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ECON3350 - Applied Econometrics for Macroeconomics and Finance

Tutorial 2: Univariate Processes – I

Tutor: Francisco Tavares Garcia

Do you have the latest versions of R and RStudio?

Install R – 4.4.3

https://cran.r-project.org/

Install RStudio – 2024.12.1+563

https://posit.co/download/rstudio-desktop/

Update all packages –

In RStudio >>

Tools >>

Check for Package Updates >>

Select All >>

Install Updates

Report 1 – due 28 March - Instructions

Instructions

The project consists of three research questions. Please answer all questions as clearly, concisely and completely as possible. Each question is worth 50 marks, for a total of 150 marks. This report will constitute 20% of your overall grade in this course.

We suggest that you use R for all empirical work involved. However, you should be able to use another statistical software (e.g. Eviews, Stata, Python, etc.) without a problem. If you do choose to work with an alternative software, please note that support for software-specific issues from the course coordinator and tutors may be very limited.

Please upload your report via the "Turnitin" submission link (in the "Assessment / Research Report" folder). Please note that hard copies *will not* be accepted. At the moment, the due date is 1:00 PM on 28 March 2025, but please check BlackBoard regularly for announcements regarding any changes to this. Your report should be a write-up of your answers (in PDF format, single-spaced, and in 12 font size).[1]

You are allowed to work on this assignment with others, i.e., you can discuss how to answer the questions with your classmate(s) and even use AI as a research tool. However, this is not a group assignment, which means that the report must be written individually and by you: you must answer all the questions in your own words and submit your report separately. The marking system will check for similarities and AI content, and UQ's student integrity and misconduct policies on plagiarism *strictly apply*.

Report 1 – due 28 March - Question 1

Questions

The dataset for Questions 1 and 2 is contained in `report1.csv`. The variables are quarterly time-series of macroeconomic indicators in Australia for the period 1995Q1—2023Q4 (116 observations). In particular, the dataset contains the following variables:

1. Use the data provided to choose three (3) ARIMA(p, d, q) models for inflation, π_t. Use each of these three models to forecast π_t for 2023 and 2024 (two years or equivalently eight quarters past the end of the sample). In doing so, please consider how such forecasts may be useful for policy, and consequently, ensure your inference is aligned with the underlying motivation. Make sure to address all potential sources of uncertainty on a conceptual level, and to the extent possible, quantitatively.

Report 1 – due 28 March - Question 2

2. Use the data provided to obtain inference on the stability of the term structure of interest rates. In particular, investigate the following questions:

   (a) Is there evidence of nonstationarity in inflation, Δp_t, or in any of the following four interest rates $\{r_{M1,t}, r_{M3,t}, r_{Y2,t}, r_{Y3,t}\}$?

   (b) Are there any identifiable equilibrium relationships among the four interest rates?

   (c) Are each of the following spreads stationary?

   - $s_{t,m3-m1} = r_{M3,t} - r_{M1,t}$,
   - $s_{t,y2-3m} = r_{Y2,t} - r_{M3,t}$,
   - $s_{t,y3-y2} = r_{Y3,t} - r_{Y2,t}$, and
   - $s_{t,y5-r} = r_{Y5,t} - r_t$

   (d) Use a regression of Δp_t on $s_{t,y5-r}$ estimated by ols to investigate support for a relationship between these two.

   In answering these questions, please consider how the answers may be useful for policy, and consequently, ensure your inference is aligned with the underlying motivation. Also, please discuss the key assumptions underlying any conclusions you obtain.

Report 1 – due 28 March - Question 3

3. The dataset for Question 3 is contained in `report2.csv`. The variables are daily time-series of two equity returns and a measure of market volatility for the period 29/06/2011—28/06/2021 (2541 observations, note the absence of weekends and holidays). The dataset contains the following variables:

   (a) Use the data provided to obtain inference on the volatility of $r_{BHP,t}$ and $r_{CBA,t}$. This should include discussion of any testing for the presence of volatility and model selection. Report only the important results that guide your conclusions, the estimated final model and estimated volatility for each process.

   (b) Compare and contrast the estimates of volatility from your models in part (a) to the $p_{VIX,t}$.

   (c) Investigate the probability of a return less than 0.01% for $r_{WES,t}$ and $r_{WBC,t}$ on each of the days 29/06/2021, 30/06/2021 and 1/07/2021.

In answering these questions, please consider how the answers may be useful for risk management, and consequently, ensure your inference is aligned with the underlying motivation. Also, please discuss the key assumptions underlying any conclusions you obtain.

Tutorial 2: Forecasting Univariate Processes - I

At the end of this tutorial you should be able to:

- derive theoretical properties of ARMA processes;
- compute the theoretical ACF and PACF for a given ARMA processes;
- use R to compute and plot the sample ACF and PACF for time series data.

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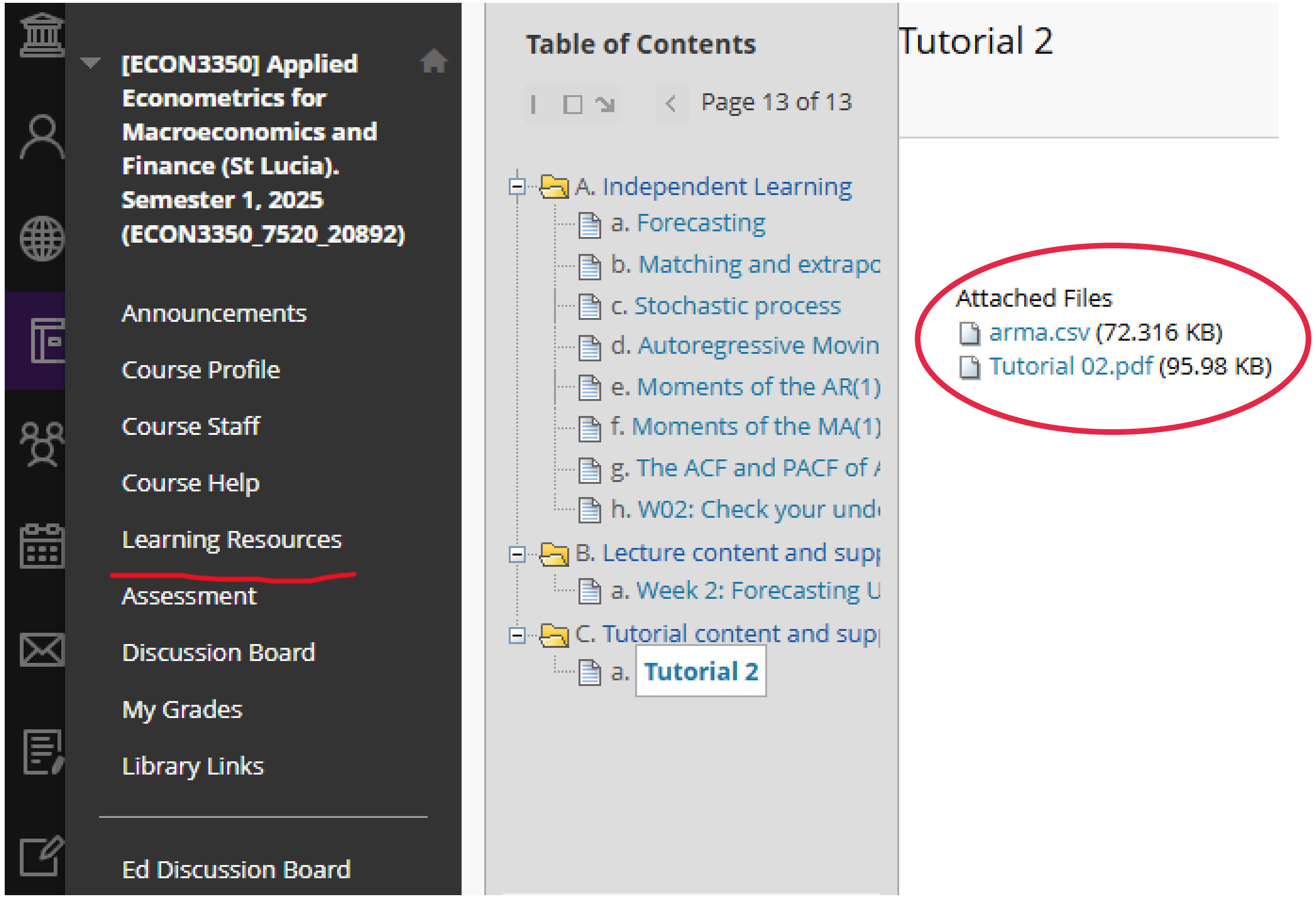
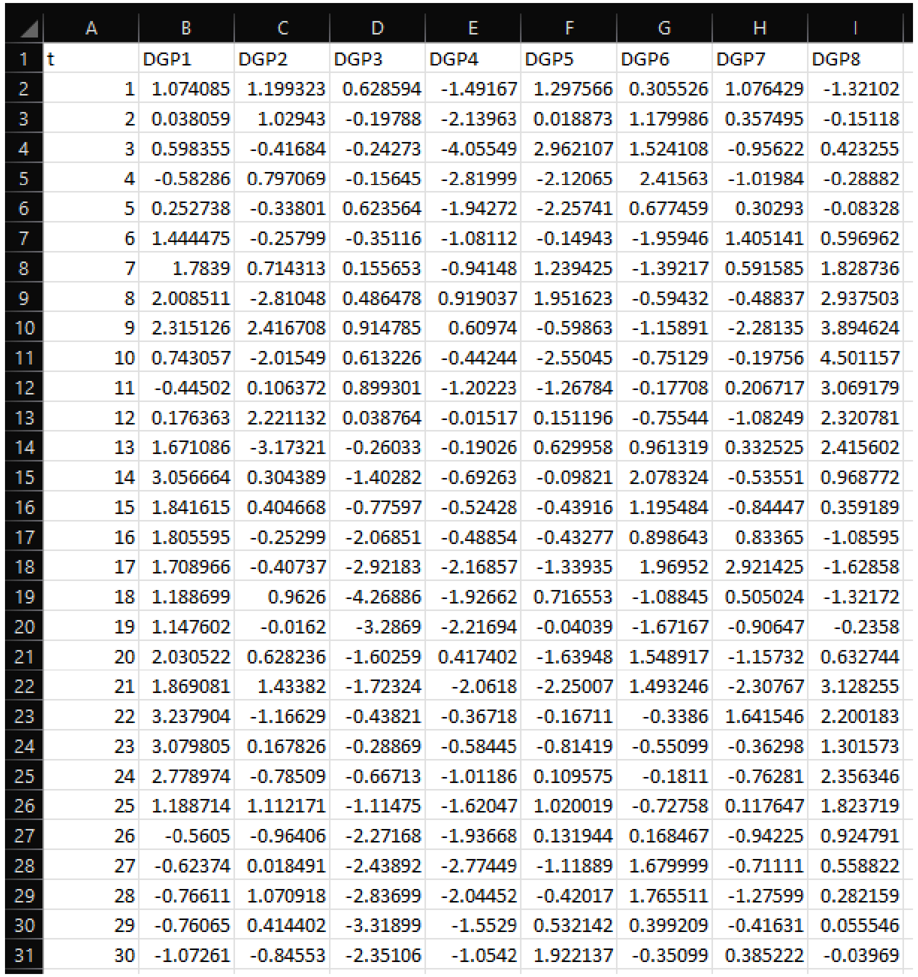
Let's download the tutorial and the dataset.





	A	B	C	D	E	F	G	H	I
1	t	DGP1	DGP2	DGP3	DGP4	DGP5	DGP6	DGP7	DGP8
2	1	1.074085	1.199323	0.628594	-1.49167	1.297566	0.305526	1.076429	-1.32102
3	2	0.038059	1.02943	-0.19788	-2.13963	0.018873	1.179986	0.357495	-0.15118
4	3	0.598355	-0.41684	-0.24273	-4.05549	2.962107	1.524108	-0.95622	0.423255
5	4	-0.58286	0.797069	-0.15645	-2.81999	-2.12065	2.41563	-1.01984	-0.28882
6	5	0.252738	-0.33801	0.623564	-1.94272	-2.25741	0.677459	0.30293	-0.08328
7	6	1.444475	-0.25799	-0.35116	-1.08112	-0.14943	-1.95946	1.405141	0.596962
8	7	1.7839	0.714313	0.155653	-0.94148	1.239425	-1.39217	0.591585	1.828736
9	8	2.008511	-2.81048	0.486478	0.919037	1.951623	-0.59432	-0.48837	2.937503
10	9	2.315126	2.416708	0.914785	0.60974	-0.59863	-1.15891	-2.28135	3.894624
11	10	0.743057	-2.01549	0.613226	-0.44244	-2.55045	-0.75129	-0.19756	4.501157
12	11	-0.44502	0.106372	0.899301	-1.20223	-1.26784	-0.17708	0.206717	3.069179
13	12	0.176363	2.221132	0.038764	-0.01517	0.151196	-0.75544	-1.08249	2.320781
14	13	1.671086	-3.17321	-0.26033	-0.19026	0.629958	0.961319	0.332525	2.415602
15	14	3.056664	0.304389	-1.40282	-0.69263	-0.09821	2.078324	-0.53551	0.968772
16	15	1.841615	0.404668	-0.77597	-0.52428	-0.43916	1.195484	-0.84447	0.359189
17	16	1.805595	-0.25299	-2.06851	-0.48854	-0.43277	0.898643	0.83365	-1.08595
18	17	1.708966	-0.40737	-2.92183	-2.16857	-1.33935	1.96952	2.921425	-1.62858
19	18	1.188699	0.9626	-4.26886	-1.92662	0.716553	-1.08845	0.505024	-1.32172
20	19	1.147602	-0.0162	-3.2869	-2.21694	-0.04039	-1.67167	-0.90647	-0.2358
21	20	2.030522	0.628236	-1.60259	0.417402	-1.63948	1.548917	-1.15732	0.632744
22	21	1.869081	1.43382	-1.72324	-2.0618	-2.25007	1.493246	-2.30767	3.128255
23	22	3.237904	-1.16629	-0.43821	-0.36718	-0.16711	-0.3386	1.641546	2.200183
24	23	3.079805	0.167826	-0.28869	-0.58445	-0.81419	-0.55099	-0.36298	1.301573
25	24	2.778974	-0.78509	-0.66713	-1.01186	0.109575	-0.1811	-0.76281	2.356346
26	25	1.188714	1.112171	-1.11475	-1.62047	1.020019	-0.72758	0.117647	1.823719
27	26	-0.5605	-0.96406	-2.27168	-1.93668	0.131944	0.168467	-0.94225	0.924791
28	27	-0.62374	0.018491	-2.43892	-2.77449	-1.11889	1.679999	-0.71111	0.558822
29	28	-0.76611	1.070918	-2.83699	-2.04452	-0.42017	1.765511	-1.27599	0.282159
30	29	-0.76065	0.414402	-3.31899	-1.5529	0.532142	0.399209	-0.41631	0.055546
31	30	-1.07261	-0.84553	-2.35106	-1.0542	1.922137	-0.35099	0.385222	-0.03969

Now, let's download the script for the tutorial.

- Copy the code from Github,
  - https://github.com/tavaresgarcia/teaching
- Save the scripts in the same folder as the data.

Before we proceed, an important disclaimer:

This is not how you will do your Reports!

- This week we know the function generating this data.
- The following weeks we will try to find this function.

3. The data file `arma.csv` contains (simulated) data for each of the DGPs in Question 2. Import the data into R. Compute, plot, and describe the behaviour of the ACF and PACF for each DGP. Discuss the effects of parameter signs. Hint: use the `acf` and `pacf` commands, respectively.

Solution Load the data using the `read.delim` command with the `sep = ","` option as it is comma delimited.

```
mydata <- read.delim("arma.csv", header = TRUE,  sep = ",")
```

The ACF and PACF plots can be generated for all eight GDPs quickly using the `for` loop. Note that we index each column in `mydata` as `1 + i` because the first column contains the time variable `t`. The option `main` is passed to `plot`—we assign it the name of a given column, which corresponds to the GDP in the loop that the sample ACF/PACF are being computed for.

```
for (i in 1:8)
{
  acf(mydata[1 + i], main = colnames(mydata[1 + i]))
  pacf(mydata[1 + i], main = colnames(mydata[1 + i]))
}
```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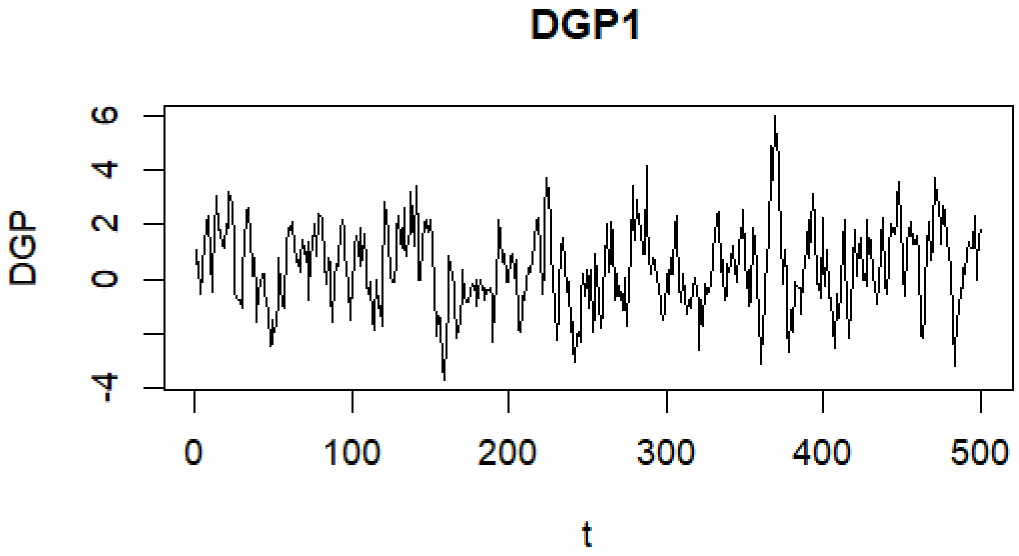
- DGP1: $y_t = 0.75 y_{t-1} + \epsilon_t$;




```
> # AR(1)
> arima(mydata$DGP1, order = c(1,0,0))

Call:
arima(x = mydata$DGP1, order = c(1, 0, 0))

Coefficients:
         ar1  intercept
      0.7485     0.4391
s.e.  0.0295     0.1714

sigma^2 estimated as 0.9397:  log likelihood = -694.33,  aic = 1394.66
```

The First-Order Autoregressive Model

One simple way to model a stochastic process is with the "regression":

$$y_t = a_0 + a_1 y_{t-1} + u_t.$$

This is called the first order auto-regressive model, or AR(1).

To make it useful in practice, we need assumptions about u_t.

The "classical regression" assumptions are:

- Mean-independence: $\mathrm{E}(u_t \mid y_{t-1}, y_{t-2}, \ldots) = 0$.
- Homoscedasticity: $\mathrm{Var}(u_t \mid y_{t-1}, y_{t-2}, \ldots) = \sigma_u^2$.

Mean-independence is crucial, but homoscedasticity can be relaxed.

Mean-independence implies zero-autocorrelation: $\mathrm{corr}(u_t, u_{t-k}) = 0$ for $k = 1, 2, \ldots$.

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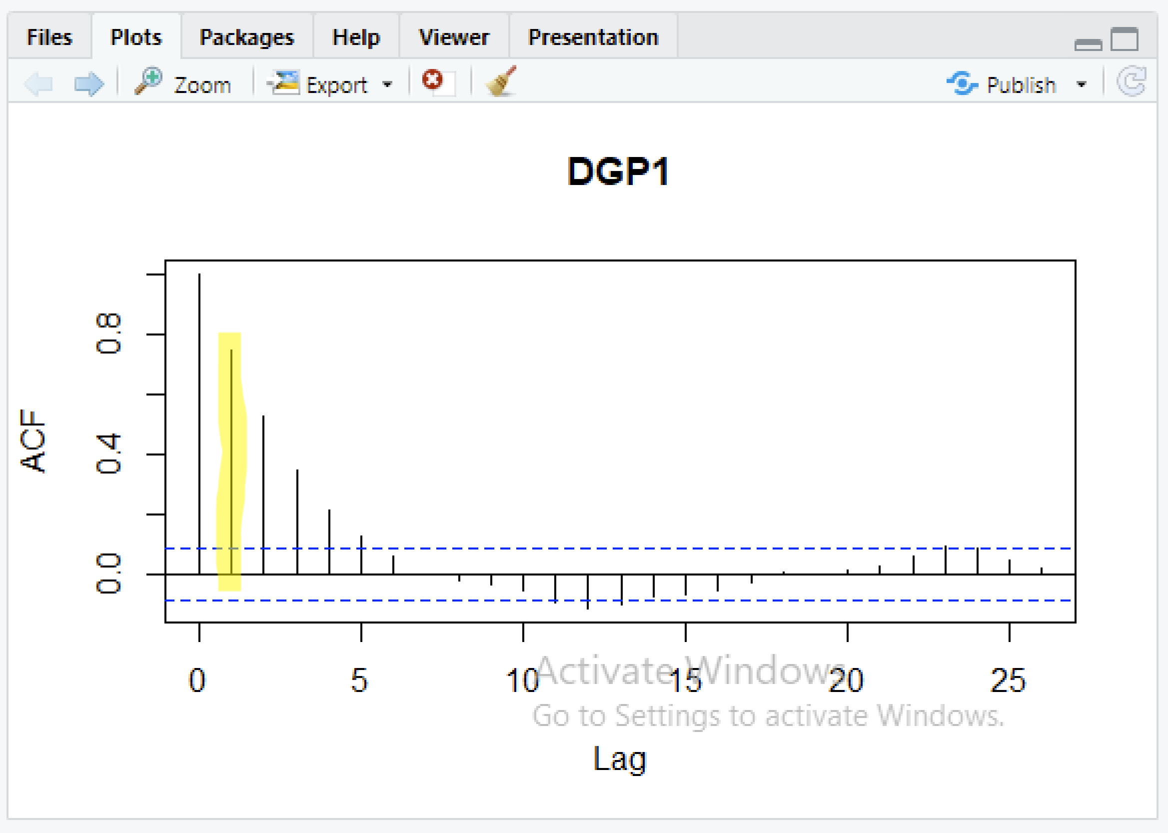
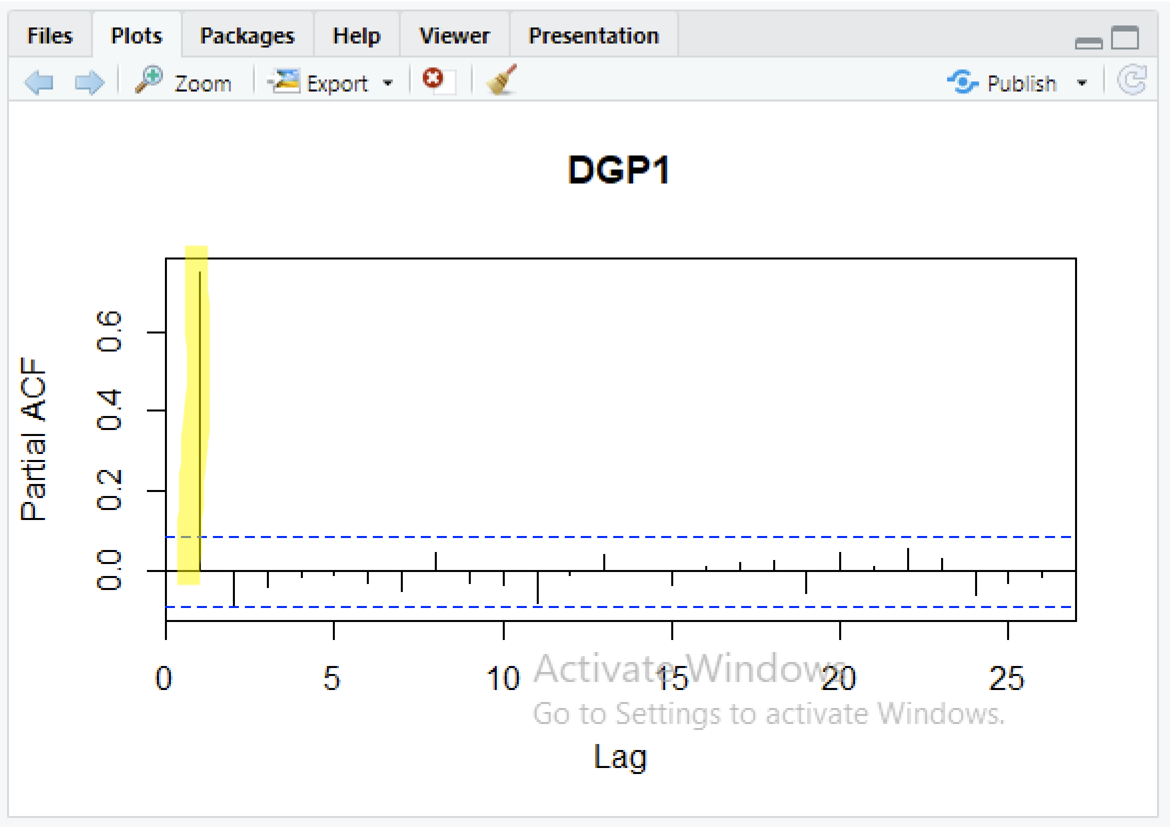




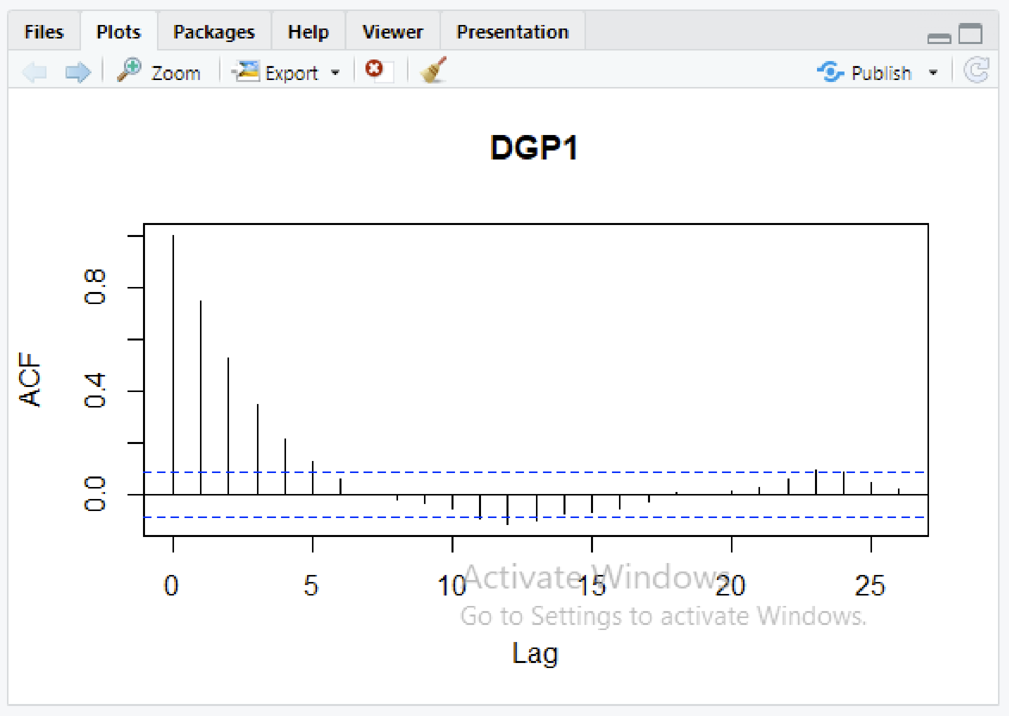
DGP1

- ACF: Decays geometrically as parameter is positive.
- PACF: One non-zero peak.

2. Compute the true ACF values for the following DGPs:

- DGP1: $y_t = 0.75 y_{t-1} + \epsilon_t$;

Solution $\rho_0 = 1,\ \rho_1 = 0.75,\ \ldots,\ \rho_k = 0.75^k$. The ACF will decay geometrically.



The ACF and PACF of an AR(1) Process

The unconditional variance and covariances are obtained in the limit as $h \longrightarrow \infty$.

If the AR(1) is unstable, then the unconditional variance and covariances do not exist. Otherwise:

- $\gamma_0 = \frac{\sigma_\varepsilon^2}{1-a_1^2}$;
- $\gamma_k = \frac{a_1^k \sigma_\varepsilon^2}{1-a_1^2},\ k = 1, 2, \ldots$;
- $\rho_k = a_1^k,\ k = 1, 2, \ldots$;
- $\phi_{11} = a_1,\ \phi_{kk} = 0$ for all $k \geq 2$.

The ACF of a stable AR(1) decays geometrically as $k \longrightarrow \infty$.

The PACF of a stable AR(1) vanishes for all $k \geq 2$.

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1. Derive the expected value, variance, covariance, autocorrelation function (ACF), and partial autocorrelation function (PACF) for the following data generating processes (DGPs):

   (a) AR(1): $y_t = a_0 + a_1 y_{t-1} + \epsilon_t$, $0 \leq |a_1| < 1$;

Important assumptions:

(1) the process is (weakly) stationary and

(2) errors are "white noise"

Stationarity

Several forms of stationarity that can be used to describe a stochastic process. We keep things simple with the following.

Definition

A stochastic process is **stationary** if and only if the mean, variance, and all covariances exist and are independent of time. Specifically, for all t,

$$\mathsf{E}(y_t) = \mu,$$
$$\mathsf{Var}(y_t) = \sigma_y^2 = \gamma_0,$$
$$\mathsf{Cov}(y_t, y_{t-k}) = \gamma_k, \qquad k \geq 1,$$

- Stationarity is a property of the stochastic process.
- Time-series data cannot be stationary or non-stationary: it is only one realisation of the process.

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The First-Order Autoregressive Model

One simple way to model a stochastic process is with the "regression":

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This is called the first order auto-regressive model, or AR(1).

To make it useful in practice, we need assumptions about u_t.

The "classical regression" assumptions are:

- Mean-independence: $\mathsf{E}(u_t \mid y_{t-1}, y_{t-2}, \dots) = 0.$
- Homoscedasticity: $\mathsf{Var}(u_t \mid y_{t-1}, y_{t-2}, \dots) = \sigma_u^2.$

Mean-independence is crucial, but homoscedasticity can be relaxed.

Mean-independence implies zero-autocorrelation: $\mathrm{corr}(u_t, u_{t-k}) = 0$ for $k = 1, 2, \dots$.

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1. Derive the expected value, variance, covariance, autocorrelation function (ACF), and partial autocorrelation function (PACF) for the following data generating processes (DGPs):

   (a) AR(1): $y_t = a_0 + a_1 y_{t-1} + \epsilon_t$, $0 \leq |a_1| < 1$;

Solution

- Expected value:

$$y_t = a_0 + a_1 y_{t-1} + \epsilon_t; \quad 0 \leq |a_1| < 1$$

$$E(y_t) = \mu = a_0 + a_1 E(y_{t-1}) + E(\epsilon_t)$$

$$\mu = \frac{a_0}{1 - a_1}; \text{ since } E(y_{t-1}) = \mu$$

In DGP1: $\mu = \dfrac{a_0}{1 - a_1} \rightarrow$

```
> mean(DGP1)
[1] 0.4272572
```

$0.4272572 = \dfrac{a_0}{1 - 0.75} \rightarrow$

$a_0 = 0.1068143$

The last video from Week 2 explains this equation (finite geometric series).

Moments of the AR(1) Process

Expected value of y_t conditional on $y_{t-h}, y_{t-h-1}, \ldots$:

$$\begin{aligned} E(y_t \mid y_{t-h}, y_{t-h-1}, \ldots) &= E(a_0 + a_1 y_{t-1} + \varepsilon_t \mid \cdot) \\ &= a_0 + a_1 E(y_{t-1} \mid \cdot) + E(\varepsilon_t \mid \cdot) \\ &= a_0 + a_1(a_0 + a_1 E(y_{t-2} \mid \cdot) + E(\varepsilon_{t-1} \mid \cdot)) \\ &\vdots \\ &= \left(1 + a_1 + a_1^2 + \cdots + a_1^{h-1}\right) a_0 + a_1^h y_{t-h} \\ &= \frac{1 - a_1^h}{1 - a_1} a_0 + a_1^h y_{t-h}. \end{aligned}$$

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Moments of the AR(1) Process

The unconditional mean $E(y_t)$ is the limiting case as $h \longrightarrow \infty$:

$$E(y_t) = \lim_{h \to \infty} E(y_t \mid y_{t-h}, y_{t-h-1}, \ldots).$$

Taking the limit yields:

- $E(y_t \mid y_{t-h}, y_{t-h-1}, \ldots) \longrightarrow \frac{a_0}{1-a_1}$ if $|a_1| < 1$;
- $E(y_t \mid y_{t-h}, y_{t-h-1}, \ldots) \longrightarrow$ indeterminate form (i.e. does not exist) if $|a_1| \geq 1$.

Hence, a finite $E(y_t)$ exists if and only if $|a_1| < 1$.

The AR(1) model with $|a_1| \geq 1$ is called unstable.

Instability implies non-stationarity, but not the other way around.

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1. Derive the expected value, variance, covariance, autocorrelation function (ACF), and partial autocorrelation function (PACF) for the following data generating processes (DGPs):

   (a) AR(1): $y_t = a_0 + a_1 y_{t-1} + \epsilon_t$, $0 \leq |a_1| < 1$;

- Variance:

$$\mathrm{Var}(y_t) = \gamma_0 = a_1^2 \mathrm{Var}(y_{t-1}) + \mathrm{Var}(\epsilon_t) + 2\mathrm{cov}(a_1 y_{t-1}, \epsilon_t)$$

$$\gamma_0 = \frac{\sigma^2}{1-a_1^2}; \text{ since } \mathrm{Var}(y_{t-1}) = \gamma_0,\ \mathrm{cov}(y_{t-1}, \epsilon_t) = 0$$

In DGP1: $\gamma_0 = \frac{\sigma^2}{1-a_1^2} \rightarrow$

```
> var(DGP1)
[1] 2.146532
```

$2.146532 = \frac{\sigma^2}{1-0.75^2} \rightarrow$

$\sigma^2 = 0.93910775$

Moments of the AR(1) Process

Variance of y_t conditional on $y_{t-h}, y_{t-h-1}, \ldots$:

$$\begin{aligned}\mathrm{Var}(y_t \mid y_{t-h}, y_{t-h-1}, \ldots) &= \mathrm{Var}\Big(\left(1 + a_1 + a_1^2 + \cdots + a_1^{h-1}\right) a_0 + a_1^h y_{t-h} \\ &\quad + \varepsilon_t + a_1\varepsilon_{t-1} + a_1^2\varepsilon_{t-2} + \cdots + a_1^{h-1}\varepsilon_{t-h+1} \mid \cdot \Big) \\ &= \mathrm{Var}(\varepsilon_t \mid \cdot) + a_1^2 \mathrm{Var}(\varepsilon_{t-1} \mid \cdot) \\ &\quad + a_1^4 \mathrm{Var}(\varepsilon_{t-2} \mid \cdot) + \cdots + a_1^{2(h-1)} \mathrm{Var}(\varepsilon_{t-h+1} \mid \cdot) \\ &= \left(1 + a_1^2 + a_1^4 + \cdots + a_1^{2(h-1)}\right)\sigma_\varepsilon^2 = \frac{1-a_1^{2h}}{1-a_1^2}\sigma_\varepsilon^2.\end{aligned}$$

Covariance between y_t and y_{t-k} conditional on $y_{t-h}, y_{t-h-1}, \ldots$:

$$\mathrm{cov}(y_t, y_{t-k} \mid y_{t-h}, y_{t-h-1}, \ldots) = \frac{1-a_1^{2(h-k)}}{1-a_1^2} a_1^k \sigma_\varepsilon^2, \qquad k < h.$$

The ACF and PACF of an AR(1) Process

The unconditional variance and covariances are obtained in the limit as $h \longrightarrow \infty$.

If the AR(1) is unstable, then the unconditional variance and covariances do not exist. Otherwise:

- $\gamma_0 = \frac{\sigma_\varepsilon^2}{1-a_1^2}$;
- $\gamma_k = \frac{a_1^k \sigma_\varepsilon^2}{1-a_1^2}$, $k = 1, 2, \ldots$;
- $\rho_k = a_1^k$, $k = 1, 2, \ldots$;
- $\phi_{11} = a_1$, $\phi_{kk} = 0$ for all $k \geq 2$.

The ACF of a stable AR(1) decays geometrically as $k \longrightarrow \infty$.

The PACF of a stable AR(1) vanishes for all $k \geq 2$.

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1. Derive the expected value, variance, covariance, autocorrelation function (ACF), and partial autocorrelation function (PACF) for the following data generating processes (DGPs):

   (a) AR(1): $y_t = a_0 + a_1 y_{t-1} + \epsilon_t$, $0 \leq |a_1| < 1$;

- Covariance:
  - Set $a_0 = 0$ without loss of generality

$$\begin{aligned}\operatorname{cov}(y_t, y_{t-k}) = \gamma_k &= \mathrm{E}(y_t y_{t-k}) \\ &= \mathrm{E}((a_1 y_{t-1} + \epsilon_t) y_{t-k})\end{aligned}$$

  - γ_1 $(k = 1)$

$$\begin{aligned}\gamma_1 &= \mathrm{E}((a_1 y_{t-1} + \epsilon_t) y_{t-1}) \\ &= a_1 \frac{\sigma^2}{1 - a_1^2} = a_1 \gamma_0\end{aligned}$$

  - γ_2 $(k = 2)$

$$\begin{aligned}\gamma_2 &= \mathrm{E}((a_1 y_{t-1} + \epsilon_t) y_{t-2}) \\ &= a_1^2 \frac{\sigma^2}{1 - a_1^2} = a_1^2 \gamma_0\end{aligned}$$

  - γ_k $(k > 2)$

$$\begin{aligned}\gamma_k &= \mathrm{E}((a_1 y_{t-1} + \epsilon_t) y_{t-k}) \\ &= a_1^k \frac{\sigma^2}{1 - a_1^2} = a_1^k \gamma_0\end{aligned}$$

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- $\phi_{11} = a_1$, $\phi_{kk} = 0$ for all $k \geq 2$.

The ACF of a stable AR(1) decays geometrically as $k \longrightarrow \infty$.

The PACF of a stable AR(1) vanishes for all $k \geq 2$.

Eric Eisenstat (School of Economics) Applied Econometrics for Macro and Finance Week 2 25 / 28

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1. Derive the expected value, variance, covariance, autocorrelation function (ACF), and partial autocorrelation function (PACF) for the following data generating processes (DGPs):

   (a) AR(1): $y_t = a_0 + a_1 y_{t-1} + \epsilon_t$, $0 \leq |a_1| < 1$;

   - Autocorrelation:
     - ρ_1 $(k = 1)$

$$\rho_1 = \frac{\gamma_1}{\gamma_0} = a_1$$

     - ρ_2 $(k = 2)$

$$\rho_2 = \frac{\gamma_2}{\gamma_0} = a_1^2$$

     - ρ_k $(k > 2)$

$$\rho_k = \frac{\gamma_k}{\gamma_0} = a_1^k$$

The ACF and PACF of an AR(1) Process

The unconditional variance and covariances are obtained in the limit as $h \longrightarrow \infty$.

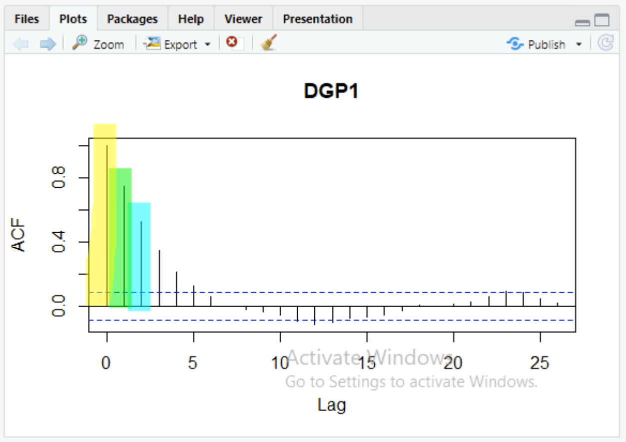
If the AR(1) is unstable, then the unconditional variance and covariances do not exist. Otherwise:

- $\gamma_0 = \frac{\sigma_\varepsilon^2}{1-a_1^2}$;
- $\gamma_k = \frac{a_1^k \sigma_\varepsilon^2}{1-a_1^2}$, $k = 1, 2, \ldots$;
- $\rho_k = a_1^k$, $k = 1, 2, \ldots$;
- $\phi_{11} = a_1$, $\phi_{kk} = 0$ for all $k \geq 2$.

The ACF of a stable AR(1) decays geometrically as $k \longrightarrow \infty$.

The PACF of a stable AR(1) vanishes for all $k \geq 2$.

Eric Eisenstat (School of Economics) Applied Econometrics for Macro and Finance Week 2 25 / 28



1. Derive the expected value, variance, covariance, autocorrelation function (ACF), and partial autocorrelation function (PACF) for the following data generating processes (DGPs):

   (a) AR(1): $y_t = a_0 + a_1 y_{t-1} + \epsilon_t$, $0 \leq |a_1| < 1$;

- Partial autocorrelation:
  - ϕ_{11}

$$\phi_{11} = \rho_1 = a_1$$

  - ϕ_{22}

$$\begin{aligned}\phi_{22} &= (\rho_2 - \rho_1^2)/(1-\rho_1^2)\\ &= (a_1^2 - a_1^2)/(1-a_1^2)\\ &= 0\end{aligned}$$

  - ϕ_{33}

$$\begin{aligned}\phi_{33} &= \frac{\rho_3 - \sum_{j=1}^{3-1}\phi_{2,j}\rho_{3-j}}{1-\sum_{j=1}^{3-1}\phi_{3-1,j}\rho_j}\\ &= \frac{a_1^3 - \phi_{21}\rho_2 - \phi_{22}\rho_1}{1-\sum_{j=1}^{3-1}\phi_{3-1,j}\rho_j}\\ &= \frac{a_1^3 - a_1 a_1^2 + 0}{1-\sum_{j=1}^{3-1}\phi_{3-1,j}\rho_j}\\ &= 0\end{aligned}$$

since

$$\begin{aligned}\phi_{21} &= \phi_{1,1} - \phi_{22}\phi_{1,1}\\ &= \phi_{1,1}\end{aligned}$$

The ACF and PACF of an AR(1) Process

The unconditional variance and covariances are obtained in the limit as $h \longrightarrow \infty$.

If the AR(1) is unstable, then the unconditional variance and covariances do not exist. Otherwise:

- $\gamma_0 = \frac{\sigma_\epsilon^2}{1-a_1^2}$;
- $\gamma_k = \frac{a_1^k \sigma_\epsilon^2}{1-a_1^2}$, $k = 1, 2, \ldots$;
- $\rho_k = a_1^k$, $k = 1, 2, \ldots$;
- $\phi_{11} = a_1$, $\phi_{kk} = 0$ for all $k \geq 2$.

The ACF of a stable AR(1) decays geometrically as $k \longrightarrow \infty$.

The PACF of a stable AR(1) vanishes for all $k \geq 2$.

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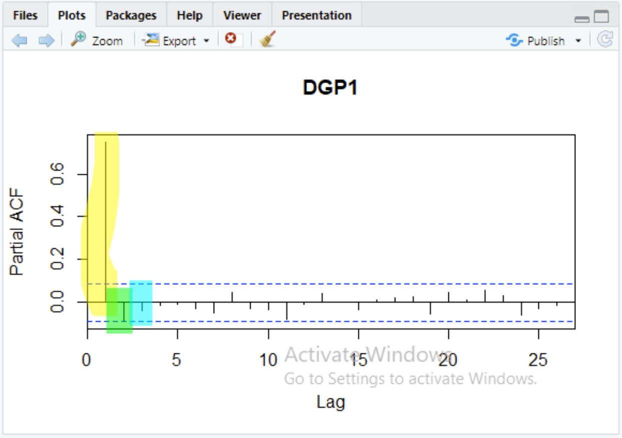
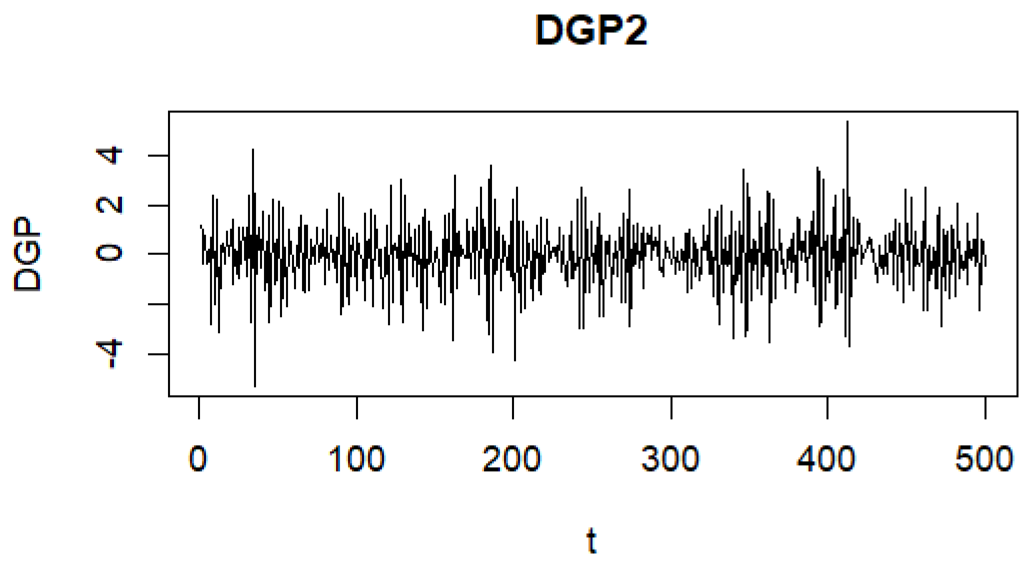


3. The data file `arma.csv` contains (simulated) data for each of the DGPs in Question 2. Import the data into R. Compute, plot, and describe the behaviour of the ACF and PACF for each DGP. Discuss the effects of parameter signs. Hint: use the `acf` and `pacf` commands, respectively.

- DGP2: $y_t = -0.75 y_{t-1} + \epsilon_t$;




```
> arima(mydata$DGP2, order = c(1,0,0))

Call:
arima(x = mydata$DGP2, order = c(1, 0, 0))

Coefficients:
          ar1  intercept
      -0.7652    -0.0121
s.e.   0.0287     0.0242

sigma^2 estimated as 0.909:  log likelihood = -686.06,  aic = 1378.13
```

The First-Order Autoregressive Model

One simple way to model a stochastic process is with the "regression":

$$y_t = a_0 + a_1 y_{t-1} + u_t.$$

This is called the first order auto-regressive model, or AR(1).

To make it useful in practice, we need assumptions about u_t.

The "classical regression" assumptions are:

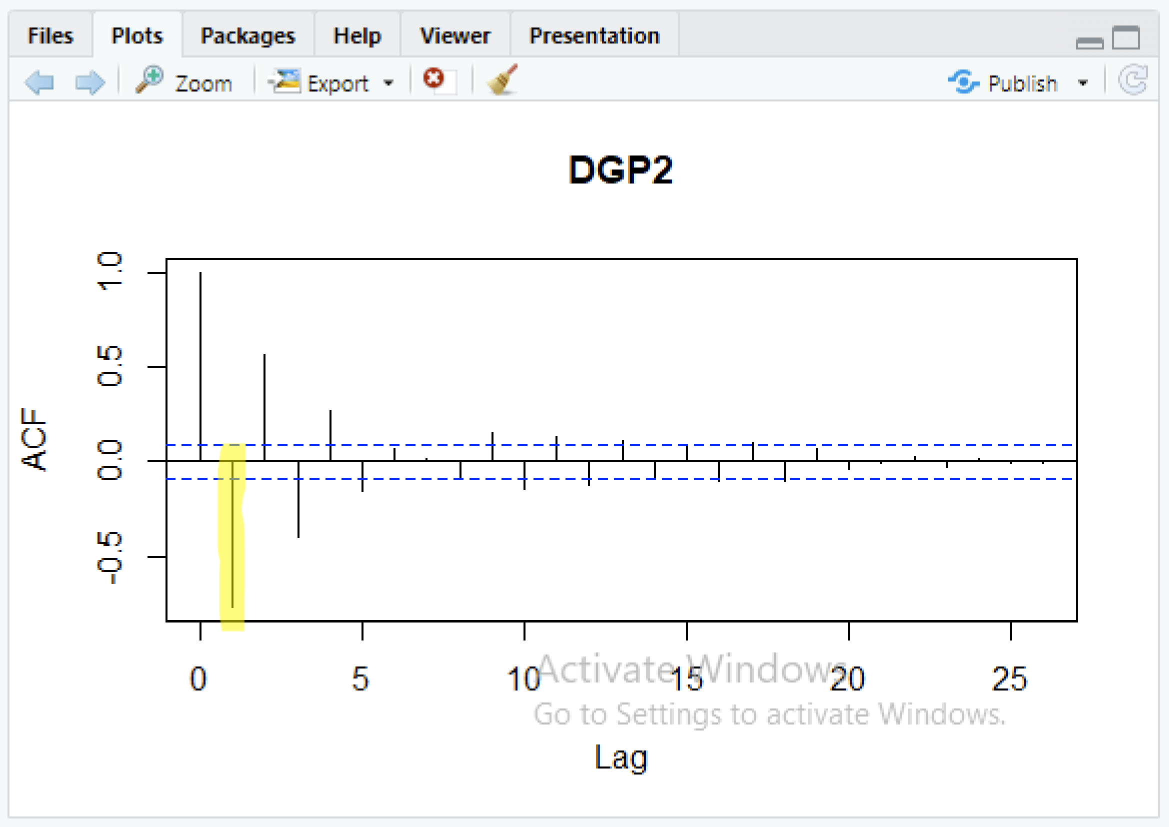
- Mean-independence: $\mathrm{E}(u_t \mid y_{t-1}, y_{t-2}, \ldots) = 0.$
- Homoscedasticity: $\mathrm{Var}(u_t \mid y_{t-1}, y_{t-2}, \ldots) = \sigma_u^2.$

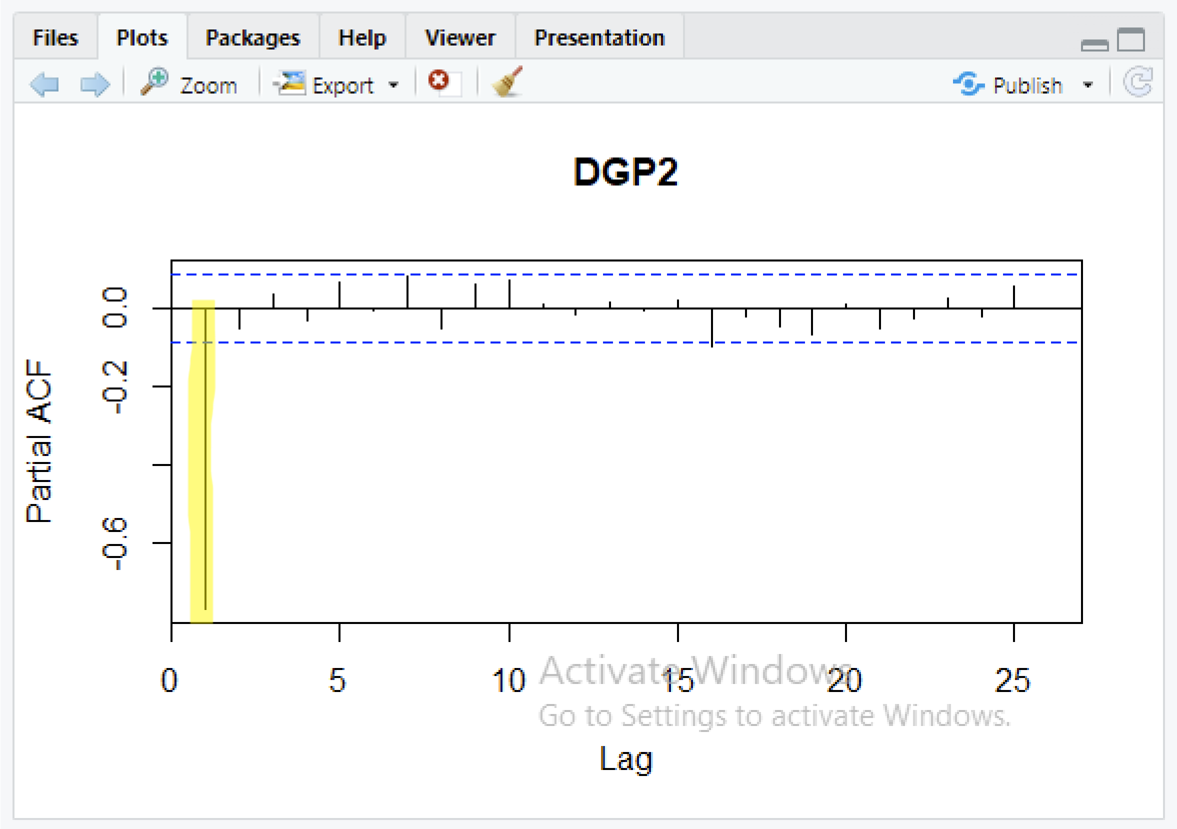
Mean-independence is crucial, but homoscedasticity can be relaxed.

Mean-independence implies zero-autocorrelation: $\mathrm{corr}(u_t, u_{t-k}) = 0$ for $k = 1, 2, \ldots$.

3. The data file `arma.csv` contains (simulated) data for each of the DGPs in Question 2. Import the data into R. Compute, plot, and describe the behaviour of the ACF and PACF for each DGP. Discuss the effects of parameter signs. Hint: use the `acf` and `pacf` commands, respectively.

- DGP2: $y_t = -0.75 y_{t-1} + \epsilon_t$;





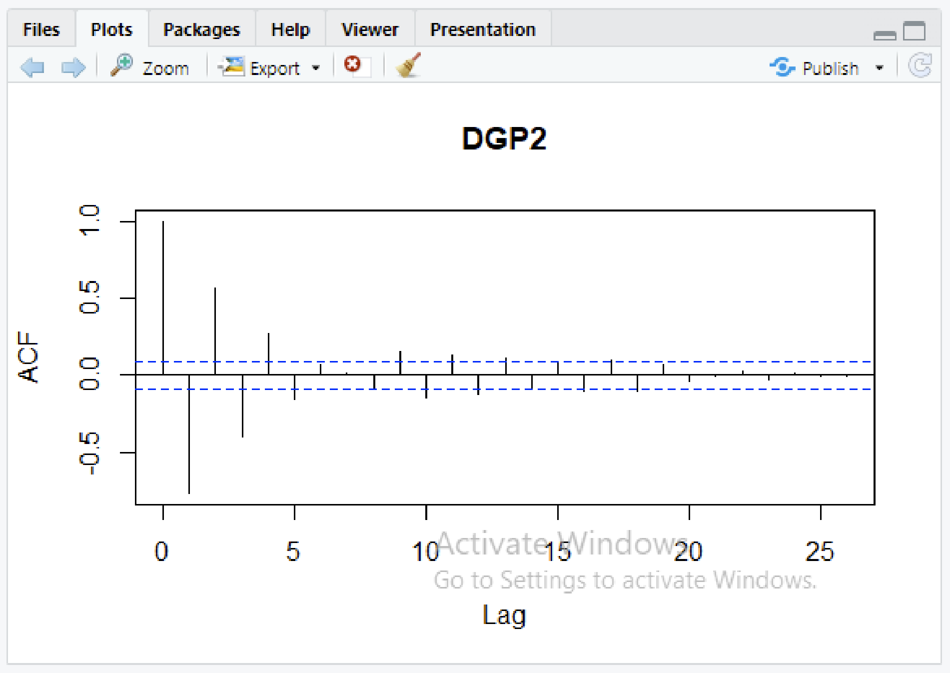
DGP2

- ACF: Decays in a dampened oscillatory path as parameter is negative.
- PACF: One non-zero peak.

2. Compute the true ACF values for the following DGPs:

- DGP1: $y_t = 0.75 y_{t-1} + \epsilon_t$;
- DGP2: $y_t = -0.75 y_{t-1} + \epsilon_t$;

Solution $\rho_0 = 1,\ \rho_1 = -0.75,\ \ldots,\ \rho_k = (-1)^k 0.75^k$. The ACF will decay in a dampened oscillatory path.



The ACF and PACF of an AR(1) Process

The unconditional variance and covariances are obtained in the limit as $h \longrightarrow \infty$.

If the AR(1) is unstable, then the unconditional variance and covariances do not exist. Otherwise:

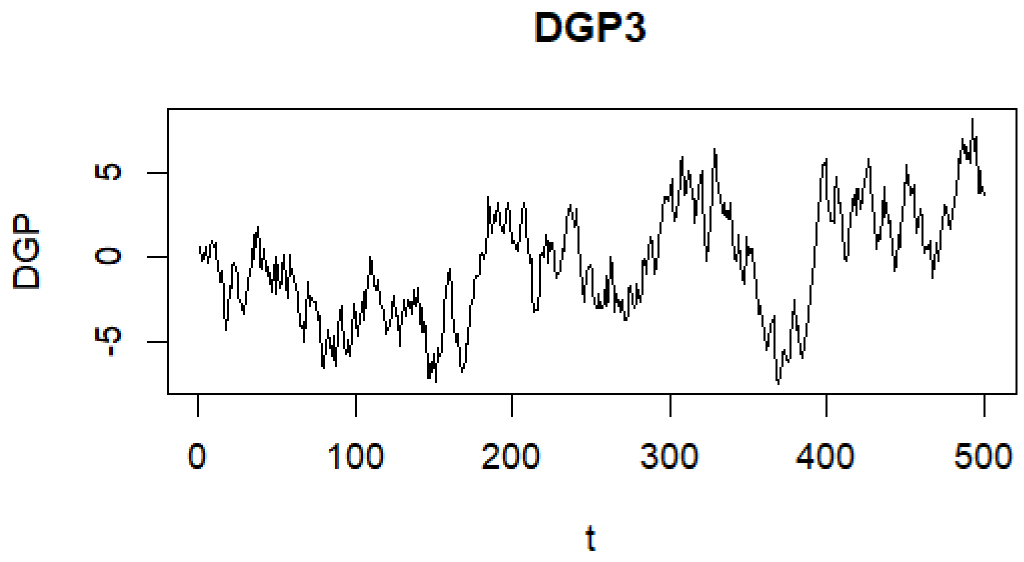
- $\gamma_0 = \frac{\sigma_\varepsilon^2}{1-a_1^2}$;
- $\gamma_k = \frac{a_1^k \sigma_\varepsilon^2}{1-a_1^2}$, $k = 1, 2, \ldots$;
- $\rho_k = a_1^k$, $k = 1, 2, \ldots$;
- $\phi_{11} = a_1$, $\phi_{kk} = 0$ for all $k \geq 2$.

The ACF of a stable AR(1) decays geometrically as $k \longrightarrow \infty$.

The PACF of a stable AR(1) vanishes for all $k \geq 2$.

3. The data file **arma.csv** contains (simulated) data for each of the DGPs in Question 2. Import the data into R. Compute, plot, and describe the behaviour of the ACF and PACF for each DGP. Discuss the effects of parameter signs. Hint: use the **acf** and **pacf** commands, respectively.

- DGP3: $y_t = 0.95 y_{t-1} + \epsilon_t$;




```
> arima(mydata$DGP3, order = c(1,0,0))

Call:
arima(x = mydata$DGP3, order = c(1, 0, 0))

Coefficients:
         ar1  intercept
      0.9519    -0.1912
s.e.  0.0134     0.8950

sigma^2 estimated as 0.996:  log likelihood = -709.65,  aic = 1425.3
```

The First-Order Autoregressive Model

One simple way to model a stochastic process is with the "regression":

$$y_t = a_0 + a_1 y_{t-1} + u_t.$$

This is called the first order auto-regressive model, or AR(1).

To make it useful in practice, we need assumptions about u_t.

The "classical regression" assumptions are:

- Mean-independence: $\mathrm{E}(u_t \mid y_{t-1}, y_{t-2}, \ldots) = 0.$
- Homoscedasticity: $\mathrm{Var}(u_t \mid y_{t-1}, y_{t-2}, \ldots) = \sigma_u^2.$

Mean-independence is crucial, but homoscedasticity can be relaxed.

Mean-independence implies zero-autocorrelation: $\mathrm{corr}(u_t, u_{t-k}) = 0$ for $k = 1, 2, \ldots$.

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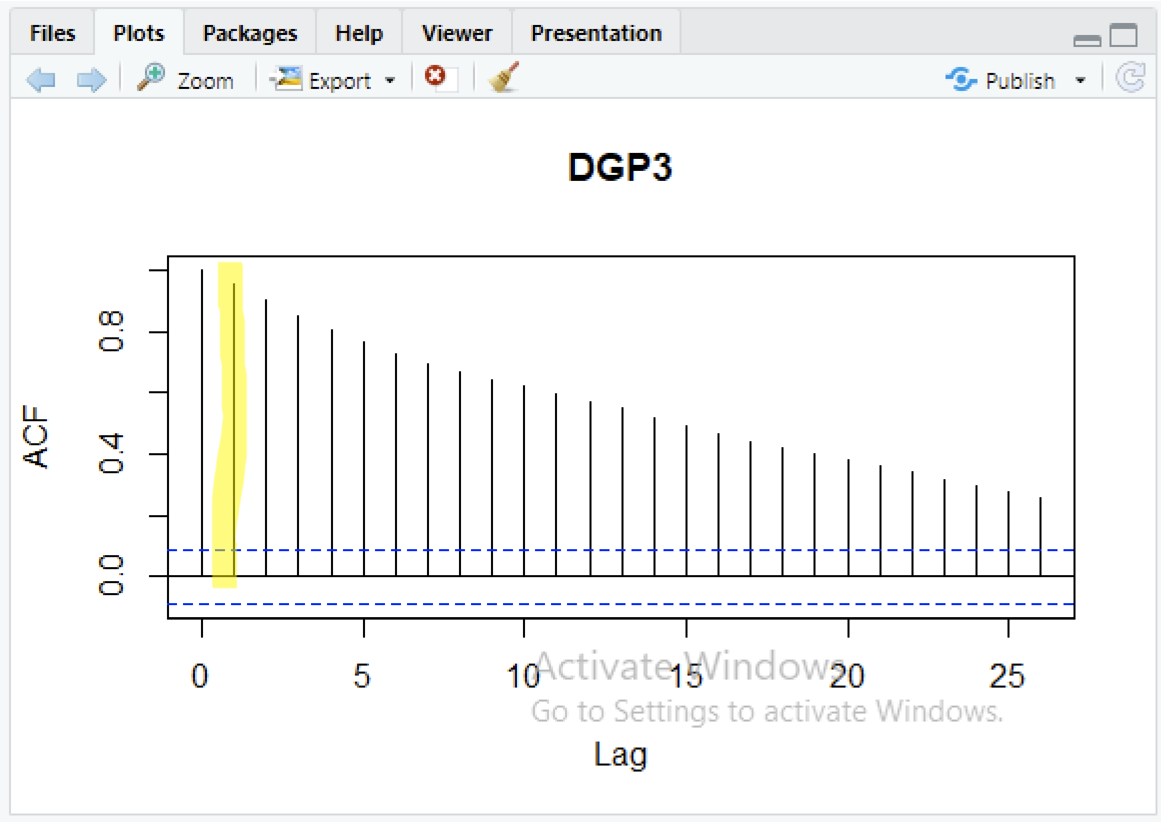
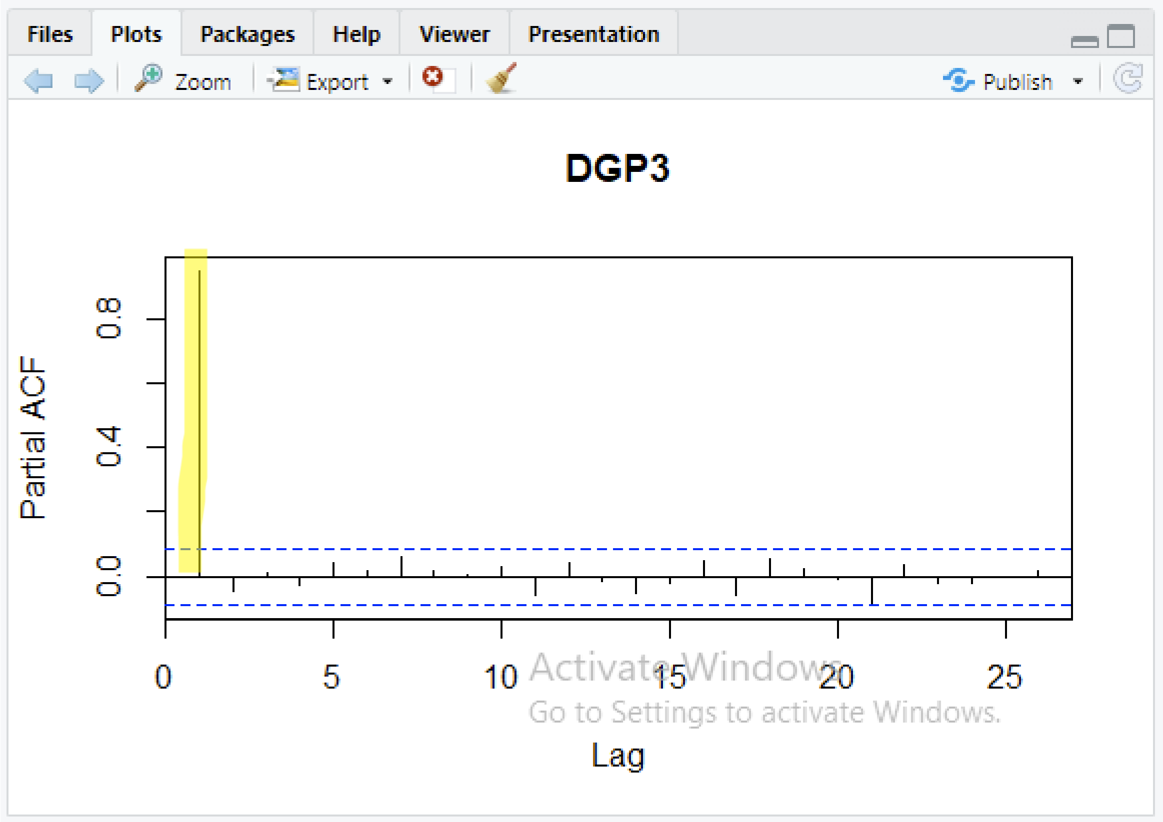
3. The data file `arma.csv` contains (simulated) data for each of the DGPs in Question 2. Import the data into R. Compute, plot, and describe the behaviour of the ACF and PACF for each DGP. Discuss the effects of parameter signs. Hint: use the `acf` and `pacf` commands, respectively.

- DGP3: $y_t = 0.95 y_{t-1} + \epsilon_t$;





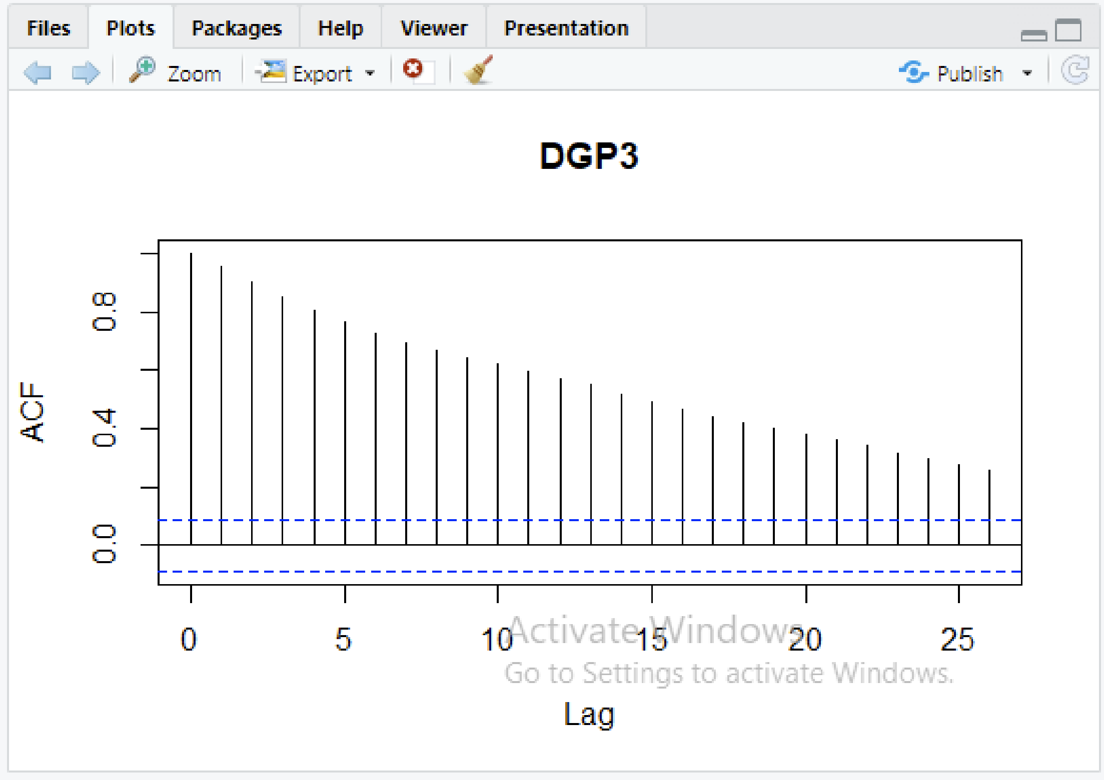
DGP3

- ACF: Decays geometrically but slower than DGP1.
- PACF: One non-zero peak.

2. Compute the true ACF values for the following DGPs:

- DGP1: $y_t = 0.75 y_{t-1} + \epsilon_t$;
- DGP2: $y_t = -0.75 y_{t-1} + \epsilon_t$;
- DGP3: $y_t = 0.95 y_{t-1} + \epsilon_t$;

Solution $\rho_0 = 1$, $\rho_1 = 0.95$, $\ldots$, $\rho_k = 0.95^k$. The ACF will decay geometrically but at a much slower rate than DGP1.



The ACF and PACF of an AR(1) Process

The unconditional variance and covariances are obtained in the limit as $h \longrightarrow \infty$.

If the AR(1) is unstable, then the unconditional variance and covariances do not exist. Otherwise:

- $\gamma_0 = \frac{\sigma_\varepsilon^2}{1-a_1^2}$;
- $\gamma_k = \frac{a_1^k \sigma_\varepsilon^2}{1-a_1^2}$, $k = 1, 2, \ldots$;
- $\rho_k = a_1^k$, $k = 1, 2, \ldots$;
- $\phi_{11} = a_1$, $\phi_{kk} = 0$ for all $k \geq 2$.

The ACF of a stable AR(1) decays geometrically as $k \longrightarrow \infty$.

The PACF of a stable AR(1) vanishes for all $k \geq 2$.

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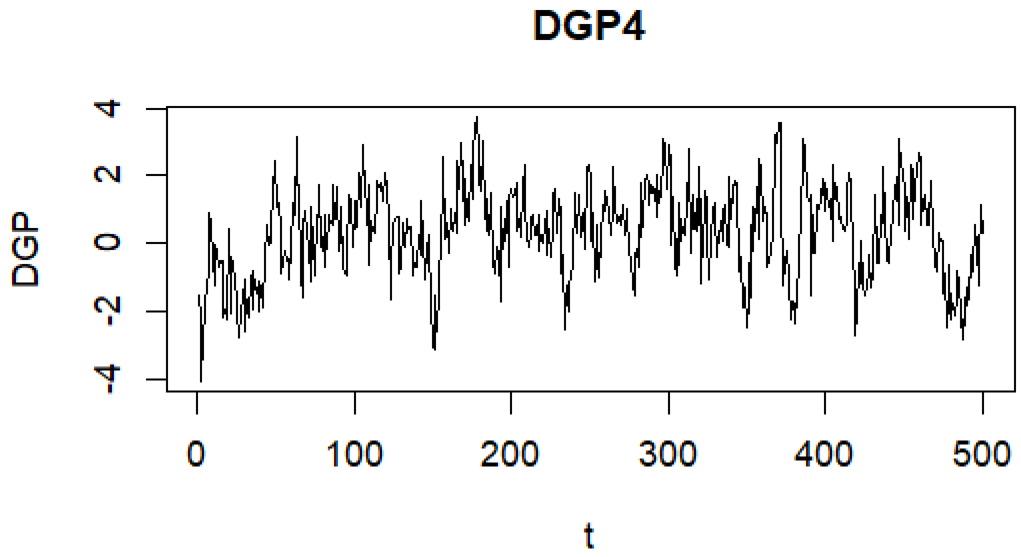
3. The data file `arma.csv` contains (simulated) data for each of the DGPs in Question 2. Import the data into R. Compute, plot, and describe the behaviour of the ACF and PACF for each DGP. Discuss the effects of parameter signs. Hint: use the `acf` and `pacf` commands, respectively.

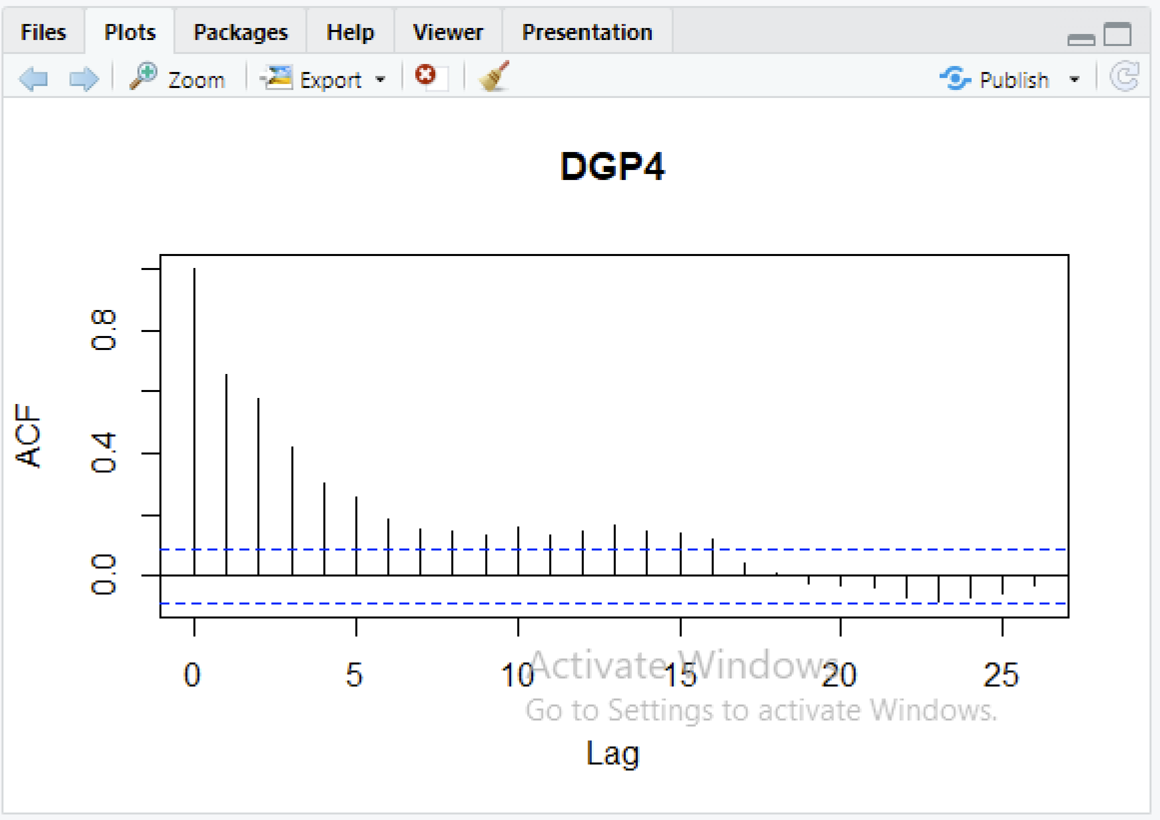
- DGP4: $y_t = 0.5y_{t-1} + 0.25y_{t-2} + \epsilon_t;$

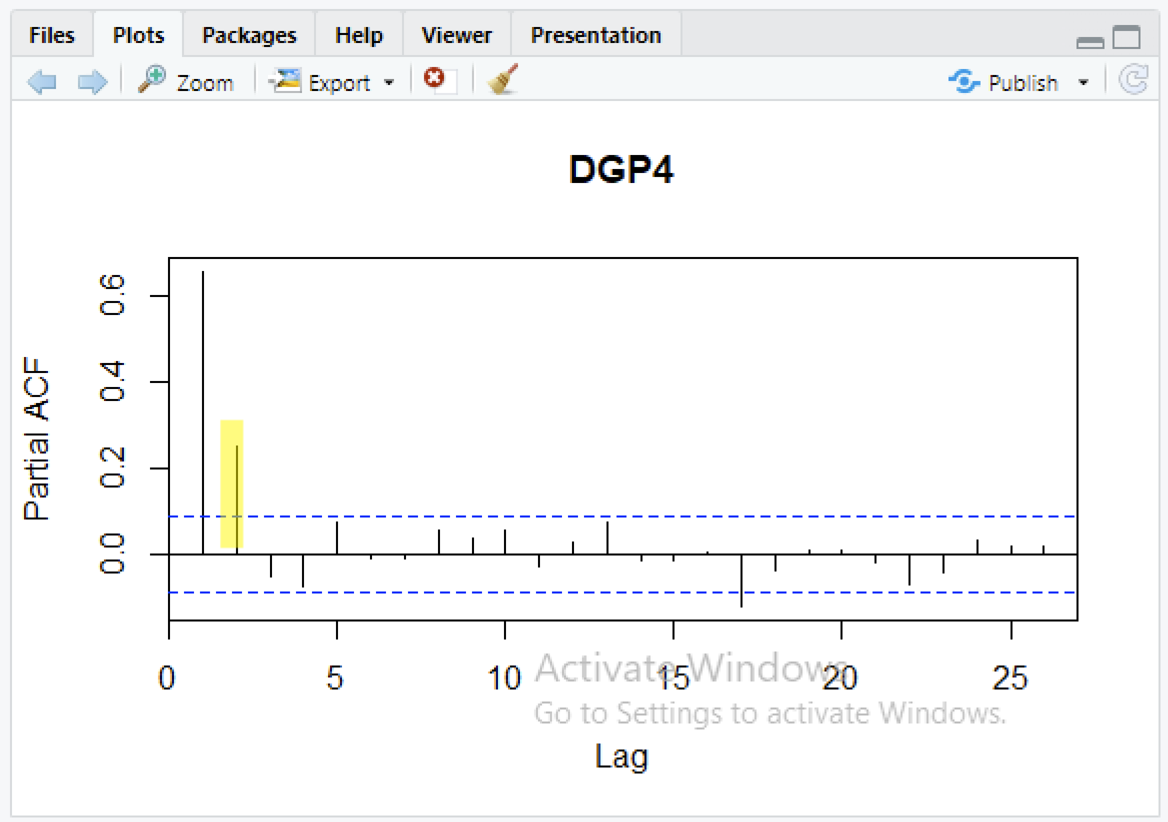



```
> arima(DGP4, order = c(2,0,0))

Call:
arima(x = DGP4, order = c(2, 0, 0))

Coefficients:
         ar1     ar2  intercept
      0.4877  0.2585     0.3194
s.e.  0.0431  0.0433     0.1719

sigma^2 estimated as 0.9668:  log likelihood = -701.37,  aic = 1410.74
```

3. The data file `arma.csv` contains (simulated) data for each of the DGPs in Question 2. Import the data into R. Compute, plot, and describe the behaviour of the ACF and PACF for each DGP. Discuss the effects of parameter signs. Hint: use the `acf` and `pacf` commands, respectively.

- DGP4: $y_t = 0.5y_{t-1} + 0.25y_{t-2} + \epsilon_t$;





DGP4
- ACF: Decays geometrically as parameter is positive.
- PACF: Two non-zero peaks.

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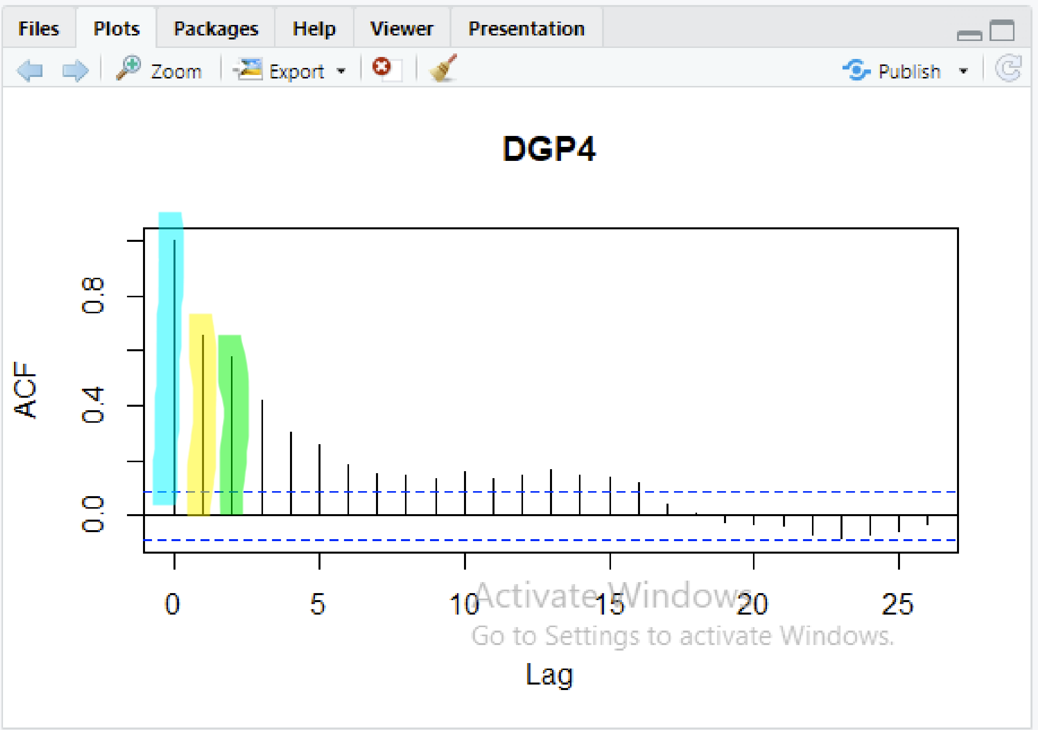
2. Compute the true ACF values for the following DGPs:

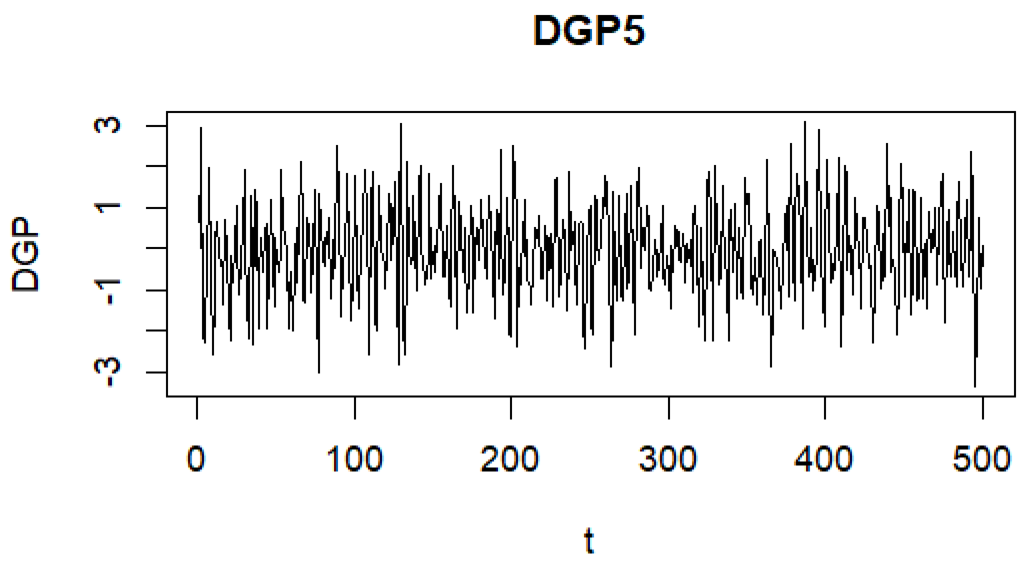
- DGP1: $y_t = 0.75y_{t-1} + \epsilon_t$;
- DGP2: $y_t = -0.75y_{t-1} + \epsilon_t$;
- DGP3: $y_t = 0.95y_{t-1} + \epsilon_t$;
- DGP4: $y_t = 0.5y_{t-1} + 0.25y_{t-2} + \epsilon_t$;

Solution For the AR(2) model $y_t = a_0 + a_1y_{t-1} + a_2y_{t-2} + \epsilon_t$, $\rho_0 = 1$, $\rho_1 = a_1/(1-a_2)$, …, $\rho_k = a_1\rho_{k-1} + a_2\rho_{k-2}$. Thus, $\rho_0 = 1$, $\rho_1 = 2/3$, $\rho_2 = 7/12$, …, $\rho_k = a_1\rho_{k-1} + a_2\rho_{k-2}$ for $k \geq 2$.



3. The data file `arma.csv` contains (simulated) data for each of the DGPs in Question 2. Import the data into R. Compute, plot, and describe the behaviour of the ACF and PACF for each DGP. Discuss the effects of parameter signs. Hint: use the `acf` and `pacf` commands, respectively.

- DGP5: $y_t = 0.25 y_{t-1} - 0.5 y_{t-2} + \epsilon_t$;




```
> arima(DGP5, order = c(2,0,0))

Call:
arima(x = DGP5, order = c(2, 0, 0))

Coefficients:
         ar1     ar2  intercept
      0.2095  -0.488    -0.0398
s.e.  0.0390   0.039     0.0354

sigma^2 estimated as 1.019:  log likelihood = -714.57,  aic = 1437.13
```

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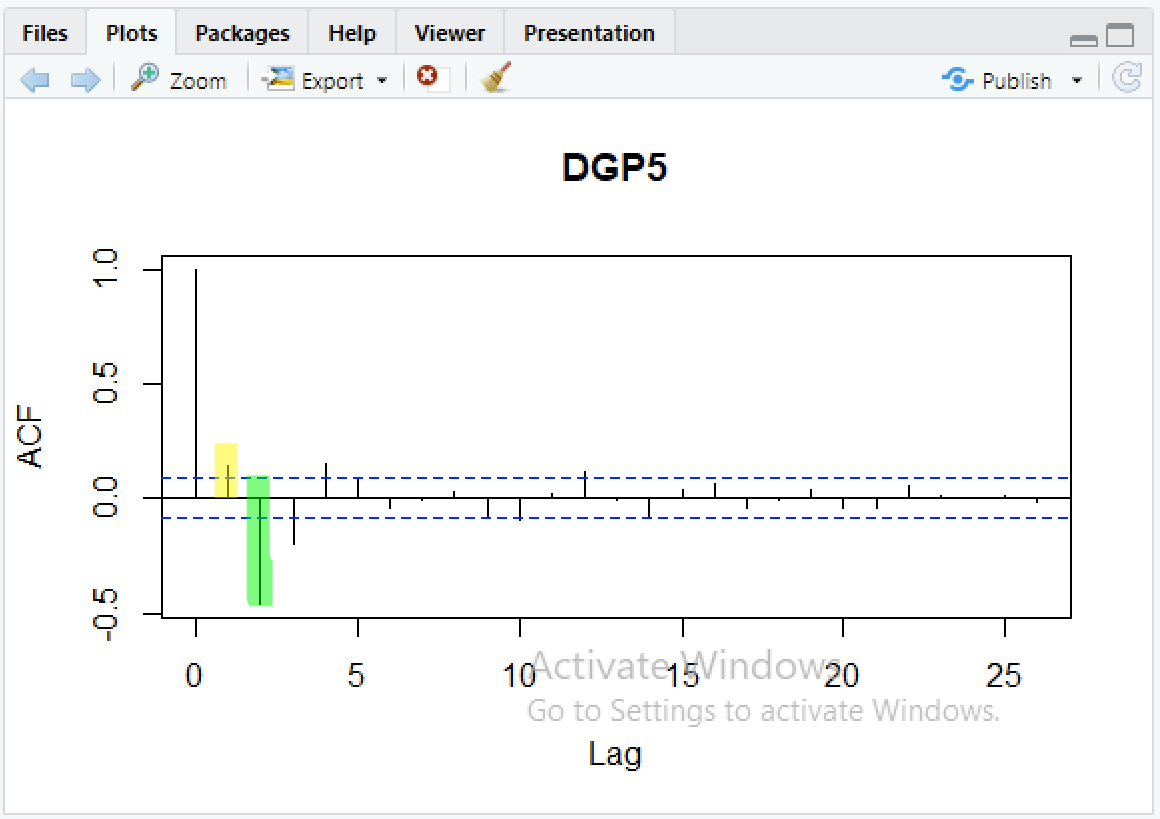
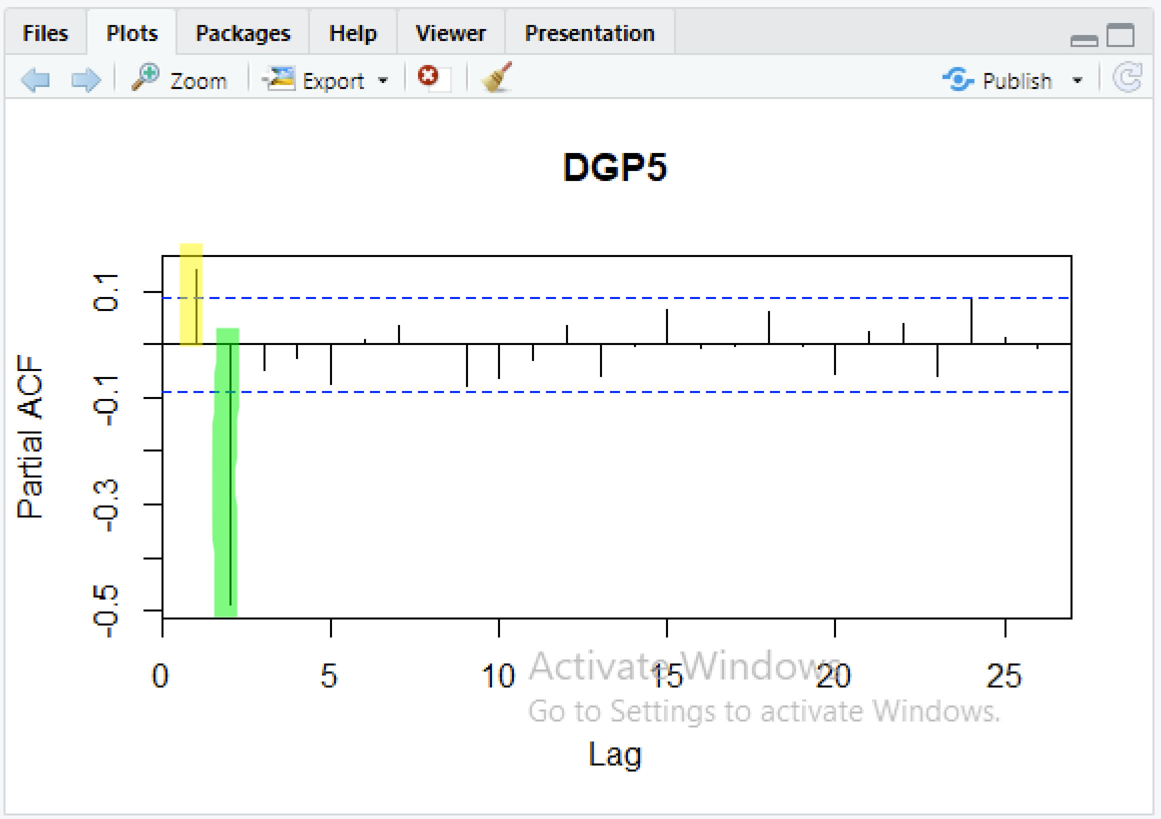
3. The data file `arma.csv` contains (simulated) data for each of the DGPs in Question 2. Import the data into R. Compute, plot, and describe the behaviour of the ACF and PACF for each DGP. Discuss the effects of parameter signs. Hint: use the `acf` and `pacf` commands, respectively.

- DGP5: $y_t = 0.25y_{t-1} - 0.5y_{t-2} + \epsilon_t$;





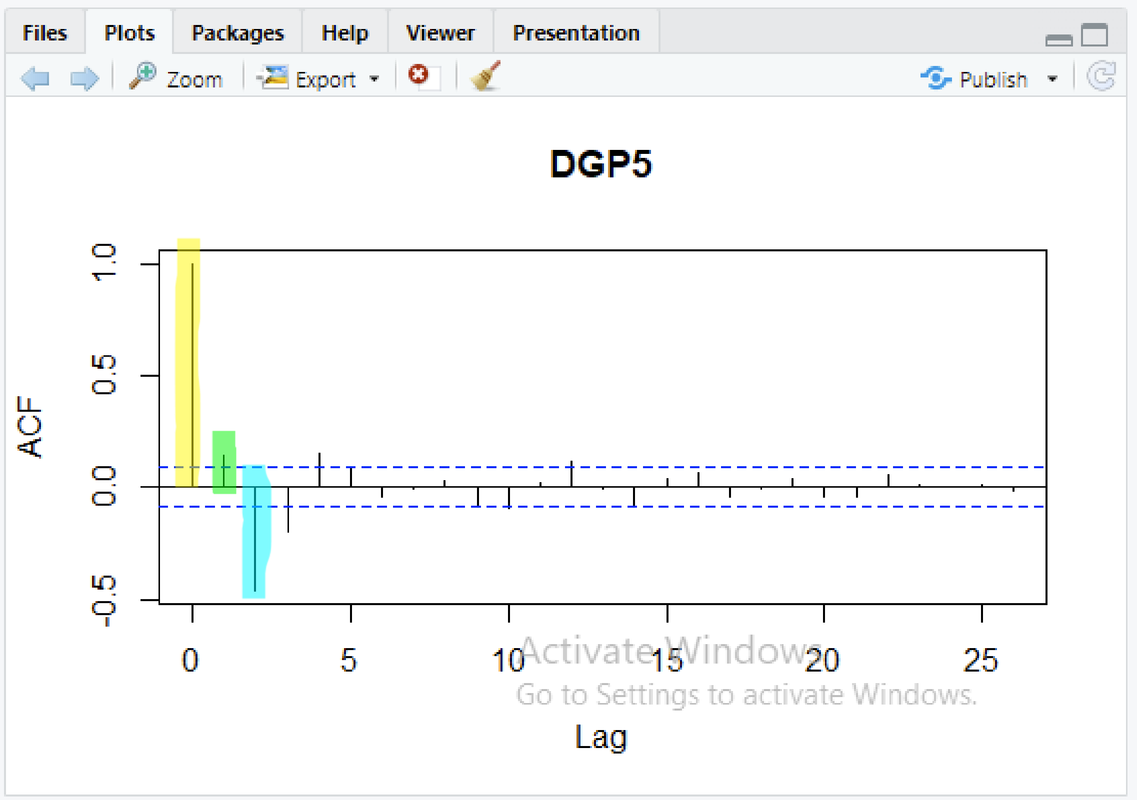
DGP5

- ACF: Decays in an oscillatory path as one parameter is negative (and large in absolute value).
- PACF: Two non-zero peaks.

2. Compute the true ACF values for the following DGPs:

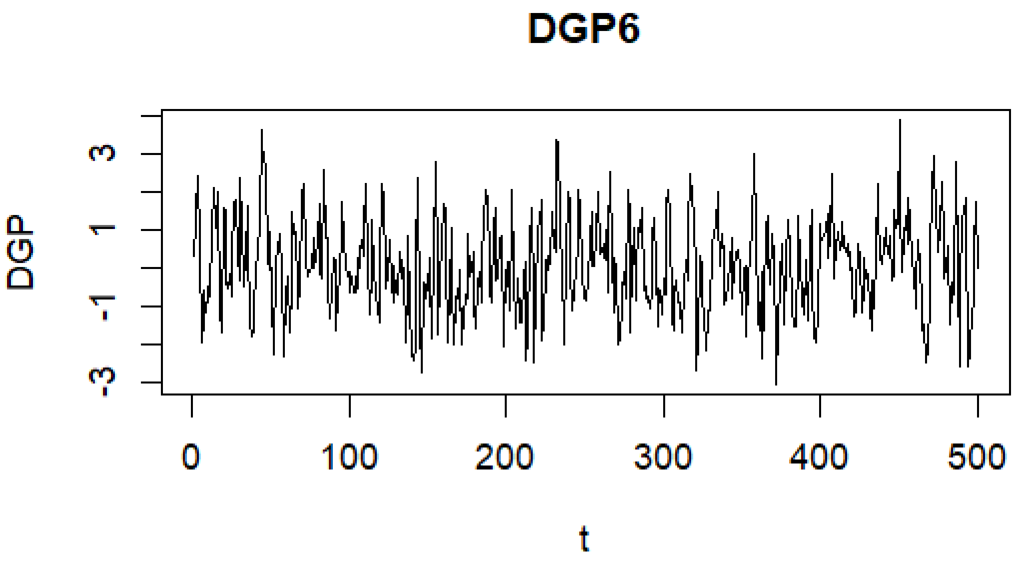
- DGP1: $y_t = 0.75y_{t-1} + \epsilon_t$;
- DGP2: $y_t = -0.75y_{t-1} + \epsilon_t$;
- DGP3: $y_t = 0.95y_{t-1} + \epsilon_t$;
- DGP4: $y_t = 0.5y_{t-1} + 0.25y_{t-2} + \epsilon_t$;
- DGP5: $y_t = 0.25y_{t-1} - 0.5y_{t-2} + \epsilon_t$;

Solution $\rho_0 = 1$, $\rho_1 = 1/6$, $\rho_2 = -11/24$, $\ldots$, $\rho_k = a_1\rho_{k-1} + a_2\rho_{k-2}$ for $k \geq 2$.



3. The data file `arma.csv` contains (simulated) data for each of the DGPs in Question 2. Import the data into R. Compute, plot, and describe the behaviour of the ACF and PACF for each DGP. Discuss the effects of parameter signs. Hint: use the `acf` and `pacf` commands, respectively.

- DGP6: $y_t = 0.75\epsilon_{t-1} + \epsilon_t$;




```
> arima(DGP, order = c(0,0,1)) # AR(2)

Call:
arima(x = DGP, order = c(0, 0, 1))

Coefficients:
         ma1  intercept
      0.7268     0.0321
s.e.  0.0297     0.0755

sigma^2 estimated as 0.9582:  log likelihood = -699.16,  aic = 1404.32
```

The First-Order Moving Average Model

What if the errors $u_1, \dots, u_T$ are also correlated?

Correlated errors could be modelled, for example, by

$$u_t = \varepsilon_t + b_1\varepsilon_{t-1},$$

where ε_t is the uncorrelated innovation in the process.

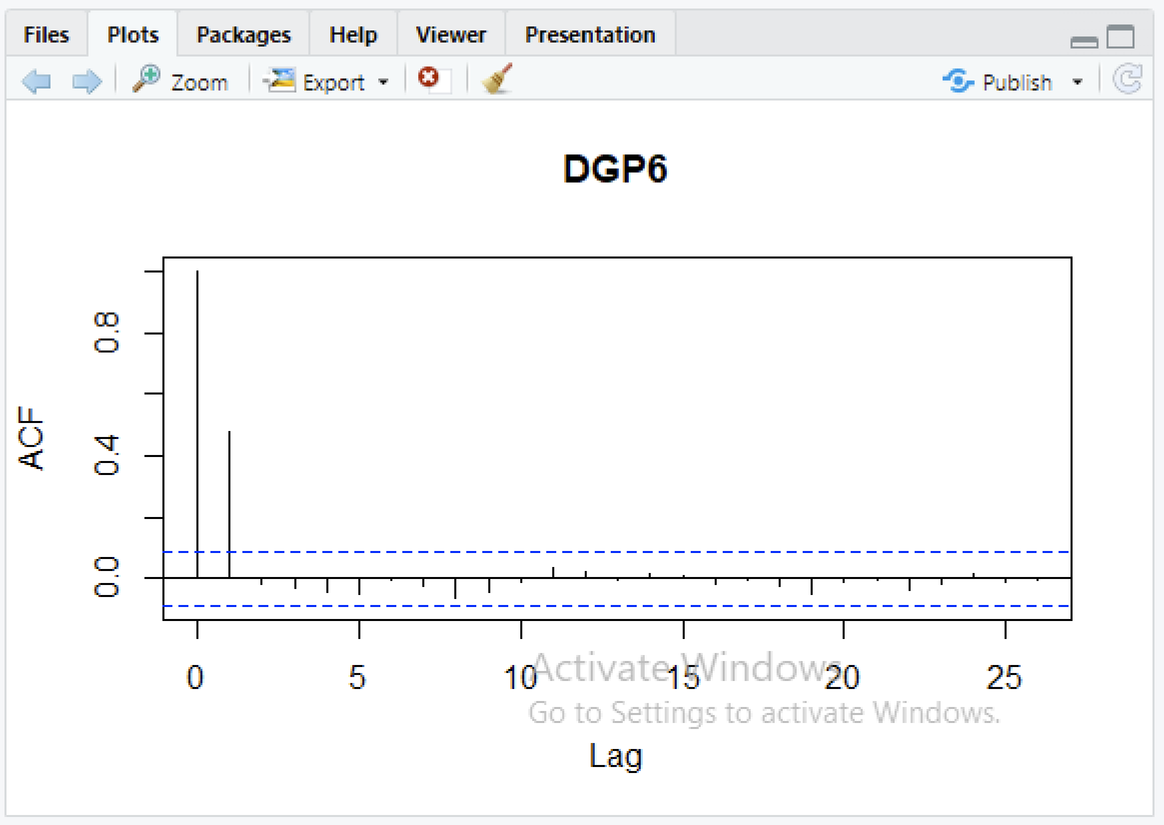
The above is called a first-order moving average MA(1) process for u_t.

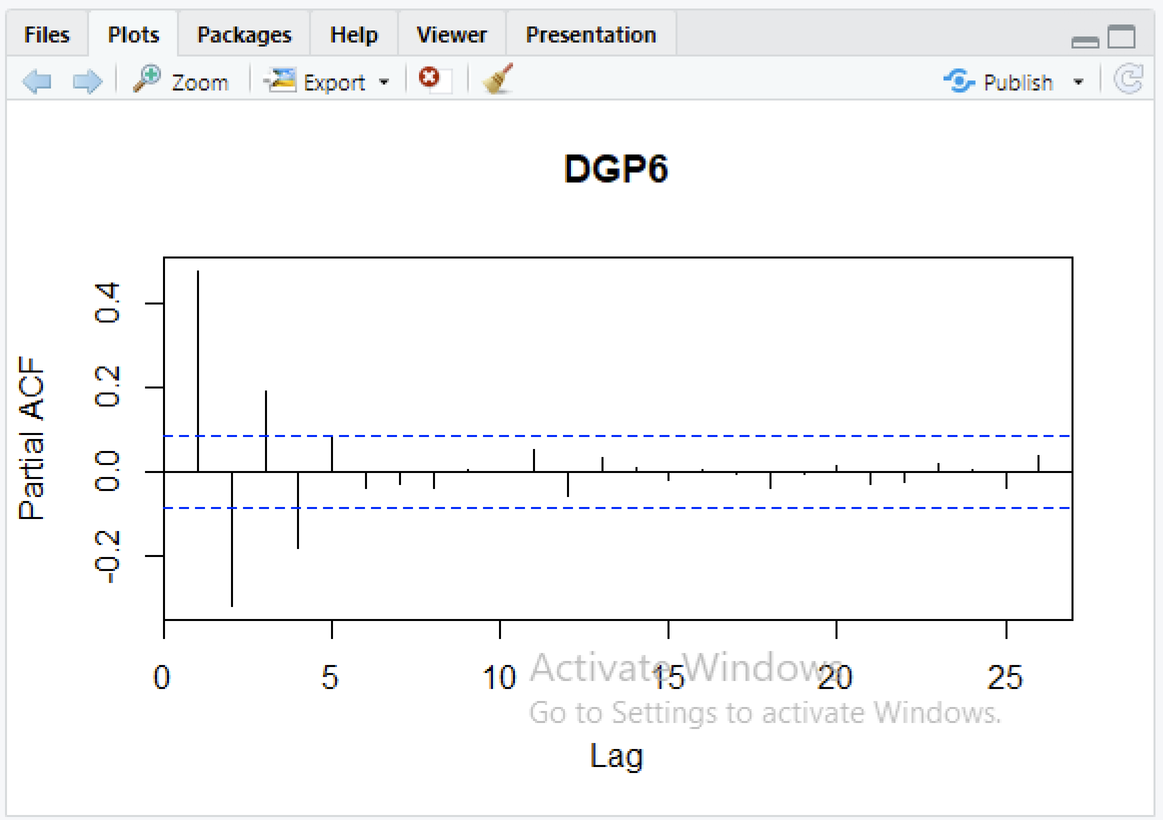
In this case, assumptions are placed on ε_t:

- Mean-independence: $\mathsf{E}(\varepsilon_t \mid y_{t-1}, y_{t-2}, \dots) = 0$.
- Homoscedasticity: $\mathsf{Var}(\varepsilon_t \mid y_{t-1}, y_{t-2}, \dots) = \sigma_\varepsilon^2$.

3. The data file **arma.csv** contains (simulated) data for each of the DGPs in Question 2. Import the data into R. Compute, plot, and describe the behaviour of the ACF and PACF for each DGP. Discuss the effects of parameter signs. Hint: use the **acf** and **pacf** commands, respectively.

- DGP6: $y_t = 0.75\epsilon_{t-1} + \epsilon_t$;





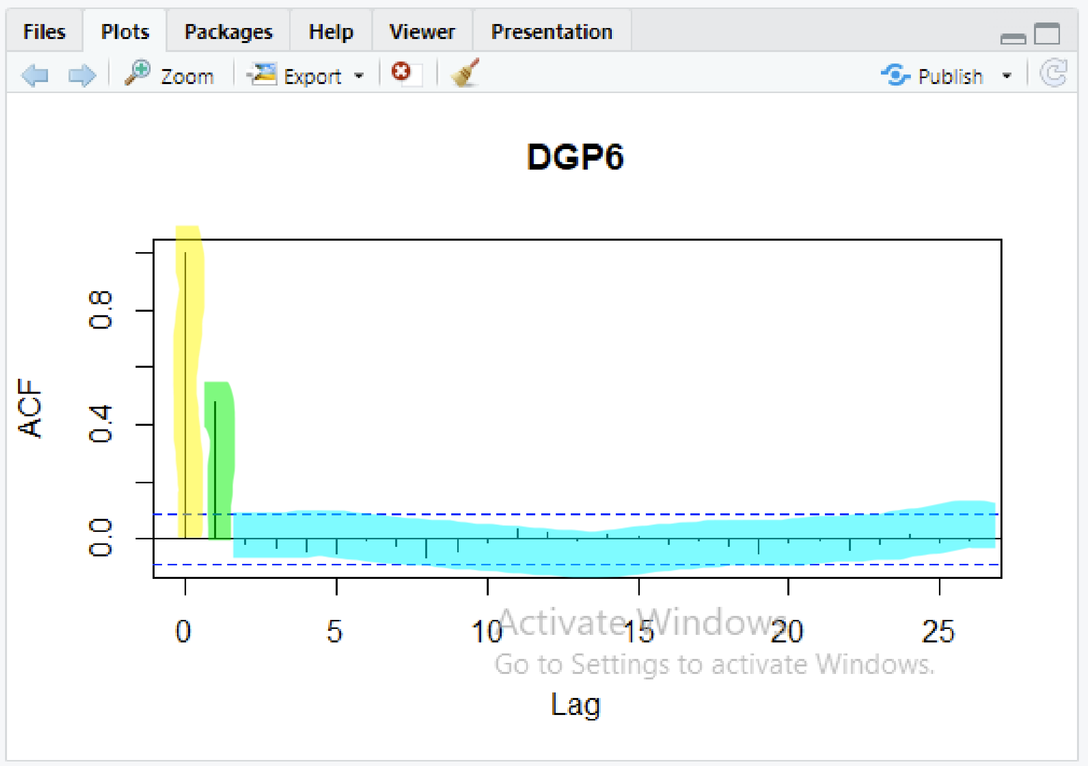
DGP6

- ACF: One non-zero peak.
- PACF: Decays in an oscillatory path.

2. Compute the true ACF values for the following DGPs:

- DGP1: $y_t = 0.75y_{t-1} + \epsilon_t$;
- DGP2: $y_t = -0.75y_{t-1} + \epsilon_t$;
- DGP3: $y_t = 0.95y_{t-1} + \epsilon_t$;
- DGP4: $y_t = 0.5y_{t-1} + 0.25y_{t-2} + \epsilon_t$;
- DGP5: $y_t = 0.25y_{t-1} - 0.5y_{t-2} + \epsilon_t$;
- DGP6: $y_t = 0.75\epsilon_{t-1} + \epsilon_t$;

Solution For the MA(q) model $y_t = b_0 + b_1\epsilon_{t-1} + \cdots + b_q\epsilon_{t-q} + \epsilon_t$, the ACF cuts off at $k = q$—i.e., $\rho_k = 0$ for all $k > q$. Thus, $\rho_0 = 1$, $\rho_1 = b_1/(1 + b_1^2) = 12/25$, $\rho_k = 0$ for $k \geq 2$.



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1. Derive the expected value, variance, covariance, autocorrelation function (ACF), and partial autocorrelation function (PACF) for the following data generating processes (DGPs):

   (a) AR(1): $y_t = a_0 + a_1 y_{t-1} + \epsilon_t$, $0 \leq |a_1| < 1$;

   (b) MA(1): $y_t = b_0 + b_1 \epsilon_{t-1} + \epsilon_t$;

- Expected value:

$$\begin{aligned} \mathrm{E}(y_t) &= b_0 + b_1 \mathrm{E}(\epsilon_{t-1}) + \mathrm{E}(\epsilon_t) \\ &= \mu \end{aligned}$$

Moments of the MA(1) Process

The unconditional mean of y_t is:

$$\mathsf{E}(y_t) = \mathsf{E}(a_0 + b_1 \varepsilon_{t-1} + \varepsilon_t) = a_0.$$

The unconditional variance of y_t is:

$$\mathsf{Var}(y_t) = \mathsf{Var}(a_0 + b_1 \varepsilon_{t-1} + \varepsilon_t) = (1 + b_1^2)\sigma_\varepsilon^2.$$

The unconditional covariance between y_t and y_{t-k} is:

$$\begin{aligned} \mathsf{cov}(y_t, y_{t-k}) &= \mathsf{cov}(a_0 + b_1 \varepsilon_{t-1} + \varepsilon_t, a_0 + b_1 \varepsilon_{t-k-1} + \varepsilon_{t-k}) \\ &= \mathsf{E}\left((b_1 \varepsilon_{t-1} + \varepsilon_t)(b_1 \varepsilon_{t-k-1} + \varepsilon_{t-k})\right) \\ &= b_1 \sigma_\varepsilon^2 \text{ if } k = 1; \text{ 0 for all } k \geq 2. \end{aligned}$$

Moments conditional on $y_{t-1}, y_{t-2}, \ldots$, etc. are complicated, but y_t is independent of $y_{t-2}, y_{t-3}, \ldots$.

Eric Eisenstat (School of Economics) Applied Econometrics for Macro and Finance Week 2 26 / 28

In DGP6:

```
> mean(DGP) # mean or expected value
[1] 0.03376414
```

$\mu = 0.03376$

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1. Derive the expected value, variance, covariance, autocorrelation function (ACF), and partial autocorrelation function (PACF) for the following data generating processes (DGPs):

   (a) AR(1): $y_t = a_0 + a_1 y_{t-1} + \epsilon_t$, $0 \leq |a_1| < 1$;

   (b) MA(1): $y_t = b_0 + b_1 \epsilon_{t-1} + \epsilon_t$;

- Variance:

$$\mathrm{Var}(y_t) = \gamma_0 = b_1^2 \mathrm{Var}(\epsilon_{t-1}) + \mathrm{Var}(\epsilon_t) + 2\mathrm{cov}(\epsilon_t, \epsilon_{t-1})$$
$$\gamma_0 = \sigma^2(1 + b_1^2)$$

Moments of the MA(1) Process

The unconditional mean of y_t is:

$$\mathsf{E}(y_t) = \mathsf{E}(a_0 + b_1\varepsilon_{t-1} + \varepsilon_t) = a_0.$$

The unconditional variance of y_t is:

$$\mathsf{Var}(y_t) = \mathsf{Var}(a_0 + b_1\varepsilon_{t-1} + \varepsilon_t) = (1 + b_1^2)\sigma_\varepsilon^2.$$

The unconditional covariance between y_t and y_{t-k} is:

$$\begin{aligned}\mathsf{cov}(y_t, y_{t-k}) &= \mathsf{cov}(a_0 + b_1\varepsilon_{t-1} + \varepsilon_t, a_0 + b_1\varepsilon_{t-k-1} + \varepsilon_{t-k}) \\ &= \mathsf{E}\left((b_1\varepsilon_{t-1} + \varepsilon_t)(b_1\varepsilon_{t-k-1} + \varepsilon_{t-k})\right) \\ &= b_1\sigma_\varepsilon^2 \text{ if } k = 1;\ 0 \text{ for all } k \geq 2.\end{aligned}$$

Moments conditional on $y_{t-1}, y_{t-2}, \ldots$, etc. are complicated, but y_t is independent of $y_{t-2}, y_{t-3}, \ldots$.

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In DGP6:

```
> var(DGP) # variance
[1] 1.49342
```

$\gamma_0 = (1 + b_1^2)\sigma^2 \rightarrow$

$1.49342 = (1 + 0.75^2)\sigma^2 \rightarrow$

$\sigma^2 = 0.9557888$

1. Derive the expected value, variance, covariance, autocorrelation function (ACF), and partial autocorrelation function (PACF) for the following data generating processes (DGPs):

   (a) AR(1): $y_t = a_0 + a_1 y_{t-1} + \epsilon_t$, $0 \leq |a_1| < 1$;

   (b) MA(1): $y_t = b_0 + b_1 \epsilon_{t-1} + \epsilon_t$;

- Covariance:
  - Set $\mu = 0$ without loss of generality

$$\begin{aligned} \text{cov}(y_t, y_{t-k}) = \gamma_k &= \text{E}(y_t y_{t-k}) \\ &= \text{E}((b_1 \epsilon_{t-1} + \epsilon_t) y_{t-k}) \end{aligned}$$

$\text{cov}(y_t, y_{t-k}) > 0$ for $k = 1$, $\text{cov}(y_t, y_{t-k}) = 0$ for $k > 1$

  - γ_1 $(k = 1)$

$$\begin{aligned} \gamma_1 &= \text{E}((b_1 \epsilon_{t-1} + \epsilon_t) y_{t-1}) \\ &= \text{E}(b_1 \epsilon_{t-1}(b_1 \epsilon_{t-2} + \epsilon_{t-1}) + \epsilon_t y_{t-1}) \\ &= b_1 \sigma^2 \\ &= \frac{b_1}{1 + b_1^2} \times \gamma_0; \text{ since } \sigma^2 = \gamma_0/(1 + b_1^2) \end{aligned}$$

  - γ_2 $(k = 2)$

$$\begin{aligned} \gamma_2 &= \text{E}((b_1 \epsilon_{t-1} + \epsilon_t) y_{t-2}) \\ &= 0; \text{ since } y_{t-2} \text{ is not a function of } \epsilon_t \text{ or } \epsilon_{t-1} \end{aligned}$$

  - γ_k $(k > 2)$

$$\begin{aligned} \gamma_k &= \text{E}((b_1 \epsilon_{t-1} + \epsilon_t) y_{t-k}) \\ &= 0 \end{aligned}$$

Moments of the MA(1) Process

The unconditional mean of y_t is:

$$\text{E}(y_t) = \text{E}(a_0 + b_1 \varepsilon_{t-1} + \varepsilon_t) = a_0.$$

The unconditional variance of y_t is:

$$\text{Var}(y_t) = \text{Var}(a_0 + b_1 \varepsilon_{t-1} + \varepsilon_t) = (1 + b_1^2)\sigma_\varepsilon^2.$$

The unconditional covariance between y_t and y_{t-k} is:

$$\begin{aligned} \text{cov}(y_t, y_{t-k}) &= \text{cov}(a_0 + b_1 \varepsilon_{t-1} + \varepsilon_t, a_0 + b_1 \varepsilon_{t-k-1} + \varepsilon_{t-k}) \\ &= \text{E}\left((b_1 \varepsilon_{t-1} + \varepsilon_t)(b_1 \varepsilon_{t-k-1} + \varepsilon_{t-k})\right) \\ &= b_1 \sigma_\varepsilon^2 \text{ if } k = 1; 0 \text{ for all } k \geq 2. \end{aligned}$$

Moments conditional on $y_{t-1}, y_{t-2}, \ldots$, etc. are complicated, but y_t is independent of $y_{t-2}, y_{t-3}, \ldots$.

Eric Eisenstat (School of Economics) | Applied Econometrics for Macro and Finance | Week 2 | 26 / 28

1. Derive the expected value, variance, covariance, autocorrelation function (ACF), and partial autocorrelation function (PACF) for the following data generating processes (DGPs):

   (a) AR(1): $y_t = a_0 + a_1 y_{t-1} + \epsilon_t$, $0 \leq |a_1| < 1$;

   (b) MA(1): $y_t = b_0 + b_1 \epsilon_{t-1} + \epsilon_t$;

- Autocorrelation:
  - ρ_1 $(k = 1)$

$$\rho_1 = \frac{\gamma_1}{\gamma_0} = \frac{b_1}{1 + b_1^2}$$

  - ρ_k $(k > 1)$

$$\rho_k = \frac{\gamma_k}{\gamma_0} = 0$$

The ACF and PACF of an MA(1) Process

The MA(1) is always stable: the unconditional mean, variances and covariances always exist (since $\text{Var}(\varepsilon_t)$ exists by assumption).

If $|b_1| > 1$, then the MA(1) is not invertible, but this does not affect the ACF (we will return to non-invertibility).

The ACF of an MA(1) always exists and is given by $\rho_1 = \frac{b_1}{1+b_1^2}$, $\rho_k = 0$ for all $k \geq 2$.

The PACF of an MA(1) always exists, with $\phi_{11} = \frac{b_1}{1+b_1^2}$ and ϕ_{kk} decaying geometrically as $k \longrightarrow \infty$.

- Recall that the PACF is computed form the ACF, so if the ACF exists, the PACF does as well.

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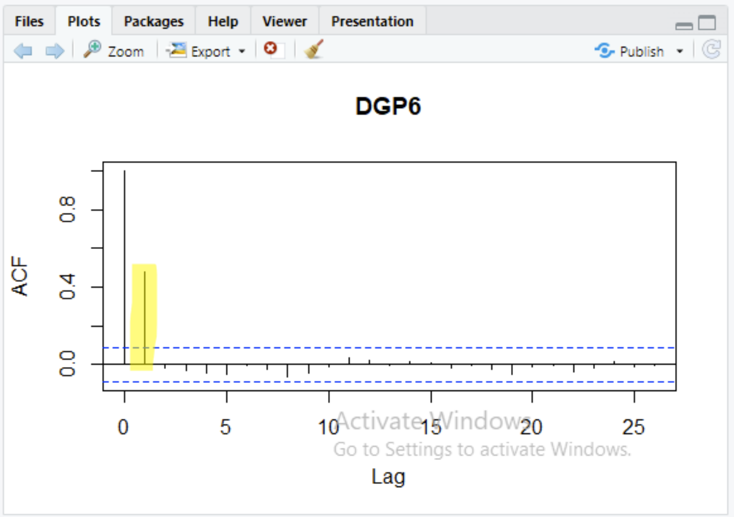

In DGP6:

$$\rho_1 = \frac{b_1}{1+b_1^2} \rightarrow$$

$$\rho_1 = \frac{0.75}{1+0.75^2} \rightarrow$$

$$\rho_1 = 0.48$$

1. Derive the expected value, variance, covariance, autocorrelation function (ACF), and partial autocorrelation function (PACF) for the following data generating processes (DGPs):

   (a) AR(1): $y_t = a_0 + a_1 y_{t-1} + \epsilon_t$, $0 \leq |a_1| < 1$;

   (b) MA(1): $y_t = b_0 + b_1 \epsilon_{t-1} + \epsilon_t$;

- Partial autocorrelation:
  - ϕ_{11}

$$\phi_{11} = \rho_1$$

  - ϕ_{22}

$$\begin{aligned}\phi_{22} &= (\rho_2 - \rho_1^2)/(1-\rho_1^2) \\ &= (0 - \rho_1^2)/(1-\rho_1^2) \\ &= -\rho_1^2/(1-\rho_1^2)\end{aligned}$$

  - ϕ_{33}

$$\begin{aligned}\phi_{33} &= \frac{\rho_3 - \sum_{j=1}^{3-1} \phi_{2,j}\rho_{3-j}}{1 - \sum_{j=1}^{3-1} \phi_{2,j}\rho_j} \\ &= \frac{\rho_3 - \phi_{21}\rho_2 - \phi_{22}\rho_1}{1 - \sum_{j=1}^{3-1} \phi_{2,j}\rho_j} \\ &= \frac{\rho_1^3/(1-\rho_1^2)}{1 - \sum_{j=1}^{3-1} \phi_{2,j}\rho_j}; \text{ since } \rho_2 = \rho_3 = 0\end{aligned}$$

The ACF and PACF of an MA(1) Process

The MA(1) is always stable: the unconditional mean, variances and covariances always exist (since $\mathrm{Var}(\varepsilon_t)$ exists by assumption).

If $|b_1| > 1$, then the MA(1) is not invertible, but this does not affect the ACF (we will return to non-invertibility).

The ACF of an MA(1) always exists and is given by $\rho_1 = \frac{b_1}{1+b_1^2}$, $\rho_k = 0$ for all $k \geq 2$.

The PACF of an MA(1) always exists, with $\phi_{11} = \frac{b_1}{1+b_1^2}$ and ϕ_{kk} decaying geometrically as $k \longrightarrow \infty$.

- Recall that the PACF is computed form the ACF, so if the ACF exists, the PACF does as well.

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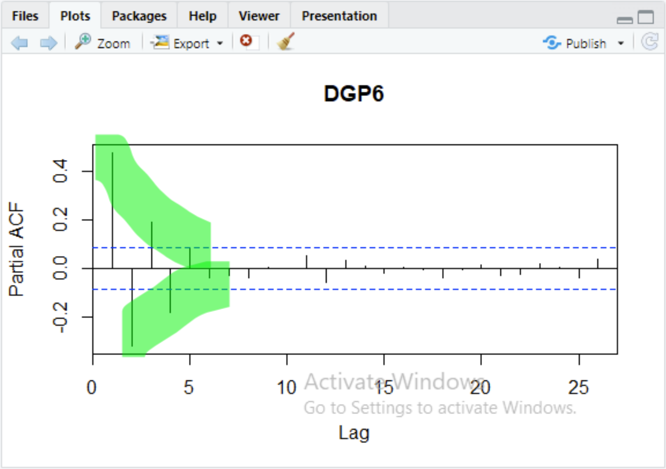
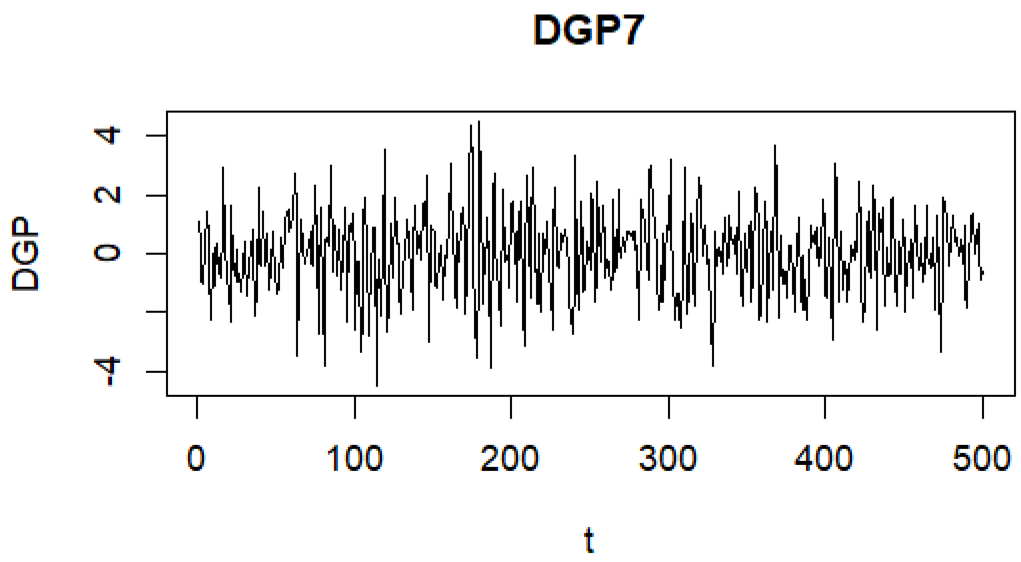


3. The data file `arma.csv` contains (simulated) data for each of the DGPs in Question 2. Import the data into R. Compute, plot, and describe the behaviour of the ACF and PACF for each DGP. Discuss the effects of parameter signs. Hint: use the `acf` and `pacf` commands, respectively.

- DGP7: $y_t = 0.75\epsilon_{t-1} - 0.5\epsilon_{t-2} + \epsilon_t$;

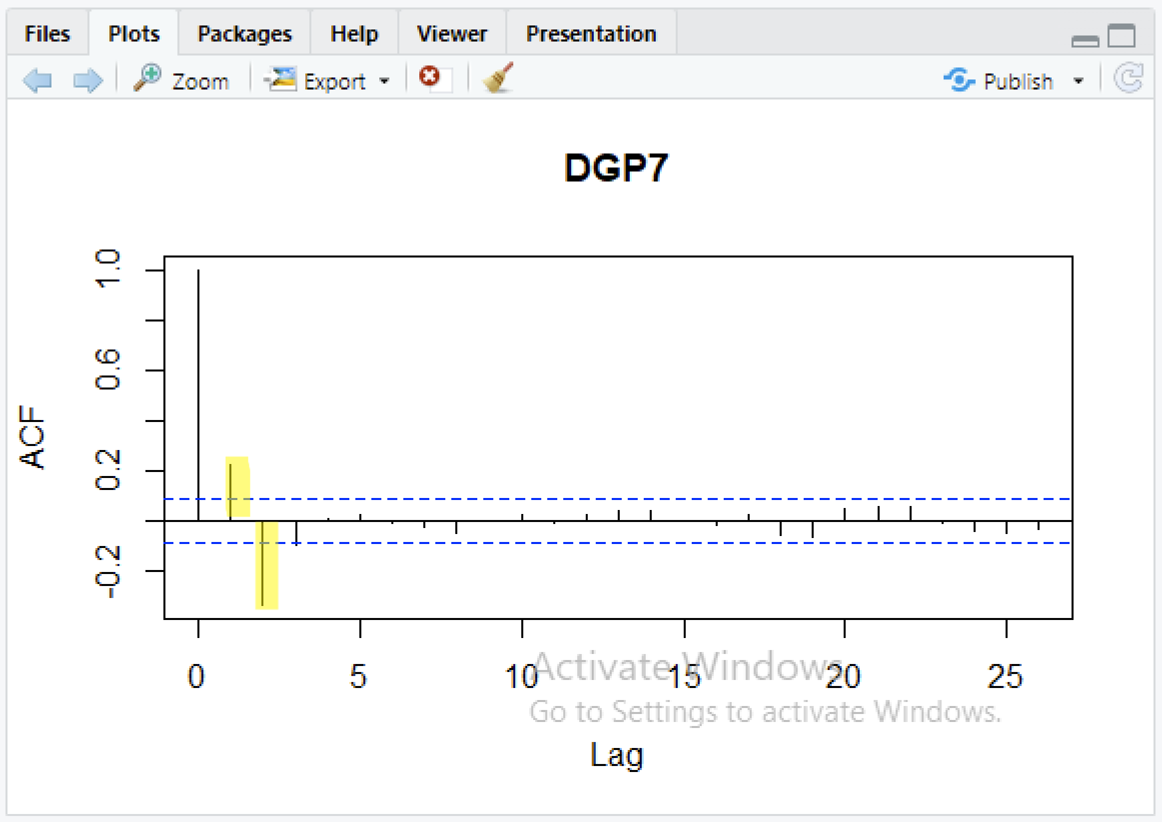


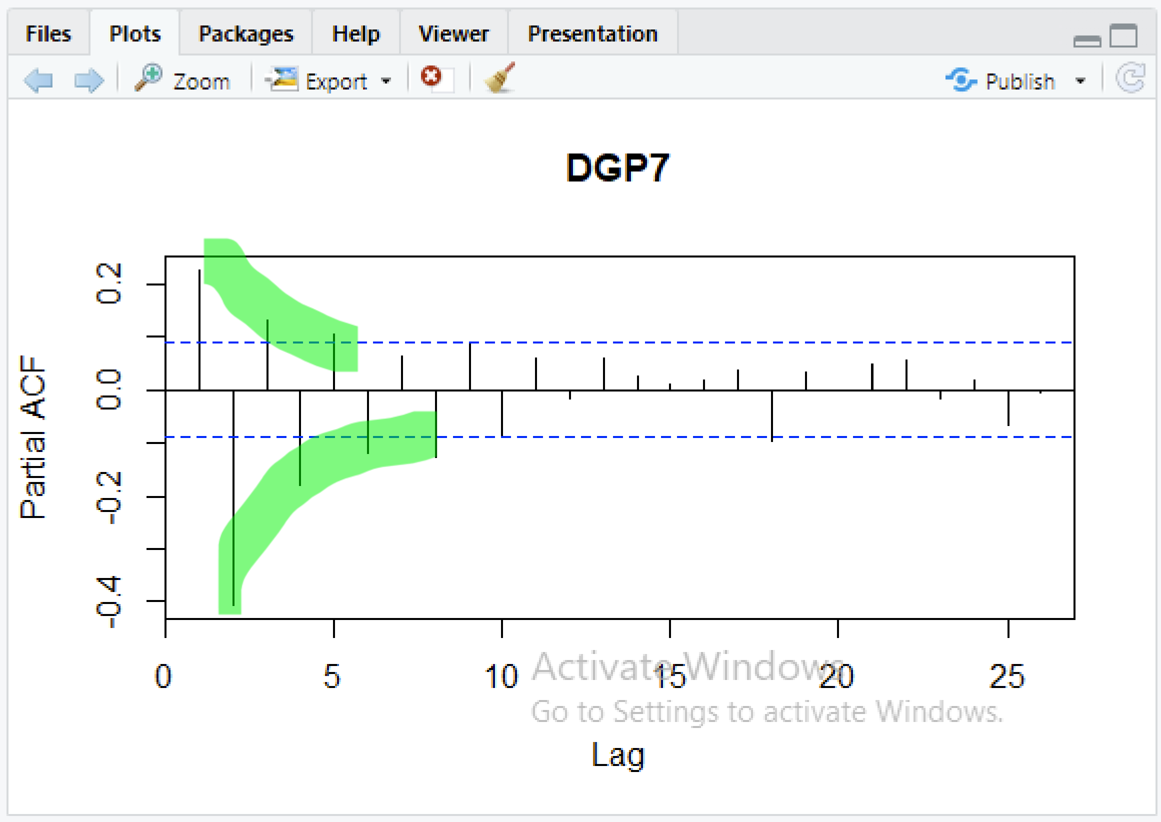

```
> arima(DGP, order = c(0,0,2)) # AR(2)

Call:
arima(x = DGP, order = c(0, 0, 2))

Coefficients:
         ma1      ma2  intercept
      0.4303  -0.3901    -0.0337
s.e.  0.0450   0.0451     0.0561

sigma^2 estimated as 1.453:  log likelihood = -803.35,  aic = 1614.69
```

Considerably less precise than previous models.

3. The data file **arma.csv** contains (simulated) data for each of the DGPs in Question 2. Import the data into R. Compute, plot, and describe the behaviour of the ACF and PACF for each DGP. Discuss the effects of parameter signs. Hint: use the **acf** and **pacf** commands, respectively.

- DGP7: $y_t = 0.75\epsilon_{t-1} - 0.5\epsilon_{t-2} + \epsilon_t$;



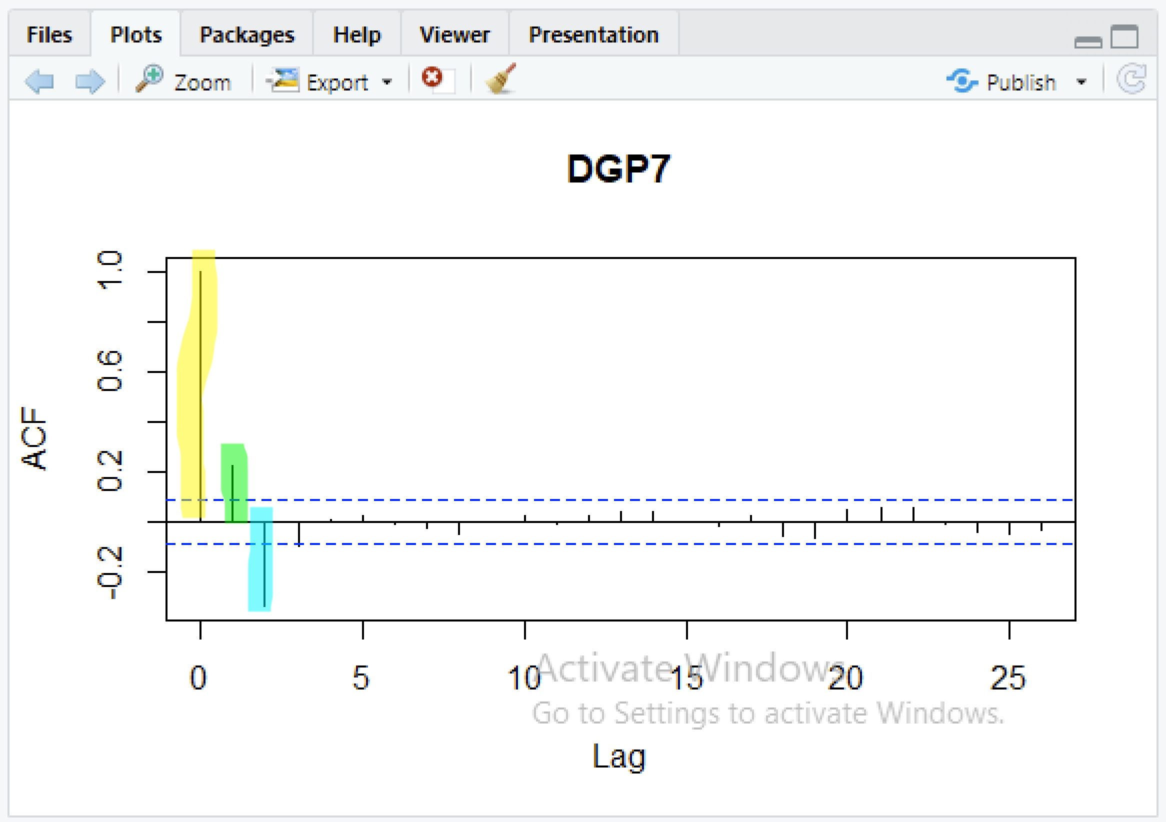


DGP7
- ACF: Two non-zero peak.
- PACF: Decays in an oscillatory path.

2. Compute the true ACF values for the following DGPs:

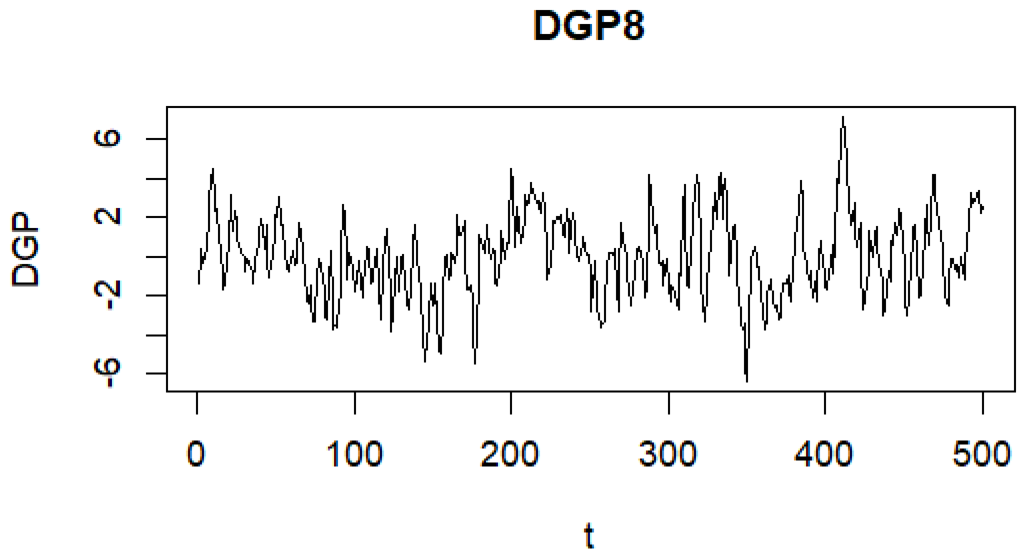
- DGP1: $y_t = 0.75y_{t-1} + \epsilon_t$;
- DGP2: $y_t = -0.75y_{t-1} + \epsilon_t$;
- DGP3: $y_t = 0.95y_{t-1} + \epsilon_t$;
- DGP4: $y_t = 0.5y_{t-1} + 0.25y_{t-2} + \epsilon_t$;
- DGP5: $y_t = 0.25y_{t-1} - 0.5y_{t-2} + \epsilon_t$;
- DGP6: $y_t = 0.75\epsilon_{t-1} + \epsilon_t$;
- DGP7: $y_t = 0.75\epsilon_{t-1} - 0.5\epsilon_{t-2} + \epsilon_t$;

Solution $\rho_0 = 1$ and $\rho_k = 0$ for $k \geq 3$. $\rho_1 = b_1(1+b_2)/(1+b_1^2+b_2^2) = 6/29$ and $\rho_2 = b_2/(1+b_1^2+b_2^2) = -8/29$.



3. The data file `arma.csv` contains (simulated) data for each of the DGPs in Question 2. Import the data into R. Compute, plot, and describe the behaviour of the ACF and PACF for each DGP. Discuss the effects of parameter signs. Hint: use the `acf` and `pacf` commands, respectively.

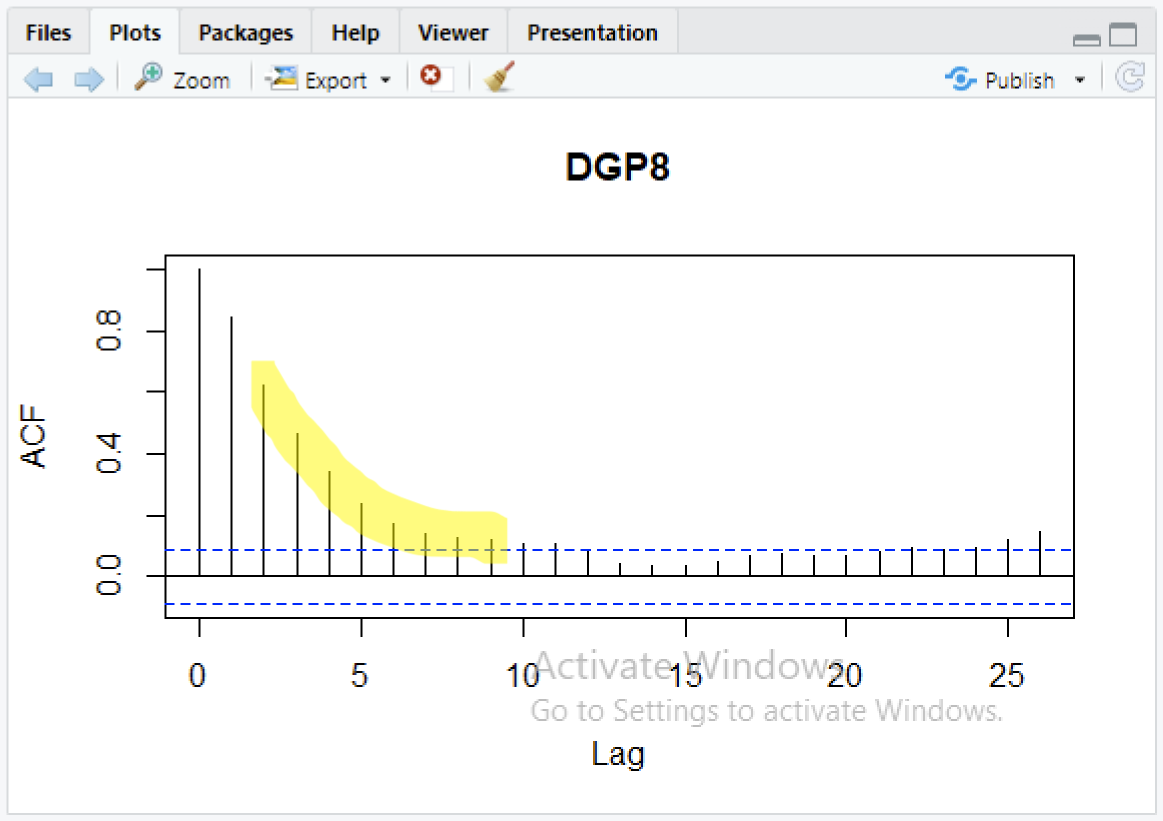
- DGP8: $y_t = 0.75 y_{t-1} + 0.5\epsilon_{t-1} + \epsilon_t$.

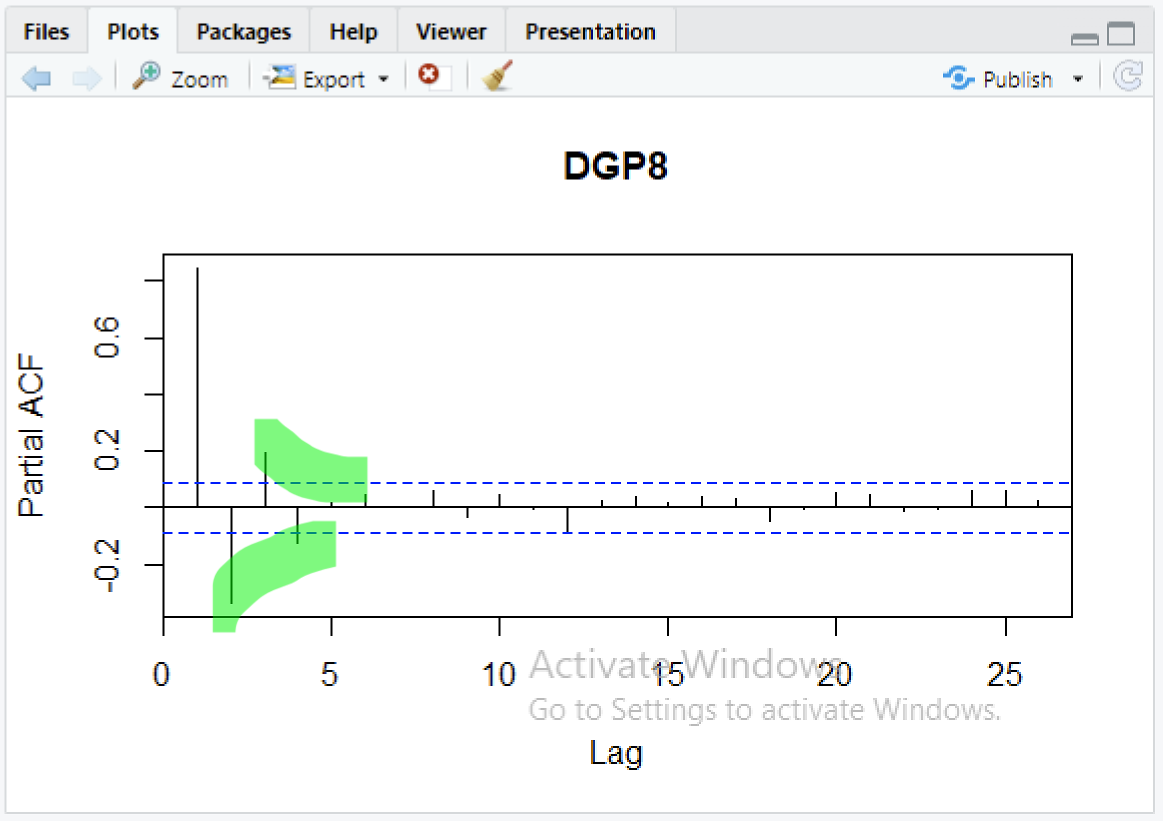



```
> arima(DGP, order = c(1,0,1)) # AR(2)

Call:
arima(x = DGP, order = c(1, 0, 1))

Coefficients:
         ar1     ma1  intercept
      0.7271  0.5088     0.0075
s.e.  0.0336  0.0404     0.2434

sigma^2 estimated as 0.9801:  log likelihood = -705.29,  aic = 1418.57
```

3. The data file `arma.csv` contains (simulated) data for each of the DGPs in Question 2. Import the data into R. Compute, plot, and describe the behaviour of the ACF and PACF for each DGP. Discuss the effects of parameter signs. Hint: use the `acf` and `pacf` commands, respectively.

- DGP8: $y_t = 0.75y_{t-1} + 0.5\epsilon_{t-1} + \epsilon_t$.





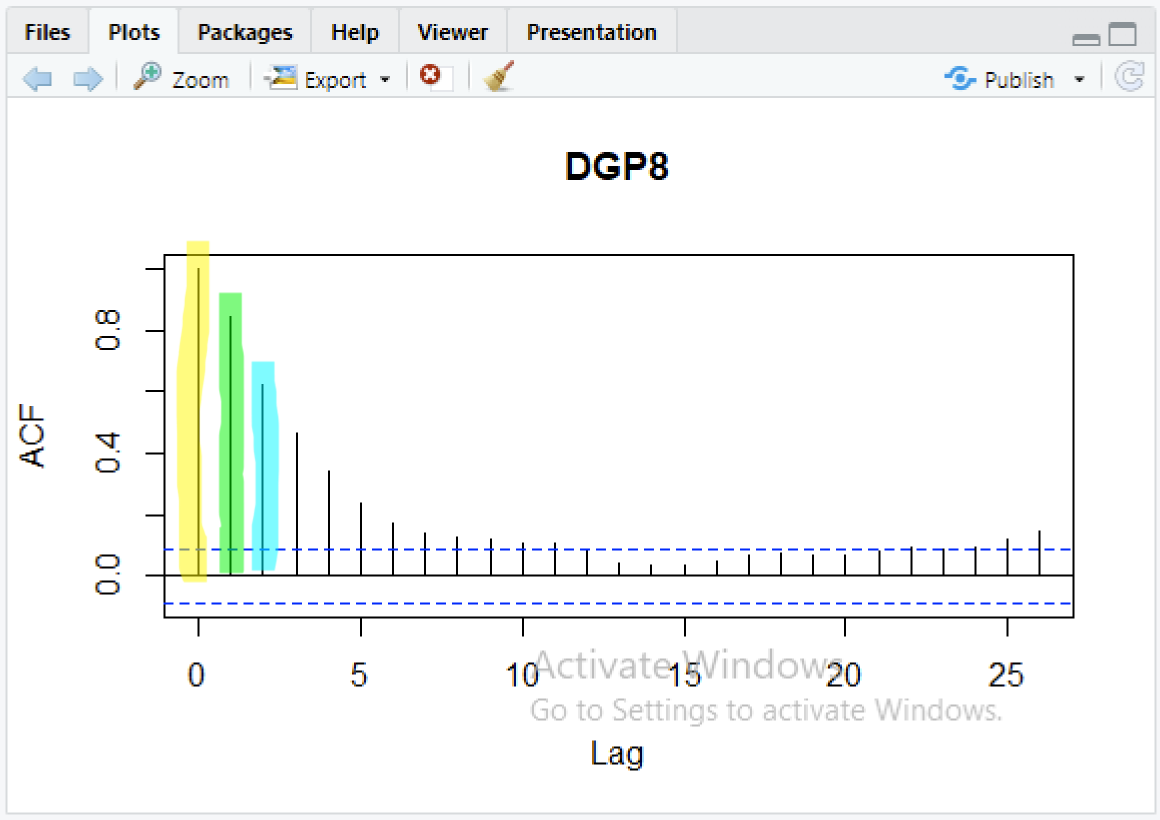
DGP8

- ACF: Decays geometrically from $k = 2$ onwards as the AR(1) component dominates.
- PACF: Decays in an oscillatory path from $k = 2$ as the MA(1) component dominates.

2. Compute the true ACF values for the following DGPs:

- DGP1: $y_t = 0.75y_{t-1} + \epsilon_t$;
- DGP2: $y_t = -0.75y_{t-1} + \epsilon_t$;
- DGP3: $y_t = 0.95y_{t-1} + \epsilon_t$;
- DGP4: $y_t = 0.5y_{t-1} + 0.25y_{t-2} + \epsilon_t$;
- DGP5: $y_t = 0.25y_{t-1} - 0.5y_{t-2} + \epsilon_t$;
- DGP6: $y_t = 0.75\epsilon_{t-1} + \epsilon_t$;
- DGP7: $y_t = 0.75\epsilon_{t-1} - 0.5\epsilon_{t-2} + \epsilon_t$;
- DGP8: $y_t = 0.75y_{t-1} + 0.5\epsilon_{t-1} + \epsilon_t$.

Solution For the ARMA(1,1) model $y_t = a_0 + a_1 y_{t-1} + b_1\epsilon_{t-1} + \epsilon_t$, $\rho_0 = 1$, $\rho_1 = (1 + a_1b_1)(a_1 + b_1)/(1 + b_1^2 + 2a_1b_1)$, $\rho_k = a_1\rho_{k-1}$ for all $k \geq 2$. Thus, $\rho_0 = 1$, $\rho_1 = 0.859$, $\rho_2 = 0.645$, ...



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1. Derive the expected value, variance, covariance, autocorrelation function (ACF), and partial autocorrelation function (PACF) for the following data generating processes (DGPs):

   (a) AR(1): $y_t = a_0 + a_1 y_{t-1} + \epsilon_t$, $0 \leq |a_1| < 1$;

   (b) MA(1): $y_t = b_0 + b_1 \epsilon_{t-1} + \epsilon_t$;

   (c) ARMA(1,1): $y_t = a_0 + a_1 y_{t-1} + b_1 \epsilon_{t-1} + \epsilon_t$, $0 \leq |a_1| < 1$.

- Expected value:

$$E(y_t) = a_0 + a_1 E(y_{t-1}) + b_1 E(\epsilon_{t-1}) + E(\epsilon_t)$$
$$\mu = \frac{a_0}{1 - a_1};\ \text{since } E(y_t) = E(y_{t-1}) = \mu$$

Moments of the AR(1) Process

The unconditional mean $\mathsf{E}(y_t)$ is the limiting case as $h \longrightarrow \infty$:

$$\mathsf{E}(y_t) = \lim_{h \to \infty} \mathsf{E}(y_t \,|\, y_{t-h}, y_{t-h-1}, \ldots).$$

Taking the limit yields:

- $\mathsf{E}(y_t \,|\, y_{t-h}, y_{t-h-1}, \ldots) \longrightarrow \frac{a_0}{1-a_1}$ if $|a_1| < 1$;
- $\mathsf{E}(y_t \,|\, y_{t-h}, y_{t-h-1}, \ldots) \longrightarrow$ indeterminate form (i.e. does not exist) if $|a_1| \geq 1$.

Hence, a finite $\mathsf{E}(y_t)$ exists if and only if $|a_1| < 1$.

The AR(1) model with $|a_1| \geq 1$ is called unstable.

Instability implies non-stationarity, but not the other way around.

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Moments of the MA(1) Process

The unconditional mean of y_t is:

$$\mathsf{E}(y_t) = \mathsf{E}(a_0 + b_1\varepsilon_{t-1} + \varepsilon_t) = a_0.$$

The unconditional variance of y_t is:

$$\mathsf{Var}(y_t) = \mathsf{Var}(a_0 + b_1\varepsilon_{t-1} + \varepsilon_t) = (1 + b_1^2)\sigma_\varepsilon^2.$$

The unconditional covariance between y_t and y_{t-k} is:

$$\begin{aligned}\mathsf{cov}(y_t, y_{t-k}) &= \mathsf{cov}(a_0 + b_1\varepsilon_{t-1} + \varepsilon_t, a_0 + b_1\varepsilon_{t-k-1} + \varepsilon_{t-k}) \\ &= \mathsf{E}\left((b_1\varepsilon_{t-1} + \varepsilon_t)(b_1\varepsilon_{t-k-1} + \varepsilon_{t-k})\right) \\ &= b_1\sigma_\varepsilon^2 \text{ if } k = 1;\ 0 \text{ for all } k \geq 2.\end{aligned}$$

Moments conditional on $y_{t-1}, y_{t-2}, \ldots$, etc. are complicated, but y_t is independent of $y_{t-2}, y_{t-3}, \ldots$.

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1. Derive the expected value, variance, covariance, autocorrelation function (ACF), and partial autocorrelation function (PACF) for the following data generating processes (DGPs):

   (a) AR(1): $y_t = a_0 + a_1 y_{t-1} + \epsilon_t$, $0 \leq |a_1| < 1$;

   (b) MA(1): $y_t = b_0 + b_1 \epsilon_{t-1} + \epsilon_t$;

   (c) ARMA(1,1): $y_t = a_0 + a_1 y_{t-1} + b_1 \epsilon_{t-1} + \epsilon_t$, $0 \leq |a_1| < 1$.

- Variance:

$$
\begin{aligned}
\mathrm{Var}(y_t) = \gamma_0 = {} & \mathrm{Var}(a_0) + a_1^2 \mathrm{Var}(y_{t-1}) + b_1^2 \mathrm{Var}(\epsilon_{t-1}) + \mathrm{Var}(\epsilon_t) \\
& + 2\mathrm{cov}(a_1 y_{t-1}, b_1 \epsilon_{t-1}) + 2\mathrm{cov}(a_1 y_{t-1}, \epsilon_t) + 2\mathrm{cov}(b_1 \epsilon_{t-1}, \epsilon_t) \\
\gamma_0 = {} & \frac{1 + b_1^2 + 2a_1 b_1}{1 - a_1^2} \sigma^2, \quad \text{since } \mathrm{cov}(a_1 y_{t-1}, b_1 \epsilon_{t-1}) = a_1 b_1 E(\epsilon_{t-1}^2)
\end{aligned}
$$

  – To show $\mathrm{cov}(a_1 y_{t-1}, b_1 \epsilon_{t-1}) = a_1 b_1 E(\epsilon_{t-1}^2)$ you can proceed as follows

$$
\begin{aligned}
\mathrm{cov}(a_1 y_{t-1}, b_1 \epsilon_{t-1}) &= E[(a_1 y_{t-1})(b_1 \epsilon_{t-1})] \\
&= \mathrm{E}([a_1(a_1 y_{t-2} + b_1 \epsilon_{t-2} + \epsilon_{t-1})](b_1 \epsilon_{t-1})) \\
&= \mathrm{E}(a_1 \epsilon_{t-1} b_1 \epsilon_{t-1})
\end{aligned}
$$

  is the only non-zero expected value.

1. Derive the expected value, variance, covariance, autocorrelation function (ACF), and partial autocorrelation function (PACF) for the following data generating processes (DGPs):

   (a) AR(1): $y_t = a_0 + a_1 y_{t-1} + \epsilon_t$, $0 \leq |a_1| < 1$;

   (b) MA(1): $y_t = b_0 + b_1 \epsilon_{t-1} + \epsilon_t$;

   (c) ARMA(1, 1): $y_t = a_0 + a_1 y_{t-1} + b_1 \epsilon_{t-1} + \epsilon_t$, $0 \leq |a_1| < 1$.

- Covariance:
  - Set $\mu = 0$ without loss of generality

$$\begin{aligned}\operatorname{cov}(y_t, y_{t-k}) = \gamma_k &= \mathrm{E}(y_t y_{t-k}) \\ &= \mathrm{E}((a_1 y_{t-1} + b_1 \epsilon_{t-1} + \epsilon_t) y_{t-k})\end{aligned}$$

  - γ_1 $(k = 1)$

$$\gamma_1 = \frac{(1 + a_1 b_1)(a_1 + b_1)}{1 - a_1^2} \sigma^2$$

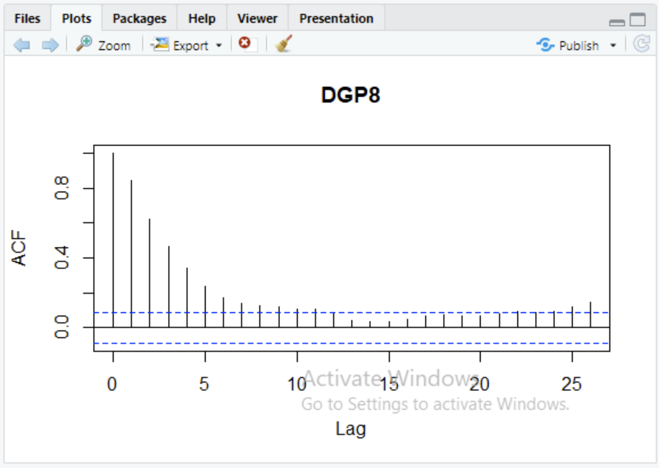
  - γ_k $(k \geq 2)$

$$\gamma_k = a_1 \gamma_{k-1}$$

1. Derive the expected value, variance, covariance, autocorrelation function (ACF), and partial autocorrelation function (PACF) for the following data generating processes (DGPs):

   (a) AR(1): $y_t = a_0 + a_1 y_{t-1} + \epsilon_t$, $0 \leq |a_1| < 1$;
   (b) MA(1): $y_t = b_0 + b_1 \epsilon_{t-1} + \epsilon_t$;
   (c) ARMA(1, 1): $y_t = a_0 + a_1 y_{t-1} + b_1 \epsilon_{t-1} + \epsilon_t$, $0 \leq |a_1| < 1$.

- Autocorrelation:
  - ρ_1 $(k = 1)$
$$\rho_1 = \frac{(1 + a_1 b_1)(a_1 + b_1)}{1 + b_1^2 + 2a_1 b_1}$$
  - ρ_k $(k \geq 2)$
$$\rho_k = a_1 \rho_{k-1}$$
  - Autoregressive pattern dominates for $k > 1$.



The ACF and PACF of AR(p), MA(q) and ARMA(p, q) Processes

In general, we can summarise ACFs of PACFs of ARMA processes as follows.

For a pure AR(p), the ACF and PACF exist if only if it is stable, in which case

- the ACF decays to zero as $k \longrightarrow \infty$;
- the PACF is given by
  - $\phi_{11}, \dots, \phi_{pp}$ computed from the ACF, with
  - $\phi_{11} = \rho_1$, $\phi_{pp} = a_p$ and
  - $\phi_{kk} = 0$ for all $k \geq p + 1$.

For a pure MA(q), the ACF and PACF always exist and

- the ACF vanishes for all $k \geq q + 1$;
- the PACF is computed from the ACF, with $\phi_{11} = \rho_1$ and ϕ_{kk} decaying as $k \longrightarrow \infty$.

For a general ARMA(p, q), the ACF and PACF exist if and only if it is stable, in which case both decay as $k \longrightarrow \infty$.

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1. Derive the expected value, variance, covariance, autocorrelation function (ACF), and partial autocorrelation function (PACF) for the following data generating processes (DGPs):
   (a) AR(1): $y_t = a_0 + a_1 y_{t-1} + \epsilon_t$, $0 \leq |a_1| < 1$;
   (b) MA(1): $y_t = b_0 + b_1 \epsilon_{t-1} + \epsilon_t$;
   (c) ARMA(1, 1): $y_t = a_0 + a_1 y_{t-1} + b_1 \epsilon_{t-1} + \epsilon_t$, $0 \leq |a_1| < 1$.

- Partial autocorrelation:
  - ϕ_{11}

$$\phi_{11} = \rho_1$$

  - ϕ_{22}

$$\begin{aligned}\phi_{22} &= (\rho_2 - \rho_1^2)/(1-\rho_1^2)\\ &= (a_1\rho_1 - \rho_1^2)/(1-\rho_1^2)\end{aligned}$$

  - ϕ_{33}

$$\begin{aligned}\phi_{33} &= \frac{\rho_3 - \sum_{j=1}^{3-1}\phi_{2,j}\rho_{3-j}}{1-\sum_{j=1}^{3-1}\phi_{2,j}\rho_j}\\ &= \frac{a_1^2\rho_1 - \phi_{21}a_1\rho_1 - \phi_{22}\rho_1}{1-\sum_{j=1}^{3-1}\phi_{2,j}\rho_j}\end{aligned}$$

where

$$\begin{aligned}\phi_{21} &= \phi_{11} - \phi_{22}\phi_{11}\\ &= \rho_1[1-(a_1\rho_1\text{-}\rho_1^2)/(1\text{-}\rho_1^2)]\end{aligned}$$

  - Moving average pattern dominates for $k > 1$.

The ACF and PACF of AR(p), MA(q) and ARMA(p,q) Processes

In general, we can summarise ACFs of PACFs of ARMA processes as follows.

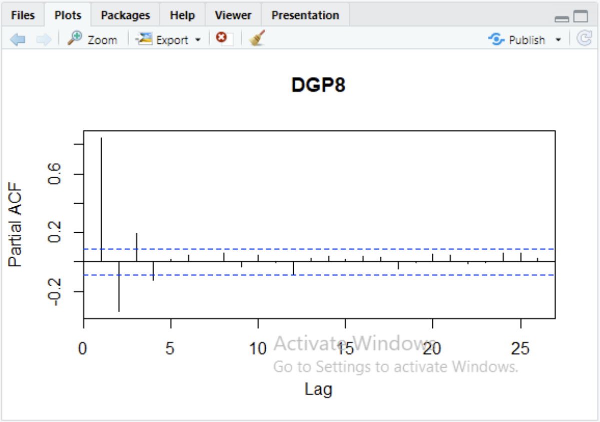
For a pure AR(p), the ACF and PACF exist if only if it is stable, in which case

- the ACF decays to zero as $k \longrightarrow \infty$;
- the PACF is given by
  - $\phi_{11}, \ldots, \phi_{pp}$ computed from the ACF, with
  - $\phi_{11} = \rho_1$, $\phi_{pp} = a_p$ and
  - $\phi_{kk} = 0$ for all $k \geq p+1$.

For a pure MA(q), the ACF and PACF always exist and

- the ACF vanishes for all $k \geq q+1$;
- the PACF is computed from the ACF, with $\phi_{11} = \rho_1$ and ϕ_{kk} decaying as $k \longrightarrow \infty$.

For a general ARMA(p,q), the ACF and PACF exist if and only if it is stable, in which case both decay as $k \longrightarrow \infty$.



Tutorial 2: Forecasting Univariate Processes - I

At the end of this tutorial you should be able to:

- derive theoretical properties of ARMA processes;
- compute the theoretical ACF and PACF for a given ARMA processes;
- use R to compute and plot the sample ACF and PACF for time series data.

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Thank you

Francisco Tavares Garcia

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Reference

Tsay, R. (2010). Analysis of Financial Time Series, 3rd Edition, John Wiley & Sons.