Exact Computation of Rotation Minimizing Frames

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Abstract

The main goal of this work is to show how to exactly compute rotation minimizing frames for an at least three times differentiable space curve. Due to their minimal twist, these frames are preferable over the usual Frenet one in many contexts, such as in motion design, sweep surface modeling, computer visualization, and in geometric considerations as well. We show that it is possible to find the angle between the principal normal and a rotation minimizing vector. This is done by first solving the problem for spherical curves and then using the concept of osculating spheres to solve the general case. Finally, since Frenet frames and osculating spheres are not well defined at the zeros of the curvature and torsion, respectively, we discuss the behavior of the angle between the principal normal and a rotation minimizing vector near these zeros. For a curve whose torsion vanishes on intervals or isolated points only, we show that a rotation minimizing frame can be globally defined using our procedure by conveniently choosing some arbitrary constants. On the other hand, we mention that rotation minimizing frames can be well defined via their Frenet frames for analytic curves. Certainly, our approach will be of great importance in those contexts where rotation minimizing frames play a role.

Keywords: curve, motion, rotation minimizing frame, spherical curve, differential geometry
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1. Introduction

The usual way of studying the geometry of curves is by means of the well known Frenet frame. Such a frame is rich in geometric informations [20], but since its principal normal always points to the center of curvature, it may result in unnecessary rotation and then making its use unsuitable in some contexts. In this respect, the consideration of *rotation minimizing frames* $\{\mathbf{t},\mathbf{n}_1,\mathbf{n}_2\}$ (RM frames, for short) is of special interest [2, 19]: the basic idea is that $\mathbf{n}_i$ rotates only the necessary amount to remain normal to the tangent $\mathbf{t}$. Due to their minimal twist, RM frames are of fundamental importance in many branches, such as in camera [12, 18] and rigid body motions [13, 14], visualization [1, 16] and deformation

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of tubes [21, 22], sweep surface modeling [4, 24, 26, 28], and in differential geometry as well [2, 6, 8], just to name a few.

While the computation of a Frenet frame can be easily done, up to now this has not been the case for RM frames. Since the inability to compute RM frames would impose severe restrictions in their applications, numerical methods and special curves for which such frames can be computed were investigated. In the numerical arena, we can mention the projection [19], rotation [3], and double reflection methods [29] (for more details on numerical approaches, see [29] and references therein). On the other hand, an important class of curves is the so called Pythagorean-hodograph (PH) curves [9] and its subset of polynomial curves that admit rational RM frames [10, 11]. The possibility of using rational function integration in order to exactly compute RM frames is of major importance and leads to many computationally attractive features.

The main goal of this work is to show that it is possible to exactly compute RM frames. Indeed, for an at least three times differentiable space curve, we show that it is possible to find the angle between the principal normal and a RM vector (the derivative of this angle gives the torsion). This is done by first showing how to solve the problem for spherical curves and subsequently using the concept of osculating spheres to solve the general case. In addition, since the Frenet frame is not well defined at an inflection point, i.e., points with zero curvature function, and the osculating sphere at zero torsion points, we discuss the behavior of the torsion near these zeros and, consequently, also of the angle between the principal normal and a RM vector.

The remaining of this work is divided as follows. In section 2 we introduce RM frames and some geometric background. In section 3 and 4 we approach the problem of finding RM frames for spherical curves and for curves in general, respectively. In section 5 we discuss the behavior of curves near inflection and zero torsion points and, finally, in section 6 we present our concluding remarks.

2. Preliminaries

Let us denote by $\mathbb{E}^3$ the three dimensional Euclidean space, i.e., $\mathbb{R}^3$ equipped with the standard metric $\langle x,y\rangle=\sum_{i=1}^{3}x_iy_i$. Given a regular curve $\alpha:I\to\mathbb{E}^3$, i.e. $\langle\alpha',\alpha'\rangle\neq0$, the usual way to introduce a moving frame along it is by means of the Frenet frame $\{\mathbf{t},\mathbf{n},\mathbf{b}\}$ [20]:

$$\mathbf{t}=\frac{\alpha'}{\Vert\alpha'\Vert}\,,\;\mathbf{b}=\frac{\alpha'\times\alpha''}{\Vert\alpha'\times\alpha''\Vert},\;\text{ and }\mathbf{n}=\mathbf{b}\times\mathbf{t}\,,\tag{1}$$

whose equation of motion is

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\begin{array}{c}\mathbf{t}\\ \mathbf{n}\\ \mathbf{b}\end{array}\right)=\Vert\alpha'\Vert\left(\begin{array}{ccc}0 & \kappa & 0\\ -\kappa & 0 & \tau\\ 0 & -\tau & 0\end{array}\right)\left(\begin{array}{c}\mathbf{t}\\ \mathbf{n}\\ \mathbf{b}\end{array}\right).\tag{2}$$

Here $\kappa=\Vert\alpha'\times\alpha''\Vert\,\Vert\alpha'\Vert^{-3}$ and $\tau=\langle\alpha',\alpha''\times\alpha'''\rangle\,\Vert\alpha'\times\alpha''\Vert^{-2}$ are the curvature function and torsion, respectively.

For simplicity, we assume that all the curves are parametrized by an arc-length s, i.e., $\|\alpha'(s)\| = 1$, unless otherwise stated (every regular curve can be parametrized in this way [20]). In addition, we say that α is a *twisted curve* if $\kappa > 0$ and $\tau \neq 0$. A point $\alpha(s^*)$ with $\kappa(s^*) = 0$ is called an *inflection point*.

Despite the importance of Frenet frames in geometric considerations, in some applications their use is unsuitable since the normal vectors $\mathbf{n}$ and $\mathbf{b}$ rotate around each other. So, one should consider other adapted orthonormal moving frames $\{\mathbf{t}(s), \mathbf{n}_1(s), \mathbf{n}_2(s)\}$ along $\alpha(s)$ with the additional property of $\mathbf{n}_i$ rotating around the unit tangent $\mathbf{t}$ only. The equation of motion of such a *rotation minimizing* (RM) moving frame is

$$\frac{\mathrm{d}}{\mathrm{d}s}\begin{pmatrix}\mathbf{t}\\ \mathbf{n}_1\\ \mathbf{n}_2\end{pmatrix} = \begin{pmatrix}0 & \kappa_1 & \kappa_2\\ -\kappa_1 & 0 & 0\\ -\kappa_2 & 0 & 0\end{pmatrix}\begin{pmatrix}\mathbf{t}\\ \mathbf{n}_1\\ \mathbf{n}_2\end{pmatrix}. \tag{3}$$

Remark 2.1. *RM frames have been independently discovered several times in the literature, see e.g. [2, 5, 19, 27]. While Bishop seems to be the first to exploit the geometric implications of such frames [2] (he named them* relatively parallel *frames), Klok seems to be the first to note their potentialities for computer aided geometric design purposes [19]. In addition, it can be proved that a RM vector field is parallel transported along $\alpha(s)$ with respect to the normal connection of the curve [7]. Observe that for a closed curve, $\alpha(s_i) = \alpha(s_f)$, $\mathbf{n}_1(s_i)$ will differ from $\mathbf{n}_1(s_f)$, by an angular amount of $\Delta\theta = \int_{s_i}^{s_f} \tau(x)\mathrm{d}x$.*

By writing $\mathbf{n}_1(s) = \cos\theta(s)\,\mathbf{n}(s) - \sin\theta(s)\,\mathbf{b}(s)$ and $\mathbf{n}_2(s) = \sin\theta(s)\,\mathbf{n}(s) + \cos\theta(s)\,\mathbf{b}(s)$, the coefficients κ_1 and κ_2 relate with the curvature function κ and torsion τ according to [2, 15]

$$\left\{\begin{array}{r}\kappa_1(s) = \kappa(s)\,\cos\theta(s)\\ \kappa_2(s) = \kappa(s)\,\sin\theta(s)\\ \theta'(s) = \tau(s)\end{array}\right. . \tag{4}$$

An advantage of a RM moving frame is that it can be globally defined even if $\kappa = 0$ at some points [2]. In addition, RM frames are not uniquely defined, since any rotation of $\mathbf{n}_i$ on the normal plane still gives a RM field, i.e., the angle θ is well defined up to an additive constant. Nonetheless, the prescription of curvatures κ_1, κ_2 still determine a curve up to rigid motions [2]. In addition, the exact computation of RM frames is a major problem in the theory. However, taking into account the above relations between (κ, τ) and (κ_1, κ_2), a strategy to find RM frames would be the computation of the angle θ between the principal normal and a RM vector [15]. This will be the approach adopted here.

Finally, we say that two regular curves curves α and β in $\mathbb{E}^3$ have a *contact of order k* if $\alpha(s_0) = \beta(s_0^*)$ and all the higher order derivatives, up to order k, also coincide: $\mathrm{d}^i\alpha(s_0)/\mathrm{d}s^i = \mathrm{d}^i\beta(s_0^*)/\mathrm{d}s^i$ for $1 \leq i \leq k$. For example, the tangent line has a contact of order 1 with its reference curve, while the osculating circle has a contact of order 2 [20]: at an inflection point the tangent line has a contact of order 2 with its reference curve, so we may say that the osculating circle at this point has an infinity radius of curvature. Further, we say that a curve α and a surface Σ has a contact of order k if there exists a curve in Σ which has a contact of order k with α and all the other curves has a lower, or equal, order of contact. For example, the osculating plane, i.e., the plane spanned by $\{\mathbf{t},\mathbf{n}\}$, has a contact of order 2 with its reference curve, while the osculating sphere has a contact of order 3: at a zero torsion point the osculating plane has a contact of order 3, so we may say that the osculating sphere at this point has an infinity radius. At a twisted point, the center and radius of the osculating sphere are respectively given by [20]

$$P_S(s_0) = \alpha + \frac{1}{\kappa}\,\mathbf{n} + \frac{1}{\tau}\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{\kappa}\right)\mathbf{b} \text{ and } R_S(s_0) = \sqrt{\frac{1}{\kappa^2} + \frac{1}{\tau^2}\left[\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{\kappa}\right)\right]^2}\,. \tag{5}$$

3. Rotation minimizing frames for spherical curves

Before attacking the general problem of determining RM frames, let us first describe the particular case of a spherical curve $\alpha \subset S^2(p,r)$. For these curves, the computation of a RM frame is pretty easy. Indeed, as pointed out by Wang *et al.* [29], the normal frame to the sphere, i.e., the normalized position vector $\mathbf{N} = (\alpha - p)/r$, minimizes rotation since $\frac{\mathrm{d}}{\mathrm{d}s}(\alpha - p)/r = (1/r)\,\alpha'$ (this is an important step in the implementation of the *double reflection method* for computing approximations of RM frames [29]).

Using the concept of osculating spheres, we would intuitively say that every curve is locally spherical. In this case, it is tempting to ask if the normals to the osculating spheres minimize rotation. Unhappily, this strategy does not work unless the curve is spherical. In fact, we have[1]

Proposition 3.1. *If $\alpha : I \to \mathbb{E}^3$ is a regular twisted curve of class C^4, then*

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\alpha(s) - P_S(s)}{R_S(s)}\right) = \frac{1}{R_S}\left[\mathbf{t} + \frac{\rho\rho'}{\tau R_S^2}\sigma\,\mathbf{n} + \left(\frac{\rho'^{\,2}}{\tau^2 R_S^2} - 1\right)\sigma\,\mathbf{b}\right]\,, \tag{6}$$

where $\rho = \kappa^{-1}$, $P_S(s)$ and $R_S(s)$ are the center and radius of the osculating sphere, respectively, and

$$\sigma(s) = \tau(s)\rho(s) + \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\rho'(s)}{\tau(s)}\right). \tag{7}$$

In addition, the normal vector field to the curve given by the normals to the osculating sphere along $\alpha(s)$ minimizes rotation if and only if α is spherical, i.e., when $\sigma \equiv 0$.

Proof. Direct computation of the derivative of $\mathbf{N} = (\alpha - P_S)/R_S$ leads to Eq. (6). Finally, by a known result of geometry, the condition to be spherical leads to $\sigma \equiv 0$ [20], which by direct examination of Eq. (6) is a necessary and sufficient condition to have $\mathbf{N}$ and $\mathbf{t}$ parallel. □

[1] The C^4 condition is needed in order to work with the derivative of τ, but this does not represent a restriction to the general theory. In fact, the goal here is to show that we can not generalize the known result for spherical curves by manipulating an osculating sphere in this way.

The above idea, i.e., using osculating spheres to extend a result valid for spherical curves, can be used by adopting a distinct strategy: we should compute the angle between the principal normal and a RM vector field, which can be done by conveniently using $\theta' = \tau$ [15]. The only drawback of such an approach is the need of a Frenet frame globally defined, i.e., no inflection point should be allowed ($\forall\, s,\ \kappa(s) \neq 0$). The next result is an important step in this direction[2].

Theorem 3.1. *Let $\alpha : I \to S^2(p,r)$ be a spherical C^3 curve parametrized by arc-length s and $J = \langle \alpha - p, \alpha' \times \alpha'' \rangle$, then*

(a) the curvature and torsion are respectively given by

$$\kappa = \frac{1}{r}\sqrt{1+J^2} \ \ \text{and} \ \ \tau = \frac{J'}{1+J^2}\,; \tag{8}$$

(b) the angle θ between a rotation minimizing vector and the principal normal satisfies

$$\theta(s_2) - \theta(s_1) = \arctan\, J(s_2) - \arctan\, J(s_1)\,. \tag{9}$$

Proof. (a) For simplicity, assume p to be the origin (the general situation is reduced to this one by studying $\tilde{\alpha} = \alpha - p$). The vectors α/r, α', and $(\alpha/r) \times \alpha'$ form an orthonormal frame along α. Let us write

$$\alpha'' = \frac{1}{r}\langle \alpha'', \alpha \rangle \frac{\alpha}{r} + \langle \alpha'', \alpha' \rangle \alpha' + \frac{1}{r}\langle \alpha'', \alpha \times \alpha' \rangle \frac{\alpha}{r} \times \alpha'\,. \tag{10}$$

Since α is parametrized by arc-length, we have $\langle \alpha'', \alpha' \rangle = 0$. In addition, from $\langle \alpha, \alpha \rangle = r^2$, it follows that $\langle \alpha'', \alpha \rangle = -\langle \alpha', \alpha' \rangle = -1$. In conclusion, the acceleration vector gives

$$\alpha'' = -\frac{1}{r}\frac{\alpha}{r} + \frac{J}{r}\frac{\alpha}{r} \times \alpha' \ \Rightarrow\ \kappa = \Vert \alpha'' \Vert = \frac{1}{r}\sqrt{1+J^2}\,. \tag{11}$$

Now, writing the normal and binormal vectors as $\mathbf{n} = \alpha''/\kappa$ and $\mathbf{b} = \alpha' \times \mathbf{n} = \alpha' \times \alpha''/\kappa$, and using the Frenet equation $\tau = -\langle \mathbf{b}', \mathbf{n} \rangle$, we have

$$\begin{aligned} \tau &= -\left\langle \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\alpha' \times \alpha''}{\kappa}\right), \frac{\alpha''}{\kappa} \right\rangle \\ &= -\frac{1}{\kappa^2}\langle \alpha' \times \alpha''', \alpha'' \rangle \\ &= -\frac{1}{r^2\kappa^2}\langle \alpha' \times \alpha''', -\alpha + J\,\alpha \times \alpha' \rangle. \end{aligned} \tag{12}$$

[2]The extension to curves with inflections points is discussed in Section 5.

Finally, using the expression for κ above, $J' = \langle \alpha, \alpha' \times \alpha''' \rangle$, and the vector identity $\langle \mathbf{A} \times \mathbf{B}, \mathbf{C} \times \mathbf{D} \rangle = \langle \mathbf{A}, \mathbf{C} \rangle \langle \mathbf{B}, \mathbf{D} \rangle - \langle \mathbf{A}, \mathbf{D} \rangle \langle \mathbf{B}, \mathbf{C} \rangle$, we get the following expression

$$\tau = \frac{J'}{1 + J^2}\,. \tag{13}$$

(b) The expression for θ is a consequence of the chain rule and of $(\arctan\, x)' = (1 + x^2)^{-1}$. $\square$

Remark 3.1. *Item (a) above shows that spherical curves do not have inflection points. So, computing RM frames using the Frenet frame can be always done. Naturally, one may argue that this is a hard way of doing a simple job: the normals to the sphere, $(\alpha - p)/r$, is RM and easy to compute. However, our approach has the advantage of being extended for a generic curve by using osculating spheres, as will be shown in the next Section.*

4. Computing rotation minimizing frames

Once we know how to solve the problem of computing RM frames for spherical curves, we can attack the general problem by exploiting the concept of osculating spheres.

Let α be a regular twisted curve and $\Sigma_s = S^2(P_S(s), R_S(s))$ be its osculating sphere at $\alpha(s)$. In a neighborhood of s_0 we can obtain a spherical curve $\beta : (s_0 - \epsilon, s_0 + \epsilon) \to \Sigma_{s_0}$ by defining

$$\beta(t) = r_0 \frac{\alpha(t) - a_0}{\|\alpha(t) - a_0\|}\,, \tag{14}$$

where $a_0 = P_S(s_0)$ and $r_0 = R_S(s_0)$.

Theorem 4.1. *The torsion τ_β of the (osculating) spherical projection β, Eq. (14), and the torsion τ_α of a C^3 regular twisted curve α coincide at s_0:*

$$\tau_\alpha(s_0) = \tau_\beta(s_0)\,. \tag{15}$$

In addition, it follows that the angle θ between a rotation minimizing vector and the principal normal satisfies

$$\theta(s_2) - \theta(s_1) = \arctan\, J(s_2) - \arctan\, J(s_1)\,, \tag{16}$$

where $J(s) = \langle \alpha(s) - P_S(s), \alpha'(s) \times \alpha''(s) \rangle$.

Proof. In order to compute τ_β it is enough to find β', β'', and β'''. Calculating the derivatives of β and taking into account the relations

$$\left\{ \begin{array}{ccc} \langle\, \alpha(s_0) - a_0, \mathbf{t}(s_0)\,\rangle & = & 0 \\ 1 + \kappa(s_0)\,\langle\, \alpha(s_0) - a_0, \mathbf{n}(s_0)\,\rangle & = & 0 \\ \kappa'(s_0) - \kappa^2(s_0)\tau(s_0)\,\langle\, \alpha(s_0) - a_0, \mathbf{b}(s_0)\,\rangle & = & 0 \end{array} \right. \tag{17}$$

satisfied by an osculating sphere [20], we obtain after some lengthy but straightforward calculations the following relations at $s = s_0$

$$\begin{cases} \beta'(s_0) &= \mathbf{t}(s_0) \\ \beta''(s_0) &= \kappa(s_0)\mathbf{n}(s_0) \\ \beta'''(s_0) &= -\kappa^2(s_0)\mathbf{t}(s_0) + \kappa'(s_0)\mathbf{n}(s_0) + \tau(s_0)\kappa(s_0)\mathbf{b}(s_0) \end{cases}, \tag{18}$$

where the functions κ, τ, and the frame $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ are the curvature, torsion, and Frenet frame of α, respectively. It follows that $\alpha(s_0) = \beta(s_0)$, $\alpha'(s_0) = \beta'(s_0)$, and $\alpha''(s_0) = \beta''(s_0)$ (this is not a surprise, since an osculating sphere has a contact of order 3 with its reference curve).

Substituting the expressions above for β', β'', and β'', in the equation for τ_β we find the desired result $\tau_\beta = \tau_\alpha$. And, finally, the result for the angle θ follows from the fact that the torsion τ_β can be computed in terms of $\langle \beta - a_0, \beta' \times \beta'' \rangle$, Theorem 3.1, and from the equalities $\alpha^{(i)}(s_0) = \beta^{(i)}(s_0)$ $(i = 0, 1, 2)$. □

In general, finding the arc-length parameter may be a difficult task, since it involves an integral: $s(t) = \int_{t_0}^{t} \|\alpha'(u)\| \, \mathrm{d}u$. For a curve with a generic regular parametrization t, we can write

$$\frac{\mathrm{d}\alpha}{\mathrm{d}s} = \frac{1}{v}\frac{\mathrm{d}\alpha}{\mathrm{d}t} \quad \text{and} \quad \frac{\mathrm{d}^2\alpha}{\mathrm{d}s^2} = -\frac{1}{v^3}\frac{\mathrm{d}v}{\mathrm{d}t}\frac{\mathrm{d}\alpha}{\mathrm{d}t} + \frac{1}{v^2}\frac{\mathrm{d}^2\alpha}{\mathrm{d}t^2}, \tag{19}$$

where $v(t) = \|\alpha(t)\|$. So, it follows that

Proposition 4.1. *Let α be a regular twisted C^3 curve with a generic regular parameter t. The angle θ between a rotation minimizing vector and the principal normal satisfies*

$$\theta(t_2) - \theta(t_1) = \frac{\arctan\, J(t_2)}{v^3(t_2)} - \frac{\arctan\, J(t_1)}{v^3(t_1)}, \tag{20}$$

where $J(t) = \langle\, \alpha(t) - P_S(t),\, \alpha'(t) \times \alpha''(t)\, \rangle$, $P_S(t)$ is the center of the osculating sphere, and $v(t) = \|\alpha'(t)\|$.

From the two theorems above we are able to exactly compute RM frames along regular twisted curves. Indeed, given α and two initial orthogonal normal vectors $\mathbf{u}_0, \mathbf{v}_0 \in \mathrm{span}(\mathbf{t}_0)^\perp$, a quick computational scheme would be:

(1) compute α', α'', and α''';

(2) compute $L = \alpha' \times \alpha''$, $d = \langle L, \alpha''' \rangle$, $\ell = \|L\|$, and $v = \|\alpha'\|$;

(3) compute $\mathbf{t} = \alpha'/v$, $\mathbf{b} = L/\ell$, $\mathbf{n} = \mathbf{b} \times \mathbf{t}$, $\kappa = \ell/v^3$, and $\tau = d/\ell^2$;

(4) compute $\rho = 1/\kappa$, ρ', $P_S = \alpha + \rho\mathbf{n} + (\rho'/\tau)\mathbf{b}$, $J = \langle \alpha - P_S, L \rangle$, $\theta = (\arctan J)/v^3$, and $\theta_0 = \langle \mathbf{u}_0, \mathbf{n}(t_0) \rangle$;

(5) a RM frame along α with initial condition $\{\mathbf{t}_0, \mathbf{u}_0, \mathbf{v}_0\}$ at $t = t_0$ is

$$\Big\{\mathbf{t},\, \cos\Big(\theta - \theta(t_0) + \theta_0\Big)\mathbf{n} - \sin\Big(\theta - \theta(t_0) + \theta_0\Big)\mathbf{b},\, \sin\Big(\theta - \theta(t_0) + \theta_0\Big)\mathbf{n} + \cos\Big(\theta - \theta(t_0) + \theta_0\Big)\mathbf{b}\Big\}.$$

5. Behavior near inflection and zero torsion points

Besides the necessity to consider curves of class C^3 (the notion of RM frame makes sense for C^1 curves), there is another drawback to our approach. Indeed, the curve should be twisted: it has neither inflections nor zero torsion points. At an inflection point the Frenet frame is not well defined and then neither the angle θ, while at a zero torsion point the osculating sphere is not well defined (in fact, the osculating plane plays the role of an osculating sphere, i.e., $R_S = +\infty$). A natural question then is if it is possible to adapt our approach in order to take into account the possibility of a curvature function κ or torsion τ with zeros.

5.1. Zero torsion points

Since our strategy to compute the angle θ between the principal normal and a RM vector is based on the manipulation of osculating spheres, we must understand the torsion behavior near its zeros. If the expression $\theta(s) = \arctan J(s) + c$, for some constant c, were to be valid at $\tau(s_0) = 0$, then we would have $J(s_0) = \pm\infty$, which is compatible with the fact that at $s = s_0$ the center of the osculating sphere P_S is located at an infinity distance from $\alpha(s_0)$: J is the volume of the tetrahedron generated by $\alpha - P_S$, α', and α''. Then, the lateral limits $\theta(s_0^-)$ or $\theta(s_0^+)$ can assume the values $\pm\pi/2 + c$.

If $\tau \equiv 0$ in a whole interval $[a, b]$, then any normal vector that makes a constant angle of the plane that contains $\alpha([a, b])$ is a RM vector field. In this case, the ambiguity in the definition of c together with the choice of an initial condition for the normal vectors at $s = a$ and $s = b$ may be adjusted in order to globally define a RM frame. Finally, the same reasoning applies for an isolated zero $\tau(s_0) = 0$: here $a = b$.

5.2. Inflection points

Since our strategy is based on the computation of θ, one must ask about the torsion behavior near an inflection point. There are three possible behaviors for τ near the points $\kappa(s_0) = 0$: the lateral limits $\tau_0^- = \lim_{s\to s_0^-} \tau(s)$ and $\tau_0^+ = \lim_{s\to s_0^+} \tau(s)$ they (i) exist and coincide; (ii) exist and do not coincide; or (iii) one of them does not exist. In the first case it is possible to extend τ to $s = s_0$ by continuity (so, also the angle θ): $\tau(s_0) \stackrel{\text{def}}{=} \tau_0^+ = \tau_0^-$.

It is possible to show that if a curve is analytic, i.e., it has a convergent Taylor series, then the lateral limits τ_0^- and τ_0^+ do exist and coincide [17] (in fact, it is possible to show that given two analytic functions $K \geq 0$ and τ, then there exists an analytic curve, up to a rigid motion of $\mathbb{E}^3$, with curvature $\kappa = \sqrt{K}$ and torsion τ [25]). This is the case for the functions commonly encountered in applications, e.g., exponential, logarithm, polynomials, and trigonometric functions. However, when we drop the analyticity assumption, one of the lateral limit may diverge [17].

Here we can furnish a partial answer to the characterization of the torsion behavior by observing that near an inflection point the lateral limits either exist and coincide or one of them is infinite. Indeed, this follows from the Darboux theorem [23], since the torsion can be seen as the derivative of a function according to Theorem 4.1 and $\theta' = \tau$.

6. Closure

In this work we showed how to exactly compute rotation minimizing frames for an at least three times differentiable curve. We proved that it is possible to find the angle between the principal normal and a rotation minimizing vector. This was done by first solving the problem for spherical curves and then using the concept of osculating spheres to solve the general case. In addition, since the Frenet frame and the osculating sphere are not well defined at a zero of the curvature function and of the torsion, respectively, we discussed the behavior of a rotation minimizing frame near these points. For a curve whose torsion vanishes on intervals or isolated points only, one can define a rotation minimizing frame by conveniently choosing some arbitrary constants. On the other hand, at an inflection point, we showed that the angle between the principal normal and a rotation minimizing vector can be well defined when the curve is analytic, which is the case for the functions commonly encountered in applications.

Finally, for a C^1 or C^2 curve one must resort to a numerical method in order to compute rotation minimizing frames (e.g., the double reflection method [29]). Anyway, the C^3 condition does not constitute a severe restriction to applications and the approach developed in this work will certainly be of great importance in the many contexts where rotation minimizing frames play a role.

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