A Lower Bound for Nonadaptive, One-Sided Error Testing of Unateness of Boolean Functions over the Hypercube

Roksana Baleshzar* Deeparnab Chakrabarty† Ramesh Krishnan S. Pallavoor*
Sofya Raskhodnikova* C. Seshadhri‡

Abstract

A Boolean function $f : \{0,1\}^d \mapsto \{0,1\}$ is unate if, along each coordinate, the function is either nondecreasing or nonincreasing. In this note, we prove that any nonadaptive, one-sided error unateness tester must make $\Omega(\frac{d}{\log d})$ queries. This result improves upon the $\Omega(\frac{d}{\log^2 d})$ lower bound for the same class of testers due to Chen et al. (STOC, 2017).

1 Introduction

We study the problem of deciding whether a Boolean function $f : \{0,1\}^d \mapsto \{0,1\}$ is *unate* in the property testing model [7, 5]. A function is unate if, for each dimension $i \in [d]$, the function is either nondecreasing along the i^{th} coordinate or nonincreasing along the i^{th} coordinate. A property tester for unateness is a randomized algorithm that takes as input a proximity parameter $\varepsilon \in (0,1)$ and has query access to a function f. If f is unate, it must accept with probability at least 2/3. If f is ε-far from unate, it must reject with probability at least 2/3. A tester has *one-sided error* if it always accepts unate functions. A tester is *nonadaptive* if it chooses all of its queries in advance; it is *adaptive* otherwise.

The problem of testing unateness was introduced by Goldreich et al. [4]. Following a result of Khot and Shinkar [6], Baleshzar et al. [1] settled the complexity of unateness testing for *real-valued functions*. Unateness can be tested with $O(\frac{d}{\varepsilon})$ queries adaptively and with $O(\frac{d \log d}{\varepsilon})$ queries nonadaptively. For constant ε, these complexities are optimal.

On the other hand, for the Boolean range, the complexity is far from settled. Baleshzar et al. [2] proved that $\Omega(\sqrt{d})$ queries are necessary for nonadaptive, one-sided error testers. Chen et al. [3] improved the lower bound for this class of testers to $\Omega(\frac{d}{\log^2 d})$. They also proved a lower bound of $\Omega(\frac{\sqrt{d}}{\log^2 d})$ for adaptive, two-sided error unateness testers.

In this note, we use a construction similar to the one used by Chen et al. [3] to get an $\Omega(\frac{d}{\log d})$ for nonadaptive, one-sided error unateness testers of Boolean functions over the hypercube. Our analysis of the lower bound construction is simpler and gives a better dependence on d. There is

*Department of Computer Science and Engineering, Pennsylvania State University. rxb5410@cse.psu.edu, ramesh@psu.edu, sofya@cse.psu.edu. Partially supported by NSF award CCF-1422975.
†Department of Computer Science, Dartmouth College. deeparnab@dartmouth.edu. Work done while at Microsoft Research, India.
‡Department of Computer Science, University of California, Santa Cruz. sesh@ucsc.edu.

still a gap of $\log^2 d$ between the query complexity of the best known algorithm for this problem (from [1]) and our lower bound.

2 The Lower Bound

In this section, we prove the following theorem.

Theorem 2.1. *Any nonadaptive, one-sided error unateness tester for functions $f : \{0,1\}^d \mapsto \{0,1\}$ with the distance parameter $\varepsilon \leq \frac{1}{8}$ must make $\Omega(\frac{d}{\log d})$ queries.*

Proof. We first define a hard distribution consisting of Boolean functions that are $\frac{1}{8}$-far from unate. By Yao's minimax principle [8], it is sufficient to give a distribution on functions for which every deterministic tester fails with high probability. A deterministic nonadaptive tester is determined by a set of query points $Q \subseteq \{0,1\}^d$. We prove that if $|Q| \leq \frac{d}{30 \log d}$, then the tester fails with probability more than 2/3 over the hard distribution.

The hard distribution $\mathcal{D}$ is defined as follows: pick 3 dimensions $a, b, c \in [d]$ uniformly at random and define $f_{a,b,c}(x) = x_a \cdot x_b + (1 - x_a) \cdot x_c$. We call a, b, c the *influential dimensions*, since the value of the function depends only on them. The coordinate x_a determines if $f_{a,b,c}(x)$ should be set to x_b or x_c. If $x_a = 1$, then $f_{a,b,c}(x) = x_b$, otherwise, $f_{a,b,c}(x) = x_c$.

There are $\binom{d}{3}$ functions in the support of $\mathcal{D}$. The next claim states that all of them are far from unate.

Claim 2.2. *Every function $f_{a,b,c}$ in the support of $\mathcal{D}$ is $\frac{1}{8}$-far from unate.*

Proof. Consider an edge (x, y) along the dimension a. We have $x_a = 0$ and $y_a = 1$, and $x_i = y_i$ for all $i \in [d] \setminus \{a\}$. By definition, $f_{a,b,c}(x) = x_c$ and $f_{a,b,c}(y) = y_b$. If $x_b = y_b = 1$ and $x_c = y_c = 0$, then $f_{a,b,c}$ is increasing along the edge (x, y). On the other hand, if $x_b = y_b = 0$ and $x_c = y_c = 1$, then $f_{a,b,c}$ is decreasing along (x, y). Thus, with respect to $f_{a,b,c}$, at least 2^{d-3} edges along the dimension a are decreasing and at least 2^{d-3} edges along the dimension a are increasing. Hence, at least 2^{d-3} function values of $f_{a,b,c}$ need to be changed to make it unate. Consequently, $f_{a,b,c}$ is $\frac{1}{8}$-far from unate. □

Note that any one-sided error tester for unateness must accept if the query answers are consistent with a unate function. Let $f_{|Q}$ denote the restriction of the function f to the points in Q. We say that $f_{|Q}$ is *extendable* to a unate function if there exists a unate function g such that $g_{|Q} = f_{|Q}$. For $f \sim \mathcal{D}$, we show that if $|Q| \leq \frac{d}{30 \log d}$, then, with high probability, $f_{|Q}$ is extendable to a unate function. Consequently, the tester accepts with high probability.

Next, we define a conjunctive normal form (CNF) formula $\phi(f_{|Q})$. Intuitively, each pair (x, y) of domain points on which f differs imposes a constraint on f (assuming that f is unate). Specifically, at least one of the dimensions on which x and y differ must be consistent (i.e., nondecreasing or nonincreasing) with the change of the function value between x and y. This constraint is formalized in the definition of $\phi(f_{|Q})$ as follows. For each dimension i, we have a variable z_i which is true if f is nondecreasing along the dimension i, and false if it is nonincreasing along that dimension. For each $x, y \in Q$ such that $f(x) = 1$ and $f(y) = 0$, create a clause (think of x, y as sets where $i \in x$ iff $x_i = 1$)

$$c_{x,y} = \bigvee_{i \in x \setminus y} z_i \vee \bigvee_{i \in y \setminus x} \overline{z_i}.$$

Set $\phi(f_{|Q}) = \bigwedge_{x,y \in Q: f(x)=1, f(y)=0} c_{x,y}$.

Observation 2.3. *The restriction $f_{|Q}$ is a certificate for non-unateness iff $\phi(f_{|Q})$ is unsatisfiable.*

Now we need to show that, with probability greater than 2/3 over $f \sim \mathcal{D}$, the CNF formula $\phi(f_{|Q})$ is satisfiable. This follows from Claims 2.4 and 2.5.

The width of a clause is the number of literals in it; the width of a CNF formula is the minimum width of a clause in it.

Claim 2.4. *With probability at least $2/3$ over $f \sim \mathcal{D}$, the width of $\phi(f_{|Q})$ is at least $3 \log d$.*

Proof. Consider a graph G with vertex set Q, and an edge between $x, y \in Q$ if $|x \Delta y| \leq 3 \log d$ (Here, $x \Delta y$ is the symmetric difference between the sets x and y). Take an arbitrary spanning forest F of G. Observe that for any edge (u, v) of G, we have $u \Delta v \subseteq \bigcup_{(x,y) \in F} x \Delta y$. Note that F has at most $\frac{d}{30 \log d}$ edges. Let $C = \bigcup_{(x,y) \in F} x \Delta y$, the set of dimensions captured by Q. We have $|C| \leq \sum_{(x,y) \in F} |x \Delta y| \leq \frac{d}{30 \log d} \cdot 3 \log d \leq \frac{d}{10}$. Over the distribution $\mathcal{D}$, the probability that at least one of the influential dimensions, $\{a, b, c\}$, is in C is at most 3/10 which is less than 1/3. Hence, with probability at least 2/3, no $(u, v) \in G$ contributes a clause to $\phi(f_{|Q})$. Therefore, the width of $\phi(f_{|Q})$ is at least $3 \log d$. □

Claim 2.5. *Any CNF that has width at least $3 \log d$ and at most d^2 clauses is satisfiable.*

Proof. Apply the probabilistic method. A clause is not satisfied by a random assignment with probability at most $1/d^3$. Hence, the expected number of unsatisfied clauses is at most $\frac{d^2}{d^3} < 1$. □

Thus, $f_{|Q}$ is a certificate for non-unateness with probability at most 1/3 when $|Q| \leq \frac{d}{30 \log d}$, which completes the proof of Theorem 2.1. □

References

[1] Roksana Baleshzar, Deeparnab Chakrabarty, Ramesh Krishnan S. Pallavoor, Sofya Raskhodnikova, and C. Seshadhri. Optimal unateness testers for real-valued functions: Adaptivity helps. In *Proceedings, International Colloquium on Automata, Languages and Processing (ICALP)*, 2017.

[2] Roksana Baleshzar, Meiram Murzabulatov, Ramesh Krishnan S. Pallavoor, and Sofya Raskhodnikova. Testing unateness of real-valued functions. *CoRR*, abs/1608.07652, 2016.

[3] Xi Chen, Erik Waingarten, and Jinyu Xie. Beyond talagrand functions: New lower bounds for testing monotonicity and unateness. *CoRR*, abs/1702.06997, 2017. To appear in STOC 2017.

[4] Oded Goldreich, Shafi Goldwasser, Eric Lehman, Dana Ron, and Alex Samorodnitsky. Testing monotonicity. *Combinatorica*, 20:301–337, 2000.

[5] Oded Goldreich, Shafi Goldwasser, and Dana Ron. Property testing and its connection to learning and approximation. *J. ACM*, 45(4):653–750, 1998.

[6] Subhash Khot and Igor Shinkar. An $\widetilde{O}(n)$ queries adaptive tester for unateness. In *Approximation, Randomization, and Combinatorial Optimization. Algorithms and Techniques, APPROX/RANDOM 2016*, pages 37:1–37:7, 2016.

[7] Ronitt Rubinfeld and Madhu Sudan. Robust characterizations of polynomials with applications to program testing. *SIAM J. Comput.*, 25(2):252–271, 1996.

[8] Andrew Chi-Chih Yao. Probabilistic computations: Toward a unified measure of complexity (extended abstract). In *Proceedings, IEEE Symposium on Foundations of Computer Science (FOCS)*, pages 222–227, 1977.