

# How to Solve Complex Problems in Parallel (Divide and Conquer *and* Task Parallelism)

Kenjiro Taura

# Contents

- ➊ Introduction
- ➋ An example :  $k$ -d tree construction
  - $k$ -d tree
- ➌ Parallelizing divide and conquer algorithms
- ➍ Reasoning about speedup
  - Work and critical path length
  - Greedy scheduler theorem
  - Calculating work and critical path
- ➎ More divide and conquer examples
  - Merge sort
  - Cholesky factorization
  - Triangular solve
  - Matrix multiply

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- 1 Introduction
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# Goals

learn:

- the power of divide and conquer paradigm, combined with task parallelism, with concrete examples,
- how to write task parallel programs (OpenMP task)
- and how to reason about the speedup of task parallel programs
  - work
  - critical path length
  - Greedy Scheduler theorem

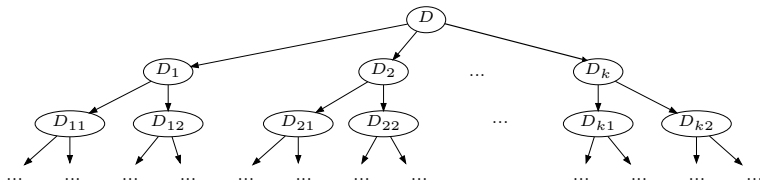
# Divide and conquer algorithms

- “Divide and conquer” is the single most important design paradigm of algorithms

```

1  answer solve( $D$ ) {
2      if ( trivial ( $D$ )) {
3          return trivially_solve ( $D$ );
4      } else {
5           $D_1, \dots, D_k = \text{divide}(D)$ ; // divide the problem into sub problems
6           $a_1 = \text{solve}(D_1)$ ; ...;  $a_k = \text{solve}(D_k)$ ; // solve them
7          return combine( $a_1, \dots, a_k$ ); // combine sub answers
8      }
9  }

```



# Benefits of “divide and conquer” thinking

Divide and conquer ...

- often helps you *come up with* an algorithm
- is easy to program, with *recursions*
- is often easy to *parallelize*, once you have a recursive formulation and a parallel programming language that support it (*task parallelism*)
- often has a good *locality* of reference, both in serial and parallel execution

# Some examples

- quick sort, merge sort
- matrix multiply, LU factorization, eigenvalue
- FFT, polynomial multiply, big int multiply
- maximum segment sum, find median
- $k$ -d tree
- ...

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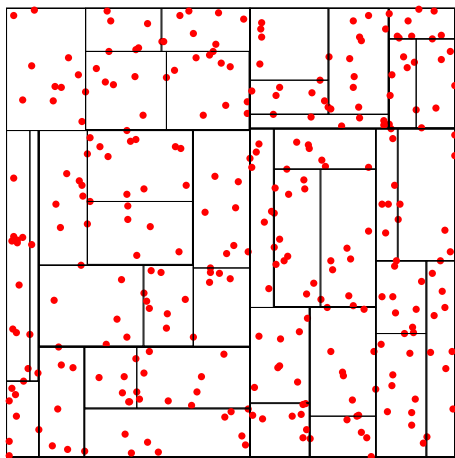


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# $k$ -d tree

- A data structure to hierarchically organize points (to facilitate “nearest neighbor” or “proximity” searches) (usually in 2D or 3D space)
- Each node represents a rectangle region



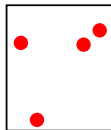
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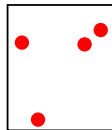
Leaf:



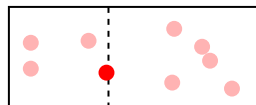
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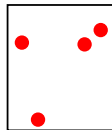
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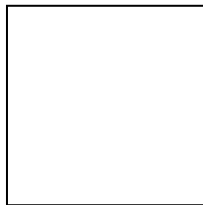
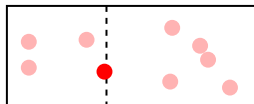
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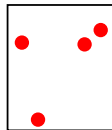
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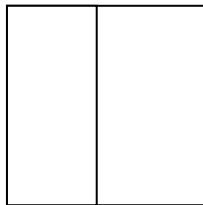
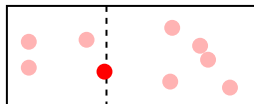
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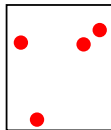
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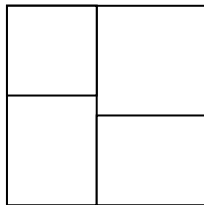
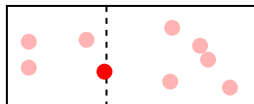
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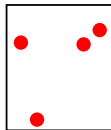




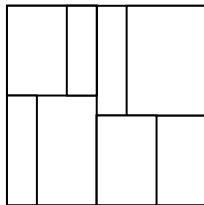
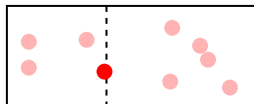
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# How to build a $k$ -d tree

Possible strategies:

- an insertion-based method
  - define a method to add a single point into a tree
  - start from an empty tree and add all points into it

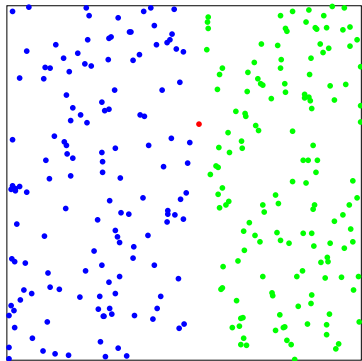
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Possible strategies:

- an insertion-based method
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  - start from an empty tree and add all points into it
- a divide and conquer method

# divide and conquer method

- to build a tree for a rectangle  $R$  and points  $P$  in  $R$ ,

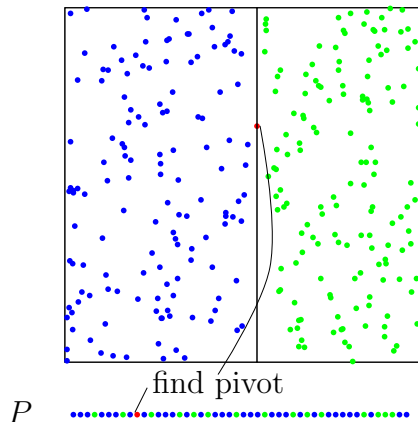


$P$



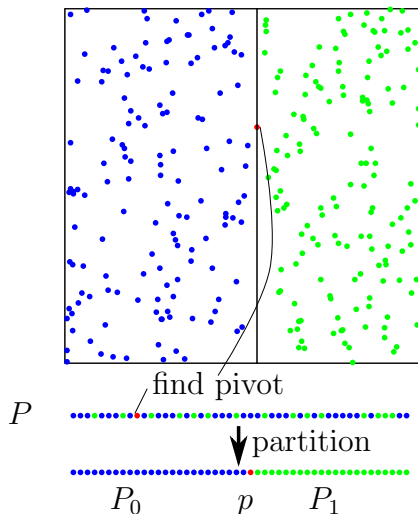
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# divide and conquer method

- to build a tree for a rectangle  $R$  and points  $P$  in  $R$ ,
- choose a point  $p \in P$  through which to split  $R$ , and
- partition  $P$  into  $P_0 + \{p\} + P_1$ 
  - let's say we split along  $x$ -axis. then
  - $P_0$  : points whose  $x$  coordinate  $< p$ 's
  - $P_1$  : points whose  $x$  coordinate  $\geq p$ 's (except  $p$ )



# divide and conquer method

```
1  /* build a k-d tree for a set of points P in a rectangular region R and return
2     the root of the tree. the node is at depth, so it should split along
3     (depth % D)th axis */
4  build(P, R, depth) {
5      if (|P| == 0) {
6          return 0; /* empty */
7      } else if (|P| <= threshold) {
8          /* small enough; leaf */
9          return make_leaf(P, R, depth);
10     } else {
11         /* find a point whose coordinate to split is near the median */
12         s = find_median(P, depth % D);
13         /* split R into two sub-rectangles */
14         R0,R1 = split_rect(R, depth % D, s.pos[depth % D]);
15         /* partition P by their coordinate lower/higher than p's coordinate */
16         P0,P1 = partition(P - { p }, depth % D, s.pos[depth % D]);
17         /* build a tree for each rectangle */
18         n0 = build(P0, R0, depth + 1);
19         n1 = build(P1, R1, depth + 1);
20         /* return a node having n0 and n1 as its children */
21         return make_node(p, n0, n1, depth);
22     }
23 }
```

# Notes on subprocedures

- $s = \text{find\_median}(P, d)$ 
  - find a point  $\in P$  whose  $d$ th coordinate is (close to) the median value among all points in  $P$
  - sample a few points and choose the median  $\Rightarrow O(1)$
- $R_0, R_1 = \text{split\_rect}(R, d, c)$ 
  - split a rectangular region  $R$  by a (hyper-)plane “ $d$ th coordinate =  $c$ ”
  - just make two rectangular regions  $\Rightarrow O(1)$
- $P_0, P_1 = \text{partition}(P, d, c)$ 
  - partition a set of points  $P$  into two subsets  $P_0$  ( $d$ th coordinate  $< c$ ) and  $P_1$  ( $d$ th coordinate  $\geq c$ )
  - $\Rightarrow O(|P|)$



# Contents

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# Parallelizing divide and conquer

- Divide and conquer algorithms are easy to parallelize if the programming language/library supports asynchronous recursive calls (*task parallel* systems)
  - OpenMP task constructs (`#pragma omp parallel, master, task, taskwait`)
  - Intel Threading Building Block (TBB)
  - Cilk, CilkPlus

# Parallelizing $k$ -d tree construction with tasks

- it's as simple as doing two recursions in parallel!
- e.g., with OpenMP tasks

```
1  build(P, R, depth) {  
2      if (|P| == 0) {  
3          return 0; /* empty */  
4      } else if (|P| <= threshold) {  
5          return make_leaf(P, R, depth);  
6      } else {  
7          s = find_median(P, depth % D);  
8          R0,R1 = split_rect(R, depth % D, s.pos[depth % D]);  
9          P0,P1 = partition(P - { p }, depth % D, s.pos[depth % D]);  
10     #pragma omp task shared(n0)  
11         n0 = build(P0, R0, depth + 1);  
12     #pragma omp task shared(n1)  
13         n1 = build(P1, R1, depth + 1);  
14     #pragma omp taskwait  
15         return make_node(p, n0, n1, depth);  
16     }  
17 }
```

- do you want to parallelize it with only parallel loops?

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# Reasoning about speedup

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- **ANSWER:** get the *work* and the *critical path length* of the computation

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# Work and critical path length

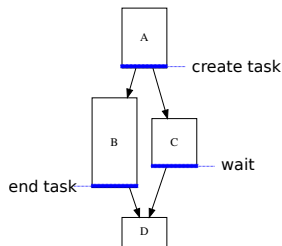
- **Work:** = the total amount of work of the computation
  - = the time it takes in a serial execution
- **Critical path length:** = the maximum length of dependent chain of computation
  - a more precise definition follows, with *computational DAGs*

# Computational DAGs

*The DAG* of a computation is a directed acyclic graph in which:

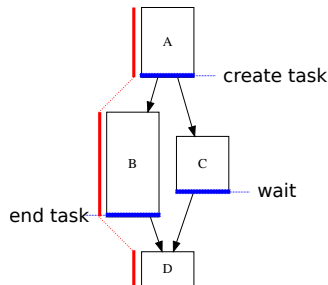
- a node = an interval of computation free of task parallel primitives
  - i.e. a node *starts* and *ends* by a task parallel primitive
  - we assume a single node is executed non-preemptively
- an edge = a dependency between two nodes, of three types:
  - parent → created child
  - child → waiting parent
  - a node → the next node in the same task

```
1 main() {  
2   A();  
3   create_task B();  
4   C();  
5   wait(); // wait for B  
6   D();  
7 }
```



# A computational DAG and critical path length

- Consider each node is augmented with a time for a processor to execute it (*the node's execution time*)
- Define *the length of a path* to be the sum of execution time of the nodes on the path



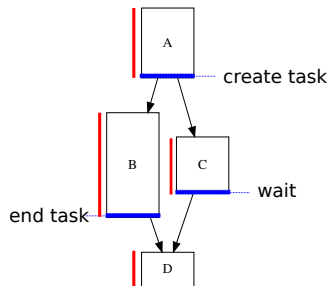
Given a computational DAG,

*critical path length = the length of the longest paths from the start node to the end node in the DAG*

(we often say *critical path* to in fact mean its length)

# A computational DAG and work

- Work, too, can be elegantly defined in light of computational DAGs

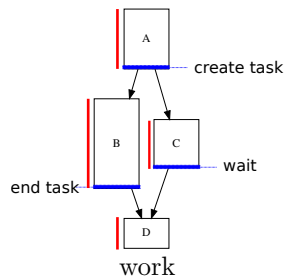
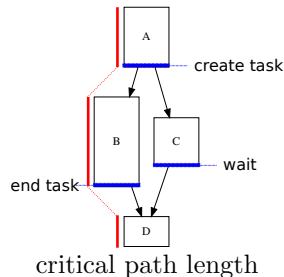


Given a computational DAG,

*work = the sum of lengths of all nodes*

# What do they intuitively mean?

- The critical path length represents the “ideal” execution time with *infinitely many* processors
  - i.e., each node is executed immediately after all its predecessors have finished
- We thus often denote it by  $T_\infty$
- Analogously, we often denote *work* by  $T_1$   
 $T_1 = \text{work}$ ,  $T_\infty = \text{critical path}$



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- **QUESTION:** Specifically, what does it tell us about performance or speedup on, say, my 64 core machines?
- **ANSWER:** A beautiful theorem (*greedy scheduler theorem*) gives us an answer

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# The greedy scheduler theorem

- Assume:
  - you have  $P$  processors
  - they are *greedy*, in the sense that a processor is *always busy* on a task whenever there is *any* runnable task in the entire system
  - an execution time of a node does not depend on which processor executed it

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- Theorem: given a computational DAG of:
  - work  $T_1$  and
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$$T_P \leq \frac{T_1 - T_\infty}{P} + T_\infty$$

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- in practice you remember a simpler form:

$$T_P \leq \frac{T_1}{P} + T_\infty$$

# The greedy scheduler theorem

- it is now a common sense in parallel computing, but the root of the idea seems:

Richard Brent. *The Parallel Evaluation of General Arithmetic Expressions*. Journal of the ACM 21(2). pp201-206. 1974

Derek Eager, John Zahorjan, and Edward Lazowska. *Speedup versus efficiency in parallel systems*. IEEE Transactions on Computers 38(3). pp408-423. 1989

- People attribute it to Brent and call it [Brent's theorem](#)
- Proof is a good exercise for you

I'll repeat! Remember it!

$$T_P \leq \frac{T_1}{P} + T_\infty$$

# A few facts to remember about $T_1$ and $T_\infty$

Consider the execution time with  $P$  processors ( $T_P$ )

- there are two obvious *lower bounds*

- $T_P \geq \frac{T_1}{P}$
- $T_P \geq T_\infty$

or more simply,

$$T_P \geq \max(\frac{T_1}{P}, T_\infty)$$

- what a greedy scheduler achieves is

$$T_P \leq \text{sum}(\frac{T_1}{P}, T_\infty)$$

- two memorable facts

- “the sum of two lower bounds is an upper bound”
- any greedy scheduler is within a factor of two of the optimal scheduler (下手な考え休むに似たり?)



## A few facts to remember about $T_1$ and $T_\infty$

- to get good (nearly perfect) speedup, we wish to have

$$\frac{T_1}{P} \gg T_\infty$$

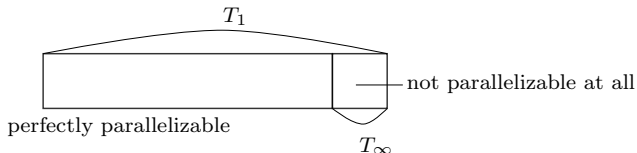
or equivalently,

$$\frac{T_1}{T_\infty} \gg P$$

- we can consider  $\frac{T_1}{T_\infty}$  to be *the average parallelism* (the speedup we would get with infinitely many processors)
- we like to make the average parallelism large enough compared to the actual number of processors

# Another way to remember the theorem

- assume a simpler case in which the entire computation (which amounts to  $T_1$ ) consists of two parts,
  - ① one completely serial (which amounts to  $T_\infty$ ), and
  - ② the other completely parallelizable (which amounts to  $(T_1 - T_\infty)$ )



- trivially, any greedy scheduler achieves

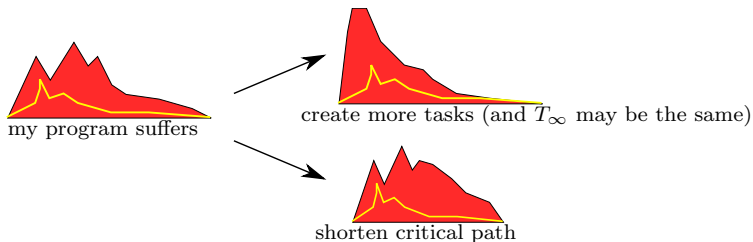
$$T_P \leq \frac{T_1 - T_\infty}{P} + T_\infty$$

- many people remember this as *Amdahl's law*
- the greedy scheduler theorem states that the same inequality holds more generally, for *any* computational DAG

# Takeaway message

*Suffer from low parallelism?  $\Rightarrow$  try to shorten its critical path*

*in contrast, people are tempted to get more speedup by creating more and more tasks; they are useless unless doing so shortens the critical path*



# What makes $T_\infty$ so useful?

$T_\infty$  is:

- a single *global metric* (just as the work is)
  - not something that fluctuates over time (cf. the number of tasks)
- *inherent to the algorithm, independent from the scheduler*
  - not something that depends on schedulers (cf. the number of tasks)
- connected to execution time with  $P$  processors in a beautiful way ( $T_P \leq T_1/P + T_\infty$ )
- *easy to estimate/calculate* (like the ordinary time complexity of serial programs)

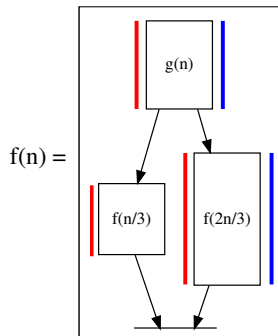
# Contents

- 1 Introduction
- 2 An example :  $k$ -d tree construction
  - $k$ -d tree
- 3 Parallelizing divide and conquer algorithms
- 4 Reasoning about speedup
  - Work and critical path length
  - Greedy scheduler theorem
  - Calculating work and critical path
- 5 More divide and conquer examples
  - Merge sort
  - Cholesky factorization
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# Calculating work and critical path

- for recursive procedures, using recurrent equations is often a good strategy
- e.g., if we have

```
1 f(n) {  
2   if (n == 1) { trivial(n); /* assume O(1) */ }  
3   else {  
4     g(n);  
5     create_task f(n/3);  
6     f(2*n/3);  
7     wait();  
8   }  
9 }
```



then

- (work)  $W_f(n) \leq W_g(n) + W_f(n/3) + W_f(2n/3)$
- (critical path)  $C_f(n) \leq C_g(n) + \max\{C_f(n/3), C_f(2n/3)\}$
- we apply this for programs we have seen

# Work of $k$ -d tree construction

```
1 build(P, R, depth) {  
2   if (|P| == 0) {  
3     return 0; /* empty */  
4   } else if (|P| <= threshold) {  
5     return make_leaf(P, R, depth);  
6   } else {  
7     s = find_median(P, depth % D);  
8     R0,R1 = split_rect(R, depth % D, s.pos[depth % D]);  
9     P0,P1 = partition(P - { p }, depth % D, s.pos[depth % D]);  
10    n0 = create_task build(P0, R0, depth + 1);  
11    n1 = build(P1, R1, depth + 1);  
12    wait();  
13    return make_node(p, n0, n1, depth);  
14  } }
```

recall that `partition` takes time proportional to  $n$  (the number of points). thus,

$$W_{\text{build}}(n) \approx 2W_{\text{build}}(n/2) + \Theta(n)$$

omitting math,

$$\therefore W_{\text{build}}(n) \in \Theta(n \log n)$$

# Remark

- the argument above is crude, as  $n$  points are not always split into two sets of equal sizes
- yet, the  $\Theta(n \log n)$  result is valid, as long as a split is guaranteed to be “never too unbalanced” (i.e., there is a constant  $\alpha < 1$ , s.t. each child gets  $\leq \alpha n$  points)



# Critical path

```
1 build(P, R, depth) {  
2   if (|P| == 0) {  
3     return 0; /* empty */  
4   } else if (|P| <= threshold) {  
5     return make_leaf(P, R, depth);  
6   } else {  
7     s = find_median(P, depth % D);  
8     R0,R1 = split_rect(R, depth % D, s.pos[depth % D]);  
9     P0,P1 = partition(P - { p }, depth % D, s.pos[depth % D]);  
10    n0 = create_task build(P0, R0, depth + 1);  
11    n1 = build(P1, R1, depth + 1);  
12    wait();  
13    return make_node(p, n0, n1, depth);  
14  } }
```

$$C_{\text{build}}(n) \approx C_{\text{build}}(n/2) + \Theta(n)$$

omitting math,

$$\therefore C_{\text{build}}(n) \in \Theta(n)$$

# Speedup of $k$ -d tree construction

- Now we have:

$$W_{\text{build}}(n) \in \Theta(n \log n),$$
$$C_{\text{build}}(n) \in \Theta(n).$$

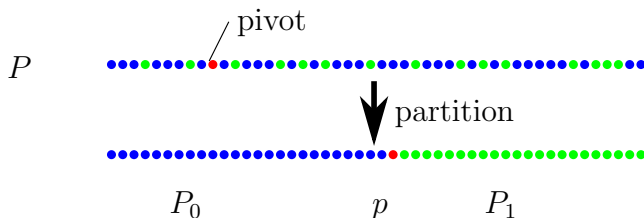
- $\Rightarrow$

$$\frac{T_1}{T_\infty} \in \Theta(\log n)$$

- not satisfactory in practice

# What the analysis tells us

- the expected speedup,  $\Theta(\log n)$ , is not satisfactory
- to improve, shorten its critical path  $\Theta(n)$ , to  $o(n)$
- where you should improve? the reason for the  $\Theta(n)$  critical path is **partition**; we should parallelize **partition**



# Contents

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- 2 An example :  $k$ -d tree construction
  - $k$ -d tree
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# Merge sort

- Input:
  - $A$ : an array
- Output:
  - $B$ : a sorted array
- Note: the result could be returned either in place or in a separate array. Assume it is “in place” in the following.

# Merge sort : serial code

```
1  /* sort a..a_end and put the result into
2     (i) a (if dest = 0)
3     (ii) t (if dest = 1) */
4  void ms(elem * a, elem * a_end,
5          elem * t, int dest) {
6      long n = a_end - a;
7      if (n == 1) {
8          if (dest) t[0] = a[0];
9      } else {
10         /* split the array into two */
11         long nh = n / 2;
12         elem * c = a + nh;
13         /* sort 1st half */
14         ms(a, c, t, 1 - dest);
15         /* sort 2nd half */
16         ms(c, a_end, t + nh, 1 - dest);
17         elem * s = (dest ? a : t);
18         elem * d = (dest ? t : a);
19         /* merge them */
20         merge(s, s + nh,
21              s + nh, s + n, d);
22     }
23 }
```

```
1  /* merge a_beg ... a_end
2     and b_beg ... b_end
3     into c */
4  void
5  merge(elem * a, elem * a_end,
6        elem * b, elem * b_end,
7        elem * c) {
8      elem * p = a, * q = b, * r = c;
9      while (p < a_end && q < b_end) {
10         if (*p < *q) { *r++ = *p++; }
11         else { *r++ = *q++; }
12     }
13     while (p < a_end) *r++ = *p++;
14     while (q < b_end) *r++ = *q++;
15 }
```

**note:** as always, actually switch to serial sort below a threshold (not shown in the code above)

# Merge sort : parallelization

```
void ms(elem * a, elem * a_end,  
        elem * t, int dest) {  
    long n = a_end - a;  
    if (n == 1) {  
        if (dest) t[0] = a[0];  
    } else {  
        /* split the array into two */  
        long nh = n / 2;  
        elem * c = a + nh;  
        /* sort 1st half */  
        create_task ms(a, c, t, 1 - dest);  
        /* sort 2nd half */  
        ms(c, a_end, t + nh, 1 - dest);  
        wait();  
        elem * s = (dest ? a : t);  
        elem * d = (dest ? t : a);  
        /* merge them */  
        merge(s, s + nh,  
              s + nh, s + n, d);  
    }  
}
```

- Will we get “good enough” speedup?



# Work of merge sort

```
void ms(elem * a, elem * a_end,  
        elem * t, int dest) {  
    long n = a_end - a;  
    if (n == 1) {  
        if (dest) t[0] = a[0];  
    } else {  
        /* split the array into two */  
        long nh = n / 2;  
        elem * c = a + nh;  
        /* sort 1st half */  
        create_task ms(a, c, t, 1 - dest);  
        /* sort 2nd half */  
        ms(c, a_end, t + nh, 1 - dest);  
        wait();  
        elem * s = (dest ? a : t);  
        elem * d = (dest ? t : a);  
        /* merge them */  
        merge(s, s + nh,  
              s + nh, s + n, d);  
    }  
}
```

$$W_{\text{ms}}(n) = 2W_{\text{ms}}(n/2) + W_{\text{merge}}(n),$$
$$W_{\text{merge}}(n) \in \Theta(n).$$

$$\therefore W_{\text{ms}}(n) \in \Theta(n \log n)$$

# Critical path of merge sort

```
void ms(elem * a, elem * a_end,  
        elem * t, int dest) {  
    long n = a_end - a;  
    if (n == 1) {  
        if (dest) t[0] = a[0];  
    } else {  
        /* split the array into two */  
        long nh = n / 2;  
        elem * c = a + nh;  
        /* sort 1st half */  
        create_task ms(a, c, t, 1 - dest);  
        /* sort 2nd half */  
        ms(c, a_end, t + nh, 1 - dest);  
        wait();  
        elem * s = (dest ? a : t);  
        elem * d = (dest ? t : a);  
        /* merge them */  
        merge(s, s + nh,  
              s + nh, s + n, d);  
    }  
}
```

$$C_{\text{ms}}(n) = C_{\text{ms}}(n/2) + C_{\text{merge}}(n),$$
$$C_{\text{merge}}(n) \in \Theta(n)$$

$$\therefore C_{\text{ms}}(n) \in \Theta(n)$$

# Speedup of merge sort

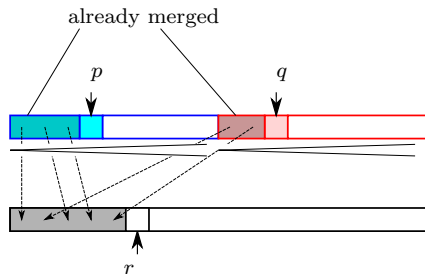
$$\begin{aligned}T_1 &= W_{\text{ms}}(n) \in \Theta(n \log n), \\T_\infty &= C_{\text{ms}}(n) \in \Theta(n).\end{aligned}$$

the average parallelism

$$T_1/T_\infty \in \Theta(\log n).$$

# How (serial) merge works

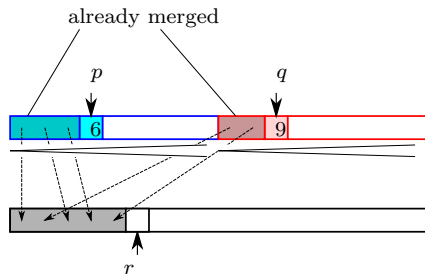
```
/* merge a_beg ... a_end
   and b_beg ... b_end
   into c */
void
merge(elem * a, elem * a_end,
      elem * b, elem * b_end,
      elem * c) {
    elem * p = a, * q = b, * r = c;
    while (p < a_end && q < b_end) {
        if (*p < *q) { *r++ = *p++; }
        else { *r++ = *q++; }
    }
    while (p < a_end) *r++ = *p++;
    while (q < b_end) *r++ = *q++;
}
```



Looks very serial ...

# How (serial) merge works

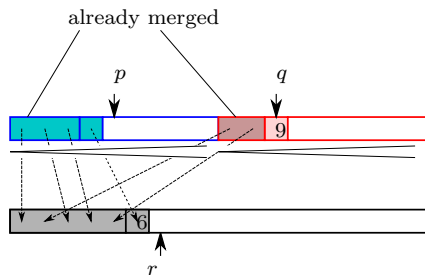
```
/* merge a_beg ... a_end
   and b_beg ... b_end
   into c */
void
merge(elem * a, elem * a_end,
      elem * b, elem * b_end,
      elem * c) {
    elem * p = a, * q = b, * r = c;
    while (p < a_end && q < b_end) {
        if (*p < *q) { *r++ = *p++; }
        else { *r++ = *q++; }
    }
    while (p < a_end) *r++ = *p++;
    while (q < b_end) *r++ = *q++;
}
```



Looks very serial ...

# How (serial) merge works

```
/* merge a_beg ... a_end
   and b_beg ... b_end
   into c */
void
merge(elem * a, elem * a_end,
      elem * b, elem * b_end,
      elem * c) {
    elem * p = a, * q = b, * r = c;
    while (p < a_end && q < b_end) {
        if (*p < *q) { *r++ = *p++; }
        else { *r++ = *q++; }
    }
    while (p < a_end) *r++ = *p++;
    while (q < b_end) *r++ = *q++;
}
```



Looks very serial ...

# How to parallelize merge?

- again, divide and conquer thinking helps
- left as an exercise

# Contents

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- 2 An example :  $k$ -d tree construction
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  - Greedy scheduler theorem
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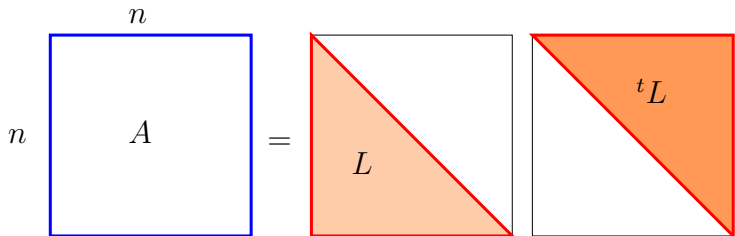


# Our running example : Cholesky factorization

- Input:
  - $A$ :  $n \times n$  positive semidefinite symmetric matrix
- Output:
  - $L$ :  $n \times n$  lower triangular matrix s.t.

$$A = L {}^tL$$

- ( ${}^tL$  is a transpose of  $L$ )



# Note : why Cholesky factorization is important?

- It is the core step when solving

$$Ax = b \quad (\text{single righthand side})$$

or, in more general,

$$AX = B \quad (\text{multiple righthand sides}),$$

as follows.

- 1 Cholesky decompose  $A = L {}^tL$  and get

$$L \underbrace{{}^tLX}_Y = B$$

- 2 Find  $X$  by solving triangular systems twice

- 1  $LY = B$

- 2  ${}^tLX = Y$

# Formulate using subproblems

$$\begin{pmatrix} A_{11} & {}^tA_{21} \\ A_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} L_{11} & O \\ L_{21} & L_{22} \end{pmatrix} \begin{pmatrix} {}^tL_{11} & {}^tL_{21} \\ O & {}^tL_{22} \end{pmatrix}$$

leads to three subproblems

- ❶  $A_{11} = L_{11} {}^tL_{11}$
- ❷  ${}^tA_{21} = L_{11} {}^tL_{21}$
- ❸  $A_{22} = L_{21} {}^tL_{21} + L_{22} {}^tL_{22}$

# Solving with recursions

$$\begin{pmatrix} A_{11} & {}^tA_{21} \\ A_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} L_{11} & O \\ L_{21} & L_{22} \end{pmatrix} \begin{pmatrix} {}^tL_{11} & {}^tL_{21} \\ O & {}^tL_{22} \end{pmatrix}$$

①  $A_{11} = L_{11} {}^tL_{11}$

②  ${}^tA_{21} = \textcolor{red}{L}_{11} {}^tL_{21}$

③  $A_{22} = \textcolor{red}{L}_{21} {}^t\textcolor{red}{L}_{21} + L_{22} {}^tL_{22}$

```
1  /* Cholesky factorization */
2  chol(A) {
3      if (n == 1) return (sqrt(a11));
4      else {
5          L11 = chol(A11);
6          /* triangular solve */
7          {}^tL21 = trsm(L11, {}^tA21);
8          L22 = chol(A22 - L21 {}^tL21);
9          return ( ( L11  {}^tL21
10                   L21  L22 )
11      }
```

# Solving with recursions

$$\begin{pmatrix} A_{11} & {}^tA_{21} \\ A_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} L_{11} & O \\ L_{21} & L_{22} \end{pmatrix} \begin{pmatrix} {}^tL_{11} & {}^tL_{21} \\ O & {}^tL_{22} \end{pmatrix}$$

①  $A_{11} = L_{11} {}^tL_{11}$   
• recursion and get  $L_{11}$

②  ${}^tA_{21} = L_{11} {}^tL_{21}$

③  $A_{22} = L_{21} {}^tL_{21} + L_{22} {}^tL_{22}$

```
1  /* Cholesky factorization */
2  chol(A) {
3      if (n == 1) return (sqrt(a11));
4      else {
5          L11 = chol(A11);
6          /* triangular solve */
7          tL21 = trsm(L11, tA21);
8          L22 = chol(A22 - L21 tL21);
9          return  $\begin{pmatrix} L_{11} & {}^tL_{21} \\ L_{21} & L_{22} \end{pmatrix}$ 
10     }
11 }
```

# Solving with recursions

$$\begin{pmatrix} A_{11} & {}^tA_{21} \\ A_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} L_{11} & O \\ L_{21} & L_{22} \end{pmatrix} \begin{pmatrix} {}^tL_{11} & {}^tL_{21} \\ O & {}^tL_{22} \end{pmatrix}$$

- ①  $A_{11} = L_{11} {}^tL_{11}$ 
  - recursion and get  $L_{11}$
- ②  ${}^tA_{21} = L_{11} {}^tL_{21}$ 
  - solve a *triangular* system and get  ${}^tL_{21}$
- ③  $A_{22} = L_{21} {}^tL_{21} + L_{22} {}^tL_{22}$

```
1  /* Cholesky factorization */
2  chol(A) {
3      if (n == 1) return (sqrt(a11));
4      else {
5          L11 = chol(A11);
6          /* triangular solve */
7          {}^tL21 = trsm(L11, {}^tA21);
8          L22 = chol(A22 - L21 {}^tL21);
9          return ( ( L11  {}^tL21
10                   L21  L22 )
11      }
```

# Solving with recursions

$$\begin{pmatrix} A_{11} & {}^tA_{21} \\ A_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} L_{11} & O \\ L_{21} & L_{22} \end{pmatrix} \begin{pmatrix} {}^tL_{11} & {}^tL_{21} \\ O & {}^tL_{22} \end{pmatrix}$$

- ①  $A_{11} = L_{11} {}^tL_{11}$ 
  - recursion and get  $L_{11}$
- ②  ${}^tA_{21} = L_{11} {}^tL_{21}$ 
  - solve a *triangular* system and get  ${}^tL_{21}$
- ③  $A_{22} = L_{21} {}^tL_{21} + L_{22} {}^tL_{22}$ 
  - recursion on  $(A_{22} - L_{21} {}^tL_{21})$  and get  $L_{22}$

```
1  /* Cholesky factorization */
2  chol(A) {
3      if (n == 1) return (sqrt(a11));
4      else {
5          L11 = chol(A11);
6          /* triangular solve */
7          tL21 = trsm(L11, tA21);
8          L22 = chol(A22 - L21 tL21);
9          return ( ( L11  tL21
10                  L21  L22 )
11      }
```

## Remark 1 : “In-place update” version

- Instead of returning the answer as another matrix, it is often possible to update the input matrix with the answer
- When possible, it is desirable, as it avoids extra copies

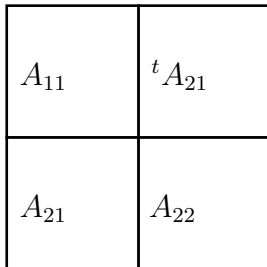
```
/* functional */
chol(A) {
  if (n = 1) return ( $\sqrt{a_{11}}$ );
  else {
    L11 = chol(A11);
    /* triangular solve */
    tL21 = trsm(L11, tA21);
    L22 = chol(A22 - L21tL21);
    return  $\begin{pmatrix} L_{11} & {}^tL_{21} \\ L_{21} & L_{22} \end{pmatrix}$ 
  }
}
```

```
1 /* in place */
2 chol(A) {
3   if (n = 1) a11 :=  $\sqrt{a_{11}}$ ;
4   else {
5     chol(A11);
6     /* triangular solve */
7     trsm(A11, A12);
8     A21 = tA12;
9     A22 -= A21A12
10    chol(A22);
11  }
12 }
```



# In-place Cholesky at work

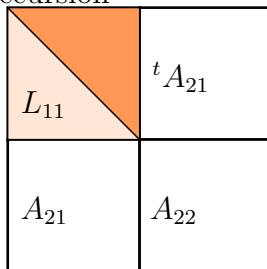
```
/* in place */  
chol(A) {  
  if (n == 1) a11 :=  $\sqrt{a_{11}}$ ;  
  else {  
    chol(A11);  
    /* triangular solve */  
    trsm(A11, A12);  
    A21 =  ${}^t A_{12}$ ;  
    A22 -= A21 A12  
    chol(A22);  
  }  
}
```



# In-place Cholesky at work

```
/* in place */  
chol(A) {  
  if (n == 1)  $a_{11} := \sqrt{a_{11}}$ ;  
  else {  
    chol( $A_{11}$ );  
    /* triangular solve */  
    trsm( $A_{11}$ ,  $A_{12}$ );  
     $A_{21} = {}^t A_{12}$ ;  
     $A_{22} -= A_{21} A_{12}$   
    chol( $A_{22}$ );  
  }  
}
```

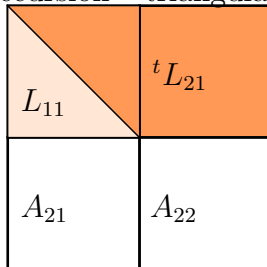
recursion



# In-place Cholesky at work

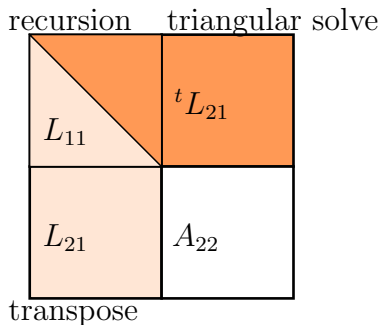
```
/* in place */  
chol(A) {  
  if (n = 1)  $a_{11} := \sqrt{a_{11}}$ ;  
  else {  
    chol( $A_{11}$ );  
    /* triangular solve */  
    trsm( $A_{11}$ ,  $A_{12}$ );  
     $A_{21} = {}^t A_{12}$ ;  
     $A_{22} -= A_{21} A_{12}$   
    chol( $A_{22}$ );  
  }  
}
```

recursion      triangular solve



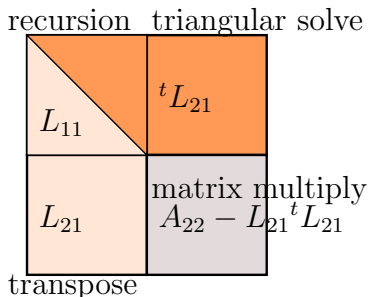
# In-place Cholesky at work

```
/* in place */  
chol(A) {  
  if (n = 1)  $a_{11} := \sqrt{a_{11}}$ ;  
  else {  
    chol( $A_{11}$ );  
    /* triangular solve */  
    trsm( $A_{11}, A_{12}$ );  
     $A_{21} = {}^t A_{12}$ ;  
     $A_{22} -= A_{21} A_{12}$   
    chol( $A_{22}$ );  
  }  
}
```



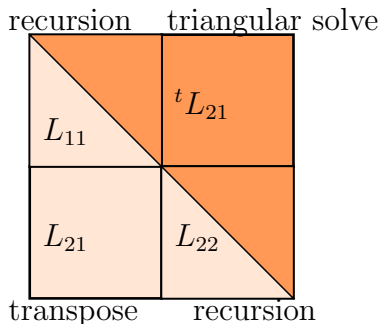
# In-place Cholesky at work

```
/* in place */
chol(A) {
  if (n = 1)  $a_{11} := \sqrt{a_{11}}$ ;
  else {
    chol( $A_{11}$ );
    /* triangular solve */
    trsm( $A_{11}, A_{12}$ );
     $A_{21} = {}^t A_{12}$ ;
     $A_{22} -= A_{21} A_{12}$ 
    chol( $A_{22}$ );
  }
}
```



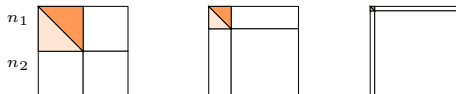
# In-place Cholesky at work

```
/* in place */  
chol(A) {  
  if (n = 1)  $a_{11} := \sqrt{a_{11}}$ ;  
  else {  
    chol( $A_{11}$ );  
    /* triangular solve */  
    trsm( $A_{11}$ ,  $A_{12}$ );  
     $A_{21} = {}^t A_{12}$ ;  
     $A_{22} -= A_{21} A_{12}$   
    chol( $A_{22}$ );  
  }  
}
```



## Remark 2 : where to decompose

- Where to partition  $A$  is *arbitrary*
- The case  $n_1 = 1$  and  $n_2 = n - 1 \approx$  loops

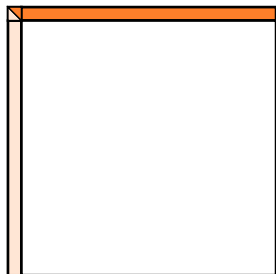


```
/* general */
chol(A) {
  if (n = 1)  $a_{11} := \sqrt{a_{11}}$ ;
  else {
    chol( $A_{11}$ );
    /* triangular solve */
    trsm( $A_{11}$ ,  $A_{12}$ );
     $A_{21} = {}^t A_{12}$ ;
     $A_{22} -= A_{21} A_{12}$ 
    chol( $A_{22}$ );
  }
}
```

```
1 /* loop-like */
2 chol(A) {
3   if (n = 1)  $a_{11} := \sqrt{a_{11}}$ ;
4   else {
5      $a_{11} := \sqrt{a_{11}}$ ;
6     /* triangular solve */
7      $\vec{a}_{12} /= a_{11}$ ;
8      $\vec{a}_{21} /= a_{11}$ ;
9      $A_{22} -= \vec{a}_{21} \vec{a}_{12}$ 
10    chol( $A_{22}$ );
11   }
12 }
```

# Recursion to loops

- The “loop-like” version (partition into  $1 + (n - 1)$ ) can be written in a true loop



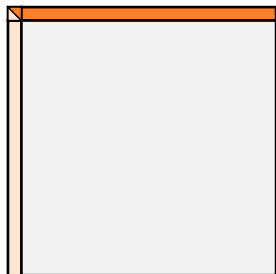
```
1  /* loop version */
2  chol_loop(A) {
3      for (k = 1; k ≤ n; k ++ ) {
4          akk := √akk;
5          Ak,k+1:n /= akk;
6          Ak+1:n,k /= akk;
7          Ak+1:n,k+1:n -= Ak:n,kAk,k:n
8      }
9  }
```

In practice, you still need to code the loop to avoid creating too small tasks



# Recursion to loops

- The “loop-like” version (partition into  $1 + (n - 1)$ ) can be written in a true loop

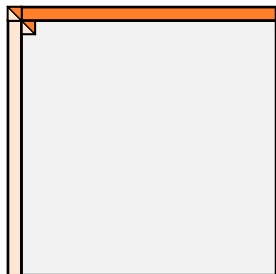


```
1  /* loop version */
2  chol_loop(A) {
3      for (k = 1; k ≤ n; k ++ ) {
4          a_kk := √a_kk;
5          A_k,k+1:n /= a_kk;
6          A_k+1:n,k /= a_kk;
7          A_k+1:n,k+1:n -= A_k:n,k A_k,k:n
8      }
9  }
```

In practice, you still need to code the loop to avoid creating too small tasks

# Recursion to loops

- The “loop-like” version (partition into  $1 + (n - 1)$ ) can be written in a true loop

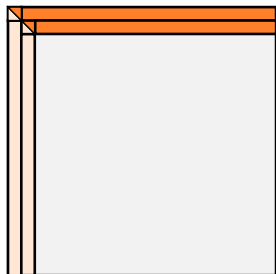


```
1  /* loop version */
2  chol_loop(A) {
3      for (k = 1; k ≤ n; k ++ ) {
4          akk := √akk;
5          Ak,k+1:n /= akk;
6          Ak+1:n,k /= akk;
7          Ak+1:n,k+1:n -= Ak:n,kAk,k:n
8      }
9  }
```

In practice, you still need to code the loop to avoid creating too small tasks

# Recursion to loops

- The “loop-like” version (partition into  $1 + (n - 1)$ ) can be written in a true loop

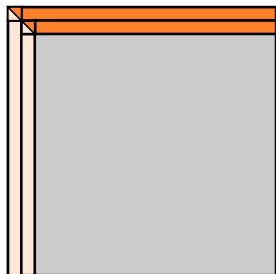


```
1  /* loop version */
2  chol_loop(A) {
3      for (k = 1; k ≤ n; k ++ ) {
4          a_kk := √a_kk;
5          A_{k,k+1:n} /= a_kk;
6          A_{k+1:n,k} /= a_kk;
7          A_{k+1:n,k+1:n} -= A_{k:n,k} A_{k,k:n}
8      }
9  }
```

In practice, you still need to code the loop to avoid creating too small tasks

# Recursion to loops

- The “loop-like” version (partition into  $1 + (n - 1)$ ) can be written in a true loop

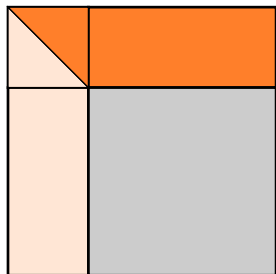


```
1  /* loop version */
2  chol_loop(A) {
3      for (k = 1; k ≤ n; k ++ ) {
4          akk := √akk;
5          Ak,k+1:n /= akk;
6          Ak+1:n,k /= akk;
7          Ak+1:n,k+1:n -= Ak:n,kAk,k:n
8      }
9  }
```

In practice, you still need to code the loop to avoid creating too small tasks

# Recursion to loops

- The “loop-like” version (partition into  $1 + (n - 1)$ ) can be written in a true loop

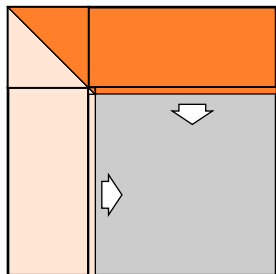


```
1  /* loop version */
2  chol_loop(A) {
3      for (k = 1; k ≤ n; k ++ ) {
4          akk := √akk;
5          Ak,k+1:n /= akk;
6          Ak+1:n,k /= akk;
7          Ak+1:n,k+1:n -= Ak:n,kAk,k:n
8      }
9  }
```

In practice, you still need to code the loop to avoid creating too small tasks

# Recursion to loops

- The “loop-like” version (partition into  $1 + (n - 1)$ ) can be written in a true loop



```
1  /* loop version */
2  chol_loop(A) {
3      for (k = 1; k ≤ n; k ++ ) {
4          akk := √akk;
5          Ak,k+1:n /= akk;
6          Ak+1:n,k /= akk;
7          Ak+1:n,k+1:n -= Ak:n,kAk,k:n
8      }
9  }
```

In practice, you still need to code the loop to avoid creating too small tasks

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- 2 An example :  $k$ -d tree construction
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  - Merge sort
  - Cholesky factorization
  - Triangular solve
  - Matrix multiply

# A subproblem 1: triangular solve

- Input:
  - $L$ :  $M \times M$  lower triangle matrix
  - $B$ :  $M \times N$  matrix
- Output:
  - $X$ :  $M \times N$  matrix  $X$  s.t.

$$LX = B$$

The diagram illustrates the matrix equation  $LX = B$ . It shows three matrices arranged horizontally, separated by an equals sign. The first matrix,  $L$ , is a square representing an  $M \times M$  lower triangular matrix, with a blue diagonal line and a blue border. The second matrix,  $X$ , is a red rectangle representing an  $M \times N$  matrix. The third matrix,  $B$ , is a blue rectangle representing an  $M \times N$  matrix. The dimensions are labeled:  $M$  for the height of all matrices,  $M$  for the width of  $L$ , and  $N$  for the width of  $X$  and  $B$ .



# Formulate using subproblems

Two ways to decompose:

- ① (split  $X$  and  $B$  vertically)

$$\begin{pmatrix} L_{11} & O \\ L_{21} & L_{22} \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} \Rightarrow$$

- $L_{11}X_1 = B_1$ , and
- $L_{21}X_1 + L_{22}X_2 = B_2$

- ② (split  $X$  and  $B$  horizontally)

$$L \begin{pmatrix} X_1 & X_2 \end{pmatrix} = \begin{pmatrix} B_1 & B_2 \end{pmatrix} \Rightarrow$$

- $LX_1 = B_1$ , and
- $LX_2 = B_2$

Choice is arbitrary, but for reasons we describe later, we decompose  $X$  and  $B$  so that their shapes are more square

# Solving with recursions

1

$$\begin{pmatrix} L_{11} & O \\ L_{21} & L_{22} \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}$$

- $L_{11}X_1 = B_1$   
recursion on  $(L_{11}, B_1)$  and get  $X_1$
- $L_{21}X_1 + L_{22}X_2 = B_2$  recursion on  
 $(L_{22}, B_2 - L_{21}X_1)$  and get  $X_2$

2

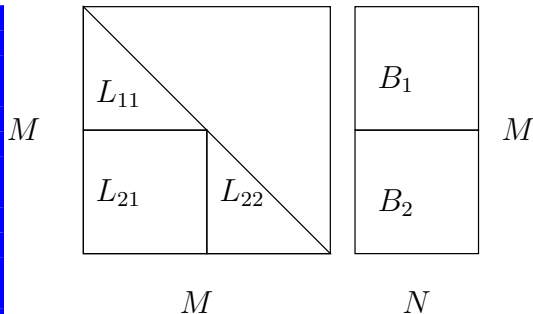
$$L \begin{pmatrix} X_1 & X_2 \end{pmatrix} = \begin{pmatrix} B_1 & B_2 \end{pmatrix} \Rightarrow$$

solve them independently (easy)

```
1  /* triangular solve LX = B.
2    replace B with X */
3  trsm(L, B) {
4    if (M = 1) {
5      B /= l11;
6    } else if (M ≥ N) {
7      trsm(L11, B1);
8      B2 -= L21B1;
9      trsm(L22, B2);
10   } else {
11     trsm(L, B1);
12     trsm(L, B2);
13   }
14 }
```

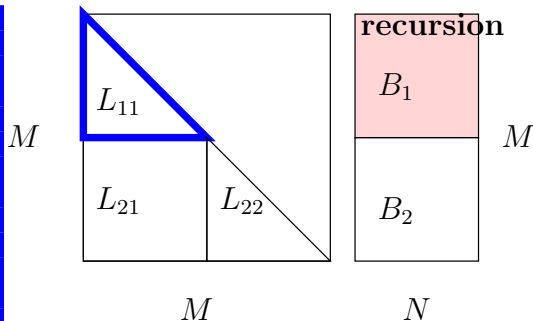
# Triangular solve at work

```
/* triangular solve  $LX = B$ .  
   replace  $B$  with  $X$  */  
trsm( $L, B$ ) {  
  if ( $M = 1$ ) {  
     $B \leftarrow l_{11}$ ;  
  } else if ( $M \geq N$ ) {  
    trsm( $L_{11}, B_1$ );  
     $B_2 \leftarrow L_{21} B_1$ ;  
    trsm( $L_{22}, B_2$ );  
  } else {  
    trsm( $L, B_1$ );  
    trsm( $L, B_2$ );  
  }  
}
```



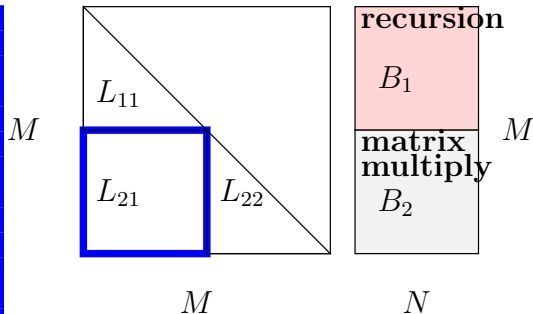
# Triangular solve at work

```
/* triangular solve  $LX = B$ .  
   replace  $B$  with  $X$  */  
trsm( $L, B$ ) {  
  if ( $M = 1$ ) {  
     $B \leftarrow l_{11}$ ;  
  } else if ( $M \geq N$ ) {  
    trsm( $L_{11}, B_1$ );  
     $B_2 \leftarrow L_{21}B_1$ ;  
    trsm( $L_{22}, B_2$ );  
  } else {  
    trsm( $L, B_1$ );  
    trsm( $L, B_2$ );  
  }  
}
```



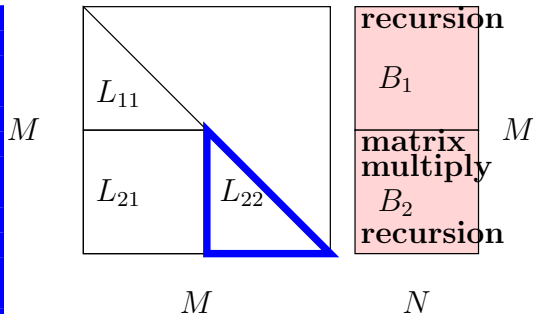
# Triangular solve at work

```
/* triangular solve  $LX = B$ .  
   replace  $B$  with  $X$  */  
trsm( $L, B$ ) {  
  if ( $M = 1$ ) {  
     $B \neq l_{11}$ ;  
  } else if ( $M \geq N$ ) {  
    trsm( $L_{11}, B_1$ );  
     $B_2 \mathrel{:=} L_{21}B_1$ ;  
    trsm( $L_{22}, B_2$ );  
  } else {  
    trsm( $L, B_1$ );  
    trsm( $L, B_2$ );  
  }  
}
```



# Triangular solve at work

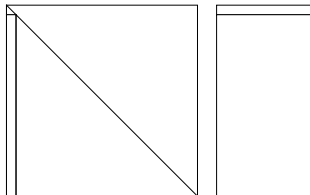
```
/* triangular solve  $LX = B$ .  
   replace  $B$  with  $X$  */  
trsm( $L, B$ ) {  
  if ( $M = 1$ ) {  
     $B \neq l_{11}$ ;  
  } else if ( $M \geq N$ ) {  
    trsm( $L_{11}, B_1$ );  
     $B_2 \mathrel{:=} L_{21}B_1$ ;  
    trsm( $L_{22}, B_2$ );  
  } else {  
    trsm( $L, B_1$ );  
    trsm( $L, B_2$ );  
  }  
}
```



# Recursions and loops

Again, partitioning is arbitrary and there is a loop-like partitioning

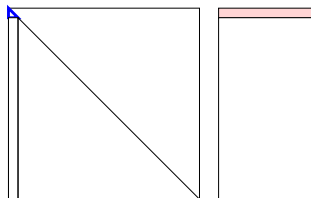
```
/* loop */  
trsm(L, B) {  
  for (k = 1; k ≤ M; k ++ ) {  
    Bk,1:M /= lkk;  
    Bk+1:M,1:M -= Lk+1:M,kBk,1:M;  
  }  
}
```



# Recursions and loops

Again, partitioning is arbitrary and there is a loop-like partitioning

```
/* loop */  
trsm(L, B) {  
  for (k = 1; k ≤ M; k ++ ) {  
    Bk,1:M /= lkk;  
    Bk+1:M,1:M -= Lk+1:M,k Bk,1:M;  
  }  
}
```

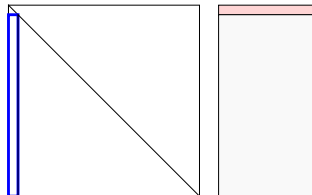




# Recursions and loops

Again, partitioning is arbitrary and there is a loop-like partitioning

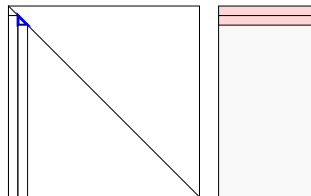
```
/* loop */  
trsm(L, B) {  
  for (k = 1; k ≤ M; k ++ ) {  
    Bk,1:M /= lkk;  
    Bk+1:M,1:M -= Lk+1:M,k Bk,1:M;  
  }  
}
```



# Recursions and loops

Again, partitioning is arbitrary and there is a loop-like partitioning

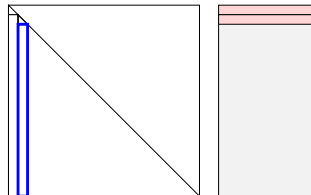
```
/* loop */  
trsm(L, B) {  
  for (k = 1; k ≤ M; k ++ ) {  
    Bk,1:M /= lkk;  
    Bk+1:M,1:M -= Lk+1:M,k Bk,1:M;  
  }  
}
```



# Recursions and loops

Again, partitioning is arbitrary and there is a loop-like partitioning

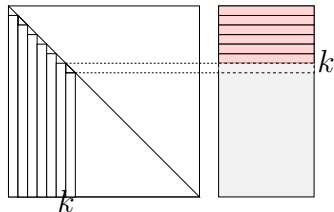
```
/* loop */  
trsm(L, B) {  
  for (k = 1; k ≤ M; k ++ ) {  
    Bk,1:M /= lkk;  
    Bk+1:M,1:M -= Lk+1:M,k Bk,1:M;  
  }  
}
```



# Recursions and loops

Again, partitioning is arbitrary and there is a loop-like partitioning

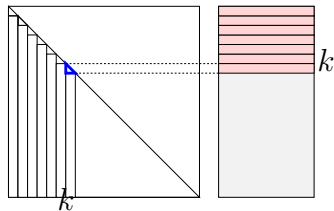
```
/* loop */  
trsm(L, B) {  
  for (k = 1; k ≤ M; k ++ ) {  
     $B_{k,1:M} \leftarrow l_{kk};$   
     $B_{k+1:M,1:M} \leftarrow L_{k+1:M,k} B_{k,1:M};$   
  }  
}
```



# Recursions and loops

Again, partitioning is arbitrary and there is a loop-like partitioning

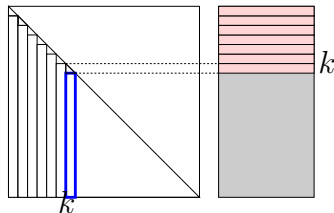
```
/* loop */  
trsm(L, B) {  
  for (k = 1; k ≤ M; k ++ ) {  
    Bk,1:M /= lkk;  
    Bk+1:M,1:M -= Lk+1:M,kBk,1:M;  
  }  
}
```



# Recursions and loops

Again, partitioning is arbitrary and there is a loop-like partitioning

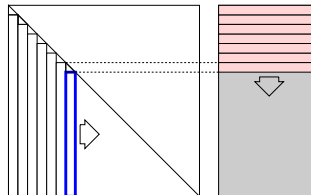
```
/* loop */  
trsm(L, B) {  
  for (k = 1; k ≤ M; k ++ ) {  
    Bk,1:M /= lkk;  
    Bk+1:M,1:M -= Lk+1:M,kBk,1:M;  
  }  
}
```



# Recursions and loops

Again, partitioning is arbitrary and there is a loop-like partitioning

```
/* loop */  
trsm(L, B) {  
  for (k = 1; k ≤ M; k ++ ) {  
    Bk,1:M /= lkk;  
    Bk+1:M,1:M -= Lk+1:M,kBk,1:M;  
  }  
}
```



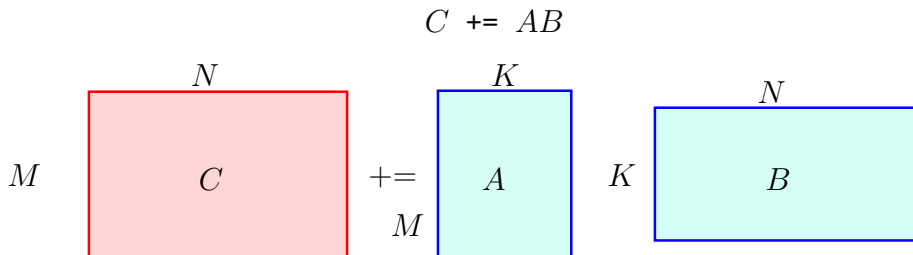
# Contents

- 1 Introduction
- 2 An example :  $k$ -d tree construction
  - $k$ -d tree
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## A subproblem 2: matrix multiply

- Input :
  - $C$ :  $M \times N$  matrix
  - $A$ :  $M \times K$  matrix
  - $B$ :  $K \times N$  matrix
- Output :



# Formulate using subproblems

Three ways to decompose

- divide  $M$  :

$$\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} += \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} B$$

$$\rightarrow C_1 += A_1 B \quad // \quad C_2 += A_2 B$$

- divide  $N$  :

$$\begin{pmatrix} C_1 & C_2 \end{pmatrix} += A \begin{pmatrix} B_1 & B_2 \end{pmatrix}$$

$$\rightarrow C_1 += AB_1 \quad // \quad C_2 += AB_2$$

- divide  $K$  :

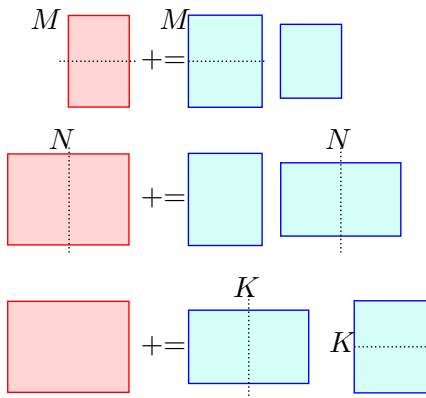
$$C += \begin{pmatrix} A_1 & A_2 \end{pmatrix} \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}$$

$$\rightarrow C += A_1 B_1 \quad ; \quad C += A_2 B_2$$

# Which decomposition should we use?

- For reasons described later, divide the largest one among  $M$ ,  $N$ , and  $K$
- Make the shape of subproblems as square as possible

# Solving using recursions



```
1  gemm(A, B, C) {  
2    if ((M, N, K) = (1, 1, 1)) {  
3       $c_{11} += a_{11} * b_{11};$   
4    } else if (M ≥ N and M ≥ K) {  
5       $A_1, A_2 = \text{split\_h}(A);$   
6       $C_1, C_2 = \text{split\_h}(C);$   
7      gemm( $A_1, B, C_1$ );  
8      gemm( $A_2, B, C_2$ );  
9    } else if (N ≥ M and N ≥ K)  
10      $B_1, B_2 = \text{split\_v}(B);$   
11      $C_1, C_2 = \text{split\_v}(C);$   
12     gemm( $A, B_1, C_1$ );  
13     gemm( $A, B_1, C_2$ );  
14   } else {  
15      $A_1, A_2 = \text{split\_v}(A);$   
16      $B_1, B_2 = \text{split\_h}(B);$   
17     gemm( $A_1, B_1, C$ );  
18     gemm( $A_2, B_2, C$ );  
19   }  
20 }
```

# Where is parallelism in our example?

## Cholesky

```
1  /* in place */
2  chol(A) {
3      if (n = 1) a11 :=  $\sqrt{a_{11}}$ ;
4      else {
5          chol(A11);
6          /* triangular solve */
7          trsm(A11, A12);
8          A21 = tA12;
9          A22 -= A21A12
10         chol(A22);
11     }
12 }
```

- data dependency prohibits any of function calls in line 5-10 to be executed in parallel

# Where is parallelism in our example?

## Triangular solve

```
1  /* triangular solve  $LX = B$ .  
2     replace  $B$  with  $X$  */  
3  trsm( $L, B$ ) {  
4      if ( $M = 1$ ) {  
5           $B /= l_{11}$ ;  
6      } else if ( $M \geq N$ ) {  
7          trsm( $L_{11}, B_1$ );  
8           $B_2 -= L_{21}B_1$ ;  
9          trsm( $L_{22}, B_2$ );  
10     } else {  
11         trsm( $L, B_1$ );  
12         trsm( $L, B_2$ );  
13     }  
14 }
```

- function calls in line 7-9 cannot be run in parallel
- two calls to trsm at line 11 and a2 *can* be run in parallel

# Where is parallelism in our example?

## Matrix multiply

```
gemm(A, B, C) {  
  if ((M, N, K) = (1, 1, 1)) {  
    c11 += a11 * b11;  
  } else if (M ≥ N and M ≥ K) {  
    A1, A2 = split_h(A);  
    C1, C2 = split_h(C);  
    gemm(A1, B, C1);  
    gemm(A2, B, C2);  
  } else if (N ≥ M and N ≥ K)  
    B1, B2 = split_v(B);  
    C1, C2 = split_v(C);  
    gemm(A, B1, C1);  
    gemm(A, B2, C2);  
  } else {  
    A1, A2 = split_v(A);  
    B1, B2 = split_h(B);  
    gemm(A1, B1, C);  
    gemm(A2, B2, C);  
  }  
}
```

- when dividing  $M$  and  $N$ , two recursive calls can be parallel
- when dividing  $K$ , they should be serial
- (alternatively, we can execute them in parallel using two different regions for  $C$  and then add them)

# That's basically it!

```
1  gemm(A, B, C) {
2      if ((M, N, K) = (1, 1, 1)) {
3          c11 += a11 * b11;
4      } else if (M ≥ N and M ≥ K) {
5          A1, A2 = split_h(A);
6          C1, C2 = split_h(C);
7          #pragma omp task
8              gemm(A1, B, C1);
9          #pragma omp task
10             gemm(A2, B, C2);
11         #pragma omp taskwait
12             } else if (N ≥ M and N ≥ K)
13                 B1, B2 = split_v(B);
14                 C1, C2 = split_v(C);
15         #pragma omp task
16             gemm(A, B1, C1);
17         #pragma omp task
18             gemm(A, B2, C2);
19         #pragma omp taskwait
20             } else {
21             // same as before
22             ...
23     }
```

```
1  /* triangular solve LX = B.
2      replace B with X */
3  trsm(L, B) {
4      if (M = 1) {
5          B /= l11;
6      } else if (M ≥ N) {
7          trsm(L11, B1);
8          B2 -= L21B1;
9          trsm(L22, B2);
10     } else {
11         #pragma omp task
12             trsm(L, B1);
13         #pragma omp task
14             trsm(L, B2);
15         #pragma omp taskwait
16     }
17 }
```



# $T_1$ and $T_\infty$ of matrix multiply

```
gemm(A, B, C) {  
  if ((M, N, K) = (1, 1, 1)) {  
    c11 += a11 * b11;  
  } else if (M ≥ N and M ≥ K) {  
    ...  
#pragma omp task  
    gemm(A1, B, C1);  
#pragma omp task  
    gemm(A2, B, C2);  
#pragma omp taskwait  
  } else if (N ≥ M and N ≥ K)  
    ...  
#pragma omp task  
    gemm(A, B1, C1);  
#pragma omp task  
    gemm(A, B1, C2);  
#pragma omp taskwait  
  } else {  
    ...  
    gemm(A1, B1, C);  
    gemm(A2, B2, C);  
  }  
}
```

Work ( $T_1$ ), written by

$W_{\text{gemm}}(M, N, K) =$

$$\begin{cases} \Theta(1) & ((M, N, K) = (1, 1, 1)) \\ 2W_{\text{gemm}}(M/2, N, K) + \Theta(1) & (M \text{ is largest}) \\ 2W_{\text{gemm}}(M, N/2, K) + \Theta(1) & (N \text{ is largest}) \\ 2W_{\text{gemm}}(M, N, K/2) + \Theta(1) & (K \text{ is largest}) \end{cases}$$

$$\Rightarrow \Theta(MNK)$$

# $T_1$ and $T_\infty$ of matrix multiply

```
gemm(A, B, C) {  
  if ((M, N, K) = (1, 1, 1)) {  
    c11 += a11 * b11;  
  } else if (M ≥ N and M ≥ K) {  
    ...  
#pragma omp task  
    gemm(A1, B, C1);  
#pragma omp task  
    gemm(A2, B, C2);  
#pragma omp taskwait  
  } else if (N ≥ M and N ≥ K)  
    ...  
#pragma omp task  
    gemm(A, B1, C1);  
#pragma omp task  
    gemm(A, B1, C2);  
#pragma omp taskwait  
  } else {  
    ...  
    gemm(A1, B1, C);  
    gemm(A2, B2, C);  
  }  
}
```

Critical path ( $T_\infty$ ), written by

$C_{\text{gemm}}(M, N, K) =$

$$\left\{ \begin{array}{l} \Theta(1) \\ ((M, N, K) = (1, 1, 1)), \\ C_{\text{gemm}}(M/2, N, K) + \Theta(1) \\ (M \text{ is largest}) \\ C_{\text{gemm}}(M, N/2, K) + \Theta(1) \\ (N \text{ is largest}) \\ 2C_{\text{gemm}}(M, N, K/2) + \Theta(1) \\ (K \text{ is largest}) \end{array} \right.$$

$\Rightarrow \Theta(\log M + \log N + K)$  (we consider it as  $\Theta(K)$  for brevity)

# $T_1$ and $T_\infty$ of triangular solve

```
/* triangular solve  $LX = B$ .  
   replace  $B$  with  $X$  */  
trsm(L, B) {  
    if ( $M = 1$ ) {  
         $B /= l_{11}$ ;  
    } else if ( $M \geq N$ ) {  
        trsm( $L_{11}$ ,  $B_1$ );  
         $B_2 -= L_{21}B_1$ ;  
        trsm( $L_{22}$ ,  $B_2$ );  
    } else {  
#pragma omp task  
        trsm( $L$ ,  $B_1$ );  
#pragma omp task  
        trsm( $L$ ,  $B_2$ );  
#pragma omp taskwait  
    }  
}
```

Work ( $T_1$ ), written by  
 $W_{\text{trsm}}(M, N) =$

$$\begin{cases} \Theta(1) & ((M, N) = (1, 1, 1)) \\ 2W_{\text{trsm}}(M/2, N) & \\ \quad + W_{\text{gemm}}(M/2, N, M/2) & \\ \quad (M \geq N) & \\ 2W_{\text{trsm}}(M, N/2) + \Theta(1) & \\ \quad (N > M) & \end{cases}$$

$$\Rightarrow \Theta(M^2N)$$

# $T_1$ and $T_\infty$ of triangular solve

```
/* triangular solve  $LX = B$ .  
   replace  $B$  with  $X$  */  
trsm(L, B) {  
  if ( $M = 1$ ) {  
     $B \neq l_{11}$ ;  
  } else if ( $M \geq N$ ) {  
    trsm( $L_{11}$ ,  $B_1$ );  
     $B_2 \mathrel{-=} L_{21}B_1$ ;  
    trsm( $L_{22}$ ,  $B_2$ );  
  } else {  
#pragma omp task  
    trsm( $L$ ,  $B_1$ );  
#pragma omp task  
    trsm( $L$ ,  $B_2$ );  
#pragma omp taskwait  
  }  
}
```

Critical path ( $T_\infty$ ), written by  
 $C_{\text{trsm}}(M, N) =$

$$\begin{cases} \Theta(1) & ((M, N) = (1, 1)), \\ 2C_{\text{trsm}}(M/2, N) & \\ \quad + C_{\text{gemm}}(M/2, N, M/2) & \\ \quad (M \geq N) & \\ C_{\text{trsm}}(M, N/2) + \Theta(1) & \\ \quad (N > M) & \end{cases}$$

$$\Rightarrow \Theta(M \log N)$$

# $T_1$ and $T_\infty$ of Cholesky

```
chol(A) {  
  if (n = 1) a11 :=  $\sqrt{a_{11}}$ ;  
  else {  
    chol(A11);  
    /* triangular solve */  
    trsm(A11, A12);  
    A21 = tA12;  
    A22 -= A21A12  
    chol(A22);  
  }  
}
```

Work ( $T_1$ ), written by  $W_{\text{chol}}(n) =$

$$\begin{cases} \Theta(1) & (n = 1), \\ 2W_{\text{chol}}(n/2) \\ \quad + W_{\text{trsm}}(n/2, n/2) \\ \quad + W_{\text{trans}}(n/2, n/2) \\ \quad + W_{\text{gemm}}(n/2, n/2, n/2) \end{cases}$$

$$\Rightarrow \Theta(n^3)$$

# $T_1$ and $T_\infty$ of Cholesky

```
chol(A) {  
  if (n = 1) a11 :=  $\sqrt{a_{11}}$ ;  
  else {  
    chol(A11);  
    /* triangular solve */  
    trsm(A11, A12);  
    A21 = tA12;  
    A22 -= A21A12  
    chol(A22);  
  }  
}
```

Critical path ( $T_\infty$ ), written by

$C_{\text{chol}}(n) =$

$$\begin{cases} \Theta(1) & (n = 1) \\ 2C_{\text{chol}}(n/2) \\ \quad + C_{\text{trsm}}(n/2, n/2) \\ \quad + C_{\text{trans}}(n/2, n/2) \\ \quad + C_{\text{gemm}}(n/2, n/2, n/2) \end{cases}$$

$\Rightarrow \Theta(n \log n)$

# Summary

For  $n \times n$  matrix,

- $T_1 \in \Theta(n^3)$
- $T_\infty \in \Theta(n \log n)$
- the average parallelism:

$$T_1/T_\infty = \frac{n^2}{\log n}$$

- this should be ample for sufficiently large  $n$
- a constant thresholding does not affect the asymptotic result;
  - you can switch to a serial loop for matrices smaller than a constant
- in practice, this threshold affects  $T_1$  and  $T_\infty$ 
  - $T_1$  will decrease (good thing)
  - $T_\infty$  will increase due to a larger serial computation at leaves