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Discrete St.pptx

Cards & pics from university, YouTube & online sources

Sets you should know ...

$\mathbb{N}$ - Natural — Counting #'s — $\{1, 2, 3, \ldots\}$

$\mathbb{W}$ - Whole — $0, \mathbb{N}$ — $\{0, 1, 2, 3, \ldots\}$

$\mathbb{Z}$ - Integers — $\pm \mathbb{W}$ — $\{\ldots -2, -1, 0, 1, 2 \ldots$

$\mathbb{Q}$ - Rational — $\frac{a}{b}$, $a, b \in \mathbb{Z}$, $b \neq 0$, $\frac{a}{b}$ is in lowest t

$\mathbb{R}$ - Real — $\mathbb{Z}^+$ positive int.

$\mathbb{C}$ - Complex — $\underline{\underline{a}} + \boxed{bi}$

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Set Identities

- Identity laws

$$A \cup \emptyset = A \qquad A \cap U = A$$

- Domination laws

$$A \cup U = U \qquad A \cap \emptyset = \emptyset$$

- Idempotent laws

$$A \cup A = A \qquad A \cap A = A$$

- Complementation law

$$\overline{(\overline{A})} = A$$

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Set Identities

- Commutative law - ORDER

$$A \cup B = B \cup A \qquad A \cap B = B \cap A$$

- Associative laws - GROUPING

$$A \cup (B \cup C) = (A \cup B) \cup C \rightarrow \text{any}$$

$$A \cap (B \cap C) = (A \cap B) \cap C \rightarrow \text{each}$$

- Distributive laws

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

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Set Identities

- De Morgan's laws

$$\overline{A \cup B} = \overline{A} \cap \overline{B}$$

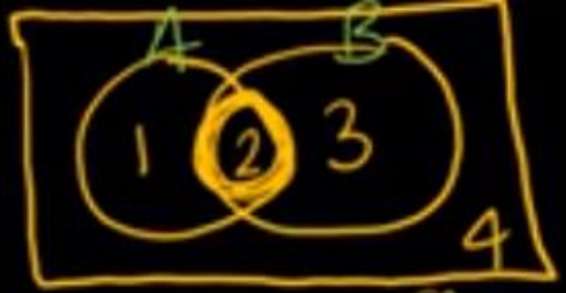
$$\overline{A \cap B} = \overline{A} \cup \overline{B}$$

$$\overline{A \cap B} = \{1, 3, 4\}$$

$$\overline{A} = \{\cancel{3}, 4\}$$

$$\overline{B} = \{\cancel{1}, 4\}$$

$$\overline{A} \cup \overline{B} = \{1, 3, 4\}$$



$A = \{1, 2\}$

$B = \{2, 3\}$

$U = \{1, \cancel{2}, \cancel{3}, 4\}$

- Absorption laws

$$A \cup (A \cap B) = A$$

$$A \cap (A \cup B) = A$$

$$\{1, 2\} \cup \{2\} = \{1, 2\} = A$$

- Complement laws

$$A \cup \overline{A} = U$$

$$A \cap \overline{A} = \emptyset$$

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More on Subsets

Show $A \subseteq B$ Show every element of A belongs to B

if $x \in A$ then $x \in B$

Show $A \nsubseteq B$ $\exists x \in A \rightarrow x \notin B$

Show x belongs to A but not set B

Show $A \subseteq B$ and $B \subseteq A$ $\boxed{A = B}$ ☆

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Floor Function $f(x) = \lfloor x \rfloor$

Largest integer less than or equal to x.

Ceiling Function $f(x) = \lceil x \rceil$

Smallest integer greater than or equal to x.

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Factorial Function

$f: \mathbb{N} \to \mathbb{Z}^+$ denoted by $\boxed{f(n) = n!}$ is the product of the first n positive integers when n is a non-negative integer.

$$f(n) = n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 3 \cdot 2 \cdot 1$$

Stirling's Formula: $$n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

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Methods of Identity Proof:

1. Prove each set in the identity is a subset of the other.
2. Use propositional logic – 2 column
3. Use a membership table showing the same combination of sets do or don't belong to the identity.

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Generalized Union and Intersection

$$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \dots \cup A_n$$

$$\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap \dots \cap A_n$$

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CONSTRUCTING COMPOUND PROPOSITIONS

A COMPOUND PROPOSITION IS COMPRISED OF PROPOSITIONS AND ONE OR MORE OF THE FOLLOWING CONNECTIVES:

1
- NEGATION $\neg$ "NOT"
- CONJUNCTION $\wedge$ "AND"
- DISJUNCTION $\vee$ "OR"

2
- IMPLICATION $\rightarrow$ "IF, THEN"
- BICONDITIONAL $\leftrightarrow$ "IF AND ONLY IF" IFF

EACH PROPOSITION IS REPRESENTED BY A PROPOSITIONAL VARIABLE ($p, q, r, s \ldots$).

EXAMPLE: $p \rightarrow q$

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CONVERSE, INVERSE, CONTRAPOSITIVE

FROM OUR IMPLICATION, $\boxed{p \to q}$, WE CAN CONSTRUCT 3 NE[W]
CONDITIONAL STATEMENTS.

- CONVERSE $q \to p$ SWITCH ORDER
- INVERSE $\neg p \to \neg q$ NEGATE
- ★ CONTRAPOSITIVE $\neg q \to \neg p$ SWITCH AND NEGATE

SAME TRUTH VALUE AS $p \to q$

★ IT IS RAINING IS A SUFFICIENT CONDITION FOR ME NOT GOING TO TOWN.

$\boxed{\text{IF}}$ IT IS RAINING, $\boxed{\text{THEN}}$ I WON'T GO TO TOWN

$\neg p$ $\neg q$

$p \to q$

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LOGICAL EQUIVALENCES

T - tautology

F - contradiction

IDENTITY LAWS

$p \wedge T \equiv p$

$p \vee F \equiv p$

DOMINATION LAWS

$p \vee T \equiv T$

$p \wedge F \equiv F$

INDEMPOTENT LAWS

$p \vee p \equiv p$

$p \wedge p \equiv p$

DOUBLE NEGATION LAW

$\neg(\neg p) \equiv p$

ABSORBTION LAWS

$p \vee (p \wedge q) \equiv p$

$p \wedge (p \vee q) \equiv p$

NEGATION LAWS

$p \vee \neg p \equiv T$

$p \wedge \neg p \equiv F$

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MORE LOGICAL EQUIVALENCES

COMMUTATIVE LAWS - ORDER

$$p \vee q \equiv q \vee p$$

$$p \wedge q \equiv q \wedge p$$

DISTRIBUTIVE LAWS

$$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$$

$$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$$

ASSOCIATIVE LAWS - GROUPING

$$(p \vee q) \vee r \equiv p \vee (q \vee r)$$

$$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$$

DE MORGAN'S LAWS

$$\neg (p \wedge q) \equiv \neg p \vee \neg q$$

$$\neg (p \vee q) \equiv \neg p \wedge \neg q$$

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A FEW MORE EQUIVALENCES

$p \rightarrow q \equiv \neg p \vee q$

$p \rightarrow q \equiv \neg q \rightarrow \neg p$

$p \vee q \equiv \neg p \rightarrow q$

$p \wedge q \equiv \neg(p \rightarrow \neg q)$

$(p \rightarrow q) \wedge (p \rightarrow r) \equiv p \rightarrow (q \wedge r)$

$(p \rightarrow r) \wedge (q \rightarrow r) \equiv (p \wedge q) \rightarrow r$

$(p \rightarrow q) \vee (p \rightarrow r) \equiv p \rightarrow (q \vee r)$

$(p \rightarrow r) \vee (q \rightarrow r) \equiv (p \wedge q) \rightarrow r$

$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$

$p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$

$p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$

$\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$

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TABLE 2.3.1 Valid Argument Forms

Modus Ponens	$p \to q$ p $\therefore q$	**Elimination**	**a.** $p \vee q$ $\sim q$ $\therefore p$ **b.** $p \vee q$ $\sim p$ $\therefore q$
Modus Tollens	$p \to q$ $\sim q$ $\therefore \sim p$	**Transitivity**	$p \to q$ $q \to r$ $\therefore p \to r$
Generalization	**a.** p $\therefore p \vee q$ **b.** q $\therefore p \vee q$	**Proof by Division into Cases**	$p \vee q$ $p \to r$ $q \to r$ $\therefore r$
Specialization	**a.** $p \wedge q$ $\therefore p$ **b.** $p \wedge q$ $\therefore q$		
Conjunction	p q $\therefore p \wedge q$	**Contradiction Rule**	$\sim p \to \mathbf{c}$ $\therefore p$

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Theorem 2.1.1 Logical Equivalences

Given any statement variables p, q, and r, a tautology $\mathbf{t}$ and a contradiction $\mathbf{c}$, the following logical equivalences hold.

1. *Commutative laws:*	$p \wedge q \equiv q \wedge p$	$p \vee q \equiv q \vee p$
2. *Associative laws:*	$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	$(p \vee q) \vee r \equiv p \vee (q \vee r)$
3. *Distributive laws:*	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
4. *Identity laws:*	$p \wedge \mathbf{t} \equiv p$	$p \vee \mathbf{c} \equiv p$
5. *Negation laws:*	$p \vee \sim p \equiv \mathbf{t}$	$p \wedge \sim p \equiv \mathbf{c}$
6. *Double negative law:*	$\sim(\sim p) \equiv p$	
7. *Idempotent laws:*	$p \wedge p \equiv p$	$p \vee p \equiv p$
8. *Universal bound laws:*	$p \vee \mathbf{t} \equiv \mathbf{t}$	$p \wedge \mathbf{c} \equiv \mathbf{c}$
9. *De Morgan's laws:*	$\sim(p \wedge q) \equiv \sim p \vee \sim q$	$\sim(p \vee q) \equiv \sim p \wedge \sim q$
10. *Absorption laws:*	$p \vee (p \wedge q) \equiv p$	$p \wedge (p \vee q) \equiv p$
11. *Negations of* **t** *and* **c**:	$\sim\mathbf{t} \equiv \mathbf{c}$	$\sim\mathbf{c} \equiv \mathbf{t}$

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1. Commutative Law:

$$p \leftrightarrow q \equiv q \leftrightarrow p$$

2. Implication Laws:

$$p \rightarrow q \equiv \sim p \vee q$$
$$\equiv \sim(p \wedge \sim q)$$

3. Exportation Law:

$$(p \wedge q) \rightarrow r \equiv p \rightarrow (q \rightarrow r)$$

4. Equivalence:

$$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$$

5. Reductio ad absurdum:

$$p \rightarrow q \equiv (p \wedge \sim q) \rightarrow c$$

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DE MORGAN'S LAWS FOR QUANTIFIERS

$$\boxed{\neg \exists_x P(x) \equiv \forall x \neg P(x)}$$

- TRUE WHEN P(x) IS FALSE FOR EVERY X
- FALSE WHEN THERE IS AN X FOR WHICH P(x) IS TRUE

$$\boxed{\neg \forall x P(x) \equiv \exists x \neg P(x)}$$

- TRUE WHEN THERE IS AN X FOR WHICH P(x) IS FALSE
- FALSE WHEN P(x) IS TRUE FOR EVERY X

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Definition: Relation

Let A and B be sets. A (binary) **relation from $\boldsymbol{A}$ to $\boldsymbol{B}$** is a subset of $\boldsymbol{A} \times \boldsymbol{B}$.

Given an ordered pair (x, y) in $A \times B$, $\boldsymbol{x}$ **is related to** $\boldsymbol{y}$ **by** $\boldsymbol{R}$, or $\boldsymbol{x}$ **is** $\boldsymbol{R}$**-related to** $\boldsymbol{y}$, written $\boldsymbol{x\ R\ y}$, iff $(\boldsymbol{x}, \boldsymbol{y}) \in \boldsymbol{R}$.

Definitions: Domain, Co-domain, Range

Let A and B be sets and R be a relation from A to B.

The **domain** of R, $Dom(R)$, is the set $\{a \in A : aRb \text{ for some } b \in B\}$.

The **co-domain** of R, $coDom(R)$, is the set B.

The **range** of R, $Range(R)$, is the set $\{b \in B: aRb \text{ for some } a \in A\}$.

Definition: Inverse of a Relation

Let R be a relation from A to B. Define the **inverse relation** R^{-1} from B to A as follows:

$$R^{-1} = \{(y, x) \in B \times A : (x, y) \in R\}.$$

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Definition: Composition of Relations

Let A, B and C be sets. Let $R \subseteq A \times B$ be a relation. Let $S \subseteq B \times C$ be a relation. The **composition of R with S**, denoted $S \circ R$, is the relation from A to C such that:

$$\forall x \in A, \forall z \in C \left(x \, S \circ R \, z \Leftrightarrow (\exists y \in B \, (xRy \wedge ySz))\right).$$

Proposition: Composition is Associative

Let A, B, C, D be sets. Let $R \subseteq A \times B, S \subseteq B \times C$ and $T \subseteq C \times D$ be relations.

$$T \circ (S \circ R) = (T \circ S) \circ R = T \circ S \circ R$$

Proposition: Inverse of Composition

Let A, B and C be sets. Let $R \subseteq A \times B$ and $S \subseteq B \times C$ be relations.

$$(S \circ R)^{-1} = R^{-1} \circ S^{-1}$$

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Definition: n-ary Relation

Given n sets $A_1, A_2, \cdots, A_n$, an $\boldsymbol{n}$**-ary relation** $\boldsymbol{R}$ on $A_1 \times A_2 \times \cdots \times A_n$ is a subset of $A_1 \times A_2 \times \cdots \times A_n$.

The special cases of 2-ary, 3-ary and 4-ary relations are called **binary**, **ternary** and **quaternary relations** respectively.

Definitions: Reflexivity, Symmetry, Transitivity

Let R be a relation on a set A.

1. R is **reflexive** iff $\forall x \in A\ (xRx)$.
2. R is **symmetric** iff $\forall x, y \in A\ (xRy \Rightarrow yRx)$.
3. R is **transitive** iff $\forall x, y, z \in A\ (xRy \wedge yRz \Rightarrow xRz)$.

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Proof of Reflexivity:

1. Let a be an arbitrarily chosen integer.
2. Now $a - a = 0$.
3. But $3 \mid 0$ (since $0 = 3 \cdot 0$), hence, $3 \mid (a - a)$.
4. Therefore, $a\ R\ a$ (by the definition of R).

Proof of Symmetry:

1. Let a and b be arbitrarily chosen integers that satisfy $a\ R\ b$.
2. Then $3|(a - b)$ (by the definition of R), hence $a - b = 3k$ for some integer k (by the definition of divisibility).
3. Multiplying both sides by -1 gives $b - a = 3(-k)$.
4. Since $-k$ is an integer, $3|(b - a)$ (by definition of divisibility).
5. Therefore $b\ R\ a$ (by the definition of R).

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Proof of Transitivity:

1. Let a, b and c be arbitrarily chosen integers that satisfy $a \mathrel{R} b$ and $b \mathrel{R} c$.
2. Then $3|(a-b)$ and $3|(b-c)$ (by the definition of R), hence $a-b=3r$ and $b-c=3s$ for some integers r and s (by the definition of divisibility).
3. Adding both equations gives $a-c=3(r+s)$.
4. Since $r+s$ is an integer, $3|(a-c)$ (by definition of divisibility).
5. Therefore $a \mathrel{R} c$ (by the definition of R).

Definition: Transitive Closure

Let A be a set and R a relation on A. The **transitive closure** of R is the relation R^t on A that satisfies the following three properties:

1. R^t is transitive.
2. $R \subseteq R^t$.
3. If S is any other transitive relation that contains R, then $R^t \subseteq S$.

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8.3 Equivalence Relations

Definition: Relation Induced by a Partition

Given a partition $\mathcal{C}$ of a set A, the relation R **induced by the partition** is defined on A as follows: $\forall x, y \in A$,

$$xRy \Leftrightarrow \exists \text{ a component } S \text{ of } \mathcal{C} \text{ s.t. } x, y \in S.$$

Definition: Equivalence Relation

Let A be a set and R a relation on A. R is an **equivalence relation** iff R is reflexive, symmetric and transitive.

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The Relation Induced by a Partition

- Let A be a set with a partition and let R be the relation induced by the partition. Then R is reflexive, symmetric, and transitive.

Proof: Suppose A is a set with a partition (finite): $A_1, A_2, \ldots, A_n$

$A_i \cap A_j \neq \emptyset$ whenever $i \neq j$ and $A_1 \cup A_2 \cup \cdots \cup A_n = A$.

For all $x, y \in A$, $x\,R\,y \Leftrightarrow$ there is a set A_i of the partition such that $x \in A_i$ and $y \in A_i$.

Proof that R is reflexive: Suppose $x \in A$. Since $A_1, A_2, \ldots, A_n$ is a partition of A, $A_1 \cup A_2 \cup \cdots \cup A_n = A$, it follows that $x \in A_i$ for some i.

There is a set A_i of the partition such that $x \in A_i$.

By definition of R, $x\,R\,x$.

The Relation Induced by a Partition

Proof that R is symmetric: Suppose x and y are elements of A such that x R y. Then there is a subset A_i of the partition such that $x \in A_i$ and $y \in A_i$ by definition of R. It follows that the statement there is a subset A_i of the partition such that $y \in A_i$ and $x \in A_i$ is also true. By definition of R, y R x.

The Relation Induced by a Partition

Proof that R is transitive: Suppose x, y, and z are in A and xRy and yRz. By definition of R, there are subsets A_i and A_j of the partition such that x and y are in A_i and y and z are in A_j.

Suppose $A_i \neq A_j$. [We will deduce a contradiction.] Then $A_i \cap A_j = \emptyset$ since $\{A_1, A_2, A_3, \ldots, A_n\}$ is a partition of A. But y is in A_i and y is in A_j also. Hence $A_i \cap A_j \neq \emptyset$. [This contradicts the fact that $A_i \cap A_j = \emptyset$.] Thus $A_i = A_j$. It follows that x, y, and z are all in A_i, and so in particular, x and z are in A_i.

Thus, by definition of R, x R z.

Equivalence Classes of an Equivalence Relation

Definition

Suppose A is a set and R is an equivalence relation on A. For each element a in A, the **equivalence class of a**, denoted **[a]** and called the **class of a** for short, is the set of all elements x in A such that x is related to a by R.

In symbols:

$$[a] = \{x \in A \mid x \, R \, a\}$$

The procedural version of this definition is

$$\boxed{\text{for every } x \in A, \quad x \in [a] \quad \Leftrightarrow \quad x \, R \, a.}$$

- Let A be a set and R an equivalence relation on A.

For any a and b elements of A, if a R b, then [a] = [b].

Proof: [a] = [b] ⇔ [a] ⊆ [b] and [b] ⊆ [a].

1. [a] ⊆ [b]

   Let x ∈ [a] iff then x R a.

   a R b by hypothesis ➔ by transitivity of R, x R b ➔ x ∈ [b]

2. [b] ⊆ [a]

   Let x ∈ [b] iff then x R b.

   b R a by hypothesis and symmetry ➔ by transitivity of R, xRa ➔ x ∈ [a]

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Definition

Suppose R is an equivalence relation on a set A and S is an equivalence class of R. A **representative** of the class S is any element a such that $[a] = S$.

Definition

Let m and n be integers and let d be a positive integer. We say that ***m* is congruent to *n* modulo *d*** and write

$$\boldsymbol{m \equiv n \pmod{d}}$$

if, and only if, $d \mid (m - n)$.

Symbolically: $m \equiv n (\text{mod } d) \quad \Leftrightarrow \quad d \mid (m - n).$

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5.1 Sequences

A mathematician, like a painter or poet, is a maker of patterns.
—G. H. Hardy, *A Mathematician's Apology*, 1940

Imagine that a person decides to count his ancestors. He has two parents, four grandparents, eight great-grandparents, and so forth. These numbers can be written in a row as

$$2, 4, 8, 16, 32, 64, 128, \ldots$$

Note The symbol "..." is called an *ellipsis*. It is shorthand for "and so forth."

To express the pattern of the numbers, suppose that each is labeled by an integer giving its position in the row.

Position in the row	1	2	3	4	5	6	7 ...
Number of ancestors	2	4	8	16	32	64	128 ...

Note Strictly speaking, the true value of A_k is less than 2^k when k is large, because ancestors from one branch of the family tree may also appear on other branches of the tree.

The number corresponding to position 1 is 2, which equals 2^1. The number corresponding to position 2 is 4, which equals 2^2. For positions 3, 4, 5, 6, and 7, the corresponding numbers are 8, 16, 32, 64, and 128, which equal 2^3, 2^4, 2^5, 2^6, and 2^7, respectively. For a general value of k, let A_k be the number of ancestors in the kth generation back. The pattern of computed values strongly suggests the following for each k:

$$A_k = 2^k.$$

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Definition: Geometric Sequence

A sequence $a_0, a_1, a_2, \cdots$ is called a **geometric sequence** (or **geometric progression**) iff there is a constant r such that

$$a_k = r a_{k-1} \qquad \text{for all integers } k \geq 1.$$

If follows that,

$$a_n = a_0 r^n \qquad \text{for all integers } n \geq 0.$$

Summing a geometric sequence of n terms $(r \neq 1)$,

$$\sum_{k=0}^{n-1} a_k = a_0 \left(\frac{1 - r^n}{1 - r} \right)$$

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Definition: Arithmetic Sequence

A sequence $a_0, a_1, a_2, \cdots$ is called an **arithmetic sequence** (or **arithmetic progression**) iff there is a constant d such that

$$a_k = a_{k-1} + d \qquad \text{for all integers } k \geq 1.$$

If follows that,

$$a_n = a_0 + dn \qquad \text{for all integers } n \geq 0.$$

Summing an arithmetic sequence of n terms:

$$\sum_{k=0}^{n-1} a_k = \frac{n}{2}(2a_0 + (n-1)d)$$

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Definition

If m and n are integers and $m \leq n$, the symbol $\sum_{k=m}^{n} a_k$, read the **summation from k equals m to n of a-sub-k**, is the sum of all the terms $a_m, a_{m+1}, a_{m+2}, \ldots, a_n$. We say that $a_m + a_{m+1} + a_{m+2} + \cdots + a_n$ is the **expanded form** of the sum, and we write

$$\sum_{k=m}^{n} a_k = a_m + a_{m+1} + a_{m+2} + \cdots + a_n.$$

We call k the **index** of the summation, m the **lower limit** of the summation, and n the **upper limit** of the summation.

Definition: Product

If m and n are integers, $m \leq n$, the symbol

$$\prod_{k=m}^{n} a_k$$

is the **product** of all the terms $a_m, a_{m+1}, a_{m+2}, \cdots, a_n$.

We write

$$\prod_{k=m}^{n} a_k = a_m \cdot a_{m+1} \cdot a_{m+2} \cdot \cdots \cdot a_n.$$

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Definition

Let n and r be integers with $0 \le r \le n$. The symbol

$$\binom{n}{r}$$

is read "**n choose r**" and represents the number of subsets of size r that can be chosen from a set with n elements.

Formula for Computing $\binom{n}{r}$

For all integers n and r with $0 \le r \le n$,

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}.$$

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Method of Proof by Mathematical Induction

Consider a statement of the form, "For all integers $n \geq a$, a property $P(n)$ is true." To prove such a statement, perform the following two steps:

Step 1 (basis step): Show that $P(a)$ is true.

Step 2 (inductive step): Show that for all integers $k \geq a$, if $P(k)$ is true then $P(k+1)$ is true. To perform this step,

suppose that $P(k)$ is true, where k is any particular but arbitrarily chosen integer with $k \geq a$.

[This suppposition is called the **inductive hypothesis.***]*

Then

show that $P(k+1)$ is true.

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Principle of Strong Mathematical Induction

Let $P(n)$ be a property that is defined for integers n, and let a and b be fixed integers with $a \leq b$. Suppose the following two statements are true:

1. $P(a), P(a+1), \ldots,$ and $P(b)$ are all true. **(basis step)**
2. For any integer $k \geq b$, if $P(i)$ is true for all integers i from a through k, then $P(k+1)$ is true. **(inductive step)**

Then the statement

$$\text{for all integers } n \geq a, \ P(n)$$

is true. (The supposition that $P(i)$ is true for all integers i from a through k is called the **inductive hypothesis.** Another way to state the inductive hypothesis is to say that $P(a), P(a+1), \ldots, P(k)$ are all true.)

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Mathematical Induction

To prove $P(x)$ is true for $x \in \mathbb{Z}^+$, where $P(x)$ is a propositional function, we complete two steps:

1) Basis step - verify $P(1)$ is true
2) Inductive step - verify if $P(k)$ is true, then $P(k+1)$ is true $\forall k \in \mathbb{Z}^+$

Inductive hypothesis: $P(k)$ is true

Must show: $P(k) \rightarrow P(k+1)$

Conclusion: $P(x)$ is true $\forall\ k \in \mathbb{Z}^+$

Back to Mathematical Induction

Let $\mathrm{P}(n)$ denote an open mathematical statement that involves one or more occurrences of the variable n, which represents a positive integer:

If $P(1)$ is true; and

If whenever $P(k)$ is true (for some particular, but arbitrarily chosen $k \in \mathbb{Z}^+$), then $P(k+1)$ is true;

then $\mathrm{P}(n)$ is true for all $n \in \mathbb{Z}^+$.

Keep in mind that $P(1)$ just represents the "least element"

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Recurrence Relation:

It defines later terms in sequence by reference to earlier terms.

5.6 Defining Sequences Recursively

5.7 Solving Recurrence Relations by Iteration

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Definition

A **recurrence relation** for a sequence $a_0, a_1, a_2, \cdots$ is a formula that relates each term a_k to certain of its predecessors $a_{k-1}, a_{k-2}, \cdots, a_{k-i}$, where i is an integer with $k - i \geq 0$.

If i is a fixed integer , the **initial conditions** for such a recurrent relation specify the values of $a_0, a_1, a_2, \cdots, a_{i-1}$.

If i depends on k, the initial conditions specify the values of $a_0, a_1, a_2, \cdots, a_m$, where m is an integer with $m \geq 0$.

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Recurrence Relations

Recall from our previous study of recursive functions and sets, that a recursive definition of a sequence $\{a_n\}$ specifies one (or more) initial terms and a rule (or equation) for determining subsequent terms using preceding terms. The rule/equation is called a recurrence relation.

The solution to a recurrence relation is a sequence whose terms satisfy the recurrence relation.

We may also wish to find the explicit function, which defines the terms of the solution (sequence) in terms of n. This will be our focus in subsequence sections of this chapter.

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5.7 Solving Recurrence Relations by Iteration

Definition

- **Goal:** Given a recursively defined sequence, we want to find an **explicit formula**.
- The basic method for finding an explicit formula is **iteration**: given a sequence $a_0, a_1, \ldots$ defined recursively, you start from the initial conditions and calculate successive terms of the sequence until you see a pattern developing. At that point you guess an explicit formula.

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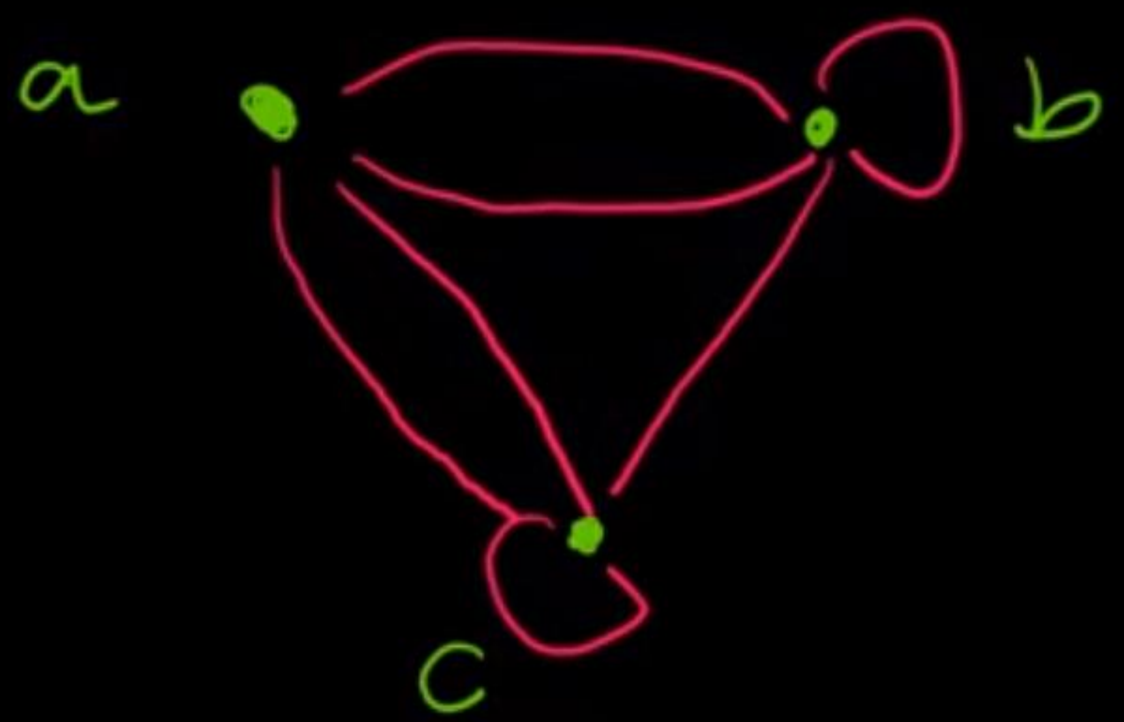
Some Terminology

Multigraph
- multiple edges connecting the same two vertices

Loop:
- An edge that connects a vertex to itself.

Pseudograph:
- may include loops as well as multiple edges.



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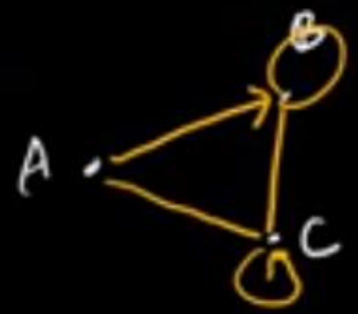
Summary

Type	Directed edges?	Multiple Edges Allowed?	Loops Allowed?
Simple Graph	NO	NO	NO
Multigraph	NO	YES	NO
Pseudograph	NO	YES	YES
Simple directed graph	YES	NO	NO
Directed Multigraph	YES	YES	NO
Mixed graph	BOTH	YES	YES



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Definitions: Walk, Trail, Path, Closed Walk, Circuit, Simple Circuit

	Repeated Edge?	Repeated Vertex?	Starts and Ends at the Same Point?	Must Contain at Least One Edge?
Walk	allowed	allowed	allowed	no
Trail	no	allowed	allowed	no
Path	no	no	no	no
Closed walk	allowed	allowed	yes	no
Circuit	no	allowed	yes	yes
Simple circuit	no	first and last only	yes	yes

Susanna Epp's book	Others
Graph	Multigraph
Simple graph	Graph
Vertex	Node
Edge	Arc
Trail	Path
Path	Simple path
Simple circuit	Cycle

Definition: Simple Graph

A **simple graph** is an undirected graph that does not have any loops or parallel edges. (That is, there is at most one edge between each pair of distinct vertices.)

Definition: Connectedness

Two vertices v and w of a graph $G=(V,E)$ are **connected** if and only if there is a walk from v to w.

The graph *G* is connected if and only if given *any* two vertices v and w in G, there is a walk from v to w. Symbolically,

G is connected iff $\forall$ vertices $v, w \in V$, $\exists$ a walk from v to w.

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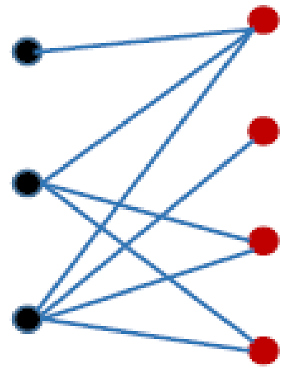
Definition: Bipartite Graph

A **bipartite graph** (or bigraph) is a simple graph whose vertices can be divided into two disjoint sets U and V such that every edge connects a vertex in U to one in V.

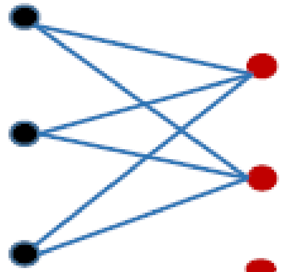


Bipartite graph

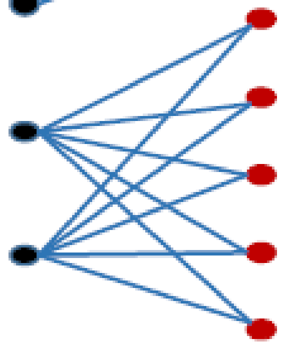
Definition: Complete Bipartite Graph

A **complete bipartite graph** is a bipartite graph on two disjoint sets U and V such that every vertex in U connects to every vertex in V.

If $|U| = m$ and $|V| = n$, the complete bipartite graph is denoted as $K_{m,n}$.



$K_{3,2}$



$K_{2,5}$

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Theorem 10.1.1 The Handshake Theorem

If G is any graph, then the sum of the degrees of all the vertices of G equals twice the number of edges of G. Specifically, if the vertices of G are $v_1, v_2, \ldots, v_n$, where $n \geq 0$, then

$$\begin{aligned} \text{The total degree of } G &= \deg(v_1) + \deg(v_2) + \ldots + \deg(v_n) \\ &= 2 \times (\text{the number of edges of } G). \end{aligned}$$

Definition: Degree of a Vertex and Total Degree of an Undirected Graph

Let G be a undirected graph and v a vertex of G. The **degree** of v, denoted $\mathbf{deg}(\boldsymbol{v})$, equals the number of edges that are incident on v, with an edge that is a loop counted twice.

The **total degree of** $\boldsymbol{G}$ is the sum of the degrees of all the vertices of G.

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For Undirected Graphs

adjacent vertices (neighbors)

incident

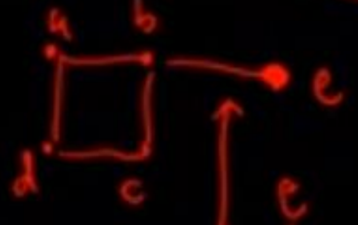
neighborhood $N(e) = \{a, b, c\}$

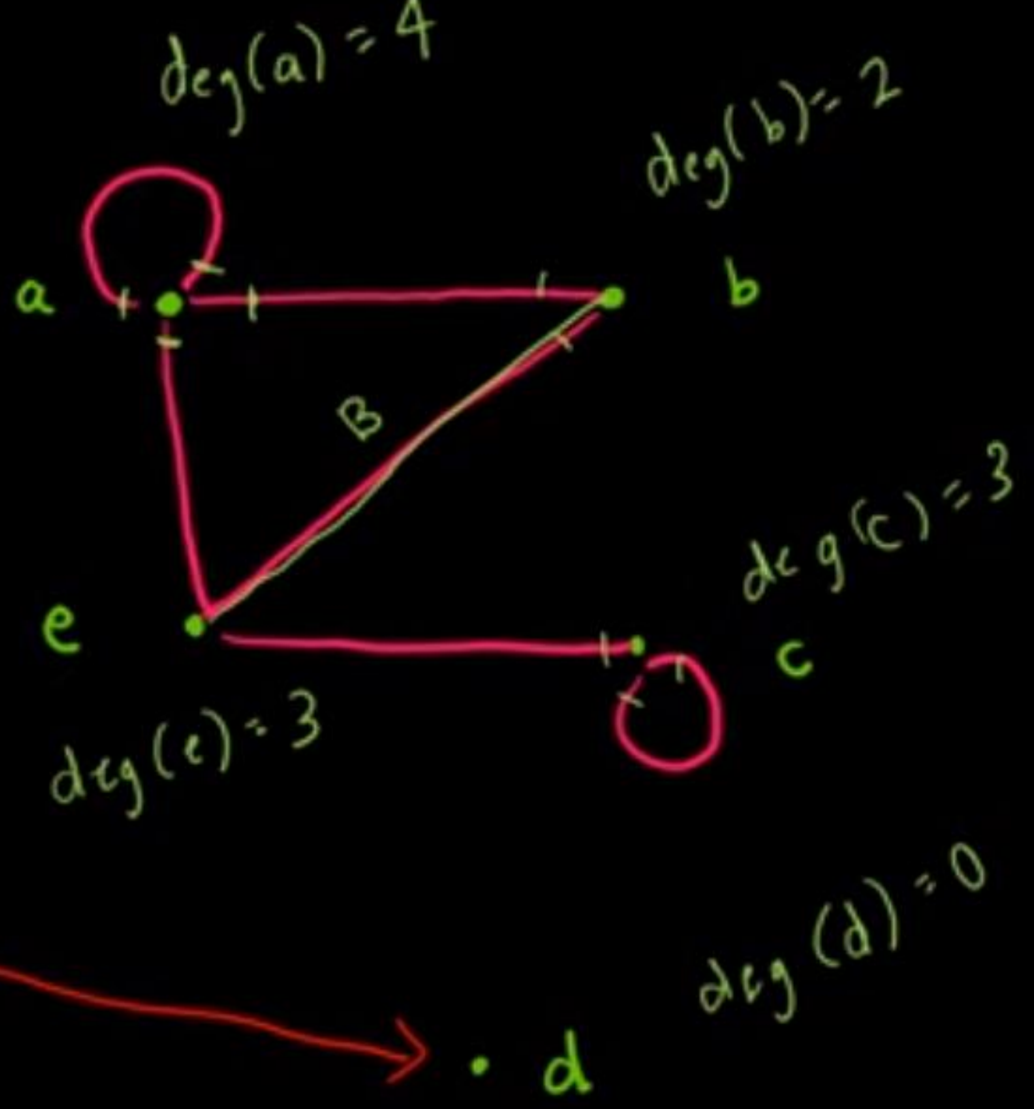
$$\boxed{N(A) = \bigcup_{v \in A} N(v)}$$

Degree

Isolated

Pendant





For Directed Graphs

$\{(a,a), \boxed{(a,b)}, \cancel{(b,a)} \ldots$

$v = \deg^-(b) = 2$

$\deg^+(b) = 1$

adjacent to a is adjacent to b

adjacent from b is adjacent from a

initial vertex a

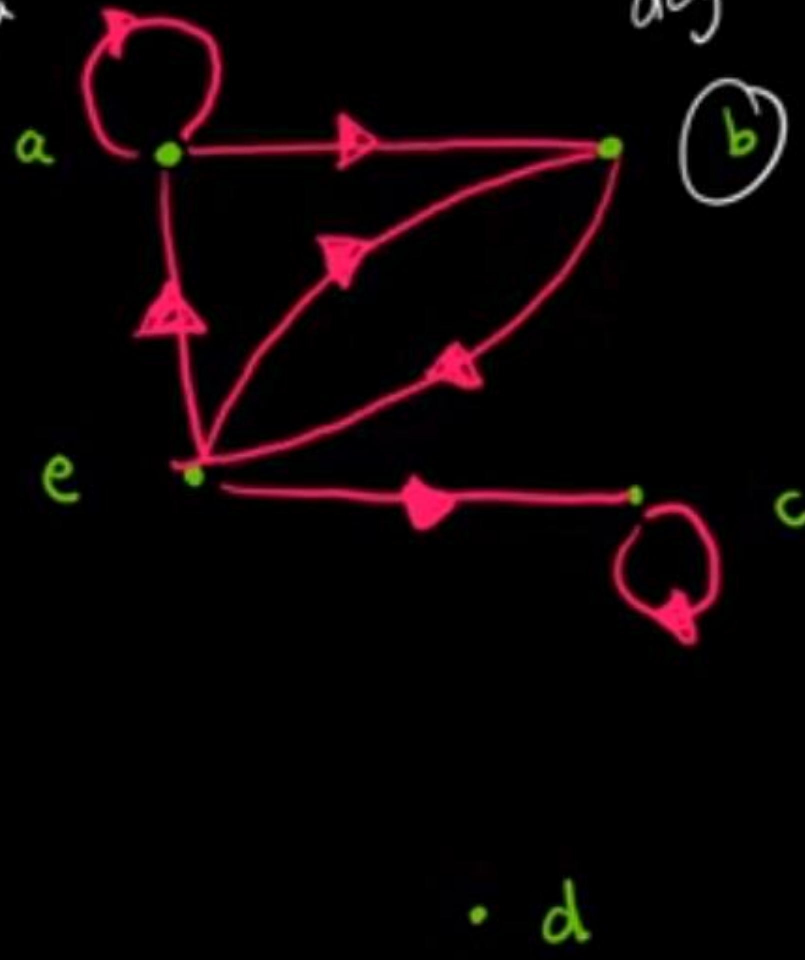
terminal / end vertex b

in-degree of vertex $\boxed{v = \deg^-(v)}$

out-degree of vertex $\boxed{v = \deg^+(v)}$

$$\boxed{\sum_{v \in V} \deg^-(v) = \sum_{v \in V} \deg^+(v) = |E|}$$

# of edges



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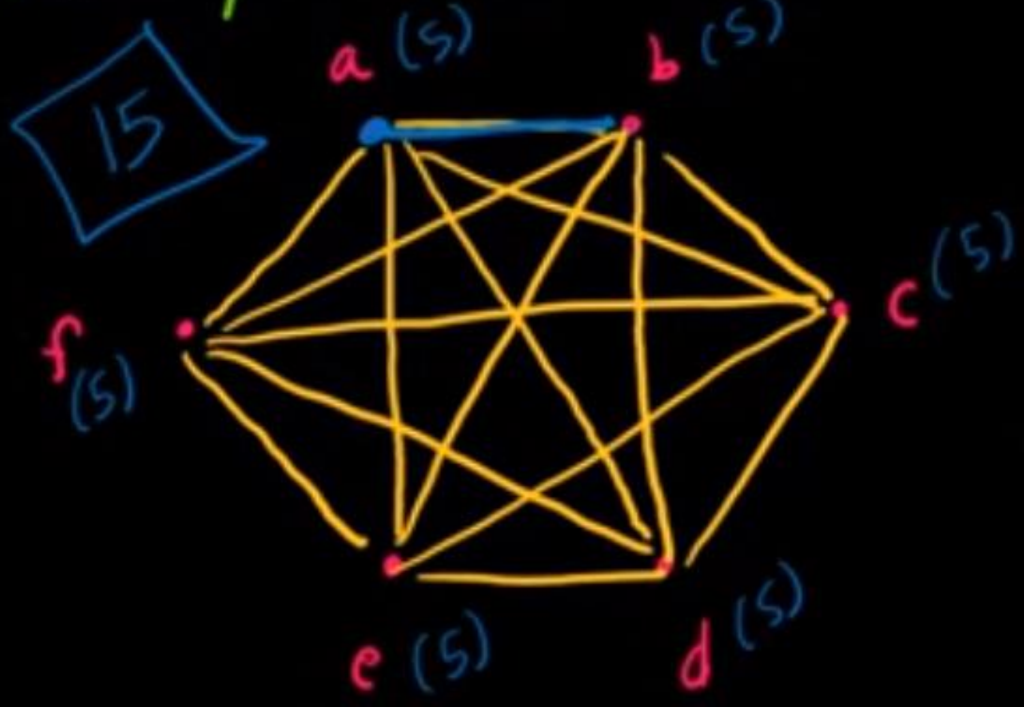
The Hand-shaking Theorem

(and what does it have to do with graph theory)

Suppose there are 6 people in a room, and each must shake hands with every other person. How many handshakes?



$G = (V, E)$ with $\underline{m}$ edges

$$2m = \sum_{v \in V} \deg(v)$$

How many edges are there if you have 10 vertices, each of degree 6?

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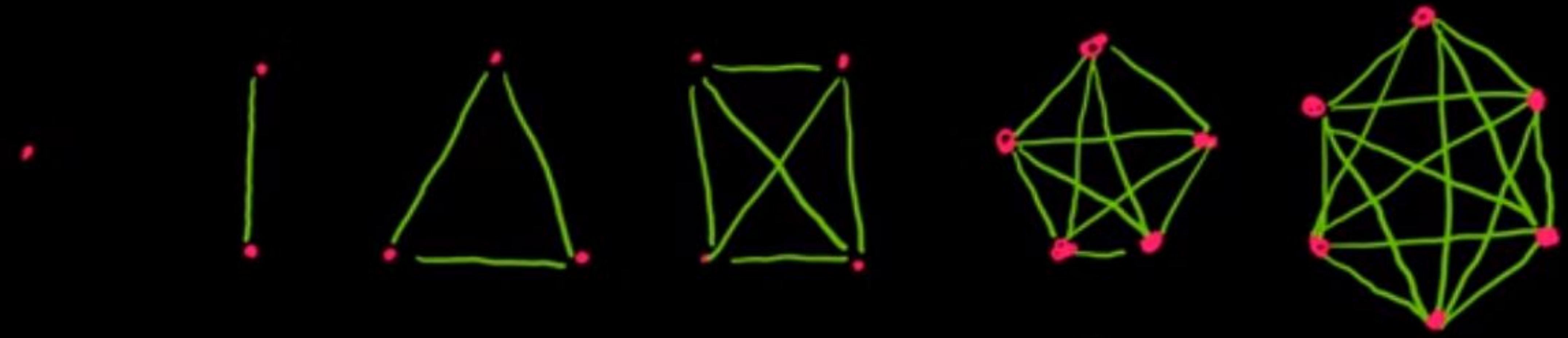
Complete Graphs

A complete graph, denoted $\boxed{K_n}$, is a simple graph that contains exactly one edge between each pair of n distinct vertices.



K_1 K_2 K_3 K_4 K_5 K_6

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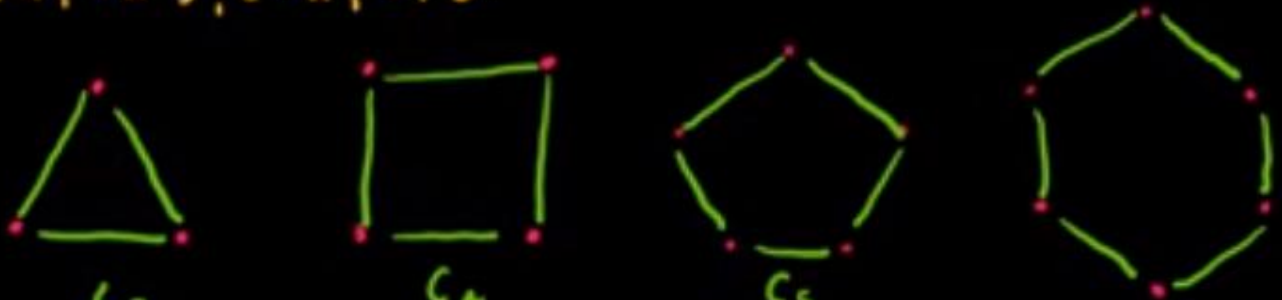
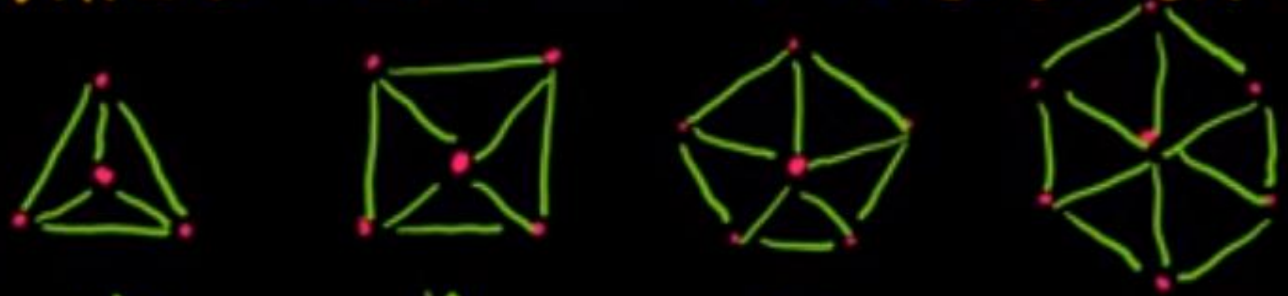
Cycles and Wheels

A cycle C_n, $\boxed{n \geq 3}$, consists of vertices $v_1, v_2 \ldots v_n$ and edges $\{v_1, v_2\}$, $\{v_2, v_3\} \ldots \{v_{n-1}, v_n\}$, $\{v_n, v_1\}$.



C_3 C_4 C_5 C_6

A wheel W_n is obtained when we add an additional vertex to C_n, and connect that vertex to each existing vertex.



W_3 W_4 W_5 W_6

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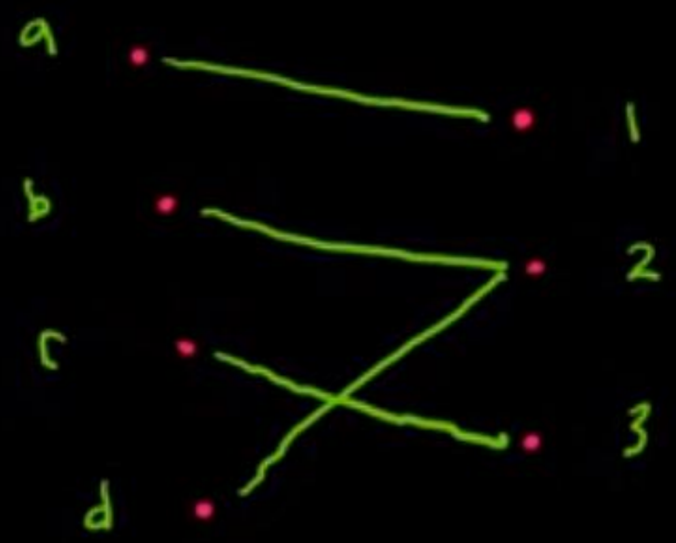
Bipartite Graphs

A simple graph is called bipartite if its vertex set, V, can be partitioned into two dijoint subsets such that each edge connects a vertex from one subsct to a vertex of the other.



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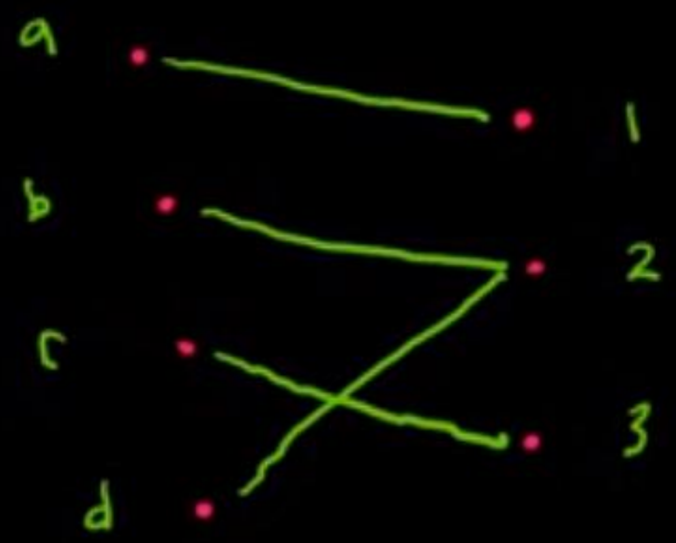

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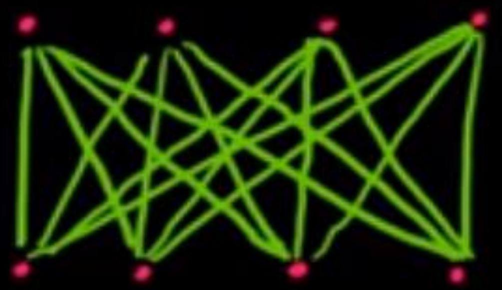
Complete Bipartite Graphs

A complete bipartite graph $\boxed{K_{m,n}}$ is a bipartite graph with subsets of m and n vertices, respectively with an edge between each pair of vertices from opposite subsets.

$K_{2,3}$



$K_{4,4}$

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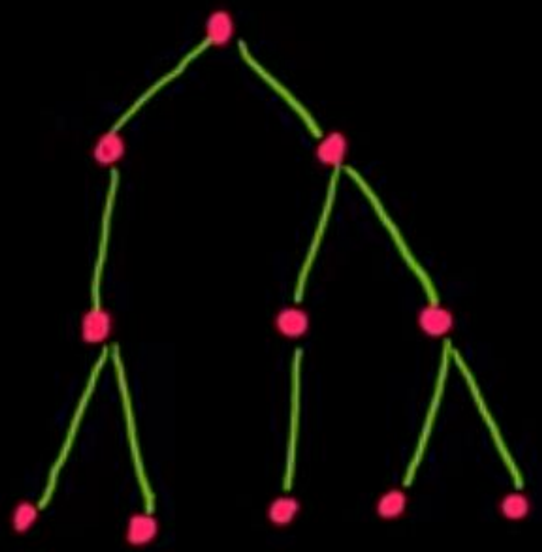
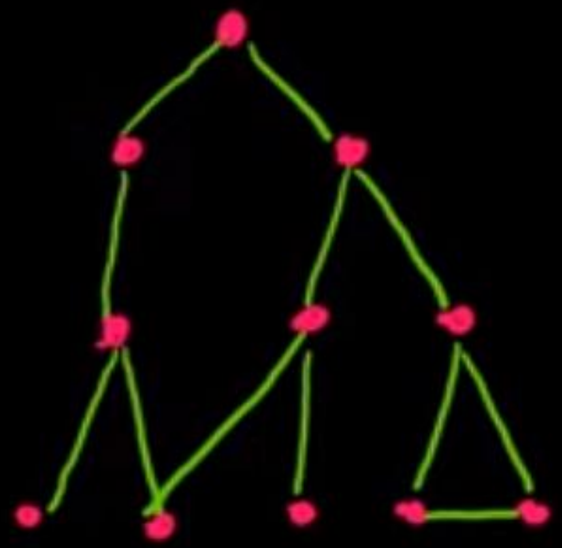
Trees

A tree is a simple, connected, undirected graph with no simple circuits. This means there is a unique simple path between any two of its vertices.



Tree



Not a Tree

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