Illustration Credit: Atomchip group, TU Wien

Scan me!

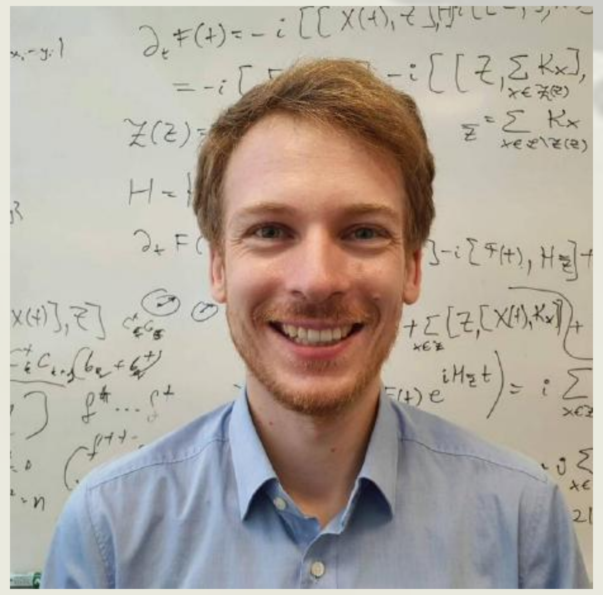
Extensive entanglement between coupled Tomonaga-Luttinger liquids in and out of equilibrium

Taufiq Murtadho*, Marek Gluza, Nelly H. Y. Ng*

arXiv 2508.20533

NANYANG TECHNOLOGICAL UNIVERSITY SINGAPORE

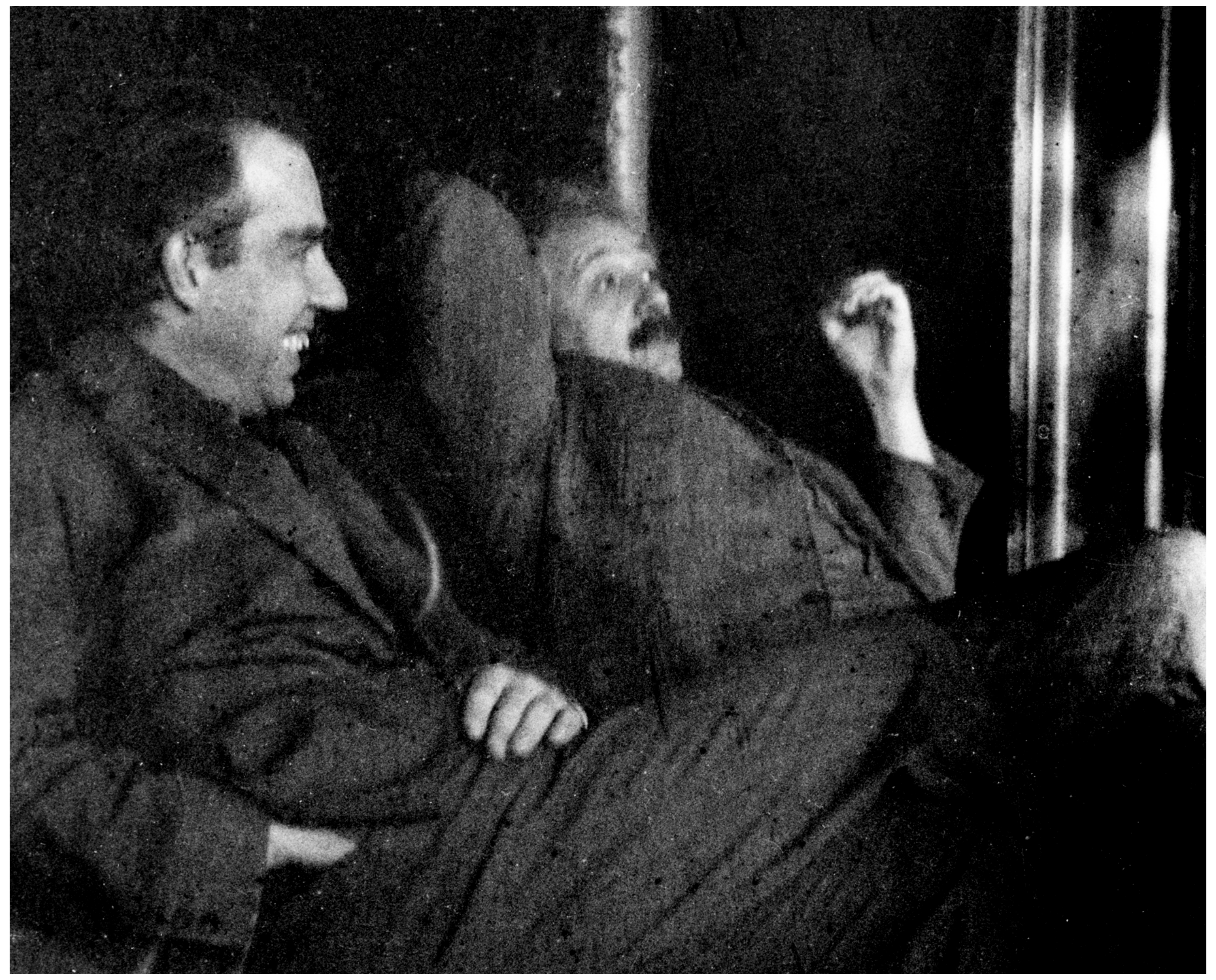
in.Q

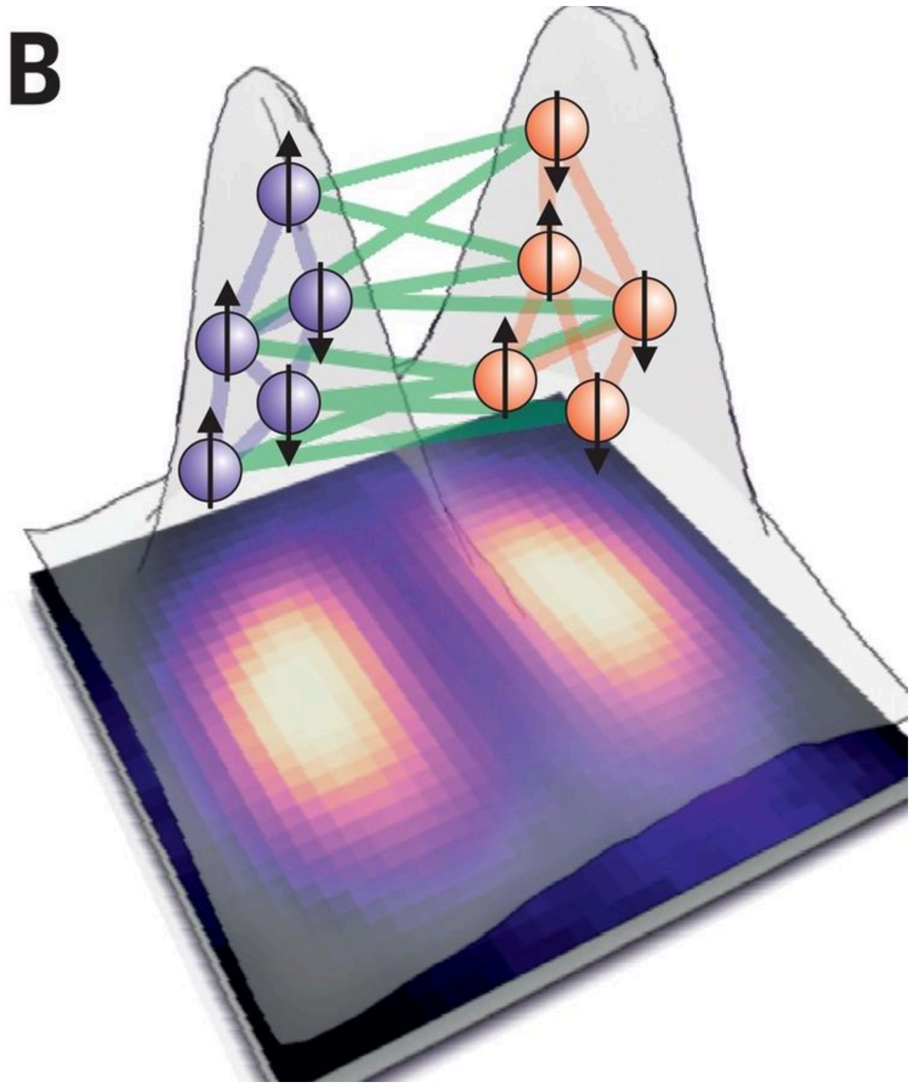


Schrödinger vs Heisenberg

Cartoon: Sergey Bubov

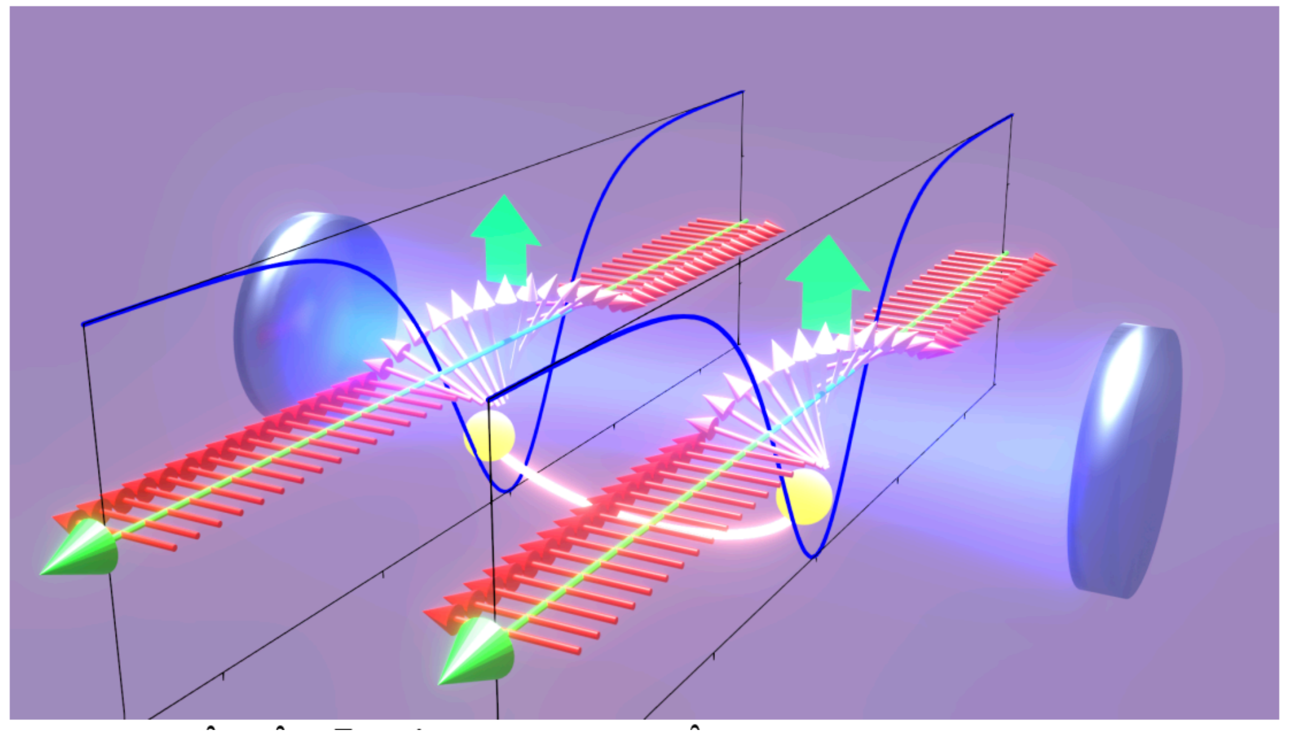
Quantum entanglement between two “large” objects

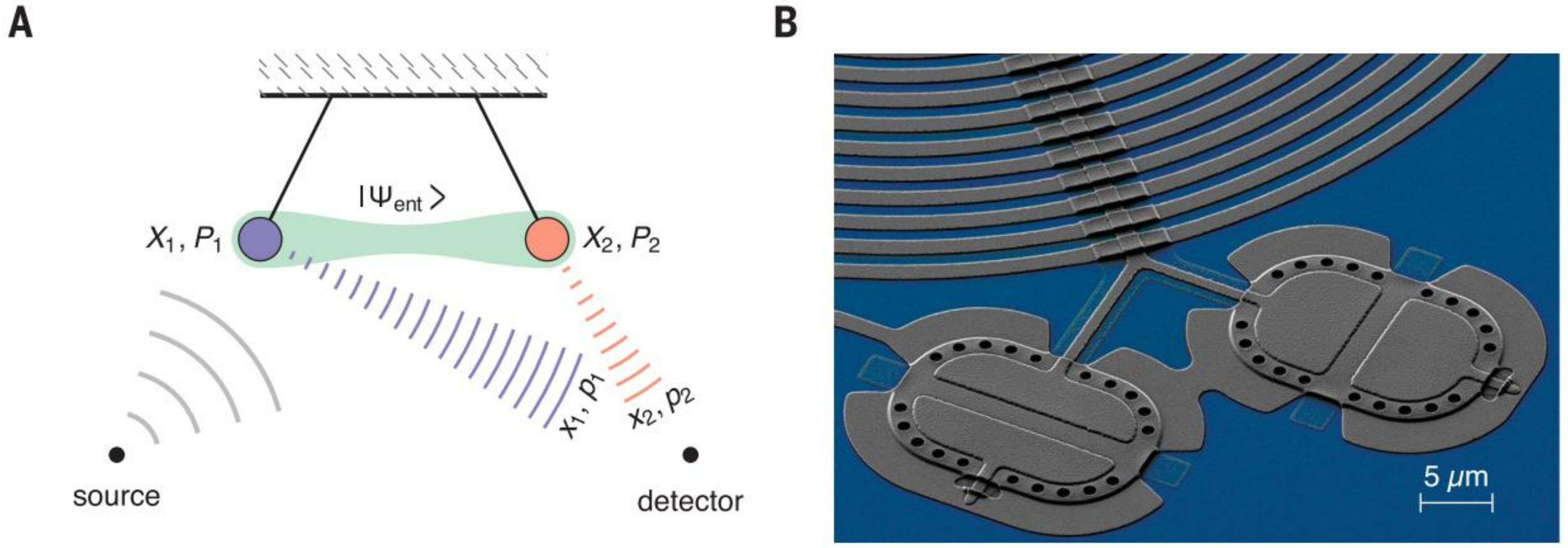




Macroscopic Entanglement between spatially-separated Bose-Einstein condensates (BEC)

Lange, Karsten, et al. *Science* 360.6387: 416-418 (2018).,





Macroscopic entanglement between mechanical oscillator

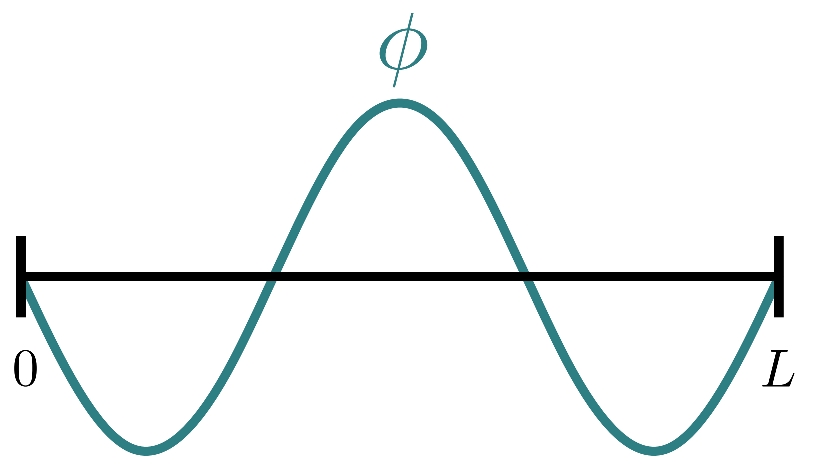
Kotler, Shlomi, et al. *Science* 372.6542: 622-625 (2021).

Macroscopic entanglement between domain walls in optical cavity

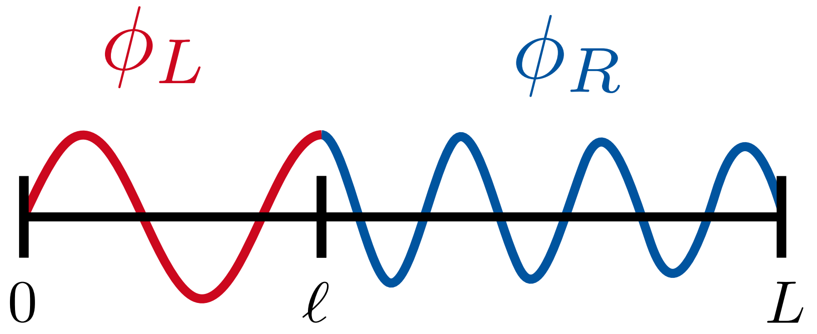
Gupta, Rahul, Huaiyang Yuan, and Himadri Shekhar Dhar. *arXiv:2508.03450* (2025).

Entanglement *between* Coupled Quantum Fields

Real Space Entanglement



Emonts, P., & Kukuljan, I. *PRR*, *4*(3), 033039 (2022).



Entanglement entropy

$$S(\rho) = -\,\mathrm{Tr}\left(\rho_L \log \rho_L\right)$$

$$\rho_L = \mathrm{Tr}_R \rho$$

Field Space Entanglement

$$\mathcal{S} = \int d^4\mathbf{x}\left(\mathscr{L}[\hat{\varphi}_1(\mathbf{x})] + \mathscr{L}[\hat{\varphi}_2(\mathbf{x})] + \mathscr{L}_{\text{int}}[\hat{\varphi}_1(\mathbf{x}), \hat{\varphi}_2(\mathbf{x})]\right)$$

$$S(\rho) = -\,\mathrm{Tr}\left(\rho_{\hat{\varphi}_1} \log \rho_{\hat{\varphi}_1}\right) \qquad \rho_{\hat{\varphi}_1} = \mathrm{Tr}_{\hat{\varphi}_2}\rho$$

$$\boxed{S(\rho) \propto V}$$

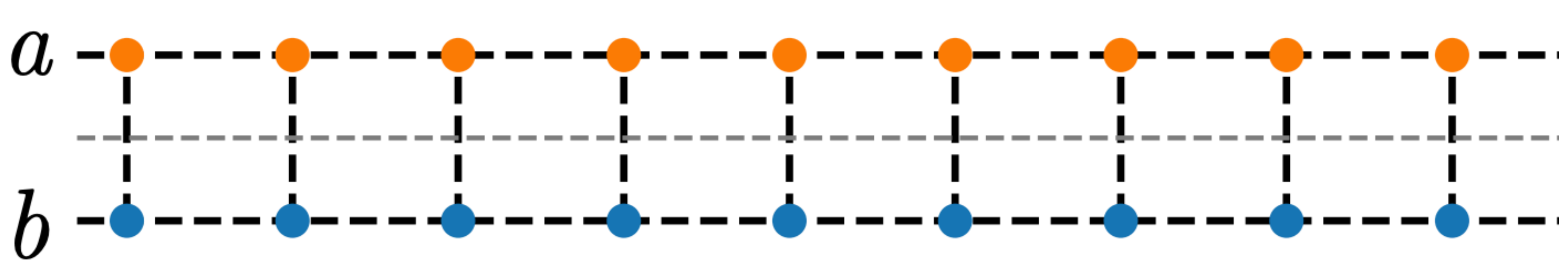
Entanglement is expected to be extensive (volume scaling), but structured (not scrambled) if interaction is local

Mollabashi, Ali, Noburo Shiba, and Tadashi Takayanagi. "Entanglement between two interacting CFTs and generalized holographic entanglement entropy." *JHEP* 2014.4: 1-36 (2014).

Potential applications: Quantum memory, Quantum metrology, test of fundamental physics, …

‘Field Space’ Entanglement in Quasi-1D systems

Spin-ladder



Continuum limit → Tomonaga-Luttinger Liquid (TLL)

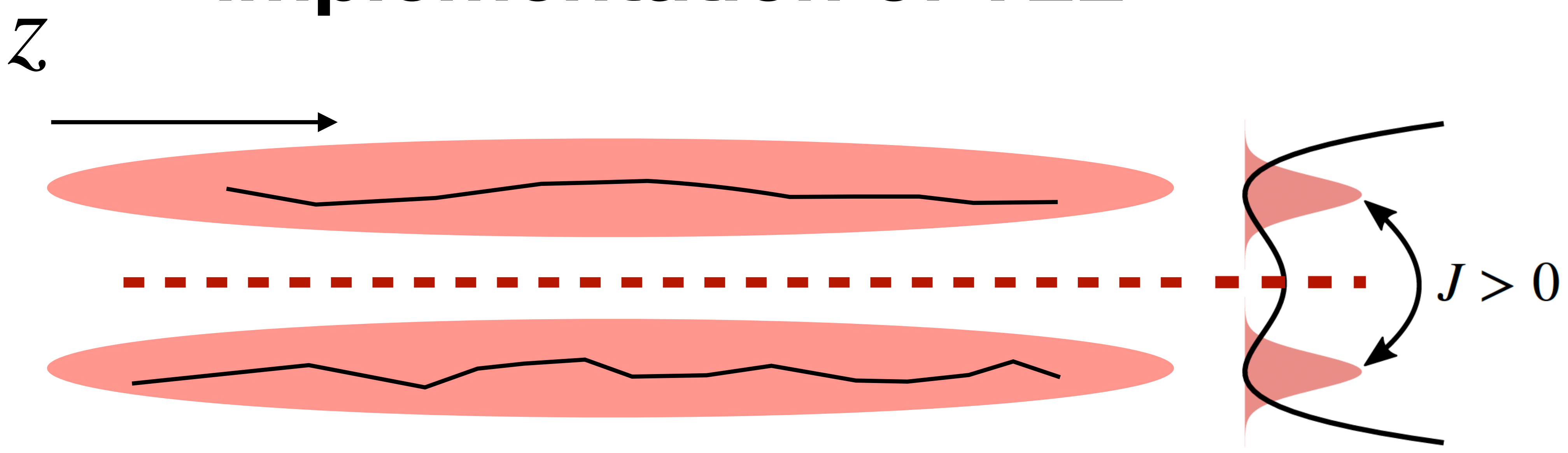
Extensive **ground-state** entanglement entropy

$$S(\rho) \propto L$$

Lundgren, Rex, et al. *PRB* 88.24: 245137 (2013).

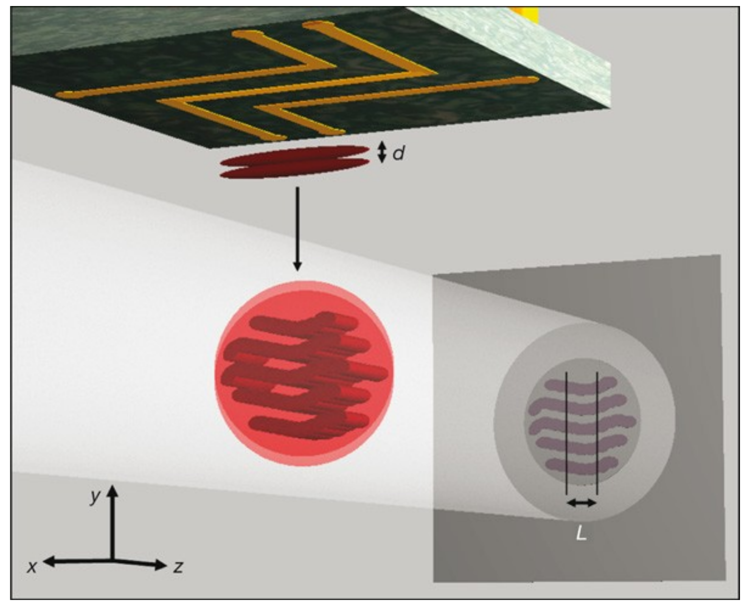
Furukawa, Shunsuke, and Yong Baek Kim. *PRB* 83.8: 085112 (2011).

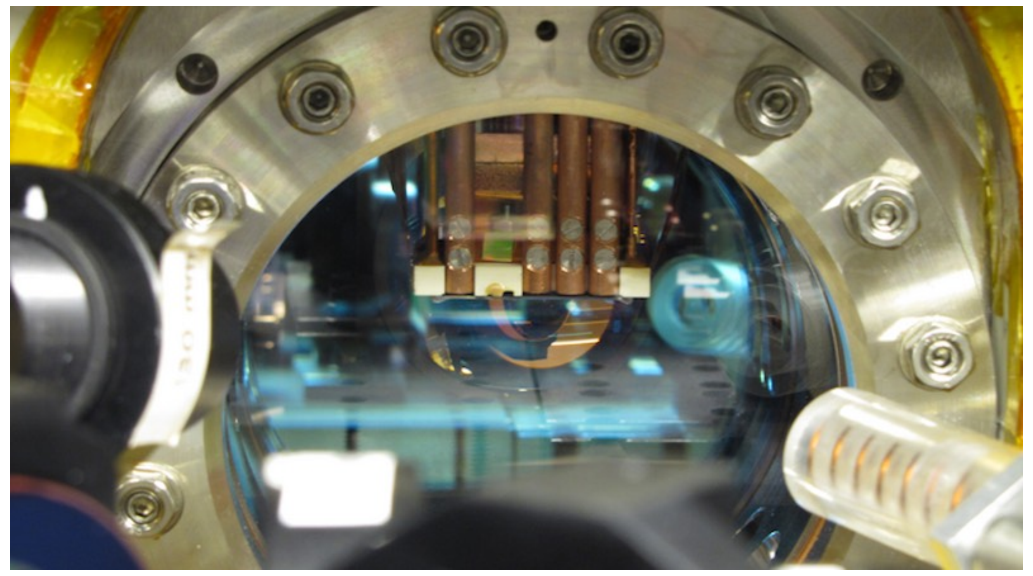
Ultracold 1D Bose Gases implementation of TLL



$N \sim 10{,}000$ ^{87}Rb atoms

$L \sim 100\ \mu\text{m}$

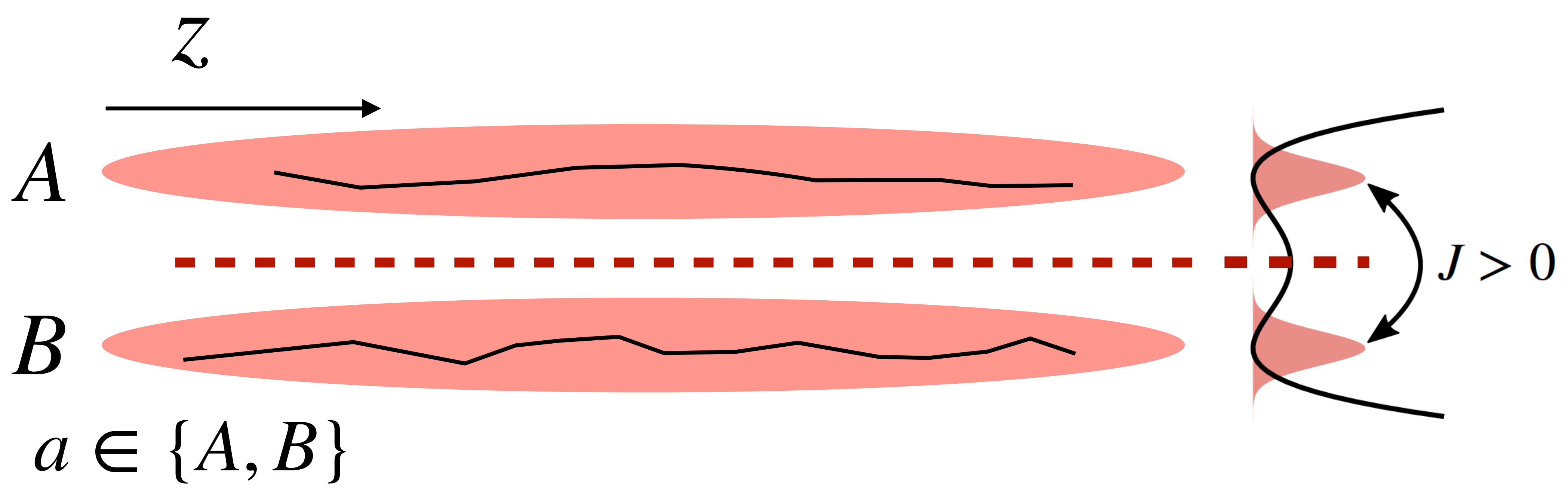




Experimentally realized at TU Wien Atomchip group

Tunnel-coupled 1D Bose Gases

Parallel 1D Bose Gases



$$\hat{\Psi}^a(z) = e^{i\hat{\phi}^a(z)}\sqrt{n_{1D} + \delta\hat{n}^a(z)}$$

$$[\delta\hat{n}^a(z), \hat{\phi}^b(z')] = i\delta(z - z')\delta_{ab}$$

Quantum + thermal fluctuations $T > 20$ nK

Low-energy Hamiltonian:
Tomonaga-Luttinger Liquid (TLL)

$$\hat{H}^a_{\text{TLL}} = \int_0^L dz \left[\frac{\hbar^2 n_{1D}}{2m}(\partial_z\hat{\phi}^a(z))^2 + \frac{g}{2}(\delta\hat{n}^a(z))^2\right]$$

$$\hat{H} = \hat{H}^A_{\text{TLL}} + \hat{H}^B_{TLL} - 2\hbar J n_{1D}\int \cos\left[\hat{\phi}^A(z) - \hat{\phi}^B(z)\right] dz$$

$$\approx \hat{H}^A_{\text{TLL}} + \hat{H}^B_{TLL} + \hbar J n_{1D}(\hat{\phi}^A(z) - \hat{\phi}^B(z))^2$$

Are fluctuations in TLL "A" and TLL "B": entangled with each other at finite temperatures achieved in the lab? If not, what is the temperature that one has to reach to observe entanglement?

- Motivation
- **Theoretical Methods**
- *Results 1*: Entanglement and Mutual Information at Finite Temperatures
- *Results 2:* Entanglement and Mutual Information after Coherent Splitting
- *Outlook:* Quantum Field Tomography & Experiments
- Take-home messages

Theoretical Methods

Characterizing Equilibrium States (Gaussian)

Use Bogoliubov approximation to diagonalize the microscopic Hamiltonian

$$\hat{H} \approx \sum_{k\neq 0} \varepsilon_k^+ (\hat{b}_k^+)^\dagger \hat{b}_k^+ + \varepsilon_k^- (\hat{b}_k^-)^\dagger \hat{b}_k^-$$

Whitlock, N. K., & Bouchoule, I. Relative phase fluctuations of two coupled one-dimensional condensates. *PRA*, *68*(5), 053609 (2003).

$$\varepsilon_k^+ = \sqrt{E_k(E_k + 2gn_{1\mathrm{D}})}$$

$$\varepsilon_k^- = \sqrt{(E_k + 2\hbar J)(E_k + 2\hbar J + 2gn_{1\mathrm{D}})}$$

$$E_k = (\hbar k)^2/2m$$

Thermal density matrix

Covariance matrix

$$\hat{\rho}_{\mathrm{th}} = \frac{1}{Z}\exp(-\beta \hat{H}) \longrightarrow \Gamma = \begin{pmatrix} \Gamma^+ & 0 \\ 0 & \Gamma^- \end{pmatrix} \xrightarrow[\text{transform.}]{\text{Symplectic}} \Gamma = \begin{pmatrix} \Gamma^{AA} & \Gamma^{AB} \\ \Gamma^{BA} & \Gamma^{BB} \end{pmatrix}$$

Entanglement and Correlation Measures for Gaussian Mixed States

Logarithmic Negativity $E_{\mathcal{N}}$

$$E_{\mathcal{N}} = \sum_k \max\{0, -\log(2\gamma_k)\}$$

γ_k : Symplectic eigenvalues of partially transposed covariance matrix Γ^{T_B}

- $E_{\mathcal{N}} > 0$ is a ***sufficient*** condition for entanglement
- It is ***necessary and sufficient*** for **two-modes entanglement**
- In our case, we have to solve **multiple independent two-modes problems** (one for each momentum mode k) $\rightarrow E_{\mathcal{N}} > 0$ is necessary and sufficient for entanglement

Mutual information $I(A:B)$

$$I(A:B) = S(A) + S(B) - S(AB)$$

$$S(\Gamma) = \sum_k \left(\sigma_k + \frac{1}{2}\right)\log\left(\sigma_k + \frac{1}{2}\right) - \left(\sigma_k - \frac{1}{2}\right)\log\left(\sigma_k - \frac{1}{2}\right)$$

σ_k : Symplectic eigenvalues of covariance matrix Γ

- $I(A:B)$ includes ***both entanglement and classical correlation in general***
- In the limit of zero temperature $I(A:B) = 2S(\rho)$ where $S(\rho)$ is the entanglement entropy.

- Motivation
- Theoretical Methods
- ***Results 1*: Entanglement and Mutual Information at Finite Temperatures**
- *Results 2:* Entanglement and Mutual Information after Coherent Splitting
- *Outlook:* Quantum Field Tomography & Experiments
- Take-home messages

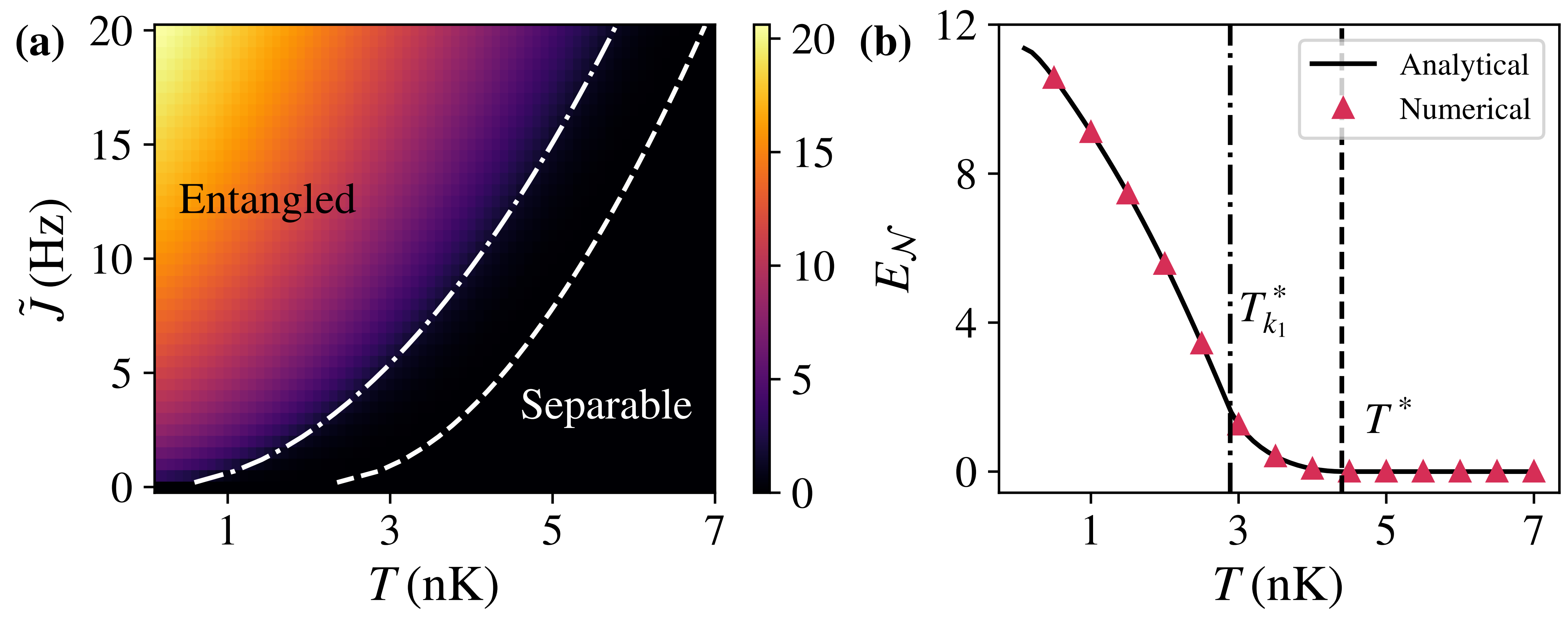
Results 1: Entanglement & Mutual Information at Finite Temperatures

Entanglement vs. Temperature - how "cold" is too "hot"?

- **We derive analytical formula for logarithmic negativity at finite temperatures**
- **Competition between tunneling strength J and temperature T**
- **We derive expression for threshold temperature T^***

$$E_{\mathcal{N}} = \sum_{k>0}^{k_\Lambda} \max \left\{ 0, -\log \sqrt{C_{k,J}(1+\eta_k^+)(1+\eta_k^-)} \right\}$$

$$C_{k,J} = \frac{\varepsilon_k^-}{\varepsilon_{k+}} \frac{E_k}{E_k + 2\hbar J} \leq 1 \qquad \eta_k^\pm = [\exp(\beta \varepsilon_k^\pm) - 1]^{-1}$$



$$T^* = \sup_{0<k\leq k_\Lambda} \left\{ T_k^* \,\middle|\, \tanh\left(\frac{\varepsilon_k^+}{2k_B T_k^*}\right) \tanh\left(\frac{\varepsilon_k^-}{2k_B T_k^*}\right) = C_{k,J} \right\}$$

For realistic experimental parameters $T^* \sim 1 - 5$ nK, about one order of magnitude colder than current state-of-the-art temperatures achieved in the lab

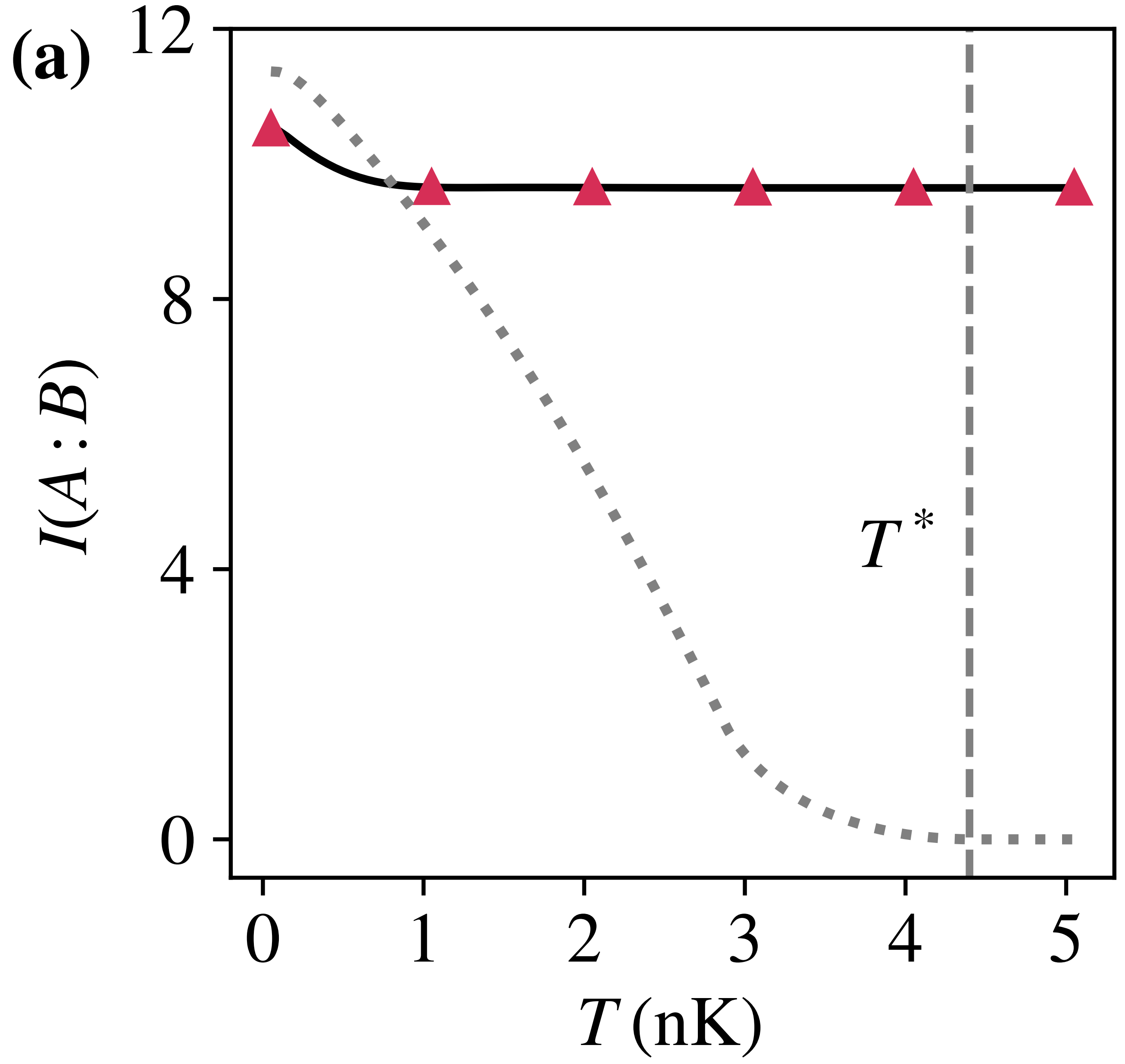
Quantum and classical correlation cross-over

- **Analytic expression for mutual information at finite temperatures**

- **Entanglement entropy at zero temperature can be recovered from the general result**

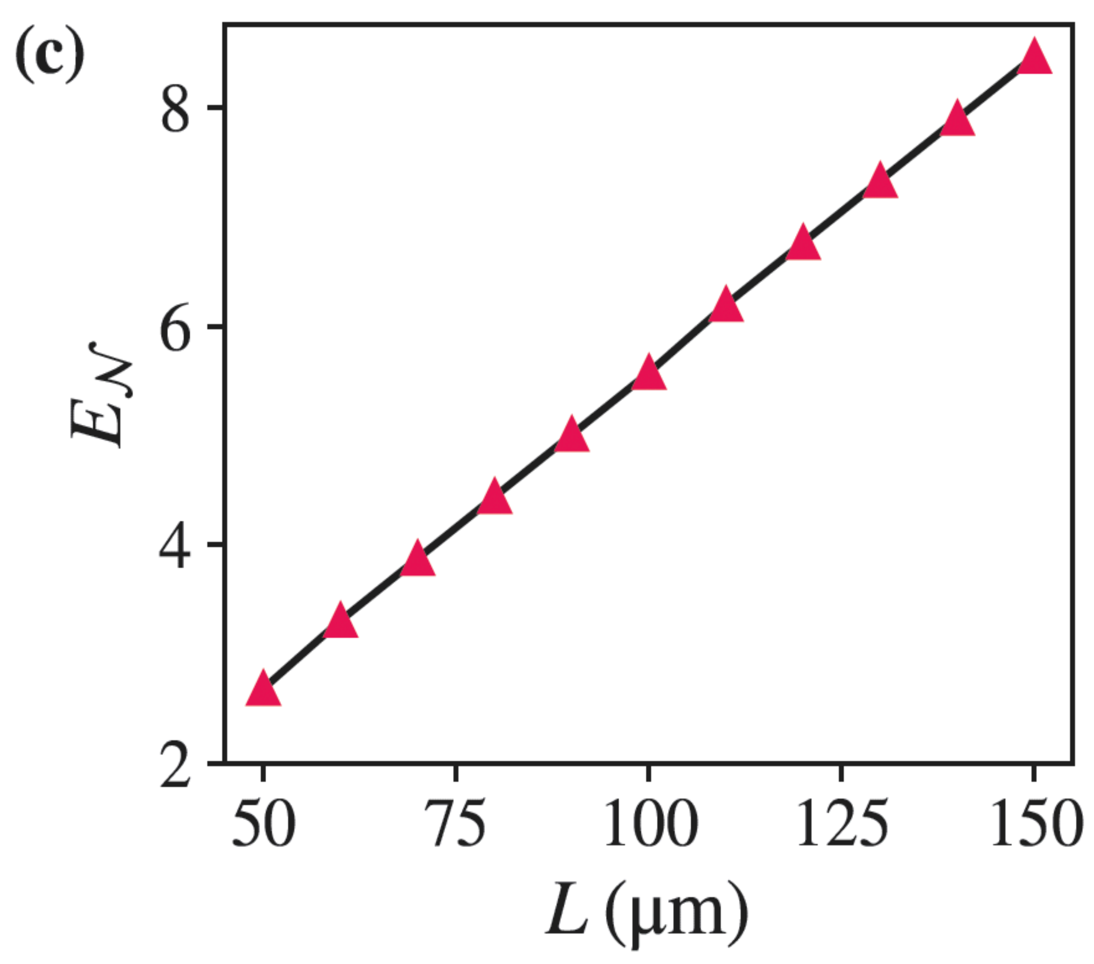
$$I(A:B) = \sum_{k>0}^{k_\Lambda} 2\left(\lambda_{k,J}+\frac{1}{2}\right)\log\left(\lambda_{k,J}+\frac{1}{2}\right) - \left(\lambda_{k,J}-\frac{1}{2}\right)\log\left(\lambda_k-\frac{1}{2}\right)$$
$$-\sum_{a=\pm}\sum_{k>0}^{k_\Lambda}\left[\left(\eta_k^a+1\right)\log\left(\eta_k^a+1\right)-\eta_k^a\log\eta_k^a\right]$$

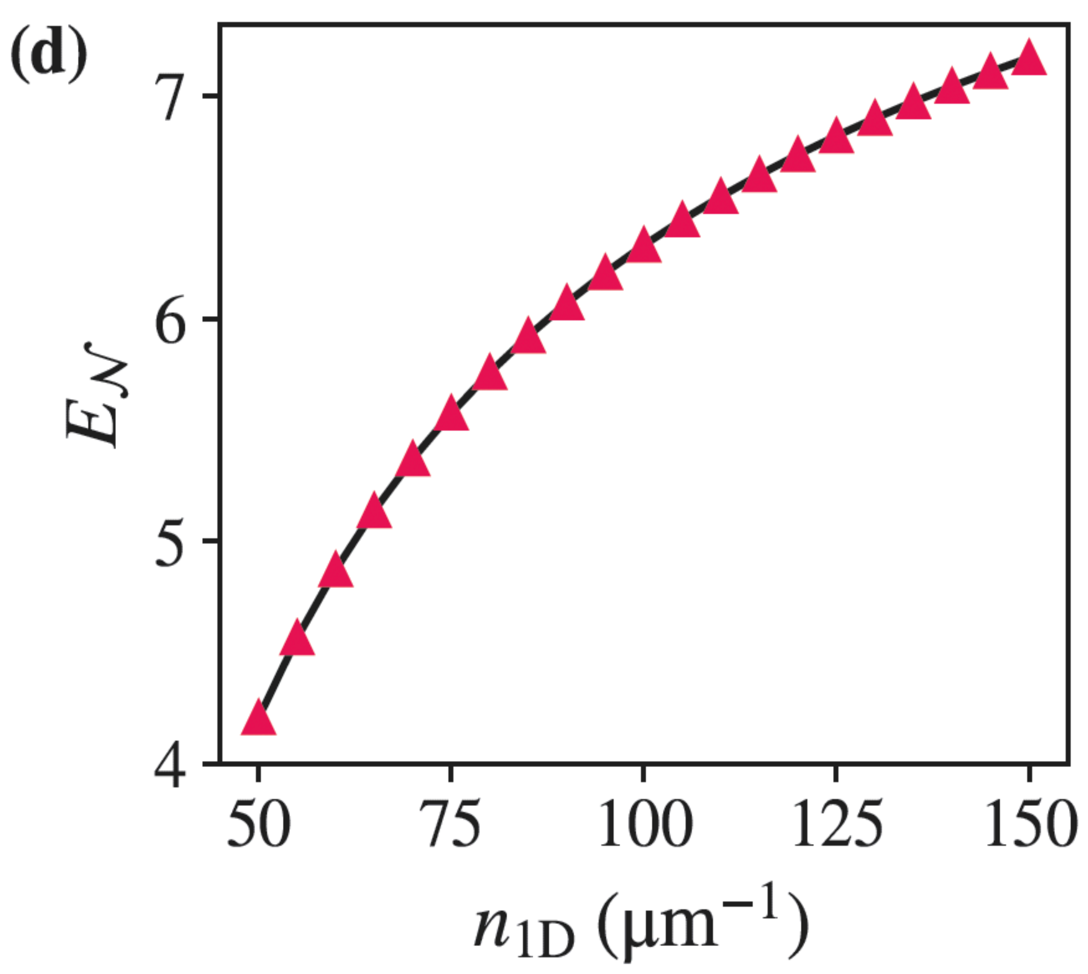
$$\lambda_{k,J} = \frac{1}{4}\sqrt{\left[(1+2\eta_k^+)+C_{k,J}(1+2\eta_k^-)\right]\left[(1+2\eta_k^+)+C_{k,J}^{-1}(1+2\eta_k^-)\right]},$$

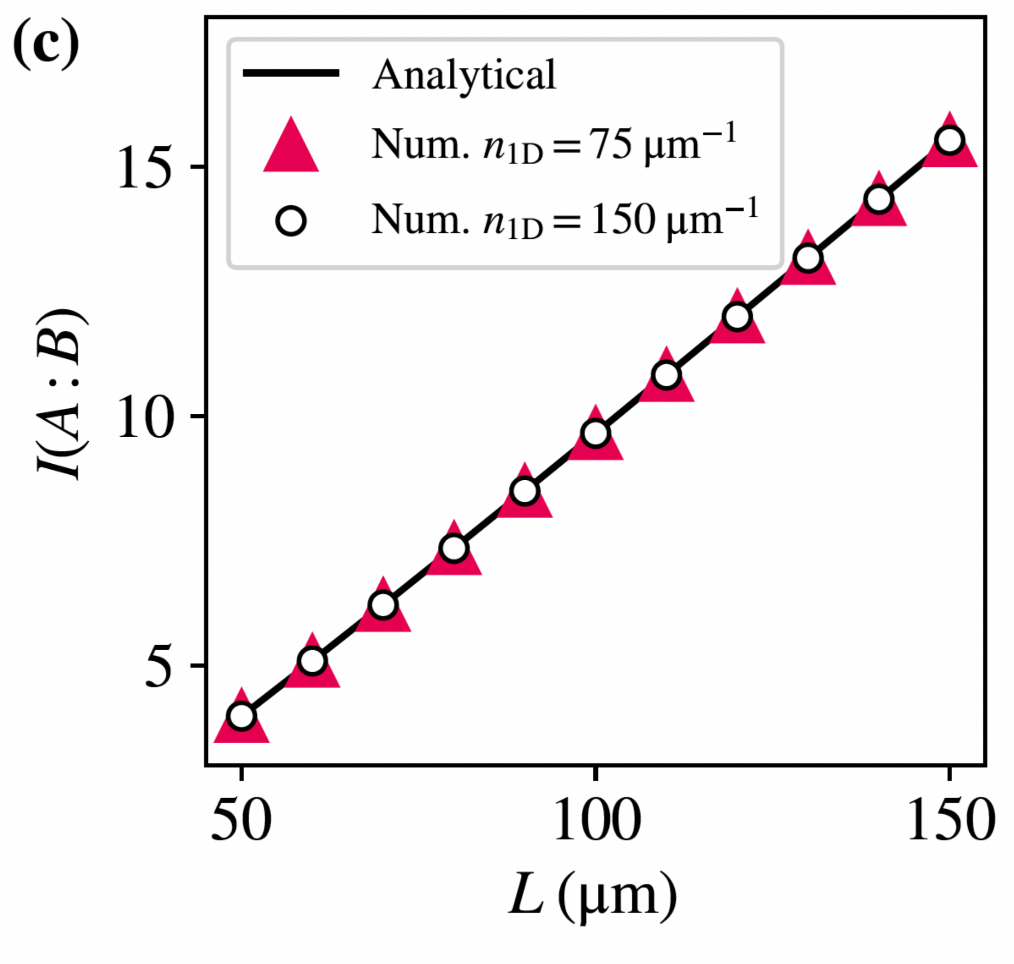


As T increases quantum correlation transmute into classical correlation to keep mutual info. constant

Scaling of entanglement and mutual information







- Both logarithmic negativity and mutual information are found to be **extensive**

$$E_{\mathcal{N}} \propto L \qquad I(A:B) \propto L$$

- The extensivity arise from the growing number of modes that are entangled/correlated

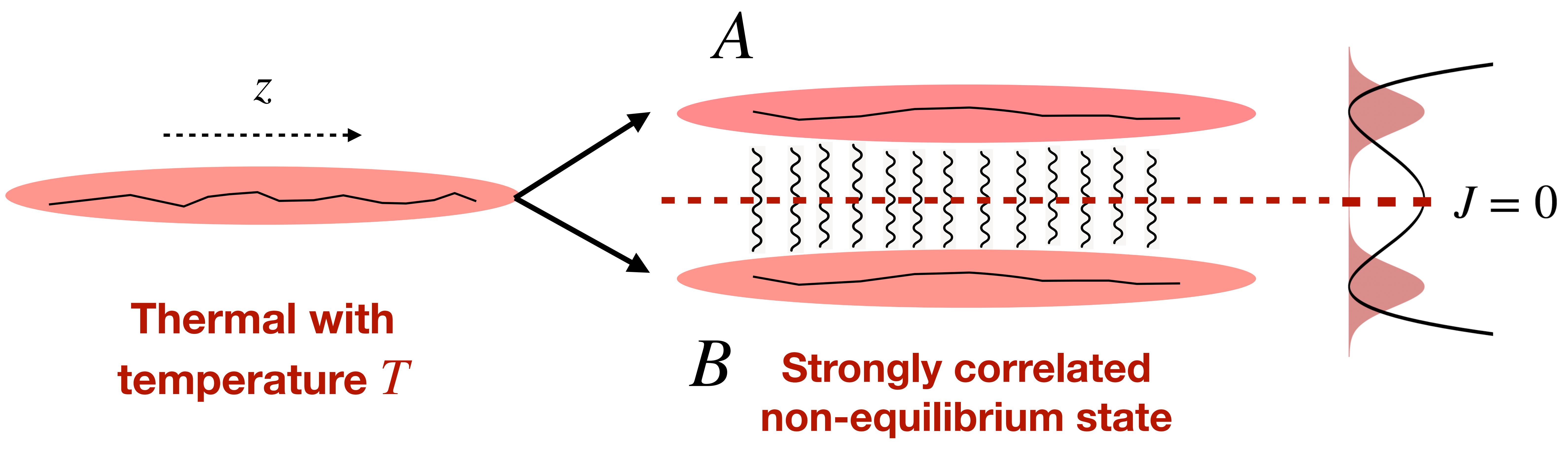
$$\sum_k \to \frac{L}{2\pi}\int dk$$

- Non-trivial scaling with density n_{1D} underscores that the entangled modes are collective

- Motivation
- Theoretical Methods
- *Results 1*: Entanglement and Mutual Information at Finite Temperatures
- ***Results 2:* Entanglement and Mutual Information after Coherent Splitting**
- *Outlook:* Quantum Field Tomography & Experiments
- Take-home messages

Results 2: Entanglement & Mutual Information after Coherent Splitting

Initial State after Coherent Splitting



Symmetric (+) and antisymmetric (-) fields

$$\delta\hat{n}^{\pm}(z) = \delta\hat{n}^{A}(z) \pm \delta\hat{n}^{B}(z)$$

$$\hat{\phi}^{\pm}(z) = \hat{\phi}^{A}(z) \pm \hat{\phi}^{B}(z)$$

Symmetric sector:
Thermal fluctuations

$$\langle \hat{\phi}_k^+ \hat{\phi}_q^+ \rangle = \frac{\delta_{kq}}{4n_{1\mathrm{D}}} \frac{\varepsilon_k^+}{E_k}(1 + 2\eta_k^+)$$

$$\langle \delta\hat{n}_k^+ \delta\hat{n}_q^+ \rangle = \delta_{kq} n_{1\mathrm{D}} \frac{E_k}{\varepsilon_k^+}(1 + 2\eta_k^+)$$

Antisymmetric sector:
Vacuum (quantum) fluctuations

$$\langle \hat{\phi}_k^- \hat{\phi}_q^- \rangle = \frac{\delta_{kq}}{2n_{1\mathrm{D}} r^2}$$

$$\langle \delta\hat{n}_k^- \delta\hat{n}_q^- \rangle = \frac{r^2 n_{1\mathrm{D}}}{2} \delta_{kq}$$

Full covariance matrix

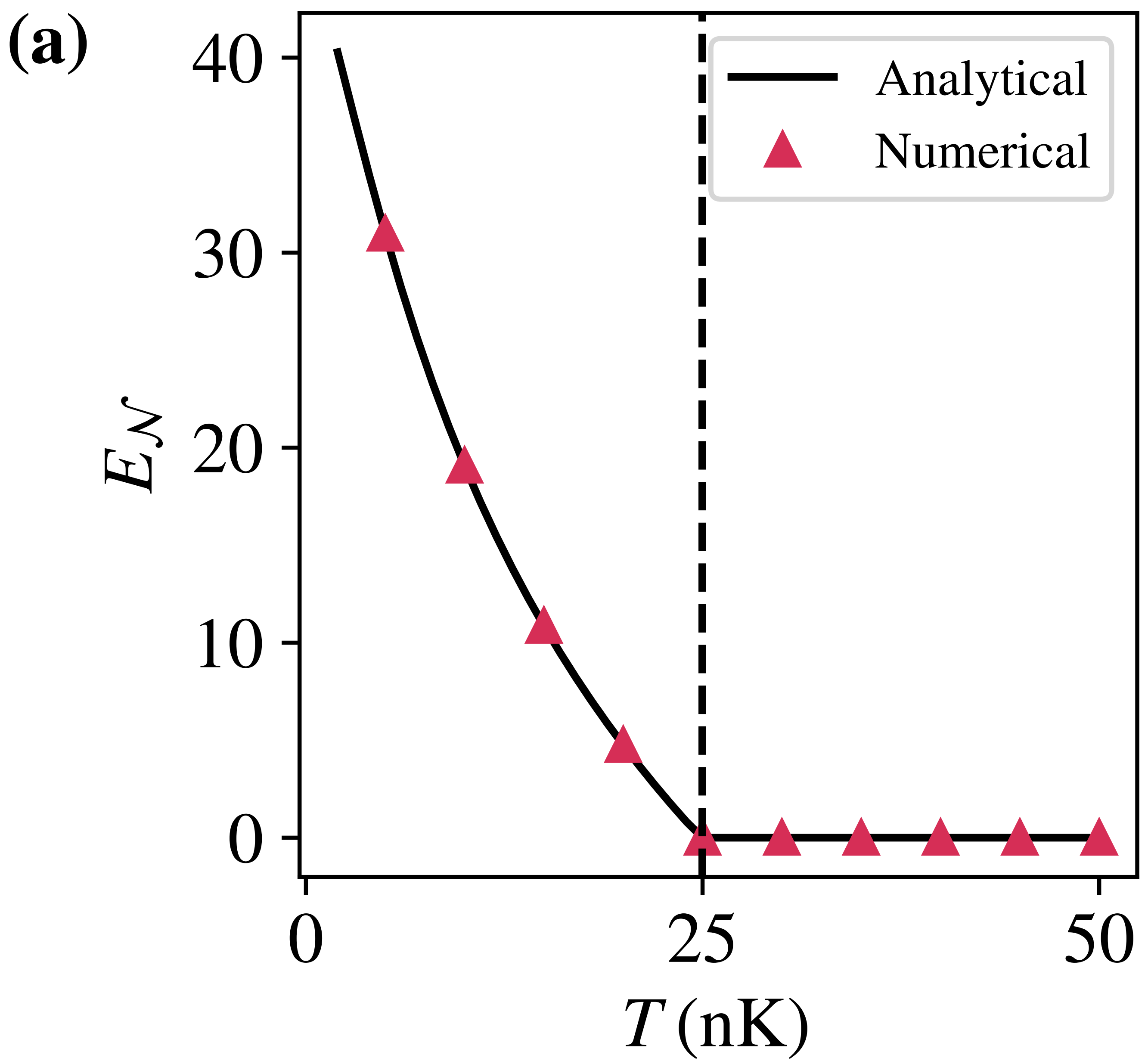
$$\Gamma = \begin{pmatrix} \Gamma^{AA} & \Gamma^{AB} \\ \Gamma^{BA} & \Gamma^{BB} \end{pmatrix}$$

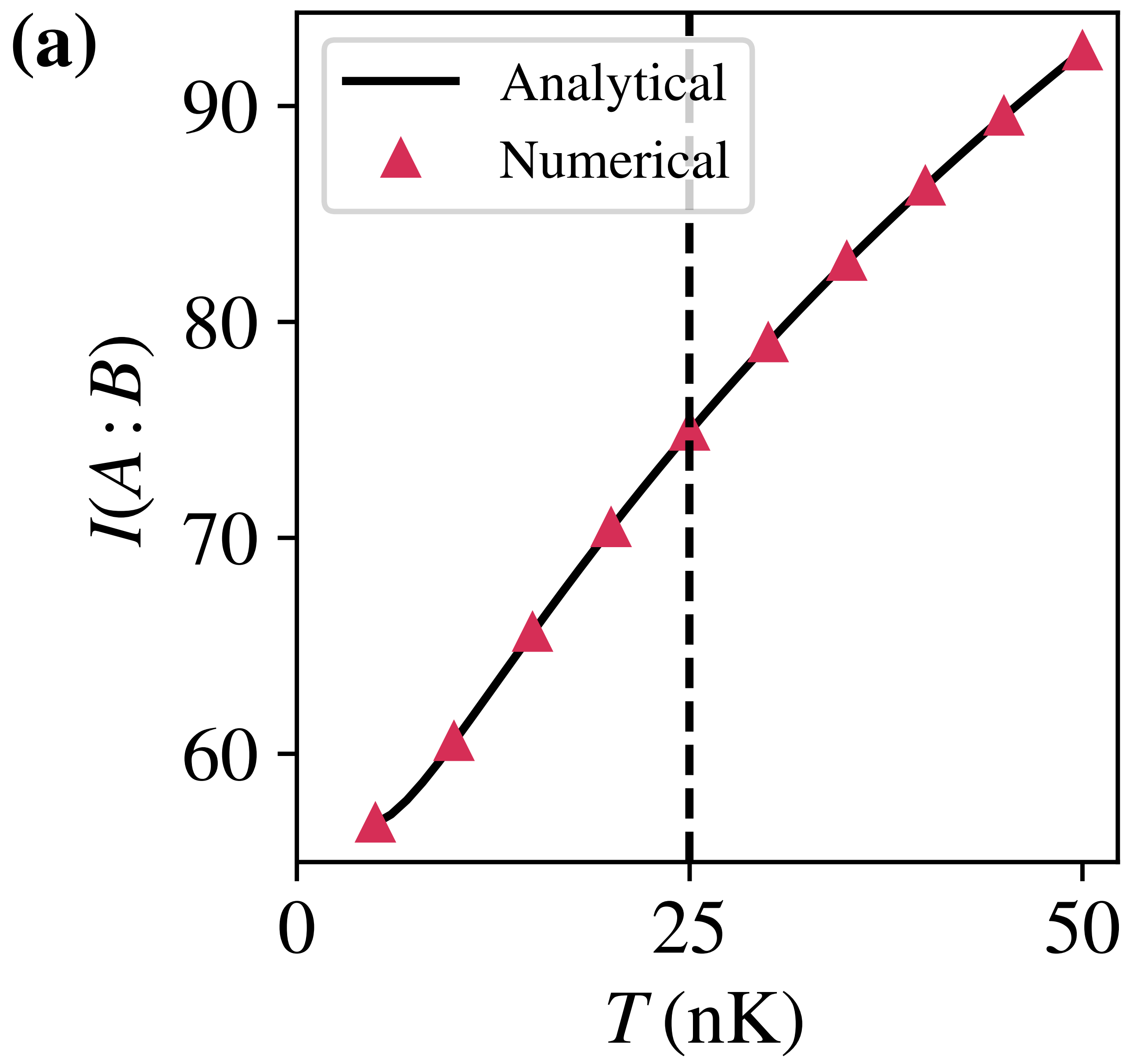
Geiger, Remi, et al. "Local relaxation and light-cone-like propagation of correlations in a trapped one-dimensional Bose gas." *New Journal of Physics* 16.5: 053034 (2014).

Entanglement & Mutual Information after Coherent Splitting

- We derived **analytical formula for logarithmic negativity, threshold temperature, and mutual information after coherent splitting** (*please check out the paper for explicit formula*)

- The threshold temperature for entanglement after coherent splitting is **20 ~ 40 nK** **→ achievable in current experiments!**



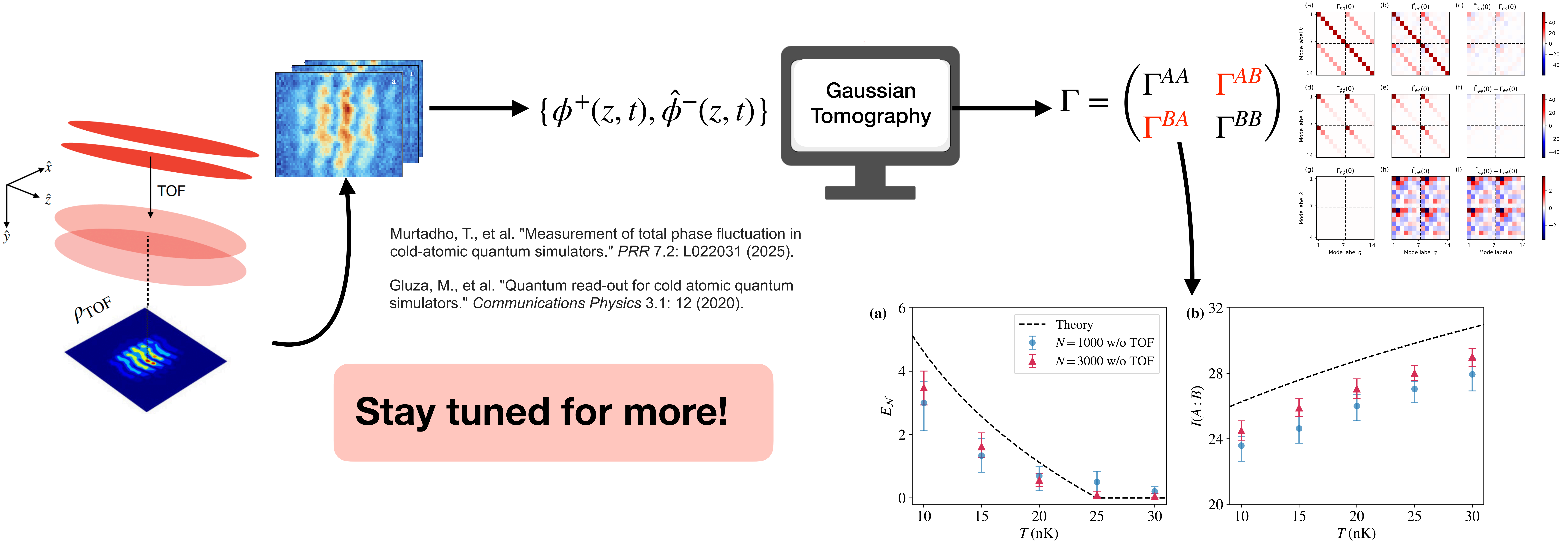


- Motivation
- Theoretical Methods
- *Results 1*: Entanglement and Mutual Information at Finite Temperatures
- *Results 2:* Entanglement and Mutual Information after Coherent Splitting
- ***Outlook:* Quantum Field Tomography & Experiments**
- Take-home messages

Outlook:
Quantum Field Tomography
& Experiments

Outlook: Quantum Field Tomography and Experimental Verification of Entanglement

Not only we know the condition and the temperature needed to observe extensive entanglement between 1D Bose gases, we also know **the measurement protocol to observe it.**



Murtadho, T., et al. "Measurement of total phase fluctuation in cold-atomic quantum simulators." *PRR* 7.2: L022031 (2025).

Gluza, M., et al. "Quantum read-out for cold atomic quantum simulators." *Communications Physics* 3.1: 12 (2020).

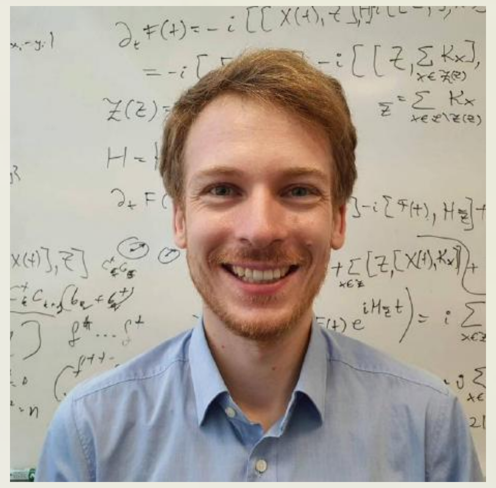
Stay tuned for more!

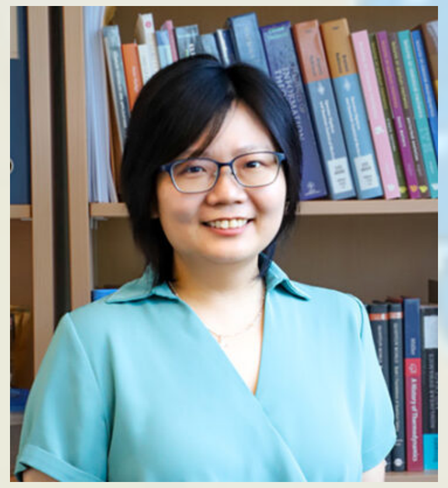
Take-home messages

Extend the study of entanglement between coupled Tomonaga-Luttinger liquids to finite temperatures and out-of-equilibrium regimes

Demonstrate that detecting extensive entanglement between interacting 1D quantum fields is within experimental reach in parallel 1D Bose gases.

Collaborators:
Marek Gluza, Nelly H. Y. Ng





Acknowledgment for feedback and discussions:
Tiang Bi Hong, Jörg Schmiedmayer, Maximilian Prüfer, Leonce Dupays, Paola Ruggiero, Giuseppe Vitagliano, Julia Mathe, Nicky Kai Hong Li.

NANYANG TECHNOLOGICAL UNIVERSITY SINGAPORE

im.Q

Thank you for your attention and happy to receive questions :)

Scan me!