Supplemental Material for "Cooperation and Competition in Synchronous Open Quantum Systems"

Taufiq Murtadho,[1, 2, 3] Sai Vinjanampathy,[4, 5, 6, *] and Juzar Thingna[1, 2, 7, †]

[1] *Center for Theoretical Physics of Complex Systems, Institute for Basic Science (IBS), Daejeon 34126, Republic of Korea.*
[2] *Basic Science Program, Korea University of Science and Technology, Daejeon 34113, Republic of Korea.*
[3] *School of Physical and Mathematical Science, Nanyang Technological University, Singapore 639798, Singapore*
[4] *Department of Physics, Indian Institute of Technology-Bombay, Powai, Mumbai 400076, India.*
[5] *Centre of Excellence in Quantum Information, Computation, Science and Technology, Indian Institute of Technology Bombay, Powai, Mumbai 400076, India.*
[6] *Centre for Quantum Technologies, National University of Singapore, 3 Science Drive 2, Singapore 117543, Singapore.*
[7] *Department of Physics and Applied Physics, University of Massachusetts, Lowell, MA 01854, USA.*

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A. Synchronization in D-level system

Here, we will derive the formula for the synchronization measure $S_{\max}$ for a D-level system from the Husimi-Q representation

$$Q[\rho] = \frac{D!}{\pi^{D-1}} \langle \alpha_D|\rho|\alpha_D\rangle\,, \tag{A1}$$

where $|\alpha_D\rangle = \sum_{n=1}^{D} \alpha_n |n\rangle$ is the SU(D) coherent state [1] with

$$\alpha_n = \begin{cases} e^{i\phi_n}\cos\theta_n \prod_{k=1}^{n-1}\sin\theta_k & 1 \le n < D \\ e^{i\phi_D}\prod_{k=1}^{D-1}\sin\theta_k & n = D. \end{cases} \tag{A2}$$

It has been implicitly assumed that for $n = 1$ the product term is just an identity. We are only concerned with the distribution of phases, so we can integrate out the polar angles $d\Theta = \prod_{l=1}^{D-1}\cos\theta_l(\sin\theta_l)^{2D-2l-1}d\theta_l$ from $Q[\rho]$ to obtain a quasi-probability distribution over a $D-1$ torus, i.e.,

$$\int_0^{\pi/2} Q[\rho]\; d\Theta = \frac{D!}{\pi^{D-1}} \sum_{n,m=1}^{D-1} \rho_{nm} \int_0^{\pi/2} \alpha_n^*\alpha_m \prod_{l=1}^{D-1} \cos\theta_l\; (\sin\theta_l)^{2D-2l-1}\; d\theta_l. \tag{A3}$$

The diagonal contribution ($n = m$) gives,

$$\int_0^{\pi/2} |\alpha_n|^2 \prod_{l=1}^{D-1} \cos\theta_l (\sin\theta_l)^{2D-2l-1}\; d\theta_l = \frac{1}{2^{D-1}D!} \qquad \forall n = 1, ..., D, \tag{A4}$$

while the off-diagonal ($n \neq m$) contribution yields,

$$\int_0^{\pi/2} \alpha_n^*\alpha_m \prod_{l=1}^{D-1} \cos\theta_n (\sin\theta_l)^{2D-2l-1}\; d\theta_l = \frac{\pi}{2^{D+1}D!} e^{i(\phi_m-\phi_n)} \qquad \forall n \neq m = 1, \ldots, D. \tag{A5}$$

Therefore, combining the diagonal and the off-diagonal contributions we obtain,

$$\int_0^{\pi/2} Q[\rho]\; d\Theta = \frac{1}{(2\pi)^{D-1}} + \frac{1}{2^{D+1}\pi^{D-2}} \sum_{n\neq m} \rho_{nm} e^{i(\phi_m-\phi_n)}. \tag{A6}$$

* sai@phy.iitb.ac.in
† juzar˙thingna@uml.edu

The first term represents the contribution from a uniform distribution, which can be eliminated by defining the phase quasi-probability distribution

$$S(\phi_1,\ldots,\phi_{D-1}) := \int_0^{\pi/2} Q[\rho]\; d\Theta - \frac{1}{(2\pi)^{D-1}} = \frac{1}{2^{D+1}\pi^{D-2}} \sum_{n\neq m}^{D} \rho_{nm} e^{i(\phi_m-\phi_n)}. \tag{A7}$$

The synchronization measure $S_{\max}$ in the main text is simply the maximum of Eq. (A7), i.e.,

$$S_{\max} \equiv \max_{\phi_1,\ldots,\phi_{D-1}} S(\phi_1,...,\phi_{D-1}) \leq \frac{1}{2^D\pi^{D-2}} C_{l_1}, \tag{A8}$$

where $C_{l_1} = \sum_{n<m} |\rho_{nm}|$ is the l_1-norm of coherence [2]. The upper-bound can be obtained by simply noting that each term in the summation of Eq. (A7) is upper-bounded by $|\rho_{nm}|$.

B. Steady-state of a generalized Scovil–Schulz-DuBois maser

The quantum master equation [Eq. (8) from the main text] can be expanded into a series of linear first-order differential equations for each density matrix element. We divide the elements into three groups: populations (ρ_{ii} with $i = 0,...,N+1$), non-degenerate coherences (ρ_{1k} with $k = 2,...,N+1$), and degenerate coherences (ρ_{lk} with $k,l = 2,...,N+1$ and $k \neq l$). We begin with the equations for the populations,

$$\frac{d\tilde\rho_{11}}{dt} = i\lambda \sum_{j=2}^{N+1} (\tilde\rho_{1j} - \tilde\rho_{j1}) - 2\gamma_c(1+n_c)\tilde\rho_{11} - 2\gamma_c n_c \sum_{j=1}^{N+1} \tilde\rho_{jj} + 2\gamma_c n_c, \tag{A9}$$

$$\frac{d\tilde\rho_{kk}}{dt} = -i\lambda(\tilde\rho_{1k} - \tilde\rho_{k1}) - 2\gamma_h(1+n_h)\tilde\rho_{kk} - 2\gamma_h n_h \sum_{j=1}^{N+1} \tilde\rho_{jj} + 2\gamma_h n_h, \tag{A10}$$

where we have eliminated $\tilde\rho_{00}$ using the trace preserving condition $\tilde\rho_{00} = 1 - \sum_{j=1}^{N+1} \tilde\rho_{jj}$. The equations for the non-degenerate and degenerate coherences read,

$$\frac{d\tilde\rho_{1k}}{dt} = -(\gamma_h(1+n_h) + \gamma_c(1+n_c))\tilde\rho_{1k} + i\lambda \left(\tilde\rho_{11} - \sum_{j=2}^{N+1} \tilde\rho_{jk}\right), \tag{A11}$$

$$\frac{d\tilde\rho_{kl}}{dt} = -2\gamma_h(1+n_h)\tilde\rho_{kl} - i\lambda(\tilde\rho_{1l} - \tilde\rho_{k1}). \tag{A12}$$

Since we are interested in the steady state we set $d\tilde\rho_{ij}/dt = 0$ and using Eq. (A11) evaluate $d\tilde\rho_{1k}/dt - d\tilde\rho_{1l}/dt = 0$ ($k \neq l$) to obtain the relation,

$$[\gamma_h(1+n_h) + \gamma_c(1+n_c)]\,(\tilde\rho^{\rm ss}_{1k} - \tilde\rho^{\rm ss}_{1l}) = i\lambda \sum_{j=2}^{N+1} (\tilde\rho^{\rm ss}_{jl} - \tilde\rho^{\rm ss}_{jk}) \qquad (k \neq l). \tag{A13}$$

We also compute $d\tilde\rho_{jl}/dt - d\tilde\rho_{jk}/dt = 0$ with $j \neq l \neq k$ using Eq. (A12) to obtain

$$\tilde\rho^{\rm ss}_{jl} - \tilde\rho^{\rm ss}_{jk} = \frac{i\lambda}{2\gamma_h(1+n_h)}(\tilde\rho^{\rm ss}_{1k} - \tilde\rho^{\rm ss}_{1l}). \tag{A14}$$

Combining Eqs. (A14) and (A13) we obtain the steady-state coherences,

$$\left[\gamma_h(1+n_h) + \gamma_c(1+n_c) + \frac{iN\lambda}{2\gamma_h(1+n_h)}\right](\tilde\rho^{\rm ss}_{1k} - \tilde\rho^{\rm ss}_{1l}) = 0, \tag{A15}$$

$$\implies \tilde\rho^{\rm ss}_{1l} = \tilde\rho^{\rm ss}_{1k} \qquad (k \neq l). \tag{A16}$$

We substitute the above result in Eq. (A13) to obtain

$$\tilde\rho^{\rm ss}_{kl} = \frac{\lambda}{\gamma_h(1+n_h)} \mathrm{Im}(\tilde\rho^{\rm ss}_{1k}). \tag{A17}$$

Since $\tilde{\rho}_{1k}^{\rm ss} = \tilde{\rho}_{1l}^{\rm ss}$ we can infer from the above that $\tilde{\rho}_{kl}^{\rm ss} = \tilde{\rho}_{lk}^{\rm ss}$ for $k \neq l$ which means all $\tilde{\rho}_{kl}^{\rm ss}$ are real. Substituting Eq. (A17) to Eq. (A13) we obtain,

$$-2(\gamma_h(1+n_h)+\gamma_c(1+n_c))\mathrm{Re}(\tilde{\rho}_{1k}^{\rm ss}) = 0, \tag{A18}$$
$$\implies \mathrm{Re}(\tilde{\rho}_{1k}^{\rm ss}) = 0. \tag{A19}$$

This clearly implies that all $\tilde{\rho}_{1k}^{\rm ss}$ are imaginary. Next, from Eq. (A10) we may calculate $d\tilde{\rho}_{kk}/dt - d\tilde{\rho}_{ll}/dt$ $(k \neq l)$ and making use of Eq. (A15) gives,

$$-2\gamma_h(1+n_h)(\tilde{\rho}_{kk}^{\rm ss} - \tilde{\rho}_{ll}^{\rm ss}) = 0, \tag{A20}$$
$$\implies \tilde{\rho}_{kk}^{\rm ss} = \tilde{\rho}_{ll}^{\rm ss} \qquad (k \neq l) \tag{A21}$$

Utilizing Eqs. (A15)-(A20) to simplify Eq. (A9) results in

$$i\lambda N \tilde{\rho}_{1k}^{\rm ss} - \gamma_c(1+2n_c)\tilde{\rho}_{11}^{\rm ss} - N\gamma_c n_c \tilde{\rho}_{kk}^{\rm ss} + \gamma_c n_c = 0. \tag{A22}$$

Similarly from Eqs. (A10) and (A11) we obtain,

$$-i\lambda\tilde{\rho}_{1k}^{\rm ss} - \gamma_h(1+n_h(1+N))\tilde{\rho}_{kk}^{\rm ss} - \gamma_h n_h \tilde{\rho}_{11}^{\rm ss} + \gamma_h n_h = 0, \tag{A23}$$
$$-\Big[\gamma_h(1+n_h)+\gamma_c(1+n_c)+\frac{\lambda^2(N-1)}{\gamma_h(1+n_h)}\Big]\tilde{\rho}_{1k}^{\rm ss} + i\lambda\tilde{\rho}_{11}^{\rm ss} - i\lambda\tilde{\rho}_{kk}^{\rm ss} = 0. \tag{A24}$$

Solving Eqs. (A22)-(A24) simultaneously gives the final solution

$$\tilde{\rho}_{1k}^{\rm ss} = \frac{i\lambda(n_c-n_h)(1+n_h)\gamma_c\gamma_h}{F(N,\lambda,\gamma_c,\gamma_h,n_h,n_c)}, \tag{A25}$$

where $F(N,\lambda,\gamma_c,\gamma_h,n_h,n_c) = AN^2 + BN + C$, with

$$A = \lambda^2 n_h(\gamma_c(1+n_c)+\gamma_h(1+n_h)), \tag{A26}$$
$$B = \lambda^2[\gamma_c(1+3n_c+2n_hn_c)+\gamma_h(1+n_h)(1+2n_h)] + n_h\gamma_h\gamma_c(1+n_h)(1+n_c)[\gamma_h(1+n_h)+\gamma_c(1+n_c)], \tag{A27}$$
$$C = \gamma_h\gamma_c(1+n_h)^2(1+2n_c)(\gamma_h(1+n_h)+\gamma_c(1+n_c)). \tag{A28}$$

After obtaining (A25), it is only a matter of substitution to solve for the other steady-state density matrix elements that read

$$\tilde{\rho}_{jl}^{\rm ss} = \frac{\lambda^2\gamma_c(n_c-n_h)}{F(N,\lambda,\gamma_c,\gamma_h,n_h,n_c)}, \tag{A29}$$
$$\tilde{\rho}_{11}^{\rm ss} = \frac{N\lambda^2(1+n_h)(n_h\gamma_h+n_c\gamma_c)+\gamma_c\gamma_h n_c(1+n_h)^2(\gamma_c(1+n_c)+\gamma_h(1+n_h))}{F(N,\lambda,\gamma_c,\gamma_h,n_h,n_c)}, \tag{A30}$$
$$\tilde{\rho}_{jj}^{\rm ss} = \frac{(N\lambda^2 n_h+\gamma_c\gamma_h n_h(1+n_h)(1+n_c))(\gamma_c(1+n_c)+\gamma_h(1+n_h))+\lambda^2(n_c-n_h)}{F(N,\lambda,\gamma_c,\gamma_h,n_h,n_c)}, \tag{A31}$$
$$\tilde{\rho}_{00}^{\rm ss} = 1 - \tilde{\rho}_{11}^{\rm ss} - \sum_{j=2}^{N+1}\tilde{\rho}_{jj}^{\rm ss}, \tag{A32}$$
$$\tilde{\rho}_{01}^{\rm ss} = \tilde{\rho}_{0j}^{\rm ss} = 0. \tag{A33}$$

Equations (A25)-(A29) are the same as Eq. (9)-(10) in the main text.

C. $S_{\max}$ calculation for generalized Scovil–Schulz-DuBois maser

1. *Refrigerator Regime $n_c > n_h$*

In this section, we analytically calculate $S_{\max}$ using the steady-state solution obtained in Supp. B. We first focus on the refrigerator regime ($n_c > n_h$). In this case, we have $\arg(\rho_{1j}) = \pi/2$ and $\arg(\rho_{jl}) = 0$ $(j \neq l)$ and by using the steady-state solutions [Eqs. (A25) and (A29)] in the phase quasi-probability distribution, Eq. (A7), we obtain,

$$S_{\max}|_{n_c>n_h} = \frac{1}{2^{N+2}\pi^N}\max_{\{\varphi_{j1}\}}\Big[-\sum_{j=2}^{N+1}|\tilde{\rho}_{1j}^{\rm ss}|\sin\varphi_{j1} + \sum_{j<l}^{N+1}|\tilde{\rho}_{jl}^{\rm ss}|\cos(\varphi_{l1}-\varphi_{j1})\Big] = \frac{1}{2^{N+2}\pi^N}\sum_{j<l}^{N+1}|\tilde{\rho}_{jl}^{\rm ss}|. \tag{A34}$$

Above $\varphi_{ij} = \phi_i - \phi_j$. The second equality is obtained by choosing optimum phases $\varphi_{j1} = 3\pi/2 \; \forall j = 2, ..., N+1$ that maximize S. Using the analytic solution obtained from Eqs. (A25)-(A29), $S_{\max}$ can be explicitly computed as,

$$S_{\max}|_{n_c > n_h} = \frac{1}{(2\pi)^N} \frac{\lambda^2 \gamma_c (n_c - n_h)(N^2 + (2k-1)N)}{8F(N, \lambda, \gamma_c, \gamma_h, n_h, n_c)}, \tag{A35}$$

where $k = \gamma_h(1+n_h)/\lambda = |\rho_{1j}^{ss}|/|\rho_{jk}^{ss}|$ is the *dissipation-to-driving* ratio. In the limit of macroscopic degeneracy $N \to \infty$, the scaled synchronization measure $\mathbb{S}_{\max} = (2\pi)^N S_{\max}$ approaches a constant value given in Eq. (13) of the main text.

2. Engine Regime $n_h > n_c$

In the engine case ($n_c < n_h$), the optimization is trickier. In this regime, the phase-matching condition breaks down due to competition between entrainment and mutual coupling as can be seen from $\arg(\tilde{\rho}_{j1}^{\rm ss}) = \pi/2$ and $\arg(\tilde{\rho}_{jl}^{\rm ss}) = \pi$ for all $j, l = 2, 3, ..., N+1$ [see Eqs. (A25) and (A29)]. The synchronization measure $S_{\max}$ can be expressed as,

$$S_{\max}|_{n_h > n_c} = \frac{1}{2^{N+2}\pi^N} \max_{\{\varphi_{j1}\}} \Big[\sum_{j=2}^{N+1} |\tilde{\rho}_{1j}^{\rm ss}| \sin\varphi_{j1} - \sum_{j<l}^{N+1} |\tilde{\rho}_{jl}^{\rm ss}| \cos(\varphi_{l1} - \varphi_{j1}) \Big]. \tag{A36}$$

Optimization of Eq. (A36) is difficult for an arbitrary N. Let us check the simplest non-trivial case of $N = 2$. In this case, $S_{\max}$ can be cast into a simple form

$$S_{\max}|_{n_h > n_c} = \frac{1}{16\pi^2} |\tilde{\rho}_{23}^{\rm ss}| \max_{\varphi_{21}, \varphi_{31}} \Big(k \sin\varphi_{21} + k \sin\varphi_{31} - \cos(\varphi_{31} - \varphi_{21}) \Big), \tag{A37}$$

where $k = |\tilde{\rho}_{1j}^{\rm ss}|/|\tilde{\rho}_{jl}^{\rm ss}| = \gamma_h(1+n_h)(1+p)/\lambda$ is dissipation-to-driving ratio. Thus, calculating $S_{\max}$ is now reduced to optimizing a two-variable function $f(x,y) \equiv k \sin x + k \sin y - \cos(x - y)$. One can easily verify

$$\max_{x,y} f(x,y) = \begin{cases} 2k - 1 & \text{if } k > 2 \\ 1 + \dfrac{k^2}{2} & \text{if } 0 \le k \le 2, \end{cases} \tag{A38}$$

with optimum points $(x, y) = (\pi/2, \pi/2)$ when $k > 2$ and $\{(\arcsin(k/2), \pi - \arcsin(k/2)), (\pi - \arcsin(k/2), \arcsin(k/2))\}$ when $k \le 2$. By substituting Eq. (A38) in Eq. (A37), one obtains Eq. (11) of the main text.

Another case of interest is the limit $N \to \infty$ where we are interested to compute the scaled synchronization measure $\mathbb{S}_{\max} = (2\pi)^N S_{\max}$,

$$\mathbb{S}_{\max} = \frac{|\tilde{\rho}_{jl}^{\rm ss}|}{4} \max_{\{\varphi_n\}} \Big[\sum_{n=1}^{N} k \sin\varphi_n - \sum_{m<n}^{N} \cos(\varphi_n - \varphi_m) \Big] \tag{A39}$$

with $|\tilde{\rho}_{jl}^{\rm ss}|$ is given in Eq. (A29). For convenience, we have relabeled $\{\varphi_{j1} | j = 2, 3, ..., N+1\}$ to $\{\varphi_n | n = 1, 2, ..., N\}$. Based on numerical simulation (Fig. 3c in the main text), we postulate that for $N \to \infty$, the optimum choice of angles is given by the uniform distribution $\varphi_n^{\rm opt} = 2\pi(n-1)/N$ (owing to phase repulsiveness). We may then explicitly compute $\mathbb{S}_{\max}$,

$$\lim_{N\to\infty} \mathbb{S}_{\max} = \frac{\Lambda}{N^2} \Big[k \ \mathrm{Im}\Big(\sum_{n=1}^{N} e^{2i\pi(n-1)/N} \Big) - \mathrm{Re}\Big(\sum_{n=1}^{N} \sum_{p=1}^{N-n} e^{2i\pi p/N} \Big) \Big] \tag{A40}$$

$$= \frac{\Lambda}{N} \mathrm{Re}\Big(\frac{1}{1 - e^{2\pi i/N}} \Big) = 0 \tag{A41}$$

with Λ being a constant that depends on the system's parameters. Thus, in contrast with the refrigerator regime, in the engine regime the scaled synchronization measure $\mathbb{S}_{\max}$ vanishes in the limit $N \to \infty$.

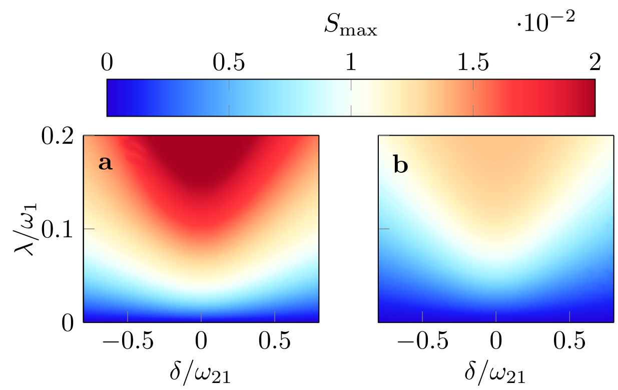

FIG. 1. Synchronization measure $S_{\max}$ as a function of drive strength λ and detuning $\delta = \Omega - \omega_{21}$ with $\omega_{21} = \omega_2 - \omega_1$ in engine (**a**, $n_h/n_c = 2$) and refrigerator (**b**, $n_h/n_c = 0.8$) regimes. The other parameter values are given by $\omega_2 = 3\omega_1$, $\gamma_h = 0.05\omega_1$, $\gamma_c = 0.2\omega_1$, and $n_c = 0.5$. The enhancement of $S_{\max}$ around zero detuning with a finite width in δ mimics the Arnold tongue behavior indicating that the system entrains with the external drive even if it is not resonant.

3. Arnold Tongue

So far, we have considered the resonant driving case, i.e. $\Omega = \omega_2 - \omega_1$. The case of finite detuning $\delta = \Omega - (\omega_2 - \omega_1) \neq 0$ is difficult to solve analytically. In Fig. 1, we numerically compute the Arnold tongue structure of synchronization measure $S_{\max}$ when varying it with respect to detuning δ and drive strength λ. The Arnold tongue clearly demonstrates the robustness of entrainment in this system even if the external drive is not fully resonant with the system energy gap.

D. $S_{\max} = 0$ if and only if ρ is diagonal ($D = 3$)

Next, we will show that in the case of $D = 3$, $S_{\max} = 0$ if and only if ρ is diagonal. We consider a three-level system with $\{|0\rangle, |1\rangle, |2\rangle\}$ representing the eigenvectors. A general expression for $S \equiv S(\phi_0, \phi_1, \phi_2)$ for such a three-level system reads,

$$S = \frac{1}{8\pi}[|\rho_{01}|\cos(\phi_1 - \phi_0 + \Phi_{01}) + |\rho_{02}|\cos(\phi_2 - \phi_0 + \Phi_{02}) + |\rho_{12}|\cos(\phi_2 - \phi_1 + \Phi_{12})], \tag{A42}$$

where $\Phi_{ij} = \arg(\rho_{ij})$. We first transform the equation by defining $\varphi_{ij} = \phi_i - \phi_j$, i.e.,

$$S = \frac{1}{8\pi}[|\rho_{01}|\cos(\varphi_{10} + \Phi_{01}) + |\rho_{02}|\cos(\varphi_{20} + \Phi_{02}) + |\rho_{12}|\cos(\varphi_{20} - \varphi_{10} + \Phi_{12})]. \tag{A43}$$

Given that the reduced density matrix ρ is diagonal, $S(\varphi_{10}, \varphi_{20})$ is zero everywhere ($\because |\rho_{ij}| = 0\ \forall i, j$) and thus it is trivial that $S_{\max} = 0$.

However, it is not trivial to show that if $S_{\max} = 0$ the ρ will be diagonal. We will prove it by contradiction. Let us assume ρ is *not* diagonal and $S_{\max} = 0$. Then, by definition, $S(\varphi_{10}, \varphi_{20})$ is zero or negative *everywhere else*. We will show below, considering all possible cases, that we can always find $\{\varphi_{10}, \varphi_{20}\}$ such that S is positive, ergo contradiction.

Case 1: Only one coherence is non-zero, let's say ρ_{01}. Then, we can choose $\varphi_{10} = -\Phi_{01}$ such that $S = |\rho_{01}| > 0$.

Case 2: Two coherences are non-zero, let's say ρ_{01} and ρ_{02}. We can then choose $\varphi_{10} = -\Phi_{01}$ and $\varphi_{20} = -\Phi_{02}$ such that $S = |\rho_{01}| + |\rho_{02}| > 0$

Case 3: All coherences are non-zero. This is a non-trivial case. First, let us choose $(\varphi_{10}, \varphi_{20}) = (\pi/2 - \Phi_{01}, \pi/2 - \Phi_{02})$ such that

$$S = \frac{1}{8\pi}|\rho_{12}|\cos(\Phi_{01} - \Phi_{02} + \Phi_{12}) > 0. \tag{A44}$$

The above is positive if the cosine term is positive. If it is negative, we can just choose $(\varphi_{10}, \varphi_{20}) = (-\pi/2 - \Phi_{01}, \pi/2 - \Phi_{02})$ such that S remains positive, i.e.,

$$S = -\frac{1}{8\pi}|\rho_{12}|\cos(\Phi_{01} - \Phi_{02} + \Phi_{12}) > 0. \tag{A45}$$

If the cosine term is zero, we choose $(\varphi_{10}, \varphi_{20}) = (-\Phi_{01}, -\Phi_{02})$ to keep S positive,

$$S = \frac{1}{8\pi} \left(|\rho_{01}| + |\rho_{02}|\right) > 0. \tag{A46}$$

Thus, we conclude that in all cases, we can always find phase configuration such that S is positive, implying that $S_{\max}$ cannot be zero if ρ is not diagonal.

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