Extending Partial Representations of Unit Circular-arc Graphs

Peter Zeman

Department of Applied Mathematics, Faculty of Mathematics and Physics,
Charles University, Czech Republic. E-mail: zeman@kam.mff.cuni.cz.

Abstract. The partial representation extension problem, introduced by Klavík et al. (2011), generalizes the recognition problem. In this short note we show that this problem is NP-complete for unit circular-arc graphs.

1 Introduction

An intersection representation $\mathcal{R}$ of a graph G is a collection of sets $\{R_v : v \in V(G)\}$ such that $R_u \cap R_v \neq \emptyset$ if and only if $uv \in E(G)$.

Interval Graphs. One of the most studied and well understood classes of intersection graphs are *interval graphs* (INT). In an *interval representation* of a graph, each set R_v is a closed interval of the real line. A graph is an interval graph if it has an interval representation; see Fig. 1a.

Circular-arc Graphs. In a *circular-arc representation*, the sets R_v are arcs of a circle; see Fig 1b.

Structure of All Representations. Despite the fact that circular-arc graphs are a straightforward generalization of interval graphs, the structure of their representations is much less understood. To understand the structure of all interval representations, the key result is the following.

Theorem 1.1 (Fulkerson and Gross [5]). *A graph G is an interval graph if and only if there exists a linear ordering $\preceq$ of its maximal cliques such that for every vertex v, the maximal cliques containing v appear consecutively in $\preceq$.*

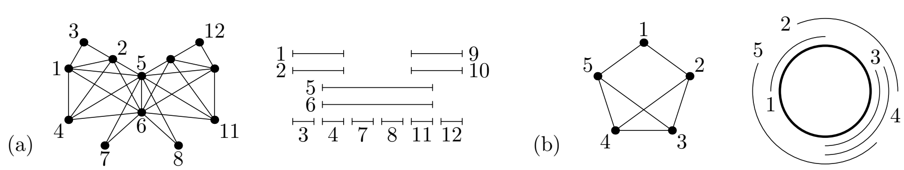

Fig. 1. (a) An interval graph and one of its interval representations. (b) A circular-arc graph and one of its representations.

Booth and Lueker used Theorem 1.1 and PQ-trees [1] to recognize interval graphs graphs in linear time. Moreover, a PQ-tree of an interval graph captures all possible orderings $\preceq$ of the maximal cliques, i.e., it stores every possible representation of the interval graph.

For circular-arc graphs, the situation is much more complicated. The main difference is that, unlike a circular-arc representation, an interval representation satisfies the Helly property: if every two intervals in a set have a nonempty intersection, then the whole set has a non-empty intersection. In particular, this means the maximal cliques of interval graphs can be associated to unique points of the line. The number of maximal cliques in an interval graph linear in the number of vertices. However, a circular-arc representation does not necessarily satisfy Helly property and the number of maximal cliques can be exponential. The complete bipartite graph $K_{n,n}$ without a matching is an example of that. It is not clear whether there exits a way to efficiently capture the structure off all representations of a circular-arc graph.

Partial Representation Extension Problem. This problem naturally generalizes the recognition problem. For a class of graphs $\mathcal{C}$, the input consists of a graph G and a *partial representation* $\mathcal{R}'$ which is a representation of some induced subgraph G' of G. The question is to decide whether there exists a representation $\mathcal{R}$ of G that *extends* $\mathcal{R}'$, i.e., $R_u = R'_u$, for every $u \in V(G')$. Note that in the case of recognition, the partial representation $\mathcal{R}'$ is empty.

Problem: Partial representation extension – $\textsc{RepExt}(\mathcal{C})$
Input: A graph G and a partial representation $\mathcal{R}'$.
Output: Is there a representation $\mathcal{R}$ of G extending $\mathcal{R}'$?

In recognition it suffices, for a given graph G, to construct a single representation of G. However, in partial representation extension, one typically needs a way to store all possible representations of G efficiently. Then we can efficiently find a representation $\mathcal{R}$ that extends $\mathcal{R}'$. For example, Klavík et. al. [10] used PQ-trees to solve RepExt(INT) in linear time.

In the past few years a lot of work was done involving the partial representation extension problem. This includes circle graphs [3], function and permutation graphs [7], unit and proper interval graphs [8], and visibility representations [4]. All of those papers use an efficient way to store all possible representations and give polynomial-time algorithms for the partial representation extension problem. For chordal graphs [9] and contact representations of planar graphs [2], the partial representation extension problem is hard.

We argued that studying the partial representation extension problem of a given class of graphs is closely related to understanding the structure of all representations. The problem RepExt can be typically solved in polynomial time if we can store all possible representations efficiently.

Unit Circular-arc Graphs. Circular-arc graphs with an intersection representation in which every arc has a unit length are called *unit circular-arc graphs* (UNIT CA). An example of a circular-arc graph that is not unit is the complete bipartite graph $K_{1,3}$.

Theorem 1.2. *The problem* RepExt(UNIT CA) *is* NP*-complete.*

Note that for unit interval graphs (defined analogously) RepExt can be solved in polynomial time [8].

2 Proof of The Main Result

We prove Theorem 1.2. The problem RepExt(UNIT CA) is clearly in NP. We show a reduction from a known NP-complete problem called 3-Partition [6]. The input of 3-Partition consists of positive integers k, M, and $A_1, \dots, A_{3k}$ such that $M/4 < A_i < M/2$, for each A_i, and $\sum A_i = kM$. The problem asks whether it is possible to partition A_i's into k triples such that the sets A_i belonging to the same triple sum up to exactly M. (Note that the size constraints on A_i's ensure that every subset that sums exactly to M, is a triplet.)

Proof (Theorem 1.2). For a given instance of 3-Partition, we construct a unit circular-arc graph G and its partial representation $\mathcal{R}'$. For technical reasons, we assume that $M \geq 8$.

Let $P_{2\ell}$ be a path of length 2ℓ. There exists a unit circular-arc representation $P_{2\ell}$ such that it spans $\ell + \varepsilon$ units, for some $\varepsilon > 0$. To see this, note that $P_{2\ell}$ has two independent sets of size ℓ and each of this independent sets needs at least $\ell + \varepsilon$. Let a, b, c be positive integers such that $a + b + c = M$. It follows that the disjoint union of P_{2a}, P_{2b}, and P_{2c} has a representation such that it spans $M + \varepsilon$ units, for some $\varepsilon > 0$, and therefore, it can be fit into $M + 1$ units; see Fig. 2.

Let $x_0, \dots, x_{k(M+2)-1}$ be points of the circle that divide it into $k(M+2)$ equal parts, i.e., vertices of a regular $k(M+2)$-gon. The graph G is a disconnected graph consisting of $4k$ connected components. For each A_i, we take the path P_{2A_i}. We further add an isolated vertex v_j, for $j = 0, \dots, k-1$. The partial representation $\mathcal{R}'$ is the collection $\{R_{v_j} : j = 0, \dots, k-1\}$, where R_{v_j} is the arc of the circle from $x_{j(M+2)}$ to $x_{j(M+2)+1}$ in the clockwise direction.

The pre-drawn arcs $R_{v_0}, \dots, R_{v_{k-1}}$ split the circle into k gaps, where each gap has exactly $M + 1$ units. By the discussion above, if the A_i's can be partitioned into k triples such that each triple sums to M, then a representation of the disjoint union of the paths corresponding to a triple can be placed in one of the k gaps. If the partial representation $\mathcal{R}'$ can be extended, then we a have partition of the A_i's into k triples such that each triple sums to M. □

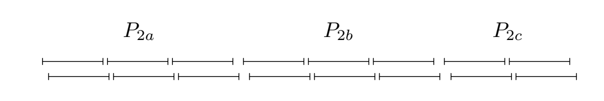

Fig. 2. A representation of the disjoint union of P_{2a}, P_{2b}, and P_{2c} fits into $M+1$ units. Here, $M = 8$, $a = 3$, $b = 3$, and $c = 2$.

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