

Tropical Combinatorial Nullstellensatz and Fewnomials Testing

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Abstract

Tropical algebra emerges in many fields of mathematics such as algebraic geometry, mathematical physics and combinatorial optimization. In part, its importance is related to the fact that it makes various parameters of mathematical objects computationally accessible. Tropical polynomials play an important role in this, especially for the case of algebraic geometry. On the other hand, many algebraic questions behind tropical polynomials remain open. In this paper we address three basic questions on tropical polynomials closely related to their computational properties:

1. Given a polynomial with a certain support (set of monomials) and a (finite) set of inputs, when is it possible for the polynomial to vanish on all these inputs?
2. A more precise question, given a polynomial with a certain support and a (finite) set of inputs, how many roots can polynomial have on this set of inputs?
3. Given an integer k , for which s there is a set of s inputs such that any non-zero polynomial with at most k monomials has a non-root among these inputs?

In the classical algebra well-known results in the direction of these questions are Combinatorial Nullstellensatz, Schwartz-Zippel Lemma and Universal Testing Set for sparse polynomials respectively. In this paper we extensively study these three questions for tropical polynomials and provide results analogous to the classical results mentioned above.

1 Introduction

A *max-plus* or a *tropical semiring* is defined by a set \mathbb{K} , which can be \mathbb{R} or \mathbb{Q} endowed with two operations, *tropical addition* \oplus and *tropical multiplication* \odot , defined in the following way:

$$x \oplus y = \max(x, y), \quad x \odot y = x + y.$$

Tropical polynomials are a natural analog of classical polynomials. In classical terms a tropical polynomial is an expression of the form $f(\vec{x}) = \max_i M_i(\vec{x})$, where each $M_i(\vec{x})$ is a linear polynomial (a tropical monomial) in variables $\vec{x} = (x_1, \dots, x_n)$, and all the coefficients of all M_i 's are nonnegative integers except for a free coefficient which can be any element of \mathbb{K} (free coefficient corresponds to a coefficient of the tropical monomial and other coefficients correspond to the powers of variables in the tropical monomial).

The degree of a tropical monomial M is the sum of its coefficients (except the free coefficient) and the degree of a tropical polynomial f denoted by $\deg(f)$ is the maximal degree of its monomials. A point $\vec{a} \in \mathbb{K}^n$ is a root of the polynomial f if the maximum $\max_i \{M_i(\vec{a})\}$ is attained on at least two different monomials M_i . We defer more detailed definitions on the basics of max-plus algebra to Preliminaries.

Tropical polynomials have appeared in various areas of mathematics and found many applications (see, for example, [15, 18, 25, 19, 20, 14, 30]). An early source of the tropical approach was the Newton's method for solving algebraic equations in Newton-Puiseux series [25]. An important advantage of tropical algebra is that it makes some properties of classical mathematical objects computationally accessible [26, 15, 18, 25]: on one hand tropical analogs reflect certain properties of classical objects and on the other hand tropical objects have much more simple and discrete structure and thus are more accessible to algorithms. One of the main goals of max-plus mathematics is to build a theory of tropical polynomials which would help to work with them and would possibly lead to new results in related areas. Computational reasons, on the other hand, make it important to keep the theory maximally computationally efficient.

The case studied best so far is the one of tropical linear polynomials and systems of tropical linear polynomials. For them an analog of a large part of the classical theory of linear polynomials was established. This includes studies of tropical analogs of the rank of a matrix and the independence of vectors [8, 16, 1], an analog of the determinant of a matrix and its properties [1, 8, 9], an analog of Gauss triangular form [9]. Also the solvability problem for tropical linear systems was studied from the complexity point of

view. Interestingly, this problem turns out to be polynomially equivalent to the mean payoff games problem [2, 11] which received considerable attention in computational complexity theory.

For tropical polynomials of arbitrary degree less is known. In [23] the radical of a tropical ideal was explicitly described. In [20, 24] a tropical version of the Bezout theorem was proved for tropical polynomial systems for the case when the number of polynomials in the system is equal to the number of variables. In [7] the Bezout bound was extended to systems with an arbitrary number of polynomials. In [12] the tropical analog of Hilbert's Nullstellensatz was established. In [5] an upper bound on the number of common nondegenerate roots for the system of sparse tropical polynomials was given. In [26] it was shown that the solvability problem for tropical polynomial systems is NP-complete.

Our results. In this paper we address several basic questions for tropical polynomials.

The first question we address is given a set S of points in \mathbb{R}^n and a set of monomials of n variables, is there a tropical polynomial with these monomials that has roots in all points of the set. In the classical case a famous result in this direction with numerous applications in Theoretical Computer Science is Combinatorial Nullstellensatz [3]. Very roughly, it states that the set of monomials of the polynomial might be substantially larger than the set of the points and the polynomial will still be non-zero on at least one of the points. In the tropical case we show that this is not the case: if the number of monomials is larger than the number of points, there is always a polynomial with roots in all points. We establish the general criteria for existence of a polynomial on a given set of monomials with roots in all points of a given set. From this criteria we deduce that if the number of points is equal to the number of monomials and the points and monomials are structured in the same way, then there is no polynomial with roots in all points. We note that the last statement for the classical case is an open question [21].

The second question is given a finite set $S \subseteq \mathbb{R}$ how many roots can a tropical polynomial of n variables and degree d have in the set S^n ? In the classical case the well-known Schwartz-Zippel lemma [31, 22] states that the maximal number of roots is $d|S|^{n-1}$. We show that in the tropical case the maximal possible number of roots is $|S|^n - (|S| - d)^n$.

The third question is related to universal testing set for tropical polynomials of n variables with at most k monomials. The universal testing set is the set of points $S \subseteq \mathbb{K}^n$ such that any nontrivial polynomial with at most k

monomials has a non-root in one of the points of S . The problem is to find a minimal size of the universal testing set for given n and k . In the classical case this problem is tightly related to the problem of interpolating a polynomial with a certain number of monomials (with a priori unknown support) given its values on some universal set of inputs. The classical problem was studied in [10, 4, 17, 13] and the minimal size of the universal testing set for the classical case turns out to be equal to k (while for the interpolation problem the size is $2k$). In the tropical case it turns out that the answer depends on which tropical semiring \mathbb{K} is considered: for $\mathbb{K} = \mathbb{R}$ we show that as in the classical case the minimal size of the universal testing set is equal to k . For $\mathbb{K} = \mathbb{Q}$ it turns out that the minimal size of the universal testing set is substantially larger. We show that its size is $\Theta(kn)$ (the constants in Θ do not depend on k and n). For $n = 2$ we find the precise size of the minimal universal testing set $s = 2k - 1$. For greater n the precise minimal size of the universal testing set still remains unclear. Finally, we establish an interesting connection of this problem to the following problem in Discrete Geometry: what is the minimal number of disjoint convex polytopes in n -dimensional space that is enough to cover any set of s points in such a way that all s points are on the boundary of the polytopes.

The rest of the paper is organized as follows. In Section 2 we introduce necessary definitions and notations. In Section 3 we give the results on the tropical analog of Combinatorial Nullstellensatz. In Section 4 we prove the tropical analog of Schwartz-Zippel Lemma. In Section 5 we give the results on tropical universal sets.

2 Preliminaries

A *max-plus* or a *tropical semiring* is defined by a set \mathbb{K} , which can be \mathbb{R} or \mathbb{Q} endowed with two operations, *tropical addition* \oplus and *tropical multiplication* \odot , defined in the following way:

$$x \oplus y = \max\{x, y\}, \quad x \odot y = x + y.$$

A tropical (or max-plus) monomial in variables $\vec{x} = (x_1, \dots, x_n)$ is defined as

$$m(\vec{x}) = c \odot x_1^{\odot i_1} \odot \dots \odot x_n^{\odot i_n}, \tag{1}$$

where c is an element of the semiring \mathbb{K} and i_1, \dots, i_n are nonnegative integers. In the usual notation the monomial is the linear function

$$m(\vec{x}) = c + i_1 x_1 + \dots + i_n x_n.$$

For $\vec{x} = (x_1, \dots, x_n)$ and $I = (i_1, \dots, i_n)$ we introduce the notation

$$\vec{x}^I = x_1^{\odot i_1} \odot \dots \odot x_n^{\odot i_n} = i_1 x_1 + \dots + i_n x_n.$$

The degree of the monomial m is defined as the sum $i_1 + \dots + i_n$. We denote this sum by $|I|$.

A *tropical polynomial* is the tropical sum of tropical monomials

$$p(\vec{x}) = \bigoplus_i m_i(\vec{x})$$

or in the usual notation $p(\vec{x}) = \max_i m_i(\vec{x})$. The *degree* of the tropical polynomial p denoted by $\deg(p)$ is the maximal degree of its monomials. A point $\vec{a} \in \mathbb{K}^n$ is a *root* of the polynomial p if the maximum $\max_i \{m_i(\vec{a})\}$ is attained on at least two distinct monomials among m_i . A polynomial p *vanishes* on the set $S \subseteq \mathbb{K}^n$ if all points of S are roots of p .

Geometrically, a tropical polynomial $p(\vec{x})$ is a convex piece-wise linear function and the roots of p are non-smoothness points of this function.

By the *product* of two tropical polynomials $p(\vec{x}) = \bigoplus_i m_i(\vec{x})$ and $q(\vec{x}) = \bigoplus_j m'_j(\vec{x})$ we naturally call a tropical polynomial $p \odot q$ that has as monomials tropical products $m_i(\vec{x}) \odot m'_j(\vec{x})$ for all i, j . We will make use of the following simple observation.

Lemma 1. *A point $\vec{a} \in \mathbb{K}^n$ is a root of $p \odot q$ iff it is a root of either $p(\vec{x})$ or $q(\vec{x})$.*

We provide the proof for the sake of completeness.

Proof. Suppose \vec{a} is a root of p . Let $m_{i_1}(\vec{x}), m_{i_2}(\vec{x})$ be two distinct monomials of p such that $m_{i_1}(\vec{a}) = m_{i_2}(\vec{a}) = \max_i m_i(\vec{a})$. Let $m'_{j_1}(\vec{x})$ be a monomial of q such that $m'_{j_1}(\vec{a}) = \max_j m'_j(\vec{a})$. Then $m_{i_1} \odot m'_{j_1}$ and $m_{i_2} \odot m'_{j_1}$ are two distinct monomials of $p \odot q$ with the maximal value on \vec{a} among all monomials of $p \odot q$. The symmetrical argument shows that any root of q is a root of $p \odot q$.

Iff \vec{a} is not a root of p and q , then there are unique i_1 and j_1 such that $m_{i_1}(\vec{a}) = \max_i m_i(\vec{a})$ and $m'_{j_1}(\vec{a}) = \max_j m'_j(\vec{a})$. Then the maximal value on \vec{a} among all monomials of $p \odot q$ is attained on a single monomial $m_{i_1} \odot m'_{j_1}$ and thus \vec{a} is not a root of $p \odot q$. \square

For two vectors $\vec{a}, \vec{b} \in \mathbb{R}^n$ throughout the paper we will denote by $\langle \vec{a}, \vec{b} \rangle$ their inner product.

3 Tropical Combinatorial Nullstellensatz

For a polynomial p denote by $\text{Supp}(p)$ the set of all $J = (j_1, \dots, j_n)$ such that p has a monomial \vec{x}^J with some coefficient.

Consider two finite sets $S, R \subseteq \mathbb{R}^n$ such that $|S| = |R|$. We call S and R *non-singular* if there is a bijection $f: S \rightarrow R$ such that $\sum_{x \in S} \langle \vec{x}, f(\vec{x}) \rangle$ is greater than the corresponding sum for all other bijections from S to R . Otherwise we say that R and S are *singular*. Note that the notion of singularity is symmetrical.

First we formulate a general criteria for vanishing polynomials with given support.

Theorem 2. *Consider support $S \subseteq \mathbb{N}^n$ and the set of points $R \subseteq \mathbb{R}^n$. There are three cases.*

- (i) *If $|R| < |S|$, then there is a polynomial p with support in S vanishing on R .*
- (ii) *If $|R| = |S|$, then there is a polynomial p with support in S vanishing on R iff S and R are singular.*
- (iii) *If $|R| > |S|$ then there is a polynomial p with support in S vanishing on R iff for any subset $R' \subset R$ such that $|R'| = |S|$ we have that R' and S are singular.*

Proof. Consider a polynomial

$$p(\vec{x}) = \bigoplus_{J \in S} c_J \odot \vec{x}^J$$

with support S . The fact that p has a root in $\vec{a} \in R$ means that the maximum in

$$\max_{J \in S} (c_J + \langle J, \vec{a} \rangle)$$

is attained on at least two monomials J_1 and J_2 . Note that once S and R are fixed this is a linear tropical equation on the coefficients $\{c_J\}_{J \in S}$ of p .

The fact that p has a root in all points of R thus means that the coefficients of p satisfy the tropical linear system with the matrix

$$(\langle J, \vec{a} \rangle)_{J \in S, \vec{a} \in R} \in \mathbb{R}^{|S| \times |R|}.$$

The tropical Cramer rule [1, 8, 9, 16] states that if the number of rows $|R|$ in such system is less then the number of columns $|S|$, then there is always

a nontrivial solution. Thus in this case there is a polynomial with roots in all points of R .

If the matrix is square, that is $|R| = |S|$, then it is known [1, 8, 9, 16] that there is a nontrivial solution iff the tropical determinant of the matrix is singular. Tropical determinant is defined completely analogously to the classical one. That is for our matrix it is given by the expression

$$\bigoplus_{f: S \rightarrow R} \left(\bigodot_{J \in S} \langle J, f(J) \rangle \right) = \max_{f: S \rightarrow R} \left(\sum_{J \in S} \langle J, f(J) \rangle \right),$$

where f ranges over all bijections from S to R . Its singularity means that the maximum is attained on two different monomials. This means that there are two bijections $f, g: S \rightarrow R$ with equal maximum sum $\sum_{J \in S} \langle J, f(J) \rangle = \sum_{J \in S} \langle J, g(J) \rangle$. Note that this is precisely the singularity of S and R .

If the number of rows $|R|$ in the matrix is greater than the number of columns $|S|$ in it, then it is known [1, 8, 9, 16] that the system has nontrivial solution iff the tropical determinant of each square submatrix of size $|S| \times |S|$ is singular. This means precisely that for any subset $R' \subset R$ such that $|R'| = |S|$ the sets R' and S are singular. \square

Now we will derive corollaries of this general criteria.

Suppose we have a set $S \subseteq \mathbb{N}^n$. Suppose we have a set of reals $\{\alpha_j^i\}$ for $i = 1, \dots, n$, $j \in \mathbb{N}$ and for each i we have

$$\alpha_0^i < \alpha_1^i < \alpha_2^i < \dots$$

Consider the set $R_S = \{(\alpha_{j_1}^1, \dots, \alpha_{j_n}^n) \mid (j_1, \dots, j_n) \in S\}$.

We consider the following question. Suppose we have a polynomial p with support $\text{Supp}(p) \subseteq S$. For which sets S' is it possible that p vanishes on $R_{S'}$?

A natural question is the case of $S = S'$. We show the following theorem.

Theorem 3. *For any S and for any non-zero tropical polynomial f such that $\text{Supp}(f) \subseteq S$ there is $\vec{r} \in R_S$ such that r is a non-root of f .*

An interesting case of this theorem is $S = \{0, 1, \dots, k\}^n$. Then the result states that any non-zero polynomial of individual degree at most k w.r.t. each variable x_i , $i = 1, \dots, n$, does not vanish on a lattice of size $k + 1$.

Theorem 2(i) and Theorem 3 answer some customary cases of our first question. We note that the situation here is quite different from the classical case. The classical analog of Theorem 3 for the case of $S = \prod_{i=1}^n \{0, 1, \dots, k_i\}$

is a simple observation. In the tropical setting it already requires some work. On the other hand, in the classical case it is known that for such S the domain of the polynomial can be substantially larger than S and still the polynomial remains non-vanishing on R_S (see Combinatorial Nullstellensatz [3]). In tropical case, however, if we extend the domain of the polynomial even by one extra monomial, then due to Theorem 2(i) there is a vanishing non-zero polynomial.

In the proof of Theorem 3 we will use the following simple technical lemma.

Lemma 4. *Consider two sequences of reals $v_1 \leq v_2 \leq \dots \leq v_l$ and $u_1 \leq u_2 \leq \dots \leq u_l$. Consider any permutation $\sigma \in \text{Sym}_l$ on l element set. Then*

$$\sum_i v_i u_i \geq \sum_i v_i u_{\sigma(i)}.$$

Moreover, the inequality is strict iff there are i, j such that $v_i < v_j$, $u_{\sigma(j)} < u_{\sigma(i)}$.

This lemma is well known, but we present the proof for the sake of completeness.

Proof. We count the number of inversions in σ : $D = |\{(i, j) \mid i < j, \sigma(j) < \sigma(i)\}|$. We show the lemma by induction on D . For the step of induction we pick one inversion (i, j) and swap it. We observe that by this we do not introduce new inversions.

We then use the following observation: if $a \leq b$ and $c \leq d$, then

$$bd + ac \geq bc + da.$$

This inequality is true since it is equivalent to

$$(b - a)(d - c) \geq 0.$$

We also observe that the inequality is strict iff both inequalities $a \leq b$ and $c \leq d$ are strict.

So, after the swap of i and j the sum in the statement of the lemma does not decrease. Thus the inequality follows.

To prove the second part of the lemma, if there is a pair i, j as stated in the lemma, just switch i and j on the first step. By this we get the strict inequality. If there is no such a pair i, j , note that we do not introduce one during the process above since we do not introduce new inversions. \square

Proof of Theorem 3. By Theorem 2 it is enough for us to show that S and R_S are non-singular.

Consider the bijection $f: S \rightarrow R_S$ given by $f(J) = \vec{\alpha}_J$. We claim that the maximum over all possible bijections g of the sum $\sum_{J \in S} \langle J, g(J) \rangle$ is attained on the bijection f and only on it.

Consider an arbitrary bijection $g: S \rightarrow R_S$. Since $R_S \subseteq \mathbb{R}^n$ it is convenient to denote $g(J) = (g_1(J), \dots, g_n(J))$ and $f(J) = (f_1(J), \dots, f_n(J))$. Consider the sum

$$\sum_{J \in S} \langle J, g(J) \rangle = \sum_{J \in S} \sum_{i=1}^n j_i g_i(J) = \sum_{i=1}^n \sum_{J \in S} j_i g_i(J)$$

We will show that for each i

$$\sum_{J \in S} j_i g_i(J) \leq \sum_{J \in S} j_i f_i(J) \quad (2)$$

and for at least one i

$$\sum_{J \in S} j_i g_i(J) < \sum_{J \in S} j_i f_i(J) \quad (3)$$

From these inequalities the theorem follows.

Consider an arbitrary i and consider projections of all points in the set S on the i -th coordinate. Enumerate there projections in the nondecreasing order:

$$j_{1,1} = \dots = j_{1,k_1} < j_{2,1} = \dots = j_{2,k_2} < \dots < j_{l,1} = \dots = j_{l,k_l}.$$

Different points in S can have the same i -th coordinate, so we split points into blocks according to their i -th coordinate. Due to the definition of R_S , the projections of its points on the i -th coordinate will have the same structure:

$$r_{1,1} = \dots = r_{1,k_1} < r_{2,1} = \dots = r_{2,k_2} < \dots < r_{l,1} = \dots = r_{l,k_l}.$$

Both bijections f and g induce bijections f' and g' from the sequence \vec{j} to the sequence \vec{r} . Moreover f induces a natural bijection: $f'(j_{i_1, i_2}) = r_{i_1, i_2}$. The inequality (2) thus follows from the first part of Lemma 4.

For inequality (3) note that since $g \neq f$ there is $J \in S$ such that $g(J) \neq \vec{\alpha}_J$. This means that there is i such that

$$g_i(J) \neq \alpha_{j_i}^i.$$

Thus for the bijection induced by g on the coordinate i we have that $g'(j_{i_1, i_2}) = r_{i'_1, i'_2}$, where $i_1 \neq i'_1$. Without loss of generality assume that $i_1 < i'_1$, the other case is symmetrical. Consider the subsequence

$$\vec{j}' = j_{i'_1, 1}, \dots, j_{i'_1, k_{i'_1}}, \dots, j_{l, 1}, \dots, j_{l, k_l}.$$

Since j_{i_1, i_2} is mapped by g' into the sequence \vec{j}' and g' is a bijection, there is j_{i_3, i_4} in \vec{j}' that is mapped outside of this sequence, that is $g(j_{i_3, i_4}) = r_{i'_3, i'_4}$, where $i'_3 < i'_1$.

Denoting $a = j_{i_1, i_2}$ and $b = j_{i_3, i_4}$ we obtain that $a < b$, but $g'(a) > g'(b)$. By Lemma 4 this gives inequality (3). \square

4 Tropical Analog of Schwartz-Zippel Lemma

Using the results of the previous section we can prove an analog of Schwartz-Zippel Lemma for tropical polynomials.

Theorem 5. *Let $S_1, S_2, \dots, S_n \subseteq \mathbb{R}$, denote $|S_i| = k_i$. Then for any $d \leq \min_i k_i$ the maximal number of roots a non-vanishing tropical polynomial p of degree d can have in $S_1 \times \dots \times S_n$ is equal to*

$$\prod_{i=1}^n k_i - \prod_{i=1}^n (k_i - d).$$

Exactly the same statement is true for the polynomials with individual degree of each variable at most d .

In particular, we have the following corollary.

Corollary 6. *Let $S \subseteq \mathbb{R}$ be a set of size k . Then for any $d \leq k$ the maximal number of roots a non-vanishing tropical polynomial p of degree d can have in S^n is equal to*

$$k^n - (k - d)^n.$$

Exactly the same statement is true for the polynomials with individual degree of each variable at most d .

Proof of Theorem 5. The upper bound is achieved on the product of d degree-1 polynomials. Indeed, denote $S_i = \{s_{i,1}, s_{i,2}, \dots, s_{i,k_i}\}$, where $s_{i,1} > s_{i,2} > \dots > s_{i,k_i}$. For $j = 1, \dots, d$ denote by p_j the following degree-1 polynomial:

$$p_j(\vec{x}) = (-s_{1,j} \odot x_1) \oplus \dots \oplus (-s_{i,j} \odot x_i) \oplus \dots \oplus (-s_{n,j} \odot x_n) \oplus 0.$$

Observe that $\vec{a} \in S_1 \times \dots \times S_n$ is a root of p_j if for some i $a_i = s_{i,j}$ and for the rest of i we have $a_i \leq s_{i,j}$.

Consider a degree d polynomial $p(\vec{x}) = \odot_{j=1}^d p_j(\vec{x})$. Then from Lemma 1 we have that $\vec{a} \in S_1 \times \dots \times S_n$ is a non-root of p iff for all i $a_i < s_{i,d}$. Thus the number of non-roots of p is $\prod_{i=1}^n (|S_i| - d)$. This proves the upper bound.

For the lower bound, suppose there is a polynomial p with individual degrees d that has more than $\prod_{i=1}^n k_i - \prod_{i=1}^n (k_i - d)$ roots. Then the number of its non-roots is at most $\prod_{i=1}^n (k_i - d) - 1$. Denote the set of all non-roots by R .

Consider a family of all the polynomials of individual degree at most $k_i - d - 1$ in variable x_i for all i . Then their support is of size $\prod_{i=1}^n (k_i - d)$. Since the size of the support is greater than R , by Theorem 2 there is a polynomial q with this support that vanishes on R .

Then, by Lemma 1 the non-zero polynomial $p \odot q$ vanishes on $S_1 \times \dots \times S_n$ and on the other hand has support $\{0, \dots, k_1 - 1\} \times \dots \times \{0, \dots, k_n - 1\}$. This contradicts Theorem 3. Thus there is no such polynomial p and the theorem follows. \square

5 Tropical Universal Testing Set

In this section we study the minimal size of the universal testing set for sparse tropical polynomials. It turns out that in the tropical case there is a big difference between testing sets over \mathbb{R} and \mathbb{Q} . We thus consider these two cases separately below.

Throughout this section we denote by n the number of variables in the polynomials, by k the number of monomials in them and by s the number of points in the universal testing set.

5.1 Testing sets over \mathbb{R}

In this section we will show that the minimal size s of the universal testing set over \mathbb{R} is equal to k .

Theorem 7. *For polynomials over \mathbb{R} the minimal size s of the universal testing set for tropical polynomials with at most k monomials is equal to k .*

Proof. First of all, it follows from Theorem 2(i) that for any set of s points there is a polynomial with $k = s + 1$ monomials that has roots in all s points. Thus the universal testing set has to contain at least as many points as there are monomials and we have the inequality $s \geq k$.

Next we show that $s \leq k$. Consider the set of s points $S = \{\vec{a}_1, \dots, \vec{a}_s\} \in \mathbb{R}^n$ that has linearly independent over \mathbb{Q} coordinates. Suppose we have a polynomial p with k monomials that has roots in all points $\vec{a}_1, \dots, \vec{a}_s$. We will show that $k \geq s + 1$. Thus we will establish that S is a universal set for $k = s$ monomials.

Suppose the monomials of p are m_1, \dots, m_k , where $m_i(\vec{x}) = c_i \odot \vec{x}^{J_i}$. Introduce the notation $p(\vec{a}_j) = \max_i(m_i(\vec{a}_j)) = p_j$. Since \vec{a}_j is a root, the value p_j is achieved on at least two monomials.

Note the monomial m_i has the value p_j in the point \vec{a}_j iff

$$\langle \vec{a}_j, J_i \rangle + c_i = p_j.$$

Now, consider a bipartite undirected graph G . The vertices in the left part correspond to monomials of p (k vertices). The vertices in the right part correspond to the points in S (s vertices). We connect vertex m_i on the left side to the vertex \vec{a}_j on the right side iff $m_i(\vec{a}_j) = p_j$.

Observe, that the degree of vertices on the right hand side is at least 2 (this means exactly that they are roots of p).

Now, we will show that there are no cycles in G . Indeed, suppose there is a cycle. For the sake of convenience of notation assume the sequence of the vertices of the cycle is

$$m_1, \vec{a}_1, m_2, \vec{a}_2, \dots, m_l, \vec{a}_l.$$

Note that since the graph is bipartite, the cycle is of even length. In particular, for all $i = 1, \dots, l$ we have $m_i(\vec{a}_i) = p_i$, that is

$$\langle \vec{a}_i, J_i \rangle + c_i = p_i. \tag{4}$$

Also for all $i = 1, \dots, l$ we have $m_{i+1}(\vec{a}_i) = p_i$ (for convenience of notation assume here $m_{l+1} = m_1$), that is

$$\langle \vec{a}_i, J_{i+1} \rangle + c_{i+1} = p_i. \tag{5}$$

Let us sum up all equations in (4) for all $i = 1, \dots, l$ and subtract from the result all the equations in (5). It is easy to see that all c_i 's and p_i 's will cancel out and thus we will have

$$\langle \vec{a}_1, J_1 \rangle - \langle \vec{a}_1, J_2 \rangle + \langle \vec{a}_2, J_2 \rangle - \langle \vec{a}_2, J_3 \rangle + \dots + \langle \vec{a}_l, J_l \rangle - \langle \vec{a}_l, J_1 \rangle = 0.$$

Since $J_1 \neq J_2$, we have a nontrivial linear combination with integer coefficients of the coordinates of vectors $\vec{a}_1, \dots, \vec{a}_l$. Since the coordinates of these

vectors are linearly independent over \mathbb{Q} , this is a contradiction. Thus we have shown that there are no cycles in G .

Thus the graph G is a forest. Consider each of the trees of the forest separately. We will show that in each of these trees T the number L of vertices in the left side is greater than the number R of vertices in the right side. Indeed, since the degree of each vertex in the right side is at least 2, the number of edges in T is at least $2R$. The number of vertices in a tree is by one greater than the number of edges. Thus there are at least $2R + 1$ vertices in T . That is

$$R + L \geq 2R + 1,$$

and thus $L \geq R + 1$. Since this holds for each tree, summing up these inequalities over all trees we have

$$k \geq s + 1.$$

Thus the set S is a universal set against polynomials with $k = s$ monomials.

Thus, the minimal size of universal set is $s = k$ over \mathbb{R} . \square

5.2 Testing sets over \mathbb{Q}

The main difference of the problem over the semiring \mathbb{Q} compared to the semiring \mathbb{R} is that now the points of the universal set have to be rational.

Before we proceed with this section we observe that we can assume that tropical polynomials can contain rational (possibly negative) powers of variables: for each such polynomial there is another polynomial with natural exponents with the same set of roots and the same number of monomials. Indeed, suppose p is a polynomial with rational exponents. Recall that

$$p(\vec{x}) = \max(m_1(\vec{x}), \dots, m_k(\vec{x})), \tag{6}$$

where m_1, \dots, m_k are monomials. Recall that each monomial is a linear function over \vec{x} . Note that if we multiply the whole expression (6) by some constant and add the same linear form $m(\vec{x})$ to all monomials, the resulting polynomial will have the same set of roots. Thus, we can get rid of rational degrees in p by multiplying p by large enough integer, and then we can get rid of negative degrees by adding linear form m with large enough coefficients to p .

Thus, throughout this section we will assume that all polynomials have rational exponents.

It will be convenient to state the results of this section using the following notation. Let $k(s, n)$ be the minimal number such that for any set S of s points in \mathbb{Q}^n there is a tropical polynomial on n variables with at most $k(s, n)$ monomials having roots in all points of S . Note that there is a universal testing set of size s for polynomials with k monomials iff $k < k(s, n)$. Thus, we can easily obtain bounds on the size of the minimal universal testing set from the bounds on $k(s, n)$.

We start with the following upper bound on $k(s, n)$.

Theorem 8. *We have $k(s, n) \leq \left\lceil \frac{2s}{(n+1)} \right\rceil + 1$.*

For the size of the minimal universal testing set the following inequality holds: $s \geq (k - 1)(n + 1)/2 + 1$.

We note that this theorem already shows the difference between universal testing sets over \mathbb{R} and \mathbb{Q} semirings.

Proof. We will show that for any set $S = \{\vec{a}_1, \dots, \vec{a}_s\} \subseteq \mathbb{Q}^n$ of size s there is a nontrivial polynomial with at most $k = \left\lceil \frac{2s}{(n+1)} \right\rceil + 1$ monomials that has roots in all of the points in S . From this the inequalities in the theorem follow.

Throughout this proof we will use the following standard facts about (classical) affine functions on \mathbb{Q}^n .

Claim 1. *Suppose π is an $(n - 1)$ dimensional hyperplane in \mathbb{Q}^n . Let P_1 be a finite set of points in one of the (open) halfspaces w.r.t. π and P_2 be a finite set of points in the other (open) halfspace. Let C_1 and C_2 be some constants. Then the following is true.*

1. *If $\vec{a}_1, \dots, \vec{a}_n \in \pi$ are points in the general position in π and p_1, \dots, p_n are some constants in \mathbb{Q} , then there is an affine function f on \mathbb{Q}^n such that $f(\vec{a}_i) = p_i$ for all i , $f(\vec{x}) > C_1$ for all $\vec{x} \in P_1$ and $f(\vec{x}) < C_2$ for all $\vec{x} \in P_2$.*
2. *If g is an affine function on \mathbb{Q}^n then there is another affine function f on \mathbb{Q}^n such that $f(\vec{x}) = g(\vec{x})$ for all $\vec{x} \in \pi$, $f(\vec{x}) > C_1$ for all $\vec{x} \in P_1$ and $f(\vec{x}) < C_2$ for all $\vec{x} \in P_2$.*

The proof of the theorem is by induction on s . The base is $s = 0$. In this case one monomial is enough (and is needed since we require polynomial to be nontrivial).

Consider the convex hull of points of S . Consider a face P of this convex hull. For simplicity of notation assume that the points from S belonging

to P are $\vec{a}_1, \dots, \vec{a}_l$. Consider a $((n-1)$ -dimensional) hyperplane π passing through $\vec{a}_1, \dots, \vec{a}_l$. Since P is a face of the convex hull of S we can pick π in such a way that all points in $S' = \{\vec{a}_{l+1}, \dots, \vec{a}_n\}$ lie in one halfspace w.r.t. π (the choice of π might be not unique since P might be of the dimension less than $n-1$).

Applying the induction hypothesis we obtain a polynomial $p'(\vec{x}) = \max_i m'_i(\vec{x})$ that has roots in all points of S' . For $j = 1, \dots, l$ introduce the notation $p_j = p'(\vec{a}_j) = \max_i m_i(\vec{a}_j)$.

We consider three cases: P contains all points of S ; P contains not all points of S and $l \leq n$; P contains not all points of S and $l > n$.

If P contains all points of S , then the polynomial p' is obtained from the base of induction and consists of one monomial m'_1 . Recall, that a monomial is just an affine function on \mathbb{Q}^n . Consider a new monomial $m(\vec{x})$ such that $m(\vec{x}) = m'_1(\vec{x})$ on the hyperplane π , but $m(\vec{b}) \neq m'_1(\vec{b})$ for some $\vec{b} \notin \pi$. Then the polynomial $p = p' \oplus m$ has roots in all points of the hyperplane π and thus in all points of S . This polynomial has $2 \leq \left\lceil \frac{2s}{(n+1)} \right\rceil + 1$ monomials.

If P contains not all points of S , then the dimension of P is $n-1$ (indeed, otherwise P is not a face).

If additionally $l \leq n$, it follows that $l = n$. Thus $\vec{a}_1, \dots, \vec{a}_n$ are points in the general position in π . Thus due to the claim above we can pick a new monomial m such that $m(\vec{a}_j) = p_j$ for all $j = 1, \dots, l$ and $m(\vec{a}_j) < p'(\vec{a}_j)$ for all $j > l$. Then the polynomial $p = p' \oplus m$ has roots in all points of S . This polynomial has $1 + \left\lceil \frac{2(s-n)}{(n+1)} \right\rceil + 1 \leq \left\lceil \frac{2s}{(n+1)} \right\rceil + 1$ monomials.

Now, if $l \geq n+1$ let $p_0 = \max_{j \leq l} p_j$. Applying the claim above take a pair of new distinct monomials m_1 and m_2 such that $m_1(\vec{x}) = m_2(\vec{x}) = p_0$ for all $\vec{x} \in \pi$ and $m_1(\vec{a}_j), m_2(\vec{a}_j) < p'(\vec{a}_j)$ for all $j > l$. Then the polynomial $p = p' \oplus m_1 \oplus m_2$ has roots in all points of S . This polynomial has $2 + \left\lceil \frac{2(s-n-1)}{(n+1)} \right\rceil + 1 = \left\lceil \frac{2s}{(n+1)} \right\rceil + 1$ monomials.

In all three cases we constructed a polynomial with the desired number of monomials. \square

The construction above leaves the room for improvement. We illustrate this for $n = 2$.

Theorem 9. *For $n = 2$ we have $k(s, 2) \leq \left\lceil \frac{s}{2} \right\rceil + 1$. For the size of the minimal universal set for polynomials in 2 variables the following inequality holds: $s \geq 2(k-1) + 1$.*

Proof. We use the same strategy as in the proof of Theorem 8. We perform the same case analysis on the induction step. Note that in the first two cases

the step of induction works.

Thus, the only remaining case is $l > 2$ and P contains not all points of S . That is there is a line π in \mathbb{Q}^2 containing points $\vec{a}_1, \dots, \vec{a}_l$ and such that all points in $S \setminus \{\vec{a}_1, \dots, \vec{a}_l\}$ are in one halfspace w.r.t. π . Consider the point of $S \setminus \{\vec{a}_1, \dots, \vec{a}_l\}$ that is the closest one to the line π . Draw the line π' parallel to π through this point. If there are several points of S on π' consider the one that does not lie between two others. To simplify the notation let this vertex be \vec{a}_{l+1} . Denote the set of remaining vertices by $S' = \{\vec{a}_{l+2}, \dots, \vec{a}_s\}$ and apply the induction hypothesis to S' . As before let $p_j = p'(\vec{a}_j)$.

Consider a new monomial m_1 (recall that the monomial is just an affine function on \mathbb{Q}^2) such that $m_1(\vec{a}_{l+1}) = p_{l+1}$, $m_1(\vec{a}_j) \leq p_j$ for all $\vec{a}_j \in S \cap \pi'$, $m_1(\vec{a}_j) \leq p_j$ for all $\vec{a}_j \in S' \setminus \pi$ and $m_1(\vec{a}_j) \geq p_j$ for all $j \leq l$. Note that this is possible since $\vec{a}_1, \dots, \vec{a}_l$ and $S' \setminus \pi$ are situated in the opposite halfplanes w.r.t. π' . Finally, pick yet another new monomial m_2 such that $m_1(\vec{x}) = m_2(\vec{x})$ for all $\vec{x} \in \pi$ and $m_2(\vec{a}_j) \leq p_j$ for all $j > l$. Then the polynomial $p = p' \oplus m_1 \oplus m_2$ has roots in all points of S . This polynomial has

$$2 + \left\lceil \frac{s-4}{2} \right\rceil + 1 = \left\lceil \frac{s}{2} \right\rceil + 1$$

monomials. □

We now proceed to lower bounds on $k(s, n)$. We start with the following non-constructive lower bound.

Theorem 10. *We have $k(s, n) \geq \left\lceil \frac{s}{n+1} \right\rceil$.*

Then for the minimal size of the universal testing set over \mathbb{Q} we have $s \leq k(n+1) + 1$.

Proof. Within this proof we will temporarily switch to polynomials over \mathbb{R} . Suppose for any set $S = \{\vec{a}_1, \dots, \vec{a}_s\} \in \mathbb{R}^n$ there is always a polynomial with k monomials that has roots in all s points.

The set of all tuples $\vec{a}_1, \dots, \vec{a}_s$ of s points in \mathbb{R}^n forms an sn dimensional space over \mathbb{R} . Suppose a polynomial p with monomials m_1, \dots, m_k has roots in all points $\vec{a}_1, \dots, \vec{a}_s$. This means that on each point \vec{a}_j there are two monomials that has two equal values. By *configuration* we call an assignment to each point \vec{a}_j of a pair of monomials m_{i_1}, m_{i_2} such that $m_{i_1}(\vec{a}_j) = m_{i_2}(\vec{a}_j)$. Note that there are finitely many configurations. Thus, we can specify each tuple $\vec{a}_1, \dots, \vec{a}_s$ by specifying k monomials and for each point \vec{a}_j specifying the one with the property $m_{i_1}(\vec{a}_j) = m_{i_2}(\vec{a}_j)$. To specify a monomial m we need to specify the vector of its exponents J and a coefficient c . Thus, to specify k monomials it is enough to give $k(n+1)$ numbers. Now, each point

\vec{a}_j should satisfy a linear restriction $m_{i_1}(\vec{a}_j) = m_{i_2}(\vec{a}_j)$. The solutions of this system form $n - 1$ dimensional space, so to specify all s points it remains to add $s(n - 1)$ coordinates. Thus, in total, for each configuration the dimension of the semi-algebraic set of tuples that can be specified by this configuration is $k(n + 1) + s(n - 1)$ and there are finitely many configurations. Thus, if there is the following inequality between dimensions

$$k(n + 1) + s(n - 1) < sn,$$

then not all tuples S can be represented by any configuration and so there is a tuple S (over \mathbb{R}) such that for any polynomial p with at most k monomials there is a non-root for p in S .

Now, suppose that we are given some set of points S . Consider a polynomial p and consider some fixed configuration. Then the restriction for monomials m_{i_1} and m_{i_2} with degrees J_1 and J_2 and coefficients c_1, c_2 respectively that $m_{i_1}(\vec{a}_j) = m_{i_2}(\vec{a}_j)$ is a linear uniform restriction in J_1, J_2, c_1 and c_2 :

$$\langle J_1, \vec{a}_j \rangle + c_1 = \langle J_2, \vec{a}_j \rangle + c_2.$$

Thus the statement that there is a polynomial that has roots in all points of S within a fixed configuration is equivalent to the solvability of a uniform linear system (with coordinates of points in S as coefficients). Such a system is solvable for some S iff all of its maximal minors are zero. Note that each minor is a polynomial over coordinates of points in S . For a configuration C denote by q_C the sum of squares of all of these minors. Thus, there is polynomial for S in configuration C iff $q_C(S) = 0$.

Let

$$q(S) = \prod_{C - \text{configuration}} q_C(S).$$

Then we have shown that there is a polynomial having roots in all points of S iff $q(S) = 0$. We also have shown before that if $s > k(n + 1)$ then there is $S \in \mathbb{R}^{sn}$ such that $q(S) \neq 0$. Since \mathbb{Q}^{sn} is everywhere dense in \mathbb{R}^{sn} , there is also $S \in \mathbb{Q}^{sn}$ such that $q(S) \neq 0$.

Thus, we have shown that if $s \geq k(n + 1) + 1$, then there is a set of s rational points S that is universal for polynomials with k monomials. \square

This theorem also leaves a room for improvement. For example, we can fix the free coefficient of one of the monomials in the proof above and to reduce the bound on s by one. Still, these improvements are minor compared to the gap between current upper and lower bounds.

The lower bound on $k(s, n)$ in Theorem 10 is not constructive. In the next section we present some constructive lower bounds. For this we establish a connection of our problem to certain questions in discrete geometry.

5.3 Constructive Lower Bounds

Suppose for some set of points $S = \{\vec{a}_1, \dots, \vec{a}_s\} \subseteq \mathbb{Q}^n$ there is a polynomial p with monomials m_1, \dots, m_k that has roots in all points of S .

Recall that the graph of p in $(n + 1)$ -dimensional space is a piece-wise linear convex function. Each linear piece corresponds to a monomial and roots of the polynomial are the points of non-smoothness of this function.

Consider the set of all roots of p in \mathbb{Q}^n . They partition the space \mathbb{Q}^n into at most k convex (possibly infinite) polytopes. Each polytope corresponds to one of the monomials.

Consider the polytope corresponding to the monomial m_i . Consider all points in S that lie on its boundary and consider their convex hull. We obtain a smaller (finite) convex polytope that we will denote by P_i .

Thus starting from p we arrive at the set of non-intersecting polytopes P_1, \dots, P_k with vertices in S not containing any points of S in the interior. The fact that p has roots in all points of S means that each point in S belong to at least two of the polytopes P_1, \dots, P_k . We call this structure by a *double covering* of points of S by convex polytopes. The *size* of the covering is the number k of polytopes in it.

Thus, if we will construct a set S of points that does not have a double covering of size k it will follow that S is a universal set for k monomials. The similar question of single covering has been studied in the literature [6] (in the single cover polytopes cannot intersect even by vertices.).

Denote by $k_1(s, n)$ the minimal number of polytopes that is enough to single cover any s points in n dimensional space. Denote by $k_2(s, n)$ the minimal number of polytopes that is enough to double cover any s points in n dimensional space. The above analysis results in the following theorem.

Theorem 11. $k(s, n) \geq k_2(s, n)$.

For single coverings the following results are known. Let $f(n)$ be the maximal number such that any large enough n -dimensional set of points S contains a convex set of $f(n)$ points that are the vertices of a convex polytope and on the other hand do not contain any other points in S . The function $f(n)$ was studied but is not well understood yet. It is known [29] that the function is at most exponential in n . We can however observe the following.

Lemma 12. *For large enough s we have that $k_1(s, n) \geq s/f(n)$.*

Proof. Consider a large enough set of s points with no empty polytopes of size $f(n) + 1$. Then in any covering each polytope can contain at most $f(n)$ points, hence the lower bound follows. \square

It is known [29] that $f(3) \geq 22$. Thus we get that $k_1(s, 3) \geq s/22$ for large enough s .

It is also known [28] that $\lceil s/2(\log_2 s + 1) \rceil \leq k_1(s, 3) \leq \lceil 2s/9 \rceil$. For $n = 2$ there are linear upper and lower bounds known [27]. For an arbitrary n in [28] an upper bound $k_1(s, n) \leq 2s/(2n+3)$ is shown and $k_1(s, n) = s/2n$ is conjectured.

We establish the following connection between $k_1(s, n)$ and $k_2(s, n)$.

Lemma 13. $k_2(s, n) \geq k_1(s, n)$. Thus for large enough s we have that $k(s, n) \geq s/f(n)$.

Proof. Consider the set of polytopes P_1, \dots, P_k for $k = k_2(s, n)$ constituting a double covering of some set of s points. Let P'_i be the convex hull of all vertices of P_i that are not contained in P_{i+1}, \dots, P_k . It is easy to see that polytopes P'_1, \dots, P'_k constitute a single covering of the same set of points. \square

Lemma 14. $k_1((n+2)s, n) \geq k_2(s, n)$.

Proof. Consider a set of s points and substitute each point by the set of vertices of a small enough n -dimensional simplex and by its center. Thus we substitute each point by $n+2$ points and obtain $(n+2)s$ points as a result. Consider a single covering of these points of size $k_1((n+2)s, n)$. None of the polytopes in this cover can contain the whole simplex and its center. Thus, each simplex contains vertices of at least two polytopes. Merging all points of each simplex back into one point results in a double covering of the original set of the same size (assuming the simplices are small enough). \square

Overall, we have a sequence of inequalities

$$k(s, n) \geq k_2(s, n) \geq k_1(s, n) \geq k_2\left(\frac{s}{n+2}, n\right).$$

We do not know how large $k(s, n)$ can be compared to $k_1(s, n)$ and $k_2(s, n)$.

However this connection helps us to show that the lower bound on the size of universal testing set we have established before for the case of $n = 2$ is tight.

Theorem 15. We have $k(s, 2) \geq k_2(s, 2) \geq \lceil \frac{s}{2} \rceil + 1$.

Corollary 16. *For $n = 2$ the size of the minimal universal testing set is equal to $s = 2k - 1$.*

Corollary 16 follows from Theorem 15 and Theorem 9 immediately.

The remaining part of the paper is devoted to the proof of Theorem 15.

It remains to show that $k_2(s, 2) \geq \lceil \frac{s}{2} \rceil + 1$.

As a universal set with s points in \mathbb{Q}^2 we will pick the set of vertices of an arbitrary convex polygon M .

Suppose we have some double covering of the vertices of M by k polygons. Among these polygons let us distinguish the set E of those that are edges of M and the set T of other polygons. Denote $|E| = k_1$ and $|T| = k_2$, thus $k = k_1 + k_2$. Denote by W the sum of the number of vertices in all polygons in T .

We will show the following lemma.

Lemma 17. *For $s \geq 2$ we have $W \leq s + 2k_2 - 2$.*

First let us show why this lemma is enough to finish the proof of the lower bound on $k_2(s, 2)$.

Note that each polygon from E has two vertices. Thus the sum of the number of vertices in all polygons in E is $2k_1$. The sum of the number of vertices in all polygons in T by definition is W . Each vertex of M should be a vertex for at least two polygons in E and T . Thus, the sum of number of vertices in all polygons in E and T is at least $2s$. Thus we get that

$$2s \leq 2k_1 + W \leq 2k_1 + s + 2k_2 - 2,$$

where the second inequality follows from Lemma 17. From this we get

$$k = k_1 + k_2 \geq \frac{s}{2} + 1.$$

Since k is an integer we have $k \geq \lceil \frac{s}{2} \rceil + 1$ and the theorem follows.

Thus it remains to prove the lemma.

Proof of Lemma 17. The proof is by induction on s .

The base case is $s = 2$ (a degenerate polygon). Then $T = \emptyset$, $k_2 = 0$, $W = 0$ and the inequality follows.

Consider $s \geq 3$. If $k_2 = 0$, then $W = 0$ and the inequality obviously holds. Suppose $k_2 \geq 1$ and pick arbitrary polygon P in T . Suppose there are r vertices in P . Then P splits the remaining part of M into r separate convex polygons (possibly degenerate, that is with just 2 vertices) M_1, \dots, M_r . Denote the number of vertices in them by s_1, \dots, s_r respectively. Note that

$$s_1 + \dots + s_r = s + r. \tag{7}$$

Suppose in polygons M_1, \dots, M_r there are t_1, \dots, t_r polygons in T respectively. Denote the sets of these polygons by T_1, \dots, T_r respectively. Then

$$t_1 + \dots + t_r = k_2 - 1. \quad (8)$$

Suppose the sum of number of vertices in T_1, \dots, T_r is W_1, \dots, W_r respectively. Then

$$W_1 + \dots + W_r = W - r. \quad (9)$$

By the induction hypothesis for any polygon M_i we have the following inequality:

$$W_i \leq s_i + 2t_i - 2. \quad (10)$$

Adding up inequality (10) for all $i = 1, \dots, r$ and using (7)-(9) we get

$$W - r \leq (s + r) + 2(k_2 - 1) - 2r,$$

i. e.

$$W \leq s + 2k_2 - 2$$

and the lemma follows. \square

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