Semester Project

Differential Equations *(Cal-II)*

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Section: BSCS - G

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INTRODUCTION

The following problem statement was presented to us for solving (parts a to e) to analyze and solve as per instructed in each part of the problem statement.

(This problem statement is only for students of section G)

Problem Statement

Consider a smooth horizontal surface with no friction on which a mass having magnitude m can have a motion between two springs and is free to move along this one-dimensional path. The mass is not attached to both of the springs. The mass goes through a simple inertial motion whenever the condition $|x| < L$ is satisfied, where x is the distance from the origin of the center point of the mass. Following Hooke's law, when $x \leq -L$, the mass experiences a force proportional to $x + L$ to the right (positive x direction) when it is in contact with the spring on the left. Similarly, When $x \geq L$, the mass experiences a force proportional to $x - L$ to the left (negative x direction) when the mass is in contact with the spring on the right.

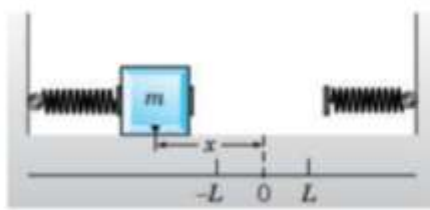


Fig 1.1) screen snip of originally allocated problem statement for the project.

Overall, the solution was seen as a piecewise function of Linear differential equations, with three variations of the harmonic spring mass motion's system in differential equations.

The derived function was analyzed, and values were put in to gain solid specific case equations. Then the method used to solve the system was found to be the *Method of Undefined Coefficients.*

Hence the handwritten solution was formulated in part **a** through **c**

. The following parts **d**, **e**, and **f** were solved via use of the *MATLAB software*. All this led to a fully solved problem statement and project solution.

SOLUTION PART A)

Analyzing the given cases and their correlation with respect to the domain, all around the given limits of "**-L-> L**" as well as beyond that, keeping in mind the proportionality shifts in the spring mass constant, the following equations were derived.

Part-a)

general form of D.E for, S.H.M

$$mx'' + bx' + kx = f(t)$$

∴ in our case,

// m - mass object //

$$f(t) = 0,$$
$$b = \gamma$$
$$(kx) = k(x \pm L)$$,{case 1&3}

∴

general equation is

$$mx'' + \gamma x' + k(x \pm L) = 0$$

or

$$\underbrace{mx''}_{F(w)} = -k(x \pm L) - \gamma x'$$

∴

Fig 1.2.1) Scan of Part a)'s handwritten solution.

general equation is

$$mx'' + \gamma x' + k(x \pm L) = 0$$

or

$$\underbrace{mx''}_{F(x)} = -k(x \pm L) - \gamma x'$$

∴ analysing

$(x \pm L)$

↳ when $x \leq -L$, ~ $m \rightarrow$ mass at left side, force towards right $\Rightarrow (x+L)$

↳ when $-L \leq x \leq 0$ ~ $m \rightarrow$ moving freely toward origin only friction... $\Rightarrow (x \pm L) = 0$
$(x \approx 0)$ $\boxed{kx = 0}$

↳ when $L \leq x$ ~ $m \rightarrow$ ~~at~~ mass at right side, force towards left $\Rightarrow (x-L)$

∴ via evaluating the above conditions we have.

$$F(x) = \begin{cases} -k(x+L) - \gamma x', & x \leq -L \\ -\gamma x', & -L \leq x \leq 0 \\ -k(x-L) - \gamma x', & L \leq x \end{cases}$$

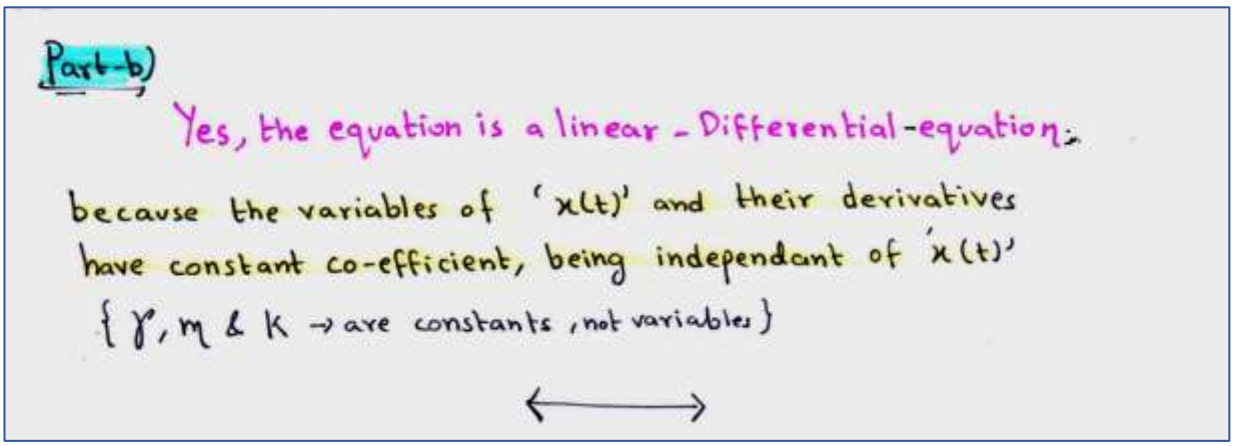
Hence Proved ✓

$\longleftrightarrow$

Fig 1.2.2) Scan of Part a)'s handwritten solution.

SOLUTION PART B)

Analyzing the derived equations with the standard conditions for linearity of a differential Equation, *it was found it was indeed a Linear Differential Equation.*



Part-b)

Yes, the equation is a linear - Differential-equation:

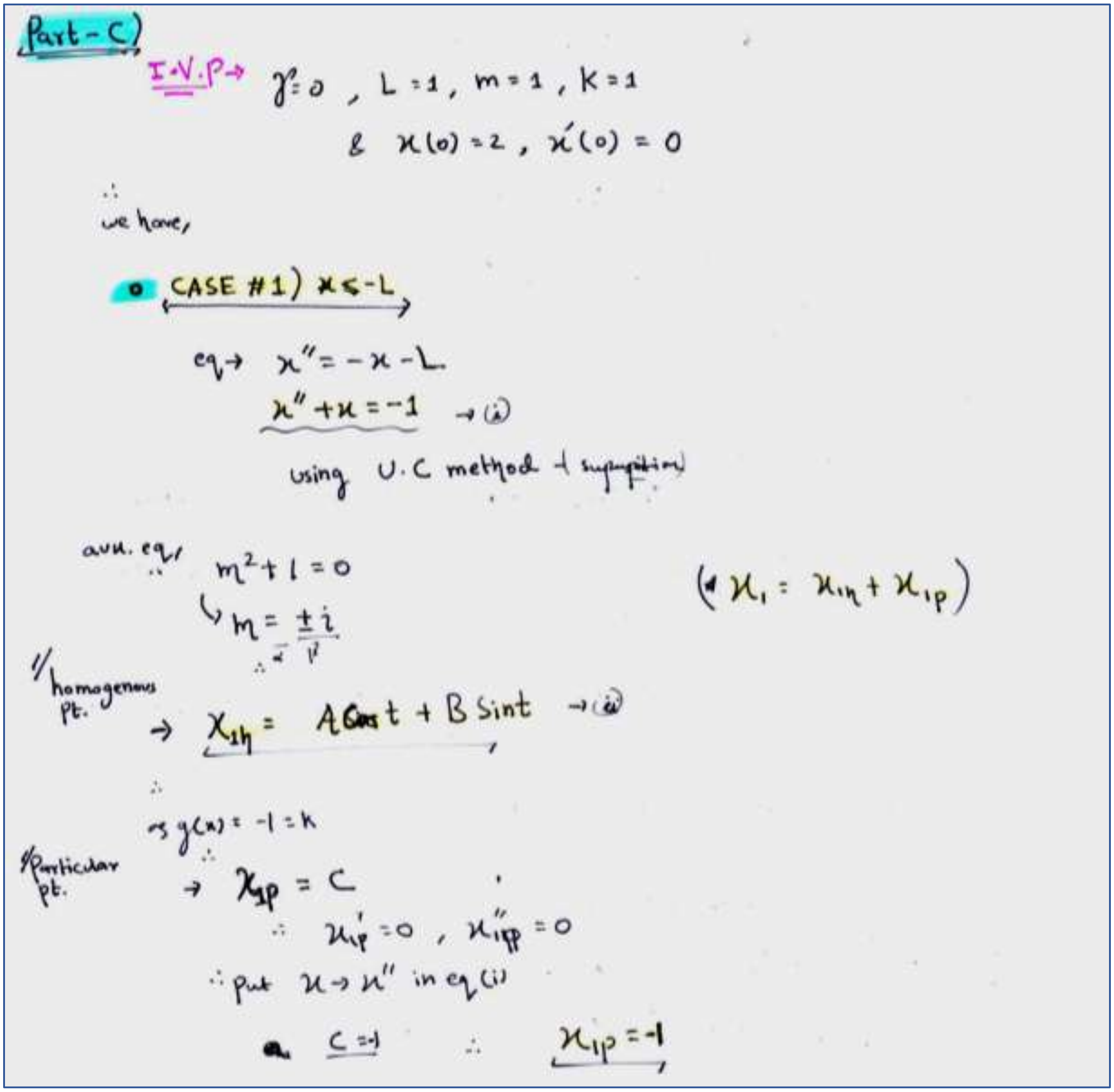
because the variables of '$x(t)$' and their derivatives have constant co-efficient, being independant of '$x(t)$'

{γ, m & K → are constants, not variables}

Fig 1.3) Scan of Part b)'s handwritten solution.

SOLUTION PART C)

Applying the presented constant values and Initial Value Conditions, we were able to solve the equation via the method of undefined Coefficients for each of the three derived situational cases.



Part-c)

I.V.P → $\gamma = 0$, $L = 1$, $m = 1$, $K = 1$

& $x(0) = 2$, $x'(0) = 0$

∴ we have,

CASE #1) $x \leq -L$

eq → $x'' = -x - L$

$x'' + x = -1 \rightarrow$ (i)

using U.C method + superposition

aux. eq, $m^2 + 1 = 0$

$\hookrightarrow m = \pm i$

(∴ $x_1 = x_{1h} + x_{1p}$)

// homogenous pt. → $x_{1h} = A \cos t + B \sin t \rightarrow$ (ii)

∴ $g(x) = -1 = k$

// Particular pt. → $x_{1p} = C$

∴ $x'_{1p} = 0$, $x''_{1p} = 0$

∴ put $x \rightarrow x''$ in eq (i)

∴ $C = -1$ ∴ $x_{1p} = -1$

Fig 1.4.1) Scan of Part c)'s handwritten solution.

∴

→ Solution for case#1,

$\cancel{x(0) =}$

$x_1 = A\cos t + B\sin t - 1$

using I.V.P

- $x(0) = 2, \Rightarrow 2 = A - 1$ $\quad$ • $x'(0) = 0, \Rightarrow 0 = -B$
- $\underline{A = 3}$ $\quad$ $\underline{B = 0}$

∴

$$\boxed{\text{Solution for case\#1)} \Rightarrow x_{(1)} = 3\cos(t) - 1}$$

(case 1)

~

- CASE #2) $-L \leq x \leq 0$;

eq →

$mx'' = \gamma x'$

$mx'' = 0$ $\quad$ (double $\int$ing b/s)

$\iint x'' = \iint 0 \, dt$

$x = C$

using I.V.P $x(0) = 2$

∴ $\underline{C = 2}$

∴

$\underline{x_2 = 2 = C}$

$$\boxed{\text{Solution for Case\#2,} \quad x_{(2)} = 2}$$

(case 2)

~

- CASE #3) $L \leq x$;

eq → $x'' = -x + L$

$x'' + x = 1$

// aux eq → $m^2 + 1 = 0$

$m = \pm i$ //

∴ U.C method –

gives us

$x_{3h} = D\cos t + E\sin t$

$x_{3p} = C_1$

2

Fig 1.4.2) Scan of Part c)'s handwritten solution.

$\Rightarrow x'_{sp} = x''_{sp} = 0$

$\therefore$

$C = 1$

$\therefore$

$\underline{x_{sp} = 1}$ & $\underline{x_{sh} = D\cos t + E\sin t}$

& so...

G.S $\Rightarrow \underline{x_s = D\cos t + E\sin t + 1}$ $\Rightarrow x'_s = -D\sin t + E\cos t$

$\rightarrow x(0) = 2$

$2 = D + 1$

$\cdot \underline{D = 1}$

$\rightarrow x'(0) = 0$

$0 = E\cdot 1$

$\cdot \underline{E = 1}$

$\therefore$

$$\boxed{\text{Solution for Case \#3)} \quad x_{(3)} = \cos(t) + 1}$$

(cases)

Solutions for Part c)

for

$$F(x) = \underbrace{\begin{cases} -k(x+L) - \gamma x' , & x \le -L \\ -\gamma x' , & -L \le x \le x \\ -k(x-L) - \gamma x' , & L \le x \end{cases}}_{\text{(derived D.Es)}} \quad \begin{matrix} \rightarrow \\ \rightarrow \\ \rightarrow \end{matrix} \quad \underbrace{\left.\begin{matrix} 3\cos(t) - 1 \\ 2 \\ \cos(t) + 1 \end{matrix}\right\}}_{\text{(Solutions)}} x(t)$$

Fig 1.4.3) Scan of Part c)'s handwritten solution

SOLUTION PART D)

MATLAB(PART D&E):

```
%Prints the information of group members.
disp("FAST CS-G");
disp("Moaz Farooq 22i-1173");
disp("Hussain Ali 22i-0902");
disp("Tauha Imran 22i-1239");
%Program Starts
str = input('Press Any Key to Countinue or 0 to
terminate','s');
%Program continues till 0 is entered
while str~="0"
%If 1 is pressed it solves d part
%If 2 is pressed it solves e part
disp("Enter 1 if you want solution to part d. ");
disp("Enter 2 if you want solution to part e. ");
choice=input('Enter your Choice: ');
if (choice==1)
    disp("For x<=-L press 1");
    disp("For -L<=x<=L press 2");
    disp("For x>=L press 3");
    choice1=input('Enter your Choice: ');
    if(choice1==1)
        Z=-1;
    end
    if(choice1==2)
        Z=0;
    end
    if(choice1==3)
        Z=1;
    end
    % Define the function for the differential equation
    f = @(t, x) [x(2); -0.1*x(2)-x(1)+Z];
    % Set up the time range and initial conditions
    tspan = [0 20];
    x0 = [0 0];
```

```
    % Using the ode45 solver to generate the solution
    [t, x] = ode45(f, tspan, x0);
    % Plot the direction field
    x1 = linspace(-2.5, 2.5, 20);
    x2 = linspace(-2.5, 2.5, 20);
    [X1, X2] = meshgrid(x1, x2);
    u = X2;
    v = -0.1*X2 - X1 + Z;
    quiver(X1, X2, u, v);
    % Add labels and legend
    xlabel('x');
    ylabel('x''');
    title('Direction Field for Selected Case');
    legend('Direction field');
end
if(choice==2)
    disp("For Undamped Press 1");
    disp("For Damped Press 2");
    choice3=input('Enter your Choice: ');
    if(choice3==1)
        disp("For IVP x(0)=2 Press 1");
        disp("For IVP x(0)=5 Press 2");
        choice2=input('Enter your Choice: ');
        %setting the constant values according to the
seleceted IVP
        if(choice2==1)
            a=3;
            b=1;
        end
        if(choice2==2)
            a=6;
            b=4;
        end
        t = linspace(0, 6*pi, 100);
        %Equation for x<=L
        x = a*cos(t) - 1;
        plot(t, x);
        xlabel('t');
        ylabel('x');
        hold on;
        %Equation for -L<=x<=0
        x2=2+t-t;
        plot(t,x2);
        %Equation for L<=x
```

```
        x3=b*cos(t)+1;
        plot(t,x3);
        legend("x<=L","-L<=x<=0","L<=x");
        title("IVP Plot");
    end
    if(choice3==2)
        disp("For IVP x(0)=2 Press 1");
        disp("For IVP x(0)=5 Press 2");
        choice2=input('Enter your Choice: ');
        %setting the constant values according to the
seleceted IVP
        if(choice2==1)
            a=3;
            b=1;
        end
        if(choice2==2)
            a=6;
            b=4;
        end
        t = linspace(0, 15*pi, 100);
        %setting the damping factor as k
        k=exp(-0.05.*t);
        %Equation for x<=L
        x = k.*a.*cos(t) - 1;
        plot(t, x);
        xlabel('t');
        ylabel('x');
        hold on;
        %Equation for -L<=x<=0
        x2=1+t-t;
        plot(t,x2);
        %Equation for L<=x
        x3=k.*b.*cos(t)+1;
        plot(t,x3);
        legend("x<=L","-L<=x<=0","L<=x");
        title("IVP Plot");
    end
end
str = input('Press Any Key to Countinue or 0 to
terminate','s');
end
```

Part-2)

I.V.P

$\hookrightarrow L=1,\ m=1,\ \gamma=0.1,\ k=1$

(damped motion → friction exists)

$\therefore$ eq is..

$$mx'' = -\gamma x' - k(x \pm L)$$

$$x'' = -0.1x' - x \pm 1$$

$\therefore$

$$\underline{x'' + 0.1x' + x = \pm 1}$$

$\therefore$ solving for all cases....

→CASE#1; $x \le -L$) (+1)

eq → $x'' + 0.1x' + x = -1$

(← U.C method using)

aux eq → $m^2 + 0.1m + 1 = 0$ // $m = \dfrac{-0.1 \pm \sqrt{(0.1)^2 - (4)(1)(1)}}{2(1)}$ // quadratic formula...

by Q.F

$$m = -\frac{0.1}{2} \pm \frac{1.989}{2} i$$

$$* \ m = \underbrace{-0.05}_{\alpha} \pm \underbrace{0.994987}_{\beta} i$$

$\therefore$

$$\boxed{\Rightarrow x_{1h} = (e^{-0.05})\{A \sin(0.99t) + B\cos(0.99t)\}}$$

(case1)

→ $g(x) = 1$

$x_p = C$

$x_{1p}' = x_{1p}'' = 0$ $\Rightarrow \underline{x_{1p} = -1}$

→ CASE #2; $-L \le x \le 0$)

$$-\gamma x' = m x''$$
$$x'' = -0.1 x'$$

(S.b.s)
$$\int x''\,dt = -0.1 \int x'\,dt$$
$$x' = -0.1 x$$
$$\boxed{x' + 0.1 x = 0} \rightarrow \text{linear D.E!}$$

$\therefore$ I.F, $\mu(t) = e^{\int 0.1\,dt} = e^{0.1}$

$\therefore$ Xply I.F on b/s of Eq.

$$e^{0.1} x' + 0.1 e^{0.1} x = 0$$
$$\int D_t[e^{0.1} x] = \int 0\,dt$$
$$e^{0.1} x = C$$
$$\boxed{G.S \Rightarrow x_2 = Ce^{-0.1}} \Rightarrow G.S$$

(case 2)

Alternate method!

$$-\gamma x' = m x''$$
$$x'' + 0.1 x' = 0$$

aux eq. $m^2 + 0.1m = 0$

$m(m + 0.1) = 0$

$\therefore m = 0,\ m = -0.1$

$\therefore x_h = C_1 e^0 + C_2 e^{-0.1t}$

$\boxed{x_h = C_1 + C_2 e^{(-0.1)t}}$

$x_c = C$, $g(x) = 0$

$x_c' = 0$ already homogeneous!

$x_c'' = 0$ ... $0 + 0 \Rightarrow 0$

$x_c = 0$

$x_2 = x_h + x_c$

$x_2 = C_1 + C_2 e^{(-0.1)t}$

→ CASE # 3; L ≤ x)

$$x'' + 0.1x' + x = 1$$

U.C method

Similarly to case 1)

$$m = -0.05 \pm 0.99i$$

$\therefore$

$$\Rightarrow x_{sh} = e^{-0.05}\left(D\cos(0.99t) + E\sin(0.99t)\right)$$

$\therefore$

$$x_{sp} = F$$

$$x'_{sp} = x''_{sp} = 0$$

$\therefore$

$$0 + 0 + F = 1$$

$F=1$ $\therefore$ $F \Rightarrow$ $\Rightarrow x_{sp} = 1$

G.S ⇒ solution (case 3)

$$\boxed{x_s = e^{-0.05}\left(D\cos(0.99t) + E\sin(0.99t)\right)}$$

(case 3)

* To plot ⇒ 7

$$x(t) = \begin{cases} e^{-0.05}\left(A\cos(0.99t) + B\sin(0.99t)\right), & x \leq -L \\ C_1 + C_2 e^{t \cdot 0.1t} & , -L \leq x \leq 0 \\ e^{-0.05}\left(D\cos(0.99t) + E\sin(0.99t)\right), & L \leq 0 \end{cases}$$

// → where the variation of the constants A, B, D & E help form level curve allowing contour plots in matlab //

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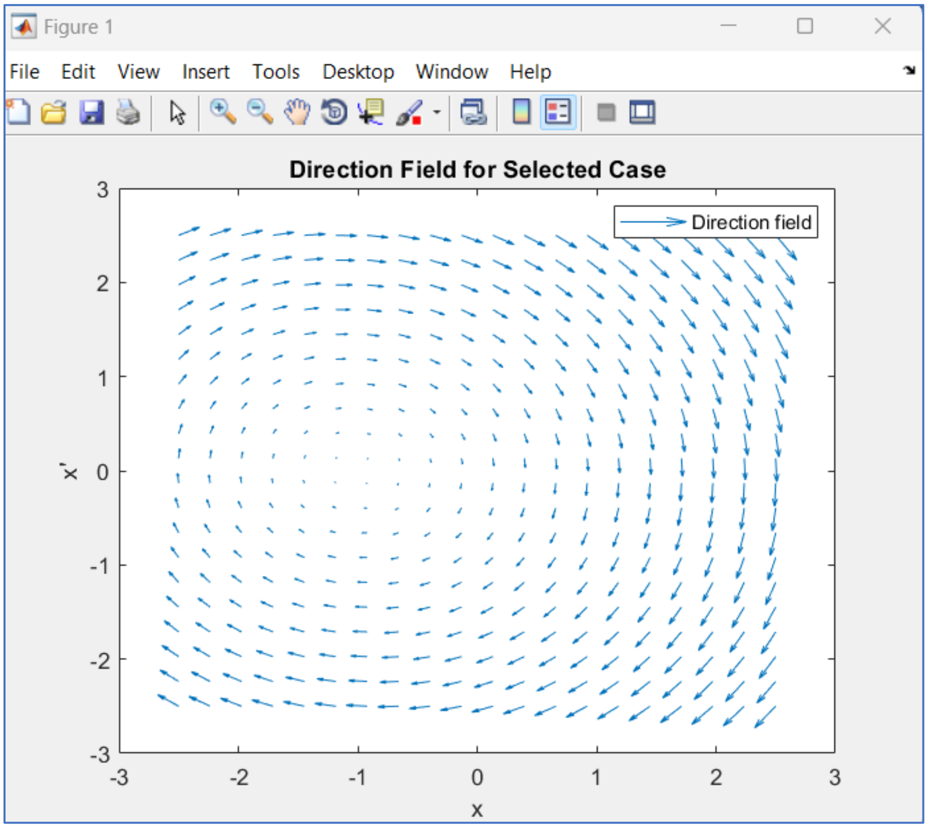
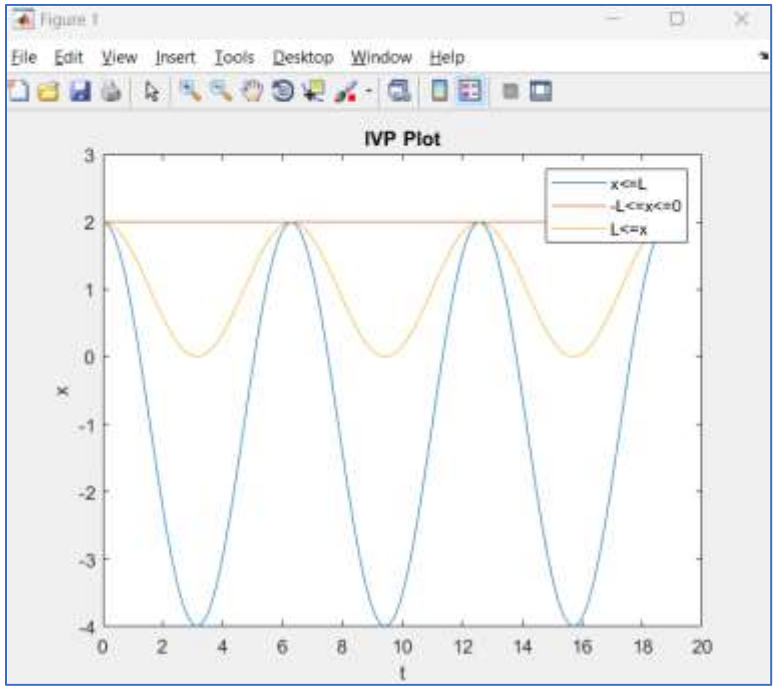
DIRECTION PLOT EXAMPLE(FIRST INTERVAL X<=L)

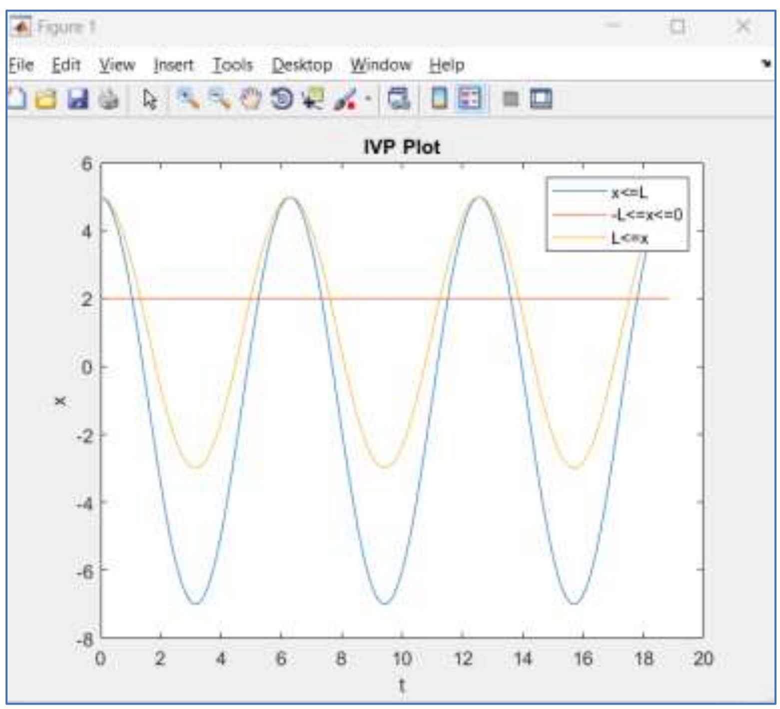


SOLUTION PART E)

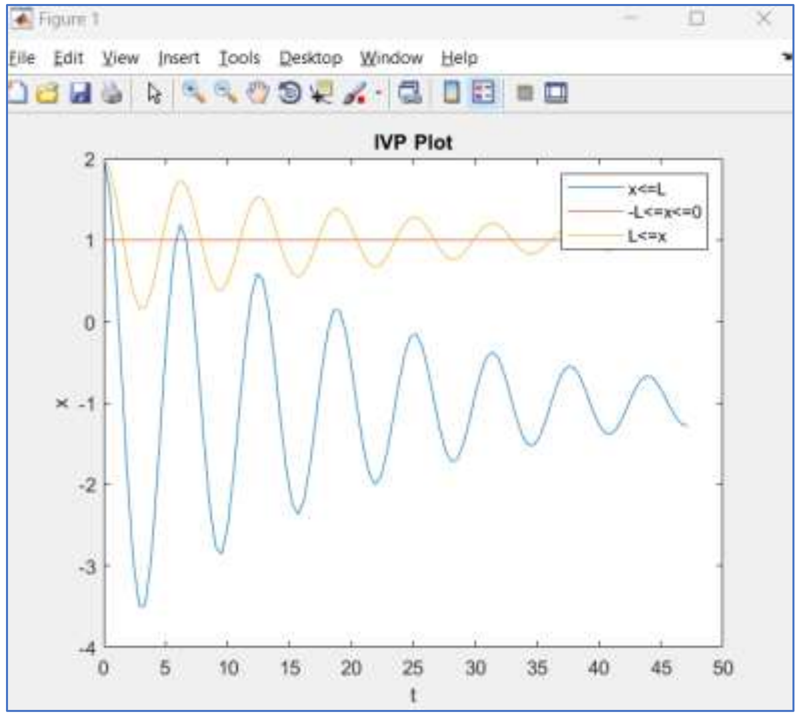
UNDAMPED(x(0) =2,x'(0)=0)



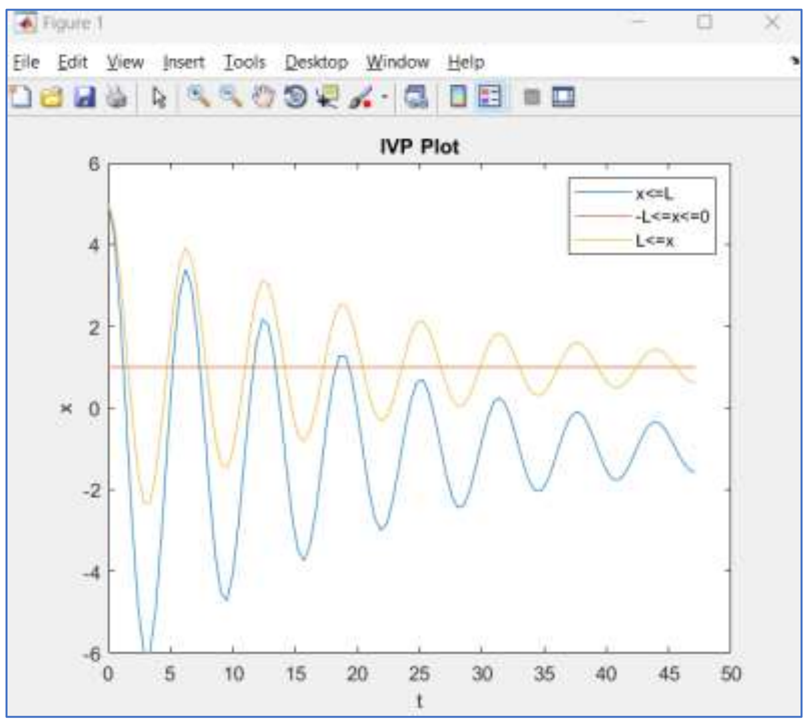
UNDAMPED(x(0) =5,x'(0)=0)



DAMPED(x(0) =2,x’(0)=0)



DAMPED(x(0) =5,x’(0)=0)



CONCLUSION

In completion of the project, we have solved all parts of the given problem statement. From the manually calculated equations to the MATLAB code used to graph vector plots as well as the other graphs for multiple conditions

By the end of this Project, was found that for part **d**) , when *Time t->∞ , the oscillation settles down as x->0 from both sides back and forth till the(amplitude of oscillation of x)->0 .* This shows us how a damped case leads to the block coming back into a state of rest after a good amount of time. In the Undamped – motions we see no change and a constant oscillation showing us how the mass is always in motion.

In Contributions for the work if this project, the work was all done equally with good cooperation, however in a broad aspect we all had our areas of expertise. All in all we found clearing up the small mistakes in the code and calculations particularly tricky when working in depth, here we assigned a person (Moaz) to over view and check on the other two working in depth , in order to maintain a smooth simultaneously checked solution.

Calculations and Solution verification – Tauha Imran (22i-1239)
MATLAB codes and plotting – Hussain Ali Zaidi (22i-0902)
Report & rechecking solutions – Moaz Farooq (22i-1173)