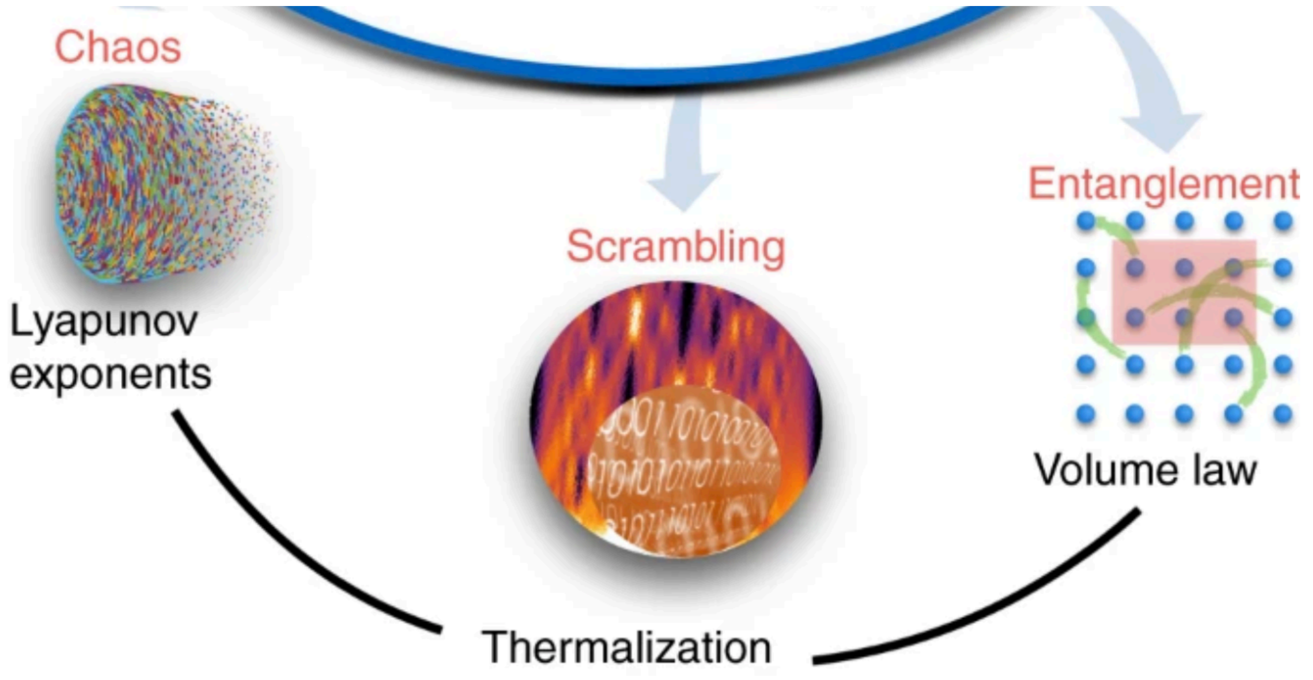
For ground state ρ_{GS} of locally interacting gapped Hamiltonian

$$S(\rho_{\mathrm{GS}}) \propto \partial V$$

Important for holography, matrix product states (MPS), etc.

Nishioka, Tatsuma. "Entanglement entropy: holography and renormalization group." *RMP* 90.3: 035007 (2018).
Schuch, Norbert, et al. "Entropy scaling and simulability by matrix product states." *PRL* 100.3: 030504 (2008).
… many more!

'Volume Law' Entanglement usually indicates scrambling, thermalization, chaos, etc.



Lewis-Swan, Robert J., et al. *Nat. Comm.* : 1581 (2019).
Bianchi, Eugenio, et al. *PRX quantum* 3.3: 030201 (2022).
… many more!

Entanglement *between* coupled quantum fields

Field Space Entanglement

$$\mathcal{S} = \int d^4\mathbf{x} \left(\mathscr{L}[\hat{\Psi}_1(\mathbf{x})] + \mathscr{L}[\hat{\Psi}_2(\mathbf{x})] + \mathscr{L}_{\mathrm{int}}[\hat{\Psi}_1(\mathbf{x}), \hat{\Psi}_2(\mathbf{x})] \right)$$

$$S(\rho) = -\,\mathrm{Tr}\left(\rho_{\hat{\Psi}_1} \log \rho_{\hat{\Psi}_1}\right) \qquad \rho_{\hat{\Psi}_1} = \mathrm{Tr}_{\hat{\Psi}_2}\rho$$

Field space entanglement is not as well understood as geometric entanglement.

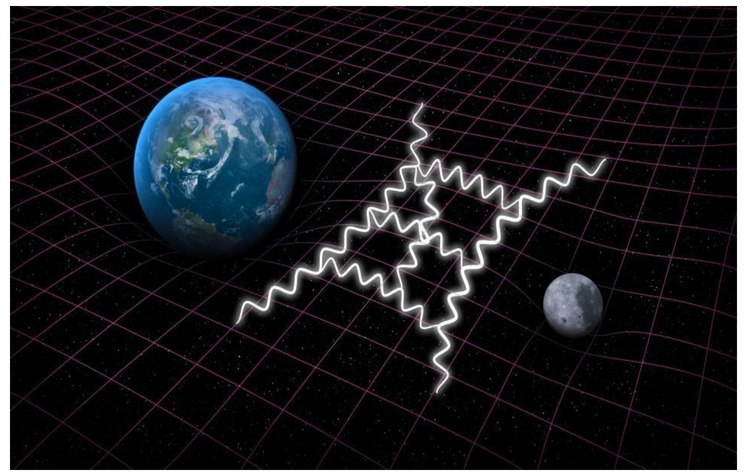
Relevant to detect (for example):

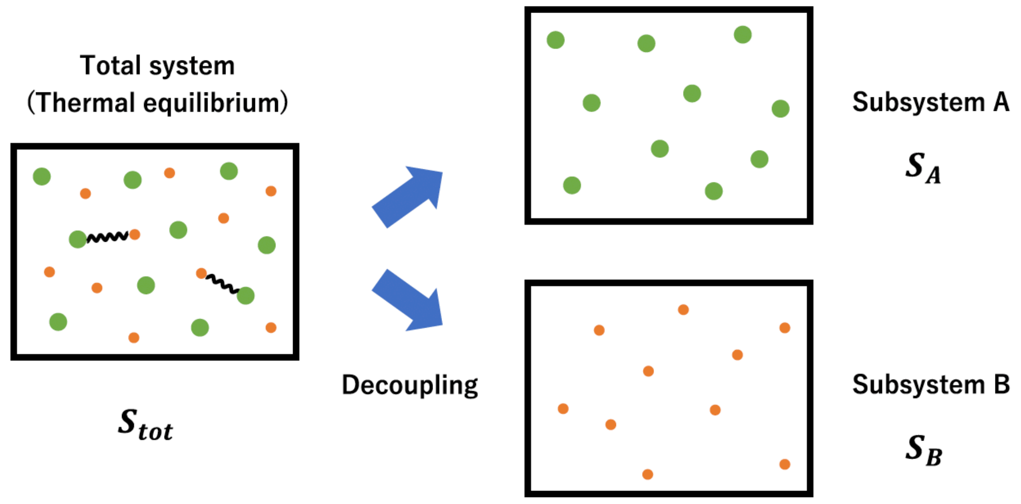
- **Entanglement between known fields and unknown fields.**
- **Decoupling between interacting fields in an expanding universe.**

Volume scaling (extensive) without scrambling

$$S(\rho) \propto V$$

Beneficial for detectability of entanglement, and also for applications in sensor, memory, etc.





Mollabashi, Ali, Nobuto Shiba, and Tadashi Takayanagi. *JHEP* 2014.4: 1-36 (2014).
Teresi, Daniele, and Giuseppe Compagno. *arXiv:1012.3915* (2010).
Nakai, Yuichiro, Noburo Shiba, and Masaki Yamada." *PRD* 96.12: 123518 (2017).

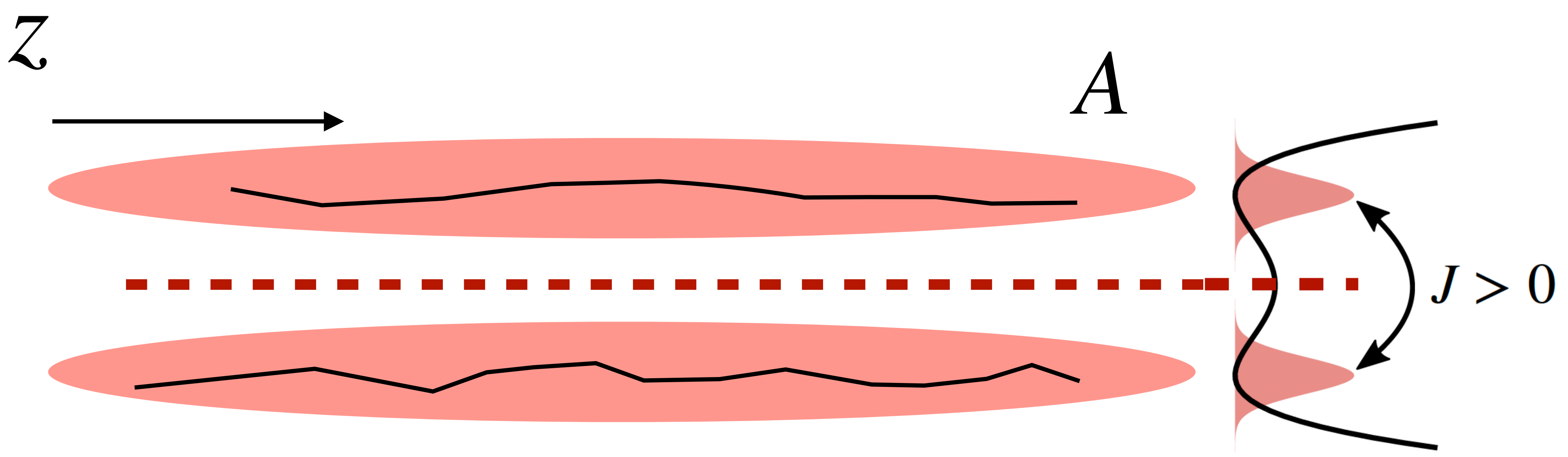
Key Research Questions:

- How can we simulate the physics of field space entanglement in laboratory (i.e. quantum field simulators)?

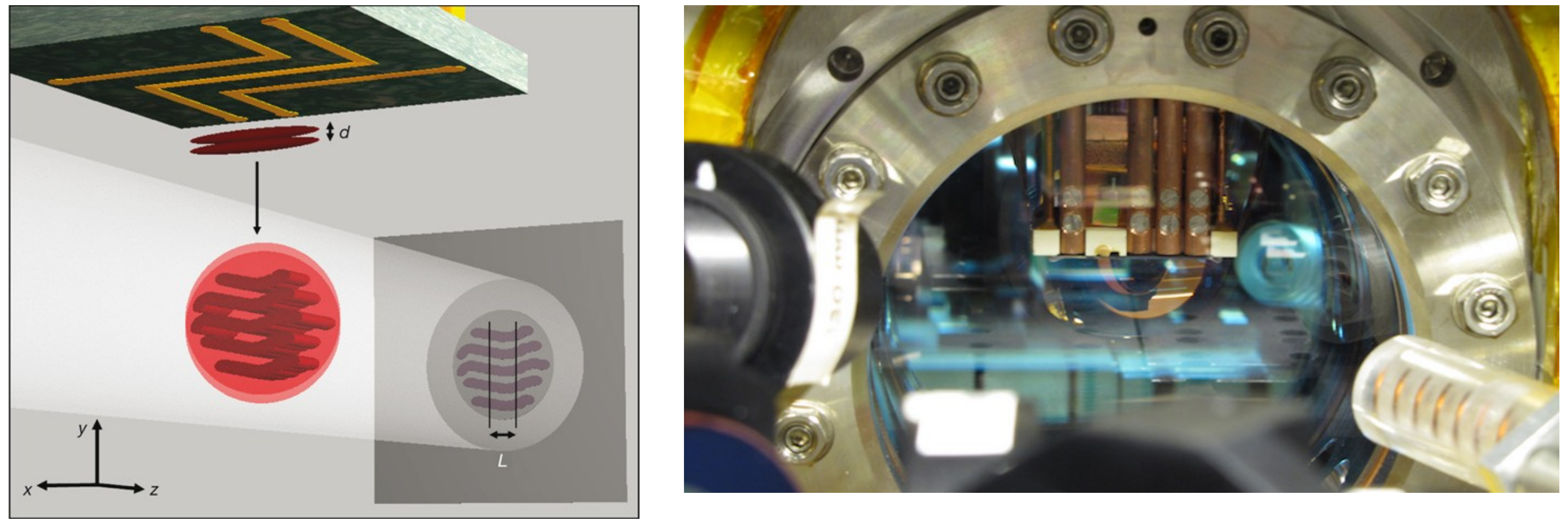
- Most theoretical studies of entanglement in field theories focus on entanglement entropy, which applies only to pure states. Meanwhile, quantum simulators in the lab are much more likely to be in mixed (e.g. thermal) states. Can we quantify and detect entanglement in quantum field simulators at finite temperatures?

Tunnel-coupled Parallel 1D Bose Gases

Parallel 1D Bose Gases



$N \sim 10{,}000$ ^{87}Rb atoms
$L \sim 80\ \mu$m



Experimentally realized at TU Wien Atomchip group

Each 1D gas is described by a 1D **bosonic field** $\hat{\Psi}_a(z)$

$$[\hat{\Psi}_a(z), \hat{\Psi}_b^\dagger(z')] = \delta_{zz'}\delta_{ab} \qquad a, b \in \{A, B\}$$

with Hamiltonian

$$\hat{H} = \sum_{a=A,B} \int dz \left[-\frac{\hbar^2}{2m} \hat{\Psi}_a^\dagger \partial_z^2 \hat{\Psi}_a + (U(z) - \mu)\hat{\Psi}_a^\dagger \hat{\Psi}_a + \frac{g}{2} \hat{\Psi}_a^\dagger \hat{\Psi}_a^\dagger \hat{\Psi}_a \hat{\Psi}_a \right]$$

$$-\hbar J \int dz \, [\hat{\Psi}_A^\dagger \hat{\Psi}_B + \hat{\Psi}_B^\dagger \hat{\Psi}_A]$$

In thermal eq., $\hat{\Psi}_a(z)$ includes **both quantum + thermal fluctuations**

Are $\hat{\Psi}_A(z)$ and $\hat{\Psi}_B(z)$ entangled?

Low-energy and Gaussian approximations

Phase-density decomposition (valid for low energy e.g. low temperature)

$$\hat{\Psi}^a(z) = e^{i\hat{\phi}^a(z)}\sqrt{n_{1\mathrm{D}} + \delta\hat{n}^a(z)} \qquad [\delta\hat{n}^a(z), \hat{\phi}^b(z')] = i\delta(z-z')\delta_{ab}$$

Low-energy Hamiltonian:

Tomonaga-Luttinger Liquid (TLL)

$$\hat{H}^a_{\mathrm{TLL}} = \int_0^L dz \left[\frac{\hbar^2 n_{1\mathrm{D}}}{2m}(\partial_z\hat{\phi}^a(z))^2 + \frac{g}{2}(\delta\hat{n}^a(z))^2\right]$$

$$\hat{H} = \hat{H}^A_{\mathrm{TLL}} + \hat{H}^B_{TLL} - 2\hbar J n_{1\mathrm{D}} \int \cos\left[\hat{\phi}^A(z) - \hat{\phi}^B(z)\right] dz$$

$$\approx \hat{H}^A_{\mathrm{TLL}} + \hat{H}^B_{TLL} + \hbar J n_{1\mathrm{D}}(\hat{\phi}^A(z) - \hat{\phi}^B(z))^2$$

Gaussian approximation, valid for strong enough J

Diagonalizing the Hamiltonian

Symmetric (common, +) and antisymmetric (relative, -) fields

$$\delta\hat{n}^{\pm}(z) = \frac{1}{\sqrt{2}}\left(\delta\hat{n}^{A}(z) \pm \delta\hat{n}^{B}(z)\right)$$

$$\hat{\phi}^{\pm}(z) = \frac{1}{\sqrt{2}}\left(\hat{\phi}^{A}(z) \pm \hat{\phi}^{B}(z)\right)$$

Bogoliubov expansion and transformation

$$\hat{H} \approx \sum_{k\neq 0} \varepsilon_k^{+}(\hat{b}_k^{+})^{\dagger}\hat{b}_k^{+} + \varepsilon_k^{-}(\hat{b}_k^{-})^{\dagger}\hat{b}_k^{-}$$

Whitlock, N. K., & Bouchoule, I. Relative phase fluctuations of two coupled one-dimensional condensates. *PRA*, *68*(5), 053609 (2003).

Mode decomposition

$$\hat{\phi}_k^{+} = \frac{1}{\sqrt{4n_{1D}}}\sqrt{\frac{\varepsilon_k^{+}}{E_k}}\left[\hat{b}_k^{+} + \mathrm{h.c.}\right] \qquad \delta\hat{n}_k^{+} = \sqrt{n_{1D}}\sqrt{\frac{E_k}{\varepsilon_k^{+}}}\left[i\hat{b}_k^{+} + \mathrm{h.c.}\right]$$

$$\hat{\phi}_k^{-} = \frac{1}{\sqrt{4n_{1D}}}\sqrt{\frac{\varepsilon_k^{-}}{E_k + 2\hbar J}}\left[\hat{b}_k^{-} + \mathrm{h.c.}\right] \qquad \delta\hat{n}_k^{-} = \sqrt{n_{1D}}\sqrt{\frac{E_k + 2\hbar J}{\varepsilon_k^{-}}}\left[i\hat{b}_k^{-} + \mathrm{h.c.}\right],$$

$$\varepsilon_k^{+} = \sqrt{E_k(E_k + 2gn_{1D})}$$

$$\varepsilon_k^{-} = \sqrt{(E_k + 2\hbar J)(E_k + 2\hbar J + 2gn_{1D})}$$

$$E_k = (\hbar k)^2/2m$$

Field Covariance Matrix

Thermal density matrix

$$\hat{\rho}_{\text{th}} = \frac{1}{Z}\exp(-\beta \hat{H})$$

Collection of two-point correlation functions e.g. $\langle \hat{\phi}_k^+ \hat{\phi}_k^+ \rangle, \ldots$

Heisenberg uncertainty relation

$$\Gamma + i(\Omega \oplus \Omega) \geq 0 \qquad \Omega = \begin{pmatrix} 0 & \mathbf{1}_\Lambda \\ -\mathbf{1}_\Lambda & 0 \end{pmatrix}$$

$$\updownarrow$$

$$\sigma_k \geq 1/2 \qquad \forall k$$

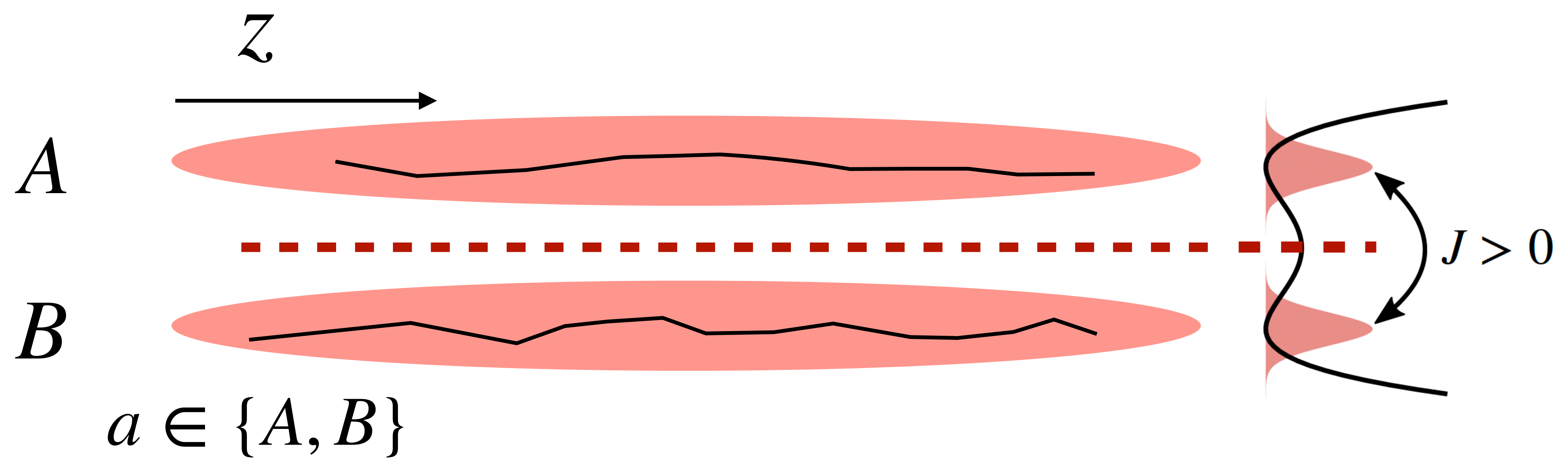
Covariance matrix

$$\Gamma = \begin{pmatrix} \Gamma^+ & 0 \\ 0 & \Gamma^- \end{pmatrix} \xrightarrow[U(\Omega \oplus \Omega)U^\top = \Omega \oplus \Omega]{\text{Symplectic transform.} \quad \tilde{\Gamma} = U\Gamma U^\top} \tilde{\Gamma} = \begin{pmatrix} \Gamma^{AA} & \Gamma^{AB} \\ \Gamma^{BA} & \Gamma^{BB} \end{pmatrix}$$

Symplectic transform leaves symplectic eigenvalues $\{\sigma_k\}$ invariant

$$\{\sigma_k\} = |\,\mathrm{eig}[\mathrm{i}(\Omega \oplus \Omega)\Gamma]\,|$$

Mutual Information $I(A:B)$ as correlation measure



$$I(A:B) = S(A) + S(B) - S(AB)$$

$$S(\Gamma) = \sum_k \left(\sigma_k + \frac{1}{2}\right) \log\left(\sigma_k + \frac{1}{2}\right) - \left(\sigma_k - \frac{1}{2}\right) \log\left(\sigma_k - \frac{1}{2}\right)$$

σ_k : Symplectic eigenvalues of covariance matrix Γ

- $I(A:B)$ ***includes both entanglement and classical correlations in general***
- In the limit of **zero temperature** $I(A:B) = 2S(\rho)$ where $S(\rho)$ is the entanglement entropy.

Logarithmic negativity $E_{\mathcal{N}}$ as entanglement measure

- **Logarithmic negativity (LN)** is directly related to **positive partial transpose (PPT)** criterion for separable states.

- **Step 1:** Partial transpose for continuous variable Gaussian states

$$\langle x_1, x_2 | \rho^{\top_B} | x_1', x_2' \rangle = \langle x_1, x_2' | \rho | x_1', x_2 \rangle$$

$$W(q_1, p_1, q_2, p_2) \to W(q_1, p_1, q_2, -p_2)$$

$$\longrightarrow \quad \tilde{\Gamma}^{\top_B} = \tilde{\Gamma}(\hat{\phi}_k^B \to -\hat{\phi}_k^B)$$

- **Step 2:** Compute symplectic eigenvalues of $\tilde{\Gamma}^{\top_B}$

$$\{\gamma_k\} = \mathrm{eig}[i(\Omega \oplus \Omega)\tilde{\Gamma}^{\top_B}]$$

Logarithmic negativity $E_{\mathcal{N}}$ as entanglement measure

- **Step 3:** Compute log. neg. by

$$E_{\mathcal{N}} = \sum_k \max\{0, -\log(2\gamma_k)\}$$

γ_k : Symplectic eigenvalues of partially transposed covariance matrix Γ^{T_B}

- $E_{\mathcal{N}} > 0$ is a ***sufficient*** condition for entanglement. It is ***necessary and sufficient*** for **two-modes entanglement**

- In our case, we have to solve **multiple independent two-mode problems** (one for each momentum mode k) $\rightarrow E_{\mathcal{N}} > 0$ is necessary and sufficient for entanglement

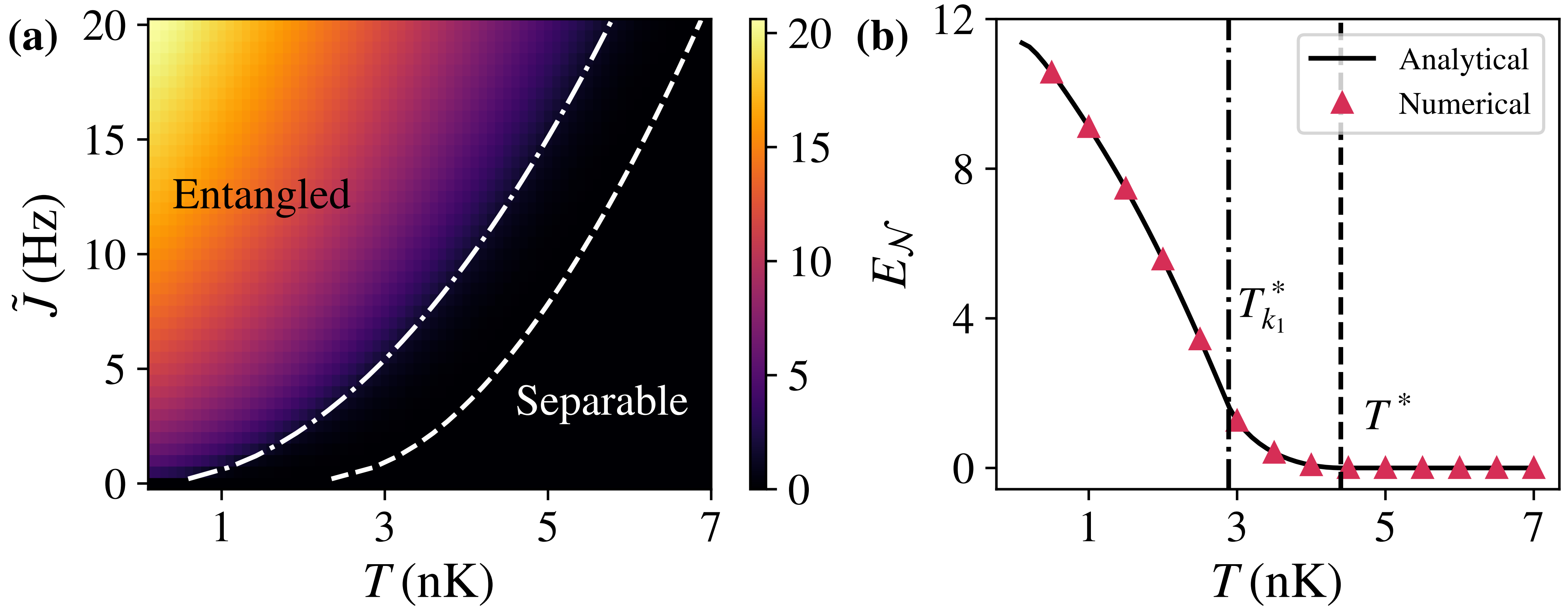
Results #1: Entanglement & Mutual Information at Finite Temperatures

Entanglement vs. Temperature - how "cold" is too "hot"?

- **We derive analytical formula for logarithmic negativity at finite temperatures**
- **Competition between tunneling strength J and temperature T**
- **We derive expression for threshold temperature T^***

$$E_{\mathcal{N}} = \sum_{k>0}^{k_\Lambda} \max\left\{0, -\log\sqrt{C_{k,J}(1+\eta_k^+)(1+\eta_k^-)}\right\}$$

$$C_{k,J} = \frac{\varepsilon_k^-}{\varepsilon_{k+}}\frac{E_k}{E_k + 2\hbar J} \leq 1 \qquad \eta_k^{\pm} = [\exp(\beta\varepsilon_k^{\pm}) - 1]^{-1}$$



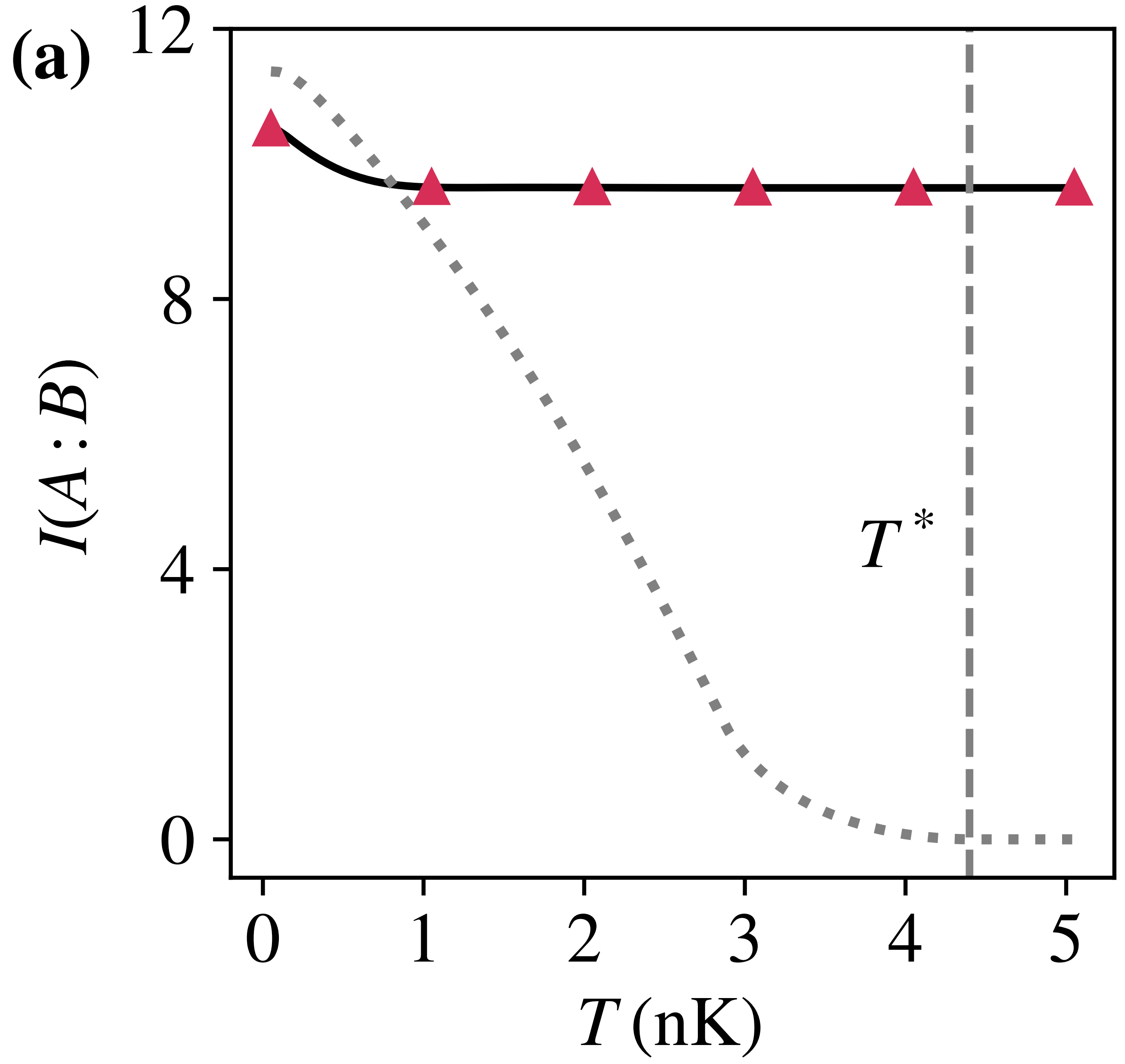
$$T^* = \sup_{0<k\leq k_\Lambda}\left\{T_k^* \,\middle|\, \tanh\left(\frac{\varepsilon_k^+}{2k_B T_k^*}\right)\tanh\left(\frac{\varepsilon_k^-}{2k_B T_k^*}\right) = C_{k,J}\right\}$$

For realistic experimental parameters $T^* \sim 1-5$ nK, about one order of magnitude colder than current state-of-the-art temperatures achieved in the lab

Quantum and classical correlation cross-over

- **Analytic expression for mutual information at finite temperatures**
- **Entanglement entropy at zero temperature can be recovered from the general result**

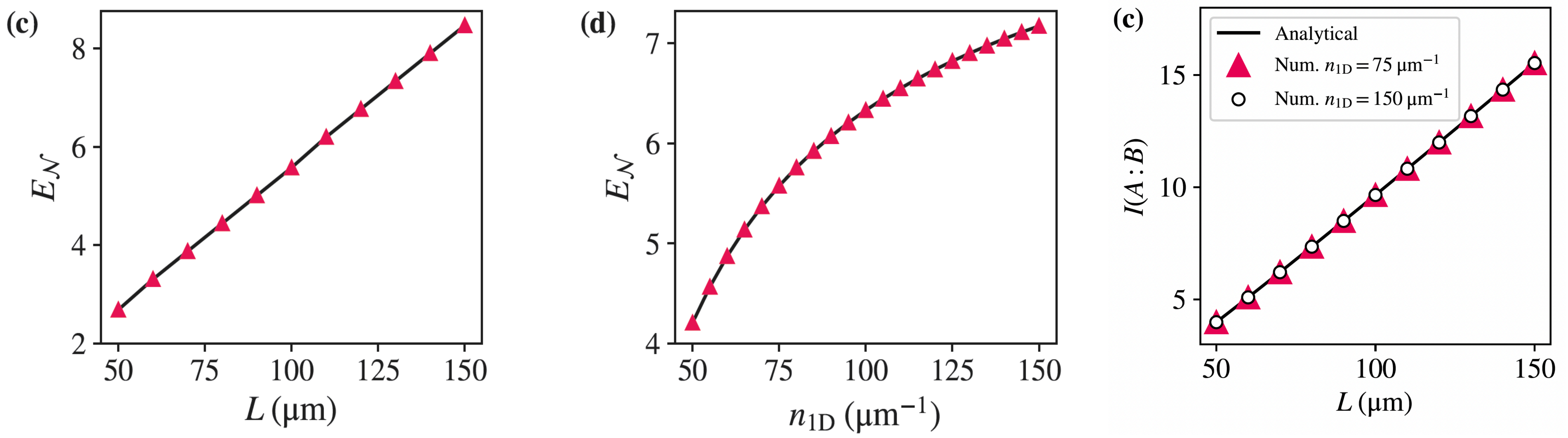
$$I(A:B) = \sum_{k>0}^{k_\Lambda} 2\left(\lambda_{k,J}+\frac{1}{2}\right)\log\left(\lambda_{k,J}+\frac{1}{2}\right) - \left(\lambda_{k,J}-\frac{1}{2}\right)\log\left(\lambda_{k,J}-\frac{1}{2}\right)$$
$$-\sum_{a=\pm}\sum_{k>0}^{k_\Lambda}\left[\left(\eta_k^a+1\right)\log\left(\eta_k^a+1\right)-\eta_k^a\log\eta_k^a\right]$$



$$\lambda_{k,J} = \frac{1}{4}\sqrt{\left[(1+2\eta_k^+)+C_{k,J}(1+2\eta_k^-)\right]\left[(1+2\eta_k^+)+C_{k,J}^{-1}(1+2\eta_k^-)\right]},$$

As T increases quantum correlation transmute into classical correlation to keep mutual info. constant

Scaling of entanglement and mutual information



- Both logarithmic negativity and mutual information are found to be **extensive**

$$E_{\mathcal{N}} \propto L \qquad\qquad I(A:B) \propto L$$

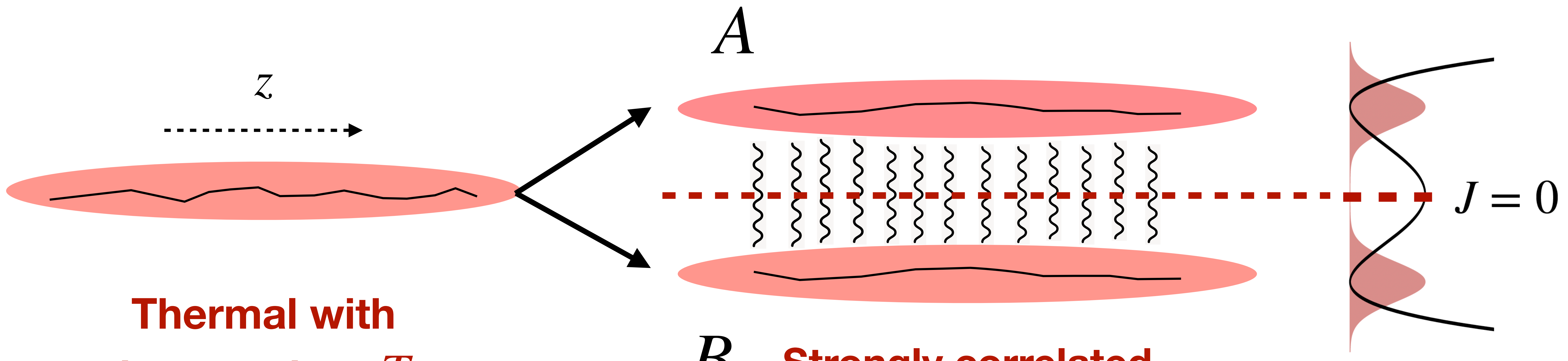
- The extensivity arise from the growing number of modes that are entangled/correlated

$$\sum_k \to \frac{L}{2\pi}\int dk$$

- Non-trivial scaling with density n_{1D} underscores that the entangled modes are collective

Results #2:
Entanglement & Mutual Information after Coherent Splitting

Initial State after Coherent Splitting



Thermal with temperature T

Strongly correlated non-equilibrium state

Symmetric (+) sector: Thermal fluctuations

$$\langle \hat{\phi}_k^+ \hat{\phi}_q^+ \rangle = \frac{\delta_{kq}}{4 n_{1\mathrm{D}}} \frac{\varepsilon_k^+}{E_k} (1 + 2\eta_k^+)$$

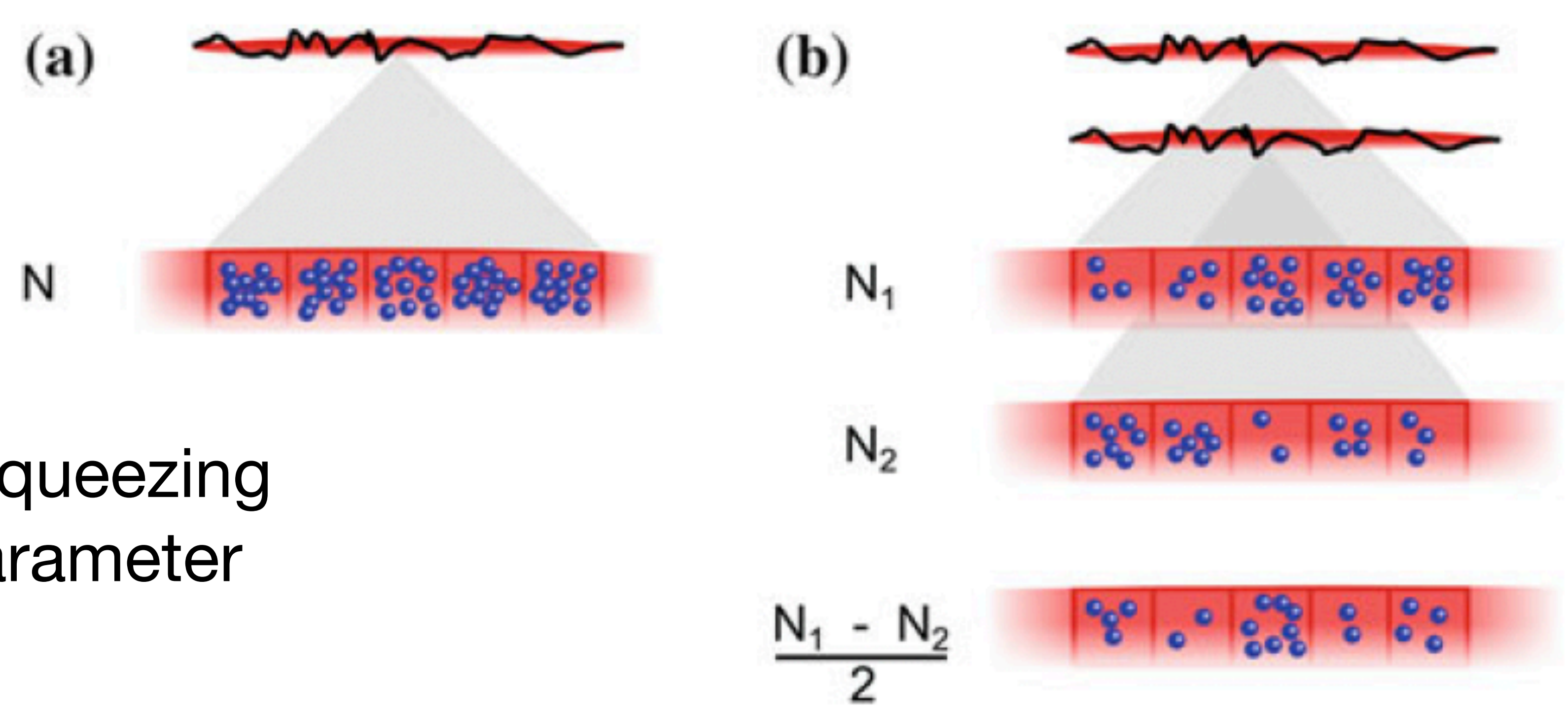
$$\langle \delta\hat{n}_k^+ \delta\hat{n}_q^+ \rangle = \delta_{kq} n_{1\mathrm{D}} \frac{E_k}{\varepsilon_k^+} (1 + 2\eta_k^+)$$

Antisymmetric (-) sector: Vacuum (quantum) fluctuations

$$\langle \hat{\phi}_k^- \hat{\phi}_q^- \rangle = \frac{\delta_{kq}}{2 n_{1\mathrm{D}} r^2}$$

$$\langle \delta\hat{n}_k^- \delta\hat{n}_q^- \rangle = \frac{r^2 n_{1\mathrm{D}}}{2} \delta_{kq}$$

$$W(N^{(1)}, N^{(2)}) = \binom{N^{(1)} + N^{(2)}}{N^{(2)}} p_1^{N^{(1)}} (1 - p_1)^{N^{(2)}},$$



r: squeezing parameter

Geiger, Remi, et al. "Local relaxation and light-cone-like propagation of correlations in a trapped one-dimensional Bose gas." *New Journal of Physics* 16.5: 053034 (2014).

Langen, Tim. *Non-equilibrium dynamics of one-dimensional Bose gases*. Springer, 2015.

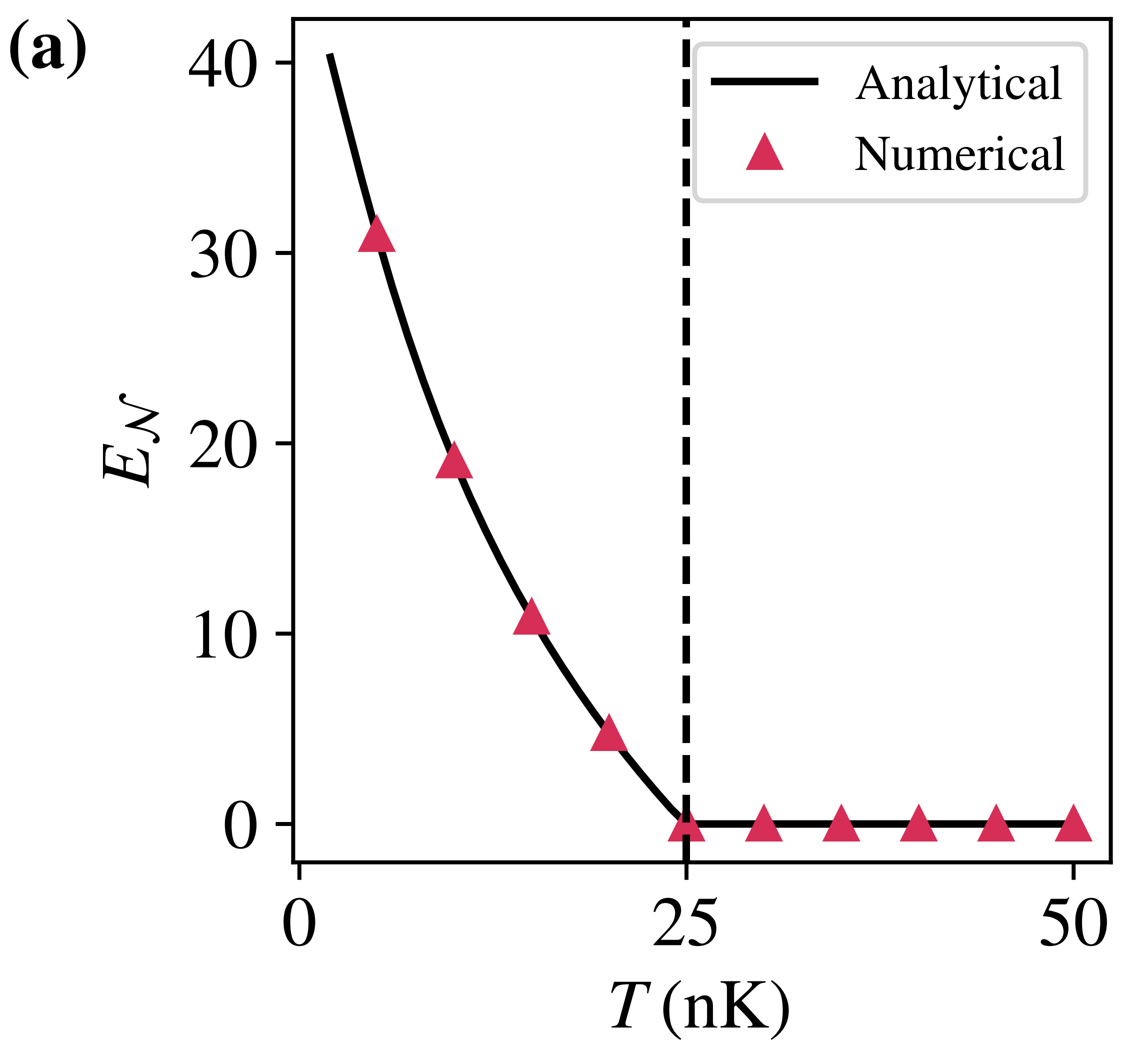
Entanglement & Threshold Temperature for Coherently Split State

- We derived **analytical formula for logarithmic negativity, threshold temperature, and mutual information after coherent splitting**

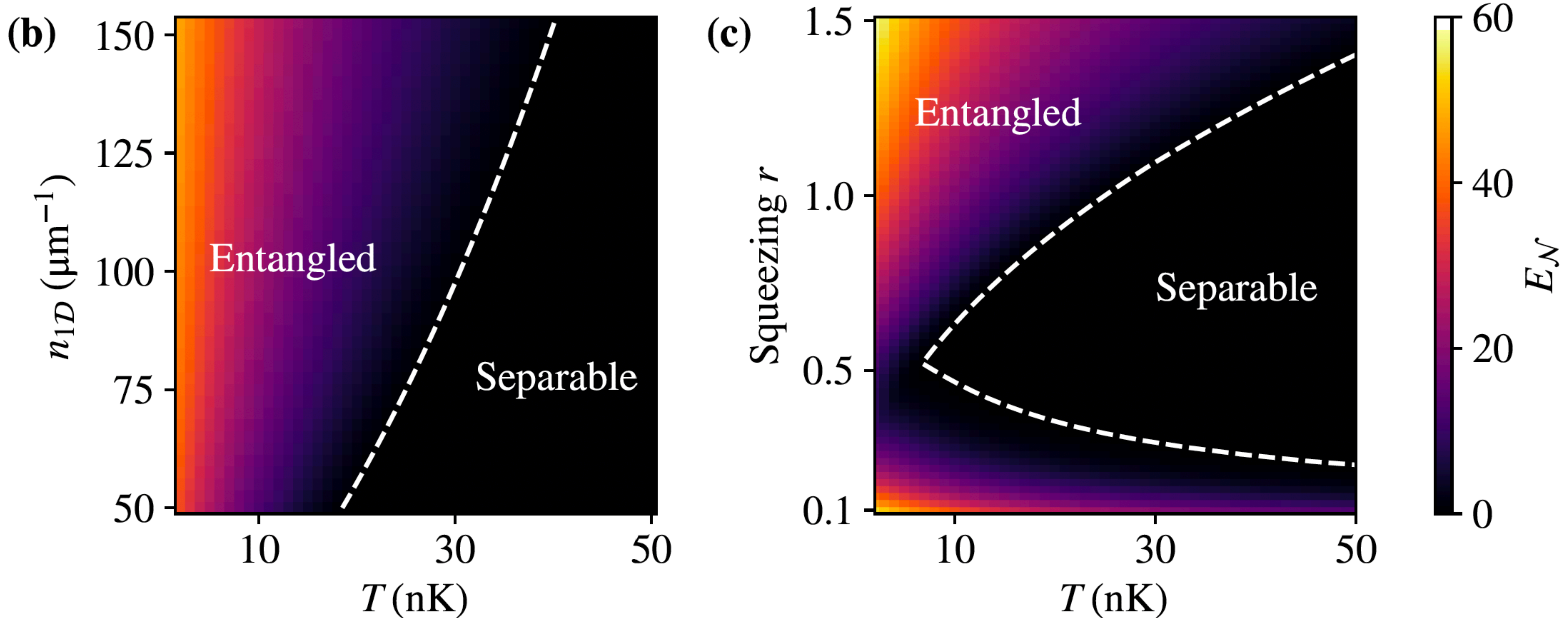
- The threshold temperature for entanglement after coherent splitting is **20 ~ 40 nK → achievable in current experiments!**

$$E_{\mathcal{N}} = \sum_{k>0}^{k_\Lambda} \max\left\{0, -\log\left[\frac{2}{r}\sqrt{\frac{E_k}{\varepsilon_k^+}\left(\eta_k^+ + \frac{1}{2}\right)}\right]\right\} + \max\left\{0, -\log\left[r\sqrt{\frac{\varepsilon_k^+}{E_k}\left(\eta_k^+ + \frac{1}{2}\right)}\right]\right\}$$

$$T^* = \sup_{0<k\leq k_\Lambda}\left\{T_k^* = \frac{\varepsilon_k^+}{k_B}\left(\ln\sqrt{\frac{F(k,r)}{F(k,r)-2}}\right)^{-1}\right\} \qquad F(k,r) = 1 + \frac{r^2}{4}\frac{\varepsilon_k^+}{E_k} + \frac{1}{r^2}\frac{E_k}{\varepsilon_k^+}$$



Entanglement & Threshold Temperature for Coherently Split State

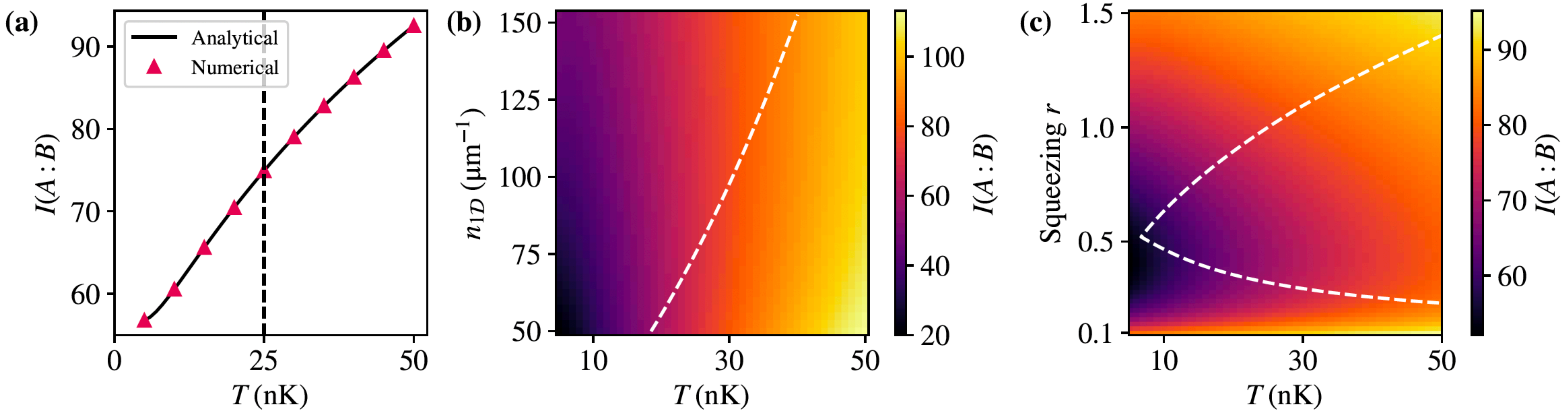


Mutual Information for Coherently Split State

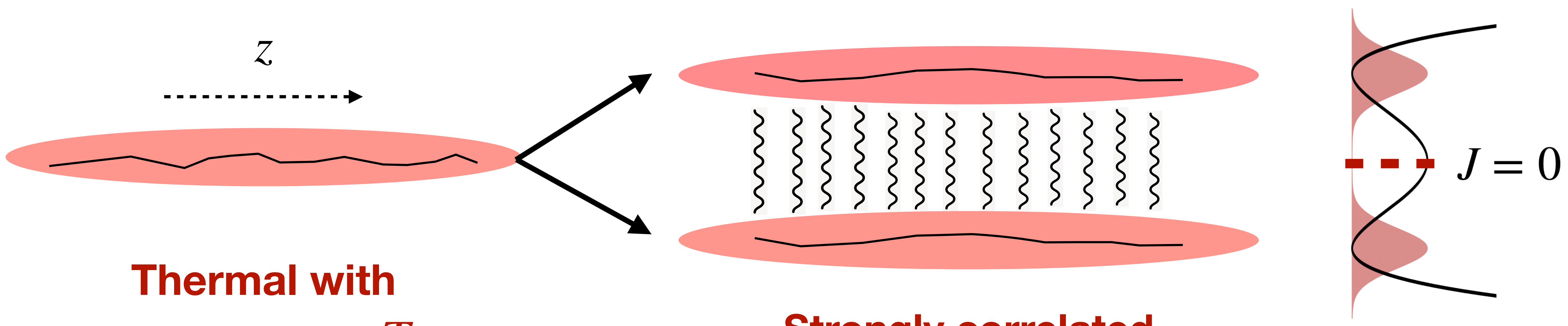
$$I(A:B) = \sum_{k>0}^{k_\Lambda} 2\left(\lambda_{k,r} + \frac{1}{2}\right)\log\left(\lambda_{k,r} + \frac{1}{2}\right) - \left(\lambda_{k,r} - \frac{1}{2}\right)\log\left(\lambda_{k,r} - \frac{1}{2}\right)$$
$$-\left[\left(\eta_k^+ + 1\right)\log\left(\eta_k^+ + 1\right) - \eta_k^+ \log \eta_k^+\right]$$

$$\lambda_{k,r} = \frac{1}{4}\sqrt{(1 + 2\eta_k^+ + C_{k,r})(1 + 2\eta_k^+ + C_{k,r}^{-1})}$$

$$C_{k,r} = \frac{r^2}{2}\frac{\varepsilon_k^+}{E_k}$$



Discussion: Relaxation of Correlation



Thermal with temperature T

Strongly correlated non-equilibrium state

Previous assumptions for describing prethermalization

1. ***Coherent splitting initial state ansatz***
2. ***Decoupled Luttinger Liquid evolution***
3. ***Relaxation toward Generalized Gibbs Ensemble (GGE) prethermalized state***

$$\langle \hat{\phi}_k^- \hat{\phi}_q^- \rangle = \frac{\delta_{kq}}{2n_{1\mathrm{D}} r^2} \qquad \langle \delta\hat{n}_k^- \delta\hat{n}_q^- \rangle = \frac{r^2 n_{1\mathrm{D}}}{2} \delta_{kq}$$

$$\hat{H} = \hat{H}^A_{\mathrm{TLL}} + \hat{H}^B_{\mathrm{TLL}}$$

$$\rho_{\mathrm{GGE}} = \frac{1}{Z} \exp(-\beta^- \hat{H}^-_{\mathrm{TLL}}) \otimes \exp(-\beta^+ \hat{H}^+_{\mathrm{TLL}})$$

Our result: these 3 asumptions **are not consistent,** because they break conservation of mutual information/ entanglement

Role of interaction and non-Gaussianity?

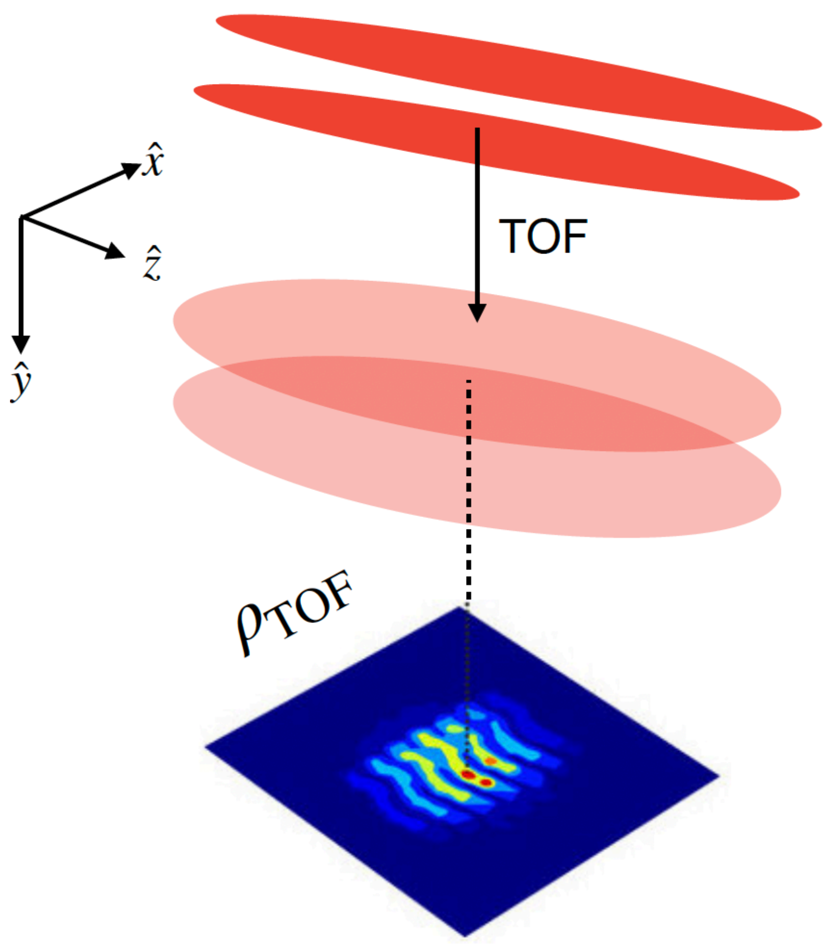
Gring, M., et al. "Relaxation and prethermalization in an isolated quantum system." *Science* 337.6100: 1318-1322 (2012).

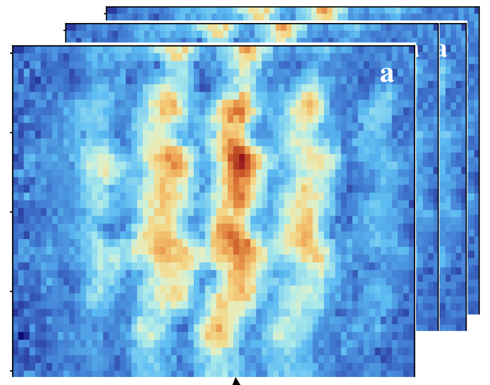
Langen, Tim, et al. "Experimental observation of a generalized Gibbs ensemble." *Science* 348.6231: 207-211 (2015).

Outlook:
Full Gaussian Tomography for Entanglement Detection

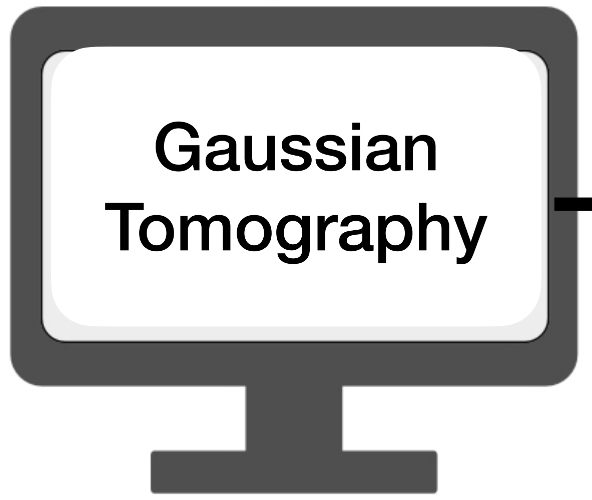
Outlook: Quantum Field Tomography and Experimental Verification of Entanglement

Not only we know the condition and the temperature needed to observe extensive entanglement between 1D Bose gases, we also know **the measurement protocol to observe it.**

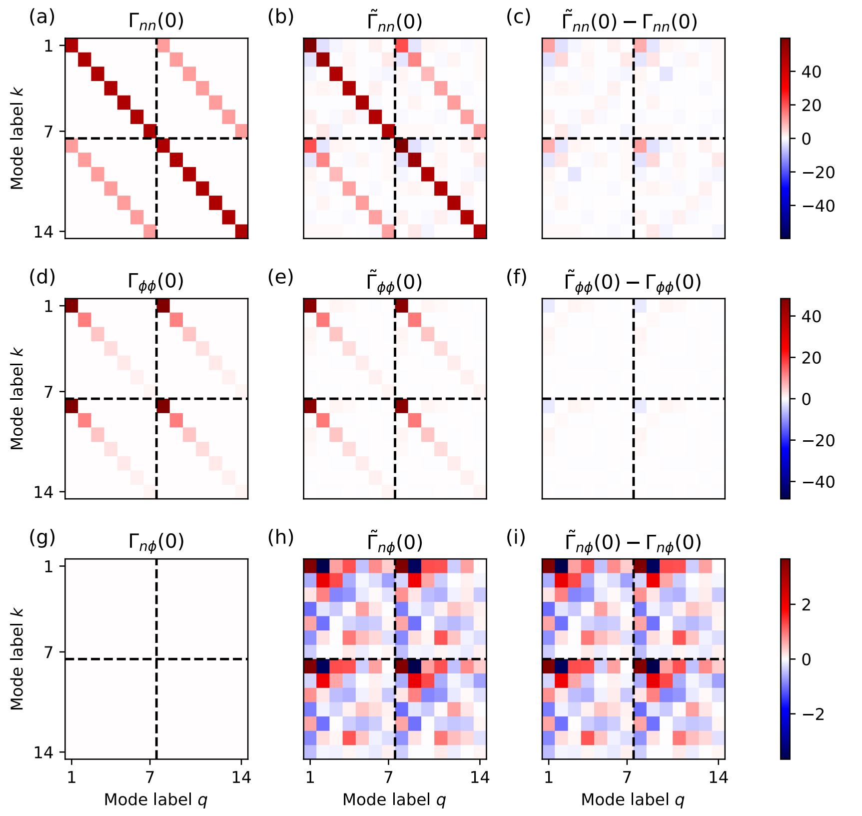




$$\{\phi^{+}(z,t), \hat{\phi}^{-}(z,t)\}$$



$$\Gamma = \begin{pmatrix} \Gamma^{AA} & \Gamma^{AB} \\ \Gamma^{BA} & \Gamma^{BB} \end{pmatrix}$$



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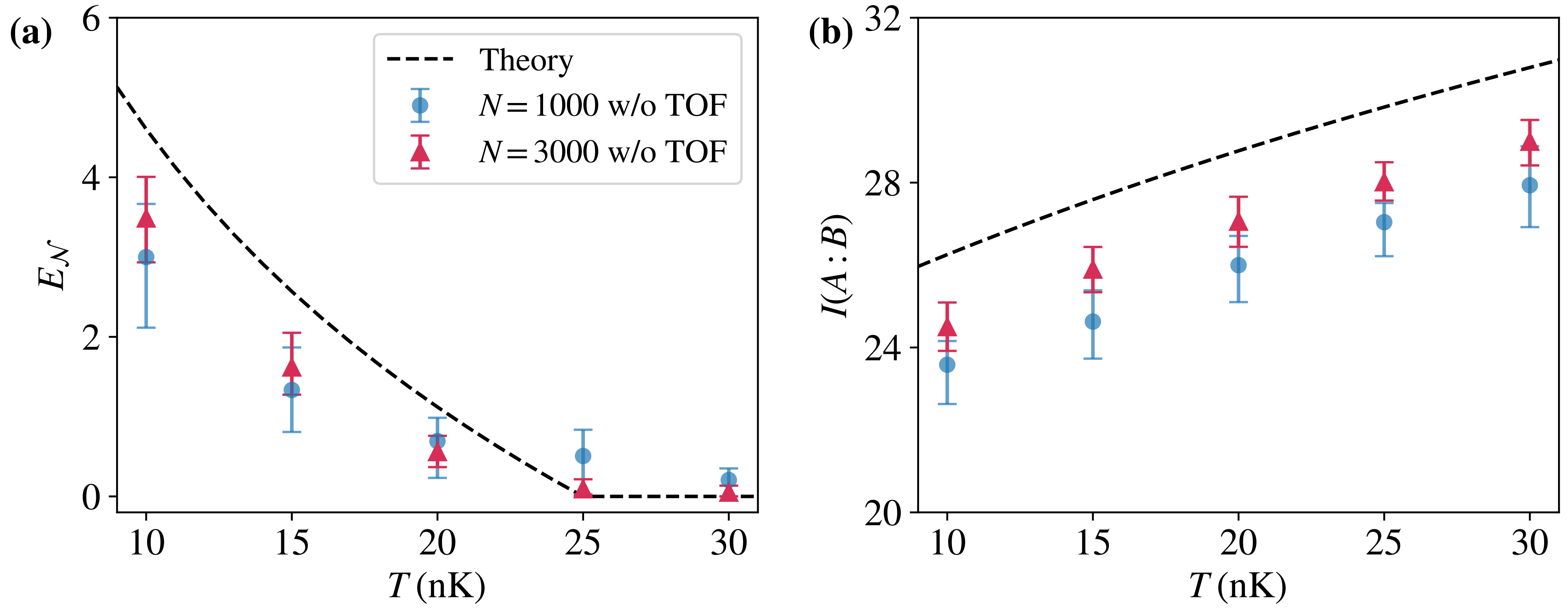
Gluza, M., et al. "Quantum read-out for cold atomic quantum simulators." *Communications Physics* 3.1: 12 (2020).



Stay tuned for more!

Take-home messages

Extend the study of entanglement between coupled 1D quantum fields to finite temperatures and out-of-equilibrium regimes.

Demonstrate that detecting extensive entanglement between interacting 1D quantum fields is within experimental reach in parallel 1D Bose gases.

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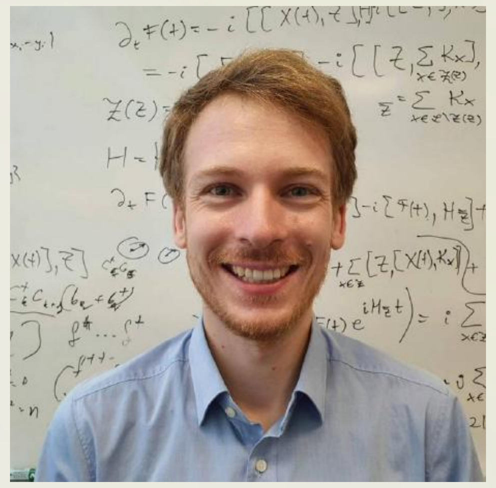

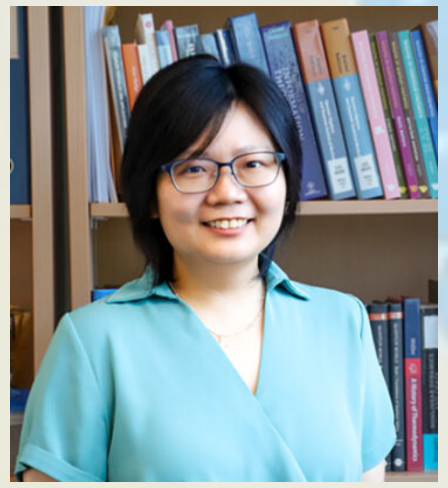

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