Minor stars in plane graphs with minimum degree five

Yangfan Li Tao Wang*
Institute of Applied Mathematics
Henan University, Kaifeng, 475004, P. R. China

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Abstract

The weight of a subgraph H in G is the degree-sum of vertices of H in G. Let Ω_Δ be the minimum integer such that there is a minor 5-star with weight at most Ω_Δ in every plane graph with minimum degree five and maximum degree Δ. Borodin and Ivanova [Discrete Math. 340 (2017) 2234–2242] proved that $\Omega_\Delta \leq \Delta + 29$ for $\Delta \geq 13$. They also asked: what's the minimum integer Δ_0 such that $\Omega_\Delta \leq \Delta + 28$ whenever $\Delta \geq \Delta_0$? In this paper, we give two descriptions of minor 5-stars in plane graphs with minimum degree at least five, the first one refines Borodin and Ivanova's result and the second one partially gives an answer of Borodin and Ivanova's question.

1 Introduction

A *normal plane map* (NMP for short) is a connected plane pseudograph in which loops and multiple edges are allowed, but the degree of each vertex and face is at least three. A 3-*polytope* is a 3-connected plane graph. Clearly, each 3-polytope is a normal plane map. The class of normal plane maps with minimum degree at least five is denoted by $\mathbf{M_5}$, and the class of 3-polytopes with minimum degree at least five is denoted by $\mathbf{P_5}$. A $(k_1, k_2, k_3, k_4, k_5)$-star is a star with $\deg(v_i) \leq k_i$, where v_is are neighbors of the center in any order. A k-star is a star with k rays. A star is *minor* if its center has degree at most five.

The *weight* of a subgraph H in G is the sum of $\deg_G(v)$ by taking over all $v \in V(H)$. The *height* of a subgraph H in G is the maximum degree of vertices in H. Let Ω_Δ be the minimum integer such that there is a minor 5-star with weight at most Ω_Δ in every plane graph with minimum degree five and maximum degree Δ.

In 1904, Wernicke [11] proved that every $\mathbf{M_5}$ has a 5-vertex adjacent to a 6^--vertex, that is a $(5, 6)$-edge. This was strengthened by Franklin [8] in 1922 to the existence of a minor $(6, 6)$-star, that is a $(6, 5, 6)$-path. In 1996, Jendrol' and Madaras [9] gave a precise description of minor 3-stars in $\mathbf{M_5}$: there is a minor $(6, 6, 6)$- or $(5, 6, 7)$-star. In 1998, Borodin and Woodall [7] showed that the minimum weight of minor 4-star in $\mathbf{M_5}$ is at most 30. Furthermore, Borodin and Ivanova [2] gave a tight description of minor 4-stars in $\mathbf{M_5}$.

In 1940, Lebesgue [10] gave an approximate description of minor 5-stars in $\mathbf{M_5}$, which implies that $\Omega_\Delta \leq \Delta + 31$, and $\Omega_\Delta \leq \Delta + 27$ for $\Delta \geq 41$. In 1998, Borodin and Woodall [7] strengthened this result to $\Omega_\Delta \leq \Delta + 30$. This result is sharp for $\Delta = 7$ due to Borodin [1] and Jendrol'–Madaras [9], $\Delta = 9$ due to Borodin–Ivanova [2], $\Delta = 10$ due to Jendrol'–Madaras [9], $\Delta = 12$ due to Borodin–Woodall [7]. Recently, Borodin and Ivanova [5] showed that $\Omega_8 = 38$ and $\Omega_{11} = 41$. Hence, Borodin and Woodall's bound $\Omega_\Delta \leq \Delta + 30$ is sharp for every integer $\Delta \in \{7, 8, \dots, 12\}$.

Recently, Borodin and Ivanova [5] strengthened the bound $\Omega_\Delta \leq \Delta + 30$ to $\Omega_\Delta \leq \Delta + 29$ for $\Delta \geq 13$.

Theorem 1.1 (Borodin and Ivanova [5])**.** Let Δ be an integer with $\Delta \geq 13$. Every 3-polytope with minimum degree five and maximum degree Δ has a minor 5-star with weight at most $\Delta + 29$.

Note that the description of minor 5-stars is unordered for the neighbors of center. Here, we want to give two descriptions of neighbors of 5-vertex in cyclic order.

A $\langle \kappa_1, \kappa_2, \kappa_3, \kappa_4, \kappa_5 \rangle$-star is a star with center having degree five and the other vertices having degrees $\leq \kappa_1, \leq \kappa_2, \leq \kappa_3, \leq \kappa_4, \leq \kappa_5$ in the cyclic order.

The first purpose of this paper is to give the following description of minor 5-stars, which refines Theorem 1.1 and include as many known results as possible.

*Corresponding author: wangtao@henu.edu.cn; iwangtao8@gmail.com

Theorem 1.2. If G is a plane graph with minimum degree five, then G contains at least one of the following minor 5-stars:

$\langle 6,6,6,6,\infty\rangle$, $\langle 5,5,10,5,12\rangle$, $\langle 5,7,8,5,11\rangle$, $\langle 5,7,5,8,8\rangle$, $\langle 6,6,6,7,8\rangle$, $\langle 6,6,7,6,8\rangle$,
$\langle 5,6,6,11,7\rangle$, $\langle 5,6,11,6,7\rangle$, $\langle 5,6,7,8,6\rangle$, $\langle 5,6,6,8,8\rangle$, $\langle 5,6,8,6,8\rangle$, $\langle 5,7,6,8,7\rangle$,
$\langle 5,6,7,7,7\rangle$, $\langle 5,6,6,7,11\rangle$, $\langle 5,6,7,6,11\rangle$, $\langle 5,7,6,7,8\rangle$, $\langle 5,7,7,6,8\rangle$, $\langle 5,7,6,6,15\rangle$,
$\langle 5,8,6,6,11\rangle$, $\langle 5,5,7,8,7\rangle$, $\langle 5,5,7,7,8\rangle$, $\langle 5,5,7,6,15\rangle$, $\langle 5,5,8,6,11\rangle$, $\langle 5,6,7,5,51\rangle$,
$\langle 5,6,8,5,18\rangle$, $\langle 5,6,9,5,10\rangle$, $\langle 5,6,11,5,9\rangle$, $\langle 5,7,7,5,21\rangle$, $\langle 5,7,11,5,8\rangle$, $\langle 5,6,5,7,15\rangle$,
$\langle 5,6,5,8,11\rangle$, $\langle 5,5,8,5,29\rangle$, $\langle 5,5,9,5,21\rangle$. □

The following six theorems are immediate consequences of Theorem 1.2.

Theorem 1.3 (Borodin and Woodall [7])**.** Every plane graph with minimum degree five has a minor 4-star with weight at most 30.

Theorem 1.4 (Jendrol' and Madaras [9])**.** Every plane graph with minimum degree five has a minor $(10,10,10,10)$-star.

Theorem 1.5 (Jendrol' and Madaras [9])**.** Every plane graph with minimum degree five has a minor $(5,6,7)$-star or a minor $(6,6,6)$-star.

Theorem 1.6 (Franklin [8])**.** Every plane graph with minimum degree five has a $(6,5,6)$-path.

Theorem 1.7 (Wernicke [11])**.** Every plane graph with minimum degree five has a $(5,6)$-edge.

Theorem 1.8 (Borodin and Ivanova [4])**.** Every plane graph with minimum degree five having neither vertices from 6 to 9 nor minor $(5,5,5,5)$-star has minimum weight at most 42 and height at most 12, where both bounds are tight.

As for Lebesgue's bound $\Omega_\Delta \le \Delta + 27$ for $\Delta \ge 41$, it was strengthened by Borodin, Ivanova and Jensen [6] to $\Omega_\Delta \le \Delta + 27$ for $\Delta \ge 28$, and further by Borodin and Ivanova [3] to $\Omega_\Delta \le \Delta + 27$ for $\Delta \ge 24$. It's natural to consider the problem $\Omega_\Delta \le \Delta + 28$, so Borodin and Ivanova [5] asked a question: what's the minimum integer Δ_0 such that $\Omega_\Delta \le \Delta + 28$ whenever $\Delta \ge \Delta_0$?

The second purpose of this paper is to give another description of minor 5-stars, which partially gives an answer of Borodin and Ivanova's question.

Theorem 1.9. If G is a plane graph with minimum degree 5, then G contains at least one of the following minor 5-stars:

$\langle 5,5,9,5,16\rangle$, $\langle 6,6,6,6,11\rangle$, $\langle 6,6,6,7,9\rangle$, $\langle 6,6,7,6,9\rangle$, $\langle 6,6,7,7,7\rangle$, $\langle 6,7,6,7,7\rangle$,
$\langle 5,6,6,8,9\rangle$, $\langle 5,6,8,6,9\rangle$, $\langle 5,6,7,7,9\rangle$, $\langle 5,6,6,7,11\rangle$, $\langle 5,6,7,6,11\rangle$, $\langle 5,7,6,7,9\rangle$,
$\langle 5,7,7,6,9\rangle$, $\langle 5,6,6,6,19\rangle$, $\langle 5,7,6,6,11\rangle$, $\langle 5,8,6,6,10\rangle$, $\langle 5,9,6,6,9\rangle$, $\langle 5,5,8,6,10\rangle$,
$\langle 5,5,9,6,9\rangle$, $\langle 5,6,6,5,\infty\rangle$, $\langle 5,6,7,5,25\rangle$, $\langle 5,6,8,5,15\rangle$, $\langle 5,6,9,5,14\rangle$, $\langle 5,9,5,6,10\rangle$,
$\langle 5,8,5,6,11\rangle$, $\langle 5,7,5,6,19\rangle$, $\langle 5,7,7,5,11\rangle$, $\langle 5,7,8,5,9\rangle$, $\langle 5,8,5,7,9\rangle$, $\langle 5,7,5,7,11\rangle$,
$\langle 5,7,5,8,10\rangle$, $\langle 5,7,5,9,9\rangle$, $\langle 5,5,7,5,\infty\rangle$, $\langle 5,5,8,5,25\rangle$, $\langle 5,5,10,5,14\rangle$, $\langle 5,5,11,5,13\rangle$. □

The following theorem is an immediate consequence of Theorem 1.9.

Theorem 1.10. If G is a plane graph with minimum degree 5 and maximum degree $\Delta \ge 17$, then G has a minor 5-star with weight at most $\Delta + 28$.

Borodin and Ivanova [5] showed that $\Omega_{13} = \Delta + 29 = 42$, which implies that $14 \le \Delta_0 \le 17$ in Borodin and Ivanova's question. Borodin, Ivanova and Jensen [6] shown that $\Omega_{20} \ge 48$, so we immediately have the exact value of Ω_{20}.

Theorem 1.11. $\Omega_{20} = \Delta + 28 = 48$.

Note that our results do not require the "3-connected" condition for the plane graphs with minimum degree at least five, so the class of graphs we considered is a little bit bigger than $\mathbf{P_5}$. We use the classic discharging method to give a proof of Theorem 1.2 in section 2 and a proof of Theorem 1.9 in section 3.

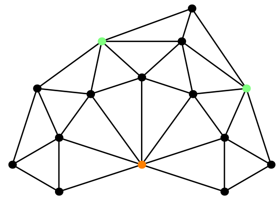

Fig. 1: Local structure of wretch

2 Proof of Theorem 1.2

Let G be a connected counterexample to Theorem 1.2 with maximum number of edges.

($*_1$) The graph G is a triangulation.

Proof of ($*_1$). Suppose that w_1, w_2, w_3, w_4 are four consecutive vertices on the boundary of a 4^+-face. Since G is a simple graph, we have that $w_1 \neq w_3$ and $w_2 \neq w_4$. Note that G is also a plane graph, thus we have that $w_1w_3 \notin E(G)$ or $w_2w_4 \notin E(G)$, otherwise the two lines representing w_1w_3 and w_2w_4 would cross each other outside the 4^+-face. But an insertion of a diagonal w_1w_3 or w_2w_4 into the 4^+-face would create a simple counterexample with more edges, which contradicts the maximality of G. □

In a triangulation, let w be a 5-vertex with neighbors w_1, w_2, w_3, w_4, w_5 in the cyclic order. The vertex w is a *wretch* if it is a weak neighbor of a 13^+-vertex w_5 and a twice-weak neighbor of a 10-vertex w_2. If w is a wretch, then we call the 5-vertex w_4 the *brother* of w. Note that w_1 and w_4 are asymmetrical, so the vertex w_1 cannot be a brother.

($*_2$) Every κ-vertex with $\kappa \geq 13$ is adjacent to at most $\frac{\kappa}{2}$ wretches. Moreover, any two wretches in the neighborhood of a 13^+-vertex are consecutive or separated by at least two non-wretches in the cyclic order

Proof of ($*_2$). Let w be a wretch with neighbors w_1, w_2, w_3, w_4, w_5 in the cyclic order. Let w_2 be a 10-vertex and w_5 be a 13^+-vertex. Let w_3 has the neighbors x, y, w_4, w, w_2 in the cyclic order. Since w is a twice-weak neighbor of w_2, the vertex x must be a 5-vertex. By the absence of $\langle 5, 5, 10, 5, 12\rangle$-stars, the vertex y must be a 13^+-vertex. Note that y and w_5 are two distinct vertices, thus w_4 has two 13^+-neighbors. This implies that a brother cannot be a wretch. Let w_4 has the neighbors w_5, w, w_3, y, z in the cyclic order. Similarly, the vertex z has two 13^+-neighbors, thus z cannot be a wretch.

Note that w_1, w, w_4 and z are the consecutive neighbors of w_5 in the cyclic order. Now, we associate each wretch in $N_G(w)$ with its brother. By the above arguments, each wretch has a brother and distinct wretch have distinct brothers. Therefore, every κ-vertex with $\kappa \geq 13$ is adjacent to at most $\frac{\kappa}{2}$ wretches.

Let w_2, w, w_5, u, v be the neighbors of w_1 in the cyclic order. If u is a wretch, then v is the center of a $\langle 5, 5, 10, 5, 10\rangle$-star, a contradiction. Hence, neither u nor z is a wretch, and any two wretches in $N_G(w_5)$ are consecutive or separated by at least two non-wretches in the cyclic order. □

The Euler's formula $|V| - |E| + |F| = 2$ can be rewritten as the following:

$$\sum_{v \in V(G)} (\deg(v) - 6) + \sum_{f \in F(G)} (2\deg(f) - 6) = -12.$$

Firstly, we give every vertex v an initial charge $\mu(v) = \deg(v) - 6$, and give every face f an initial charge $\mu(f) = 2\deg(f) - 6$. Note that every face has an initial charge zero and every vertex has a nonnegative initial charge except the 5-vertices. Secondly, we redistribute the charges among 5-vertices and 7^+-vertices such that the final charge $\mu'(v)$ of every vertex v is nonnegative, which contradicts the fact that the sum of the initial charges is negative.

2.1 Discharging rules

(R1a) Each 7-vertex sends $\frac{1}{3}$ to each strong 5-neighbor.

(R1b) Each 7-vertex sends $\frac{1}{6}$ to each non-strong 5-neighbor.

(R2a) Each 8-vertex sends $\frac{1}{2}$ to each strong 5-neighbor.

(R2b) Each 8-vertex sends $\frac{3}{8}$ to each semi-strong 5-neighbor.

(R2c) Each 8-vertex sends $\frac{1}{4}$ to each weak 5-neighbor.

(R3a) Each 9-vertex sends $\frac{2}{3}$ to each strong 5-neighbor.

(R3b) Each 9-vertex sends $\frac{1}{2}$ to each semi-strong 5-neighbor.

(R3c) Each 9-vertex sends $\frac{1}{3}$ to each weak 5-neighbor.

(R4) Let w be a κ-vertex with $\kappa = 10, 11$. Each such vertex w sends $\frac{2}{5}$ to each adjacent vertex. Let w_0, w_1, w_2 be three consecutive neighbors of w in the cyclic order. Suppose that w_0 is a 6^+-vertex and w_1 is a 5-vertex.

(a) If w_2 is a 6^+-vertex, then w_0 transfers a charge of $\frac{1}{5}$ to w_1.

(b) If w_2 is a 5-vertex, then w_0 transfers a charge of $\frac{1}{10}$ to each of w_1 and w_2.

(R5) Each 11-vertex additionally sends $\frac{1}{10}$ to each twice-weak 5-neighbor.

(R6a) Each 12-vertex or 13-vertex sends 1 to each strong 5-neighbor.

(R6b) Each 12-vertex or 13-vertex sends $\frac{3}{4}$ to each semi-strong 5-neighbor.

(R6c) Each 12-vertex or 13-vertex sends $\frac{1}{2}$ to each weak 5-neighbor.

(R7a) Each κ-vertex with $\kappa \geq 14$ sends $2\left(\frac{19}{20} - \frac{6}{\kappa}\right)$ to each strong 5-neighbor.

(R7b) Each κ-vertex with $\kappa \geq 14$ sends $\frac{3}{2}\left(\frac{19}{20} - \frac{6}{\kappa}\right)$ to each semi-strong 5-neighbor.

(R7c) Each κ-vertex with $\kappa \geq 14$ sends $(\frac{19}{20} - \frac{6}{\kappa})$ to each weak 5-neighbor.

(R8) Each 13^+-vertex additionally sends $\frac{1}{10}$ to each adjacent wretch.

(R9) Suppose that w is a 5-vertex with neighbors w_0, w_1, w_2, w_3, w_4 in the cyclic order, and w_0, w_1, w_2, w_3, w_4 have degrees $\kappa_0, 5, \kappa_2, \kappa_3, 5$, respectively.

(R9a) If $\kappa_2, \kappa_3 \geq 9$ and $\kappa_0 = 11, 13$, then w returns $\frac{1}{2}$ to w_0.

(R9b) If $\kappa_2, \kappa_3 \geq 9$ and $\kappa_0 = 7$, then w returns $\frac{1}{6}$ to w_0.

(R9c) If $\kappa_2, \kappa_3 \geq 8$ and $\kappa_0 = 11$, then w returns $\frac{1}{4}$ to w_0.

(R10) Suppose that w is a 5-vertex with neighbors w_1, w_2, w_3, w_4, w_5 in the cyclic order, and w_2, w_3, w_4, w_5 are $13^+, 5, 5, 13^+$-vertices respectively. If w_1 is a 6^+-vertex, then w returns $\frac{1}{4}$ to each of w_2 and w_5.

Remark 1. By (R4), each 10-vertex sends $\frac{4}{5}$ to each strong 5-neighbor, and sends $\frac{2}{5}$ to each adjacent wretch, and sends at least $\frac{1}{2}$ to any other 5-neighbor.

Remark 2. By (R4) and (R5), each 11-vertex sends $\frac{4}{5}$ to each strong 5-neighbor, and sends at least $\frac{1}{2}$ to any other 5-neighbor.

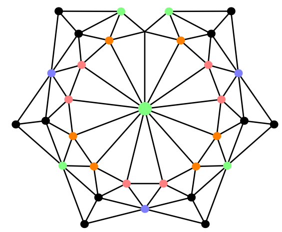

Fig. 2: Black dots have degree five, blue dots have degree ten, green dots have degree at least 13, red dots are wretches, and orange dots are brothers.

2.2 The final charge of every vertex is nonnegative

Case 1. If v is a κ-vertex with $\kappa \geq 14$, then $\mu'(v) \geq \kappa - 6 - \frac{\kappa}{2} \cdot \frac{1}{10} - \kappa \cdot \left(\frac{19}{20} - \frac{6}{\kappa}\right) = 0$.

Case 2. The vertex v is a 13-vertex.

If v is adjacent to at most five wretches, then $\mu'(v) \geq 13 - 6 - 13 \cdot \frac{1}{2} - 5 \cdot \frac{1}{10} = 0$. So we may assume that v is adjacent to exactly six wretches and exactly six brothers. By $(*_2)$, two wretches are consecutive or separated by at least two non-wretches in the cyclic order, so we may assume that $v_2, v_3, v_6, v_7, v_{10}, v_{11}$ are wretches, while the set of "brothers" in $N_G(v)$ is $\{v_1, v_4, v_5, v_8, v_9, v_{12}\}$, see Fig. 2. If v_{13} is a 5-vertex, then v_{13} returns $\frac{1}{2}$ to v by (R9a), which implies that $\mu'(v) \geq 13 - 6 - 13 \cdot \frac{1}{2} - 6 \cdot \frac{1}{10} + \frac{1}{2} \geq 0$. If v_{13} is a 6^+-vertex, then each of v_1 and v_{12} returns $\frac{1}{4}$ to v by (R10), which again implies that $\mu'(v) \geq 13 - 6 - 13 \cdot \frac{1}{2} - 6 \cdot \frac{1}{10} + 2 \cdot \frac{1}{4} \geq 0$.

Case 3. If v is a 12-vertex, then $\mu'(v) \geq 12 - 6 - 12 \cdot \frac{1}{2} = 0$.

Case 4. The vertex v is an 11-vertex.

If v has a 6^+-neighbor, then it has at most six twice-weak neighbor, which implies that $\mu'(v) \geq 11 - 6 - 11 \cdot \frac{2}{5} - 6 \cdot \frac{1}{10} = 0$. It remains to assume that v has eleven 5-neighbors. If v is involved as the receiver in (R9a), then $\mu'(v) \geq 11 - 6 - 11 \cdot (\frac{2}{5} + \frac{1}{10}) + \frac{1}{2} = 0$. In the final case, the vertex v is in a $\langle 5, 8, 8, 5, 11\rangle$-star due to the oddness of 11, so we have that $\mu'(v) \geq 11 - 6 - 11 \cdot (\frac{2}{5} + \frac{1}{10}) + 3 \cdot \frac{1}{4} \geq 0$ by (R9c); otherwise, there is a $\langle 5, 7, 8, 5, 11\rangle$-star.

Case 5. If v is a 10-vertex, then $\mu'(v) \geq 10 - 6 - 10 \cdot \frac{2}{5} = 0$.

Case 6. If v is a 9-vertex, then $\mu'(v) \geq 9 - 6 - 9 \cdot \frac{1}{3} = 0$.

Case 7. If v is an 8-vertex, then $\mu'(v) \geq 8 - 6 - 8 \cdot \frac{1}{4} = 0$.

Case 8. The vertex v is a 7-vertex.

If v has at most three 5-neighbors, then $\mu'(v) \geq 7 - 6 - 3 \cdot \frac{1}{3} = 0$. If v has exactly four 5-neighbors, then v has exactly three 6^+-vertices and has at most two strong 5-neighbors, which implies that $\mu'(v) \geq 7 - 6 - 2 \cdot \frac{1}{3} - 2 \cdot \frac{1}{6} = 0$. If v has exactly five 5-neighbors, then v has at most one strong 5-neighbor, which implies that $\mu'(v) \geq 7 - 6 - \frac{1}{3} - 4 \cdot \frac{1}{6} = 0$. If v has exactly six 5-neighbors, then v has no strong 5-neighbor, which implies that $\mu'(v) \geq 7 - 6 - 6 \cdot \frac{1}{6} = 0$. If v has seven 5-neighbors, then (R9b) is involved and $\mu'(v) \geq 7 - 6 - 7 \cdot \frac{1}{6} + \frac{1}{6} = 0$, otherwise there is a $\langle 5, 7, 5, 8, 8\rangle$-star.

Case 9. The vertex v is a 5-vertex with neighbors v_1, v_2, v_3, v_4, v_5 in the cyclic order. Suppose that v_1, v_2, v_3, v_4, v_5 have degrees $\kappa_1, \kappa_2, \kappa_3, \kappa_4, \kappa_5$ respectively.

If v is the sender in (R9a), then $\mu'(v) \geq 5 - 6 + 2 \cdot \frac{1}{2} + \frac{1}{2} - \frac{1}{2} = 0$. If v is the sender in (R9b), then $\mu'(v) \geq 5 - 6 + 2 \cdot \frac{1}{2} + \frac{1}{6} - \frac{1}{6} = 0$. If v is the sender in (R9c), then $\mu'(v) \geq 5 - 6 + 2 \cdot \frac{3}{8} + \frac{1}{2} - \frac{1}{4} = 0$. If v is the sender in (R10), then $\mu'(v) \geq 5 - 6 + 2 \cdot \frac{3}{4} - 2 \cdot \frac{1}{4} = 0$.

So we may assume that the 5-vertex v is just a receiver in what follows. By the absence of $\langle 6, 6, 6, 6, \infty\rangle$-stars, the vertex v has at least two 7^+-neighbors.

Subcase 9.1. The 5-vertex v has no 5-neighbor.

If v has at least three 7^+-neighbors, then $\mu'(v) \geq 5-6+3\cdot\frac{1}{3} = 0$. If v has at least two 8^+-neighbors, then $\mu'(v) \geq 5-6+2\cdot\frac{1}{2} = 0$. Hence, the vertex v has exactly three 6-neighbors and at most one 8^+-neighbor, which implies that $\mu'(v) \geq 5-6+\frac{1}{3}+\frac{2}{3} = 0$, otherwise there is a $\langle 6,6,6,7,8\rangle$-, or $\langle 6,6,7,6,8\rangle$-star.

Subcase 9.2. The 5-vertex v has precisely one 5-neighbor v_1.

By symmetry, we may assume that $\kappa_3 \leq \kappa_4$. If $\kappa_4 \geq 12$, then $\mu'(v) \geq 5-6+1 = 0$ and we are done.

Suppose that $\kappa_4 \in \{9,10,11\}$. If $\kappa_3 \geq 7$, then $\mu'(v) \geq 5-6+\frac{2}{3}+\frac{1}{3} = 0$. If $\min\{\kappa_2,\kappa_5\} \geq 7$, then $\mu'(v) \geq 5-6+\frac{2}{3}+2\cdot\frac{1}{6} = 0$. If $\min\{\kappa_2,\kappa_5\} = \kappa_3 = 6$, then $\mu'(v) \geq 5-6+\frac{2}{3}+\frac{3}{8} \geq 0$, otherwise there is a $\langle 5,6,6,11,7\rangle$- or $\langle 5,6,11,6,7\rangle$-star.

Suppose that v_4 is an 8-vertex. If $\kappa_3 \geq 8$, then $\mu'(v) \geq 5-6+2\cdot\frac{1}{2} = 0$. If $\kappa_3 = 7$, then $\mu'(v) \geq 5-6+\frac{1}{2}+\frac{1}{3}+\frac{1}{6} = 0$, for otherwise there is a $\langle 5,6,7,8,6\rangle$-star. If $\min\{\kappa_2,\kappa_5\} = \kappa_3 = 6$, then $\mu'(v) \geq 5-6+2\cdot\frac{1}{2} = 0$, otherwise there is a $\langle 5,6,6,8,8\rangle$-, $\langle 5,6,8,6,8\rangle$-star. If $\kappa_3 = 6$ and $\min\{\kappa_2,\kappa_5\} \geq 7$, then $\mu'(v) \geq 5-6+\frac{1}{2}+\frac{1}{6}+\frac{3}{8} \geq 0$, otherwise there is a $\langle 5,7,6,8,7\rangle$-star.

Suppose that $\kappa_3 = \kappa_4 = 7$. If $\min\{\kappa_2,\kappa_5\} \geq 7$, then $\mu'(v) \geq 5-6+2\cdot\frac{1}{3}+2\cdot\frac{1}{6} = 0$. If $\min\{\kappa_2,\kappa_5\} = 6$, then $\mu'(v) \geq 5-6+2\cdot\frac{1}{3}+\frac{3}{8} \geq 0$, otherwise there is a $\langle 5,6,7,7,7\rangle$-star.

Suppose that $\kappa_3 = 6$ and $\kappa_4 = 7$. If $\min\{\kappa_2,\kappa_5\} = 6$, then $\mu'(v) \geq 5-6+\frac{1}{3}+\frac{3}{4} \geq 0$, otherwise there is a $\langle 5,6,6,7,11\rangle$-, or $\langle 5,6,7,6,11\rangle$-star. If $\min\{\kappa_2,\kappa_5\} = 7$, then $\mu'(v) \geq 5-6+\frac{1}{3}+\frac{1}{6}+\frac{1}{2} = 0$, otherwise there is a $\langle 5,7,6,7,8\rangle$-, or $\langle 5,7,7,6,8\rangle$-star. If $\min\{\kappa_2,\kappa_5\} \geq 8$, then $\mu'(v) \geq 5-6+\frac{1}{3}+2\cdot\frac{3}{8} \geq 0$.

Suppose that $\kappa_3 = \kappa_4 = 6$. If $\min\{\kappa_2,\kappa_5\} = 7$, then $\mu'(v) \geq 5-6+\frac{1}{6}+\frac{3}{2}(\frac{19}{20}-\frac{6}{16}) \geq 0$, otherwise there is a $\langle 5,7,6,6,15\rangle$-star. If $\min\{\kappa_2,\kappa_5\} = 8$, then $\mu'(v) \geq 5-6+\frac{3}{8}+\frac{3}{4} \geq 0$, otherwise there is a $\langle 5,8,6,6,11\rangle$-star. If $\min\{\kappa_2,\kappa_5\} \geq 9$, then $\mu'(v) \geq 5-6+2\cdot\frac{1}{2} = 0$.

Subcase 9.3. The 5-vertex v has precisely two 5-neighbors v_1 and v_2.

If $\kappa_4 \geq 12$, then $\mu'(v) \geq 5-6+1 = 0$ and we are done. Suppose that $\kappa_4 \in \{9,10,11\}$. If $\min\{\kappa_3,\kappa_5\} = 6$, then $\mu'(v) \geq 5-6+\frac{2}{3}+\frac{3}{8} \geq 0$, otherwise there is a $\langle 5,6,6,11,7\rangle$-star. If $\min\{\kappa_3,\kappa_5\} \geq 7$, then $\mu'(v) \geq 5-6+\frac{2}{3}+2\cdot\frac{1}{6} = 0$.

Suppose that v_4 is an 8-vertex. If $\min\{\kappa_3,\kappa_5\} = 6$, then $\mu'(v) \geq 5-6+2\cdot\frac{1}{2} = 0$, otherwise there is a $\langle 5,6,6,8,8\rangle$-star. If $\min\{\kappa_3,\kappa_5\} \geq 7$, then $\mu'(v) \geq 5-6+\frac{1}{2}+\frac{1}{6}+\frac{3}{8} \geq 0$, otherwise there is a $\langle 5,5,7,8,7\rangle$-star.

Suppose that v_4 is a 7-vertex. If $\min\{\kappa_3,\kappa_5\} = 6$, then $\mu'(v) \geq 5-6+\frac{1}{3}+\frac{3}{4} \geq 0$, otherwise there is a $\langle 5,6,6,7,11\rangle$-star. If $\min\{\kappa_3,\kappa_5\} = 7$, then $\mu'(v) \geq 5-6+\frac{1}{3}+\frac{1}{6}+\frac{1}{2} = 0$, otherwise there is a $\langle 5,5,7,7,8\rangle$-star. If $\min\{\kappa_3,\kappa_5\} \geq 8$, then $\mu'(v) \geq 5-6+\frac{1}{3}+2\cdot\frac{3}{8} \geq 0$.

Suppose that v_4 is a 6-vertex. If $\min\{\kappa_3,\kappa_5\} = 7$, then $\mu'(v) \geq 5-6+\frac{1}{6}+\frac{3}{2}(\frac{19}{20}-\frac{6}{16}) \geq 0$, otherwise there is a $\langle 5,5,7,6,15\rangle$-star. If $\min\{\kappa_3,\kappa_5\} = 8$, then $\mu'(v) \geq 5-6+\frac{3}{8}+\frac{3}{4} \geq 0$, otherwise there is a $\langle 5,5,8,6,11\rangle$-star. If $\min\{\kappa_3,\kappa_5\} \geq 9$, then $\mu'(v) \geq 5-6+2\cdot\frac{1}{2} = 0$.

Subcase 9.4. The 5-vertex v has precisely two 5-neighbors v_1 and v_3.

As before, we may assume that $\kappa_4 \leq \kappa_5$. If $\kappa_4 \geq 9$, then $\mu'(v) \geq 5-6+2\cdot\frac{1}{2} = 0$.

Suppose that v_4 is a 6-vertex. If $\kappa_5 = 7$, then $\mu'(v) \geq 5-6+\frac{1}{6}+(\frac{19}{20}-\frac{6}{52}) \geq 0$, otherwise there is a $\langle 5,6,7,5,51\rangle$-star. If $\kappa_5 = 8$, then $\mu'(v) \geq 5-6+\frac{3}{8}+(\frac{19}{20}-\frac{6}{19}) \geq 0$, otherwise there is a $\langle 5,6,8,5,18\rangle$-star. If $\kappa_5 = 9$, then $\mu'(v) \geq 5-6+2\cdot\frac{1}{2} = 0$, otherwise there is a $\langle 5,6,9,5,10\rangle$-star. If $\kappa_5 \geq 16$, then $\mu'(v) \geq 5-6+\frac{3}{2}(\frac{19}{20}-\frac{6}{16})+\frac{1}{6} \geq 0$. If $12 \leq \kappa_5 \leq 15$, then $\mu'(v) \geq 5-6+\frac{3}{4}+\frac{1}{4} = 0$, otherwise there is a $\langle 5,7,6,6,15\rangle$-star. The remaining case is $\kappa_5 = 10, 11$. By the absence of $\langle 5,6,11,5,9\rangle$-stars, we have that $\kappa_2 \geq 10$. If v receives at least $\frac{1}{2}$ from v_2, then $\mu'(v) \geq 5-6+2\cdot\frac{1}{2} = 0$. So we may assume that $\kappa_2 = 10$ and v is a twice-weak neighbor of v_2. Let x, y, v_2, v, v_5 be the neighbors of v_1 in the cyclic order. Since v is a twice-weak neighbor of v_2, the vertex y must be a 5-vertex. Note that v_1 is not the center of a $\langle 5,5,10,5,12\rangle$-star, thus x cannot be a 5-vertex. By (R4) and (R5), we have that $\mu'(v) \geq 5-6+\frac{2}{5}+(\frac{2}{5}+2\cdot\frac{1}{10}) = 0$.

Suppose that v_4 is a 7-vertex. If $\kappa_5 = 7$, then $\mu'(v) \geq 5-6+2\cdot\frac{1}{6}+(\frac{19}{20}-\frac{6}{22}) \geq 0$, otherwise there is a $\langle 5,7,7,5,21\rangle$-star. If $\kappa_5 = 8$, then $\mu'(v) \geq 5-6+\frac{1}{6}+\frac{3}{8}+\frac{1}{2} \geq 0$, otherwise there is a $\langle 5,7,8,5,11\rangle$-star. If $9 \leq \kappa_5 \leq 11$, then $\mu'(v) \geq 5-6+\frac{1}{6}+\frac{1}{2}+\frac{1}{3} = 0$, otherwise there is a $\langle 5,7,11,5,8\rangle$-star. If $12 \leq \kappa_5 \leq 15$, then $\mu'(v) \geq 5-6+2\cdot\frac{1}{6}+\frac{3}{4} \geq 0$, otherwise there is a $\langle 5,6,5,7,15\rangle$-star. If $\kappa_5 \geq 16$, then $\mu'(v) \geq 5-6+\frac{1}{6}+\frac{3}{2}(\frac{19}{20}-\frac{6}{16}) \geq 0$.

Suppose that v_4 is an 8-vertex. If $\kappa_2 \geq 8$, then $\mu'(v) \geq 5-6+2\cdot\frac{3}{8}+\frac{1}{4} = 0$. If $\kappa_2 = 7$, then $\mu'(v) \geq 5-6+\frac{1}{6}+\frac{3}{8}+\frac{1}{2} \geq 0$, otherwise there is a $\langle 5,7,5,8,8\rangle$-star. If $\kappa_2 = 6$, then $\mu'(v) \geq 5-6+\frac{3}{8}+\frac{3}{4} \geq 0$, otherwise there is a $\langle 5,6,5,8,11\rangle$-star.

Subcase 9.5. The 5-vertex v has precisely three 5-neighbors v_1, v_2 and v_3.

Recall that v has at least two 7^+-neighbors, so we have that $\min\{\kappa_4, \kappa_5\} \geq 7$. If $\min\{\kappa_4, \kappa_5\} = 7$, then $\mu'(v) \geq 5 - 6 + \frac{1}{6} + \frac{3}{2}\left(\frac{19}{20} - \frac{6}{16}\right) \geq 0$, otherwise there is a $\langle 5, 6, 5, 7, 15\rangle$-star. If $\min\{\kappa_4, \kappa_5\} = 8$, then $\mu'(v) \geq 5 - 6 + \frac{3}{8} + \frac{3}{4} \geq 0$, otherwise there is a $\langle 5, 6, 5, 8, 11\rangle$-star. If $\min\{\kappa_4, \kappa_5\} \geq 9$, then $\mu'(v) \geq 5 - 6 + 2 \cdot \frac{1}{2} = 0$.

Subcase 9.6. The 5-vertex v has precisely three 5-neighbors v_1, v_2 and v_4.

If $\min\{\kappa_3, \kappa_5\} \geq 11$, then $\mu'(v) \geq 5 - 6 + 2 \cdot \frac{1}{2} = 0$. If $\min\{\kappa_3, \kappa_5\} = 7$, then $\mu'(v) \geq 5 - 6 + \frac{1}{6} + (\frac{19}{20} - \frac{6}{52}) \geq 0$, otherwise there is a $\langle 5, 6, 7, 5, 51\rangle$-star. If $\min\{\kappa_3, \kappa_5\} = 8$, then $\mu'(v) \geq 5 - 6 + \frac{1}{4} + (\frac{19}{20} - \frac{6}{30}) = 0$, otherwise there is a $\langle 5, 5, 8, 5, 29\rangle$-star. If $\min\{\kappa_3, \kappa_5\} = 9$, then $\mu'(v) \geq 5 - 6 + \frac{1}{3} + (\frac{19}{20} - \frac{6}{22}) \geq 0$, otherwise there is a $\langle 5, 5, 9, 5, 21\rangle$-star.

It suffices to consider $\min\{\kappa_3, \kappa_5\} = \kappa_3 = 10$. By the absence of $\langle 5, 5, 10, 5, 12\rangle$-stars, we have that $\kappa_5 \geq 13$. If v is a wretch, then $\mu'(v) \geq 5 - 6 + \frac{2}{5} + (\frac{1}{2} + \frac{1}{10}) = 0$; otherwise we have that $\mu'(v) \geq 5 - 6 + 2 \cdot \frac{1}{2} = 0$.

Remark 3. In fact, the $(6, 6, 6, 6, \infty)$-star in Theorem 1.2 can be refined as $\langle 6, 6, 6, 6, 11\rangle$-, $\langle 5, 6, 6, 6, 21\rangle$-, and $\langle 5, 6, 6, 5, \infty\rangle$-star by the discharging rules.

3 Proof of Theorem 1.9

Let G be a connected counterexample to Theorem 1.9 with maximum number of edges.

($*_3$) The graph G is a triangulation.

Proof of ($*_3$). Suppose that w_1, w_2, w_3, w_4 are four consecutive vertices on the boundary of a 4^+-face. Since G is a simple graph, we have that $w_1 \neq w_3$ and $w_2 \neq w_4$. Note that G is also a plane graph, thus we have that $w_1w_3 \notin E(G)$ or $w_2w_4 \notin E(G)$, otherwise the two lines representing w_1w_3 and w_2w_4 would cross each other outside the 4^+-face. But an insertion of a diagonal w_1w_3 or w_2w_4 into the 4^+-face would create a simple counterexample with more edges, which contradicts the assumption of G. □

In a triangulation, let w be a 5-vertex with neighbors w_1, w_2, w_3, w_4, w_5 in the cyclic order. The vertex w is a *wretch* if it is a weak neighbor of a 17^+-vertex w_5 and a twice-weak neighbor of a 9-vertex w_2. If w is a wretch, then we call the 5-vertex w_4 the *brother* of w. Note that w_1 and w_4 are asymmetrical, so the vertex w_1 cannot be a brother.

($*_4$) Every κ-vertex with $\kappa \geq 17$ is adjacent to at most $\frac{\kappa}{2}$ wretches.

Proof of ($*_4$). Let w be a wretch with neighbors w_1, w_2, w_3, w_4, w_5 in the cyclic order. Let w_2 be a 9-vertex and w_5 be a 17^+-vertex. Let w_3 has the neighbors x, y, w_4, w, w_2 in the cyclic order. Since w is a twice-weak neighbor of w_2, the vertex x must be a 5-vertex. By the absence of $\langle 5, 5, 9, 5, 16\rangle$-stars, the vertex y must be a 17^+-vertex. Note that y and w_5 are two distinct vertices, thus w_4 has two 17^+-neighbors. This implies that a brother cannot be a wretch. Let w_4 has the neighbors w_5, w, w_3, y, z in the cyclic order. Similarly, the vertex z has two 17^+-neighbors, thus z cannot be a wretch.

Note that w_1, w, w_4 and z are the consecutive neighbors of w_5 in the cyclic order. Now, we associate each wretch in $N_G(w)$ with its brother. By the above arguments, each wretch has a brother and distinct wretch have distinct brothers. Therefore, every κ-vertex with $\kappa \geq 17$ is adjacent to at most $\frac{\kappa}{2}$ wretches.

□

The Euler's formula $|V| - |E| + |F| = 2$ can be rewritten as the following:

$$\sum_{v \in V(G)} (\deg(v) - 6) + \sum_{f \in F(G)} (2\deg(f) - 6) = -12.$$

Firstly, we give every vertex v an initial charge $\mu(v) = \deg(v) - 6$, and give every face f an initial charge $\mu(f) = 2\deg(f) - 6$. Note that every face has an initial charge zero and every vertex has an nonnegative initial charge except the 5-vertices. Secondly, we redistribute the charges among 5-vertices and 7^+-vertices such that the final charge $\mu'(v)$ of every vertex v is nonnegative, which contradicts the sum the the initial charges is negative.

3.1 Discharging rules

(R1) Each 7-vertex sends $\frac{1}{4}$ to each non-weak 5-neighbor.

(R2a) Each 8-vertex sends $\frac{1}{2}$ to each strong 5-neighbor.

(R2b) Each 8-vertex sends $\frac{3}{8}$ to each semi-strong 5-neighbor.

(R2c) Each 8-vertex sends $\frac{1}{4}$ to each weak 5-neighbor.

(R3) Each 9-vertex sends $\frac{1}{3}$ to each adjacent vertex. Let w_0, w_1, w_2 be three consecutive neighbors of a 9-vertex in the cyclic order. Suppose that w_0 is a 6^+-vertex and w_1 is a 5-vertex.

(a) If w_2 is a 6^+-vertex, then w_0 transfers a charge of $\frac{1}{6}$ to w_1.

(b) If w_2 is a 5-vertex, then w_0 transfers a charge of $\frac{1}{12}$ to each of w_1 and w_2.

(R4a) Each κ-vertex with $10 \leq \kappa \leq 16$ sends $\frac{2(\kappa-6)}{\kappa}$ to each strong 5-neighbor.

(R4b) Each κ-vertex with $10 \leq \kappa \leq 16$ sends $\frac{3(\kappa-6)}{2\kappa}$ to each semi-strong 5-neighbor.

(R4c) Each κ-vertex with $10 \leq \kappa \leq 16$ sends $\frac{\kappa-6}{\kappa}$ to each weak 5-neighbor.

(R5a) Each 17-vertex with sends $\frac{34}{27}$ to each strong 5-neighbor.

(R5b) Each 17-vertex with sends $\frac{17}{18}$ to each semi-strong 5-neighbor.

(R5c) Each 17-vertex with sends $\frac{17}{27}$ to each weak 5-neighbor.

(R6a) Each κ-vertex with $\kappa \geq 18$ sends $2\left(\frac{53}{54} - \frac{6}{\kappa}\right)$ to each strong 5-neighbor.

(R6b) Each κ-vertex with $\kappa \geq 18$ sends $\frac{3}{2}\left(\frac{53}{54} - \frac{6}{\kappa}\right)$ to each semi-strong 5-neighbor.

(R6c) Each κ-vertex with $\kappa \geq 18$ sends $\left(\frac{53}{54} - \frac{6}{\kappa}\right)$ to each weak 5-neighbor.

(R7) Each 17^+-vertex additionally sends $\frac{1}{27}$ to each adjacent wretch.

Remark 4. By (R3), each 9-vertex sends $\frac{2}{3}$ to each strong 5-neighbor, sends $\frac{1}{3}$ to each twice-weak neighbor, and sends at least $\frac{5}{12}$ to any other 5-neighbor.

3.2 The final charge of every vertex is nonnegative

Case 1. If v is a κ-vertex with $\kappa \geq 18$, then $\mu'(v) \geq \kappa - 6 - \frac{\kappa}{2} \cdot \frac{1}{27} - \kappa \cdot \left(\frac{53}{54} - \frac{6}{\kappa}\right) = 0$.

Case 2. If v is a 17-vertex, then it is adjacent to at most eight wretches, and then $\mu'(v) \geq 17 - 6 - 17 \cdot \frac{17}{27} - 8 \cdot \frac{1}{27} = 0$.

Case 3. If v is a κ-vertex with $8 \leq \kappa \leq 16$, then $\mu'(v) \geq \kappa - 6 - \kappa \cdot \frac{\kappa-6}{\kappa} = 0$.

Case 4. The vertex v is a 7-vertex.

If v has at most four 5-neighbors, then $\mu'(v) \geq 7 - 6 - 4 \cdot \frac{1}{4} = 0$. If v has at least five 5-neighbors, then it has at most two 6^+-vertices and at most four non-weak 5-neighbors, which also implies that $\mu'(v) \geq 7 - 6 - 4 \cdot \frac{1}{4} = 0$.

Case 5. The vertex v is a 5-vertex with neighbors v_1, v_2, v_3, v_4, v_5 in the cyclic order. Suppose that v_1, v_2, v_3, v_4 and v_5 have degrees $\kappa_1, \kappa_2, \kappa_3, \kappa_4$ and κ_5 respectively.

Subcase 5.1. The 5-vertex v has no 5-neighbor.

If v has four 6-neighbors, then $\mu'(v) \geq 5 - 6 + 1 = 0$, otherwise there is a $\langle 6,6,6,6,11\rangle$-star. If v has at least two 8^+-neighbors, then $\mu'(v) \geq 5 - 6 + 2 \cdot \frac{1}{2} = 0$. If v has exactly three 6-neighbors and one 7-neighbor, then $\mu'(v) \geq 5 - 6 + \frac{1}{4} + \frac{4}{5} \geq 0$, otherwise there is a $\langle 6,6,6,7,9\rangle$-star or a $\langle 6,6,7,6,9\rangle$-star. If v has exactly two 6-neighbors, then $\mu'(v) \geq 5 - 6 + 2 \cdot \frac{1}{4} + \frac{1}{2} = 0$, otherwise there is a $\langle 6,6,7,7,7\rangle$-star or a $\langle 6,7,6,7,7\rangle$-star. If v has at most one 6-neighbor, then $\mu'(v) \geq 5 - 6 + 4 \cdot \frac{1}{4} = 0$.

Subcase 5.2. The 5-vertex v has precisely one 5-neighbor v_1.

By symmetry, we may assume that $\kappa_3 \leq \kappa_4$. If $\kappa_3 \geq 8$, then v receives at least $\frac{1}{2}$ from each of v_3 and v_4, which implies that $\mu'(v) \geq 5 - 6 + 2 \cdot \frac{1}{2} = 0$. If $\kappa_4 \in \{10, 11\}$, then $\mu'(v) \geq 5 - 6 + \frac{4}{5} + \frac{1}{4} \geq 0$, otherwise there is a $\langle 6,6,6,6,11\rangle$-star. If $\kappa_4 \geq 12$, then $\mu'(v) \geq 5 - 6 + 1 = 0$ and we are done. So we may assume that $\kappa_3 \leq 7$ and $\kappa_4 \leq 9$.

Suppose that v_4 is a 9-vertex. If $\kappa_3 = 7$, then $\mu'(v) \geq 5 - 6 + \frac{2}{3} + 2 \cdot \frac{1}{4} \geq 0$, otherwise there is a $\langle 6,5,6,7,9\rangle$-star. If $\min\{\kappa_2, \kappa_5\} = \kappa_3 = 6$, then $\mu'(v) \geq 5 - 6 + \frac{2}{3} + \frac{3}{8} \geq 0$, otherwise there is a $\langle 6,6,6,7,9\rangle$-, or $\langle 6,6,7,6,9\rangle$-star. If $\min\{\kappa_2, \kappa_5\} \geq 7$, then $\mu'(v) \geq 5 - 6 + \frac{2}{3} + 2 \cdot \frac{1}{4} \geq 0$.

Suppose that v_4 is an 8-vertex. If $\kappa_3 = 7$, then $\mu'(v) \geq 5-6+\frac{1}{2}+2\cdot\frac{1}{4} = 0$, otherwise there is a $\langle 6,6,6,7,9\rangle$-star. If $\kappa_3 = 6$ and $\min\{\kappa_2, \kappa_5\} \geq 7$, then $\mu'(v) \geq 5-6+\frac{1}{2}+2\cdot\frac{1}{4} = 0$. If $\min\{\kappa_2, \kappa_5\} = \kappa_3 = 6$, then $\mu'(v) \geq 5-6+\frac{1}{2}+\frac{3}{5} \geq 0$, otherwise there is a $\langle 5,6,6,8,9\rangle$- or $\langle 5,6,8,6,9\rangle$-star.

Suppose that $\kappa_3 = \kappa_4 = 7$. If $\min\{\kappa_2, \kappa_5\} \geq 7$, then $\mu'(v) \geq 5 - 6 + 4 \cdot \frac{1}{4} = 0$. If $\min\{\kappa_2, \kappa_5\} = 6$, then $\mu'(v) \geq 5 - 6 + 2 \cdot \frac{1}{4} + \frac{3}{5} \geq 0$, otherwise there is a $\langle 5,6,7,7,9\rangle$-star.

Suppose that $\kappa_4 = 7$ and $\kappa_3 = 6$. If $\min\{\kappa_2, \kappa_5\} = 6$, then $\mu'(v) \geq 5-6+\frac{1}{4}+\frac{3}{4} = 0$, otherwise there is a $\langle 5,6,6,7,11\rangle$- or $\langle 5,6,7,6,11\rangle$-star. If $\min\{\kappa_2, \kappa_5\} = 7$, then $\mu'(v) \geq 5 - 6 + 2 \cdot \frac{1}{4} + \frac{3}{5} \geq 0$, otherwise there is a $\langle 5,7,6,7,9\rangle$- or $\langle 5,7,7,6,9\rangle$-star. If $\min\{\kappa_2, \kappa_5\} \geq 8$, then $\mu'(v) \geq 5 - 6 + 2 \cdot \frac{3}{8} + \frac{1}{4} = 0$.

Suppose that $\kappa_3 = \kappa_4 = 6$. If $\min\{\kappa_2, \kappa_5\} = 6$, then $\mu'(v) \geq 5-6+\frac{3}{2}(\frac{53}{54}-\frac{6}{20}) \geq 0$, otherwise there is a $\langle 5,6,6,6,19\rangle$-star. If $\min\{\kappa_2, \kappa_5\} = 7$, then $\mu'(v) \geq 5 - 6 + \frac{1}{4} + \frac{3}{4} = 0$, otherwise there is a $\langle 5,7,6,6,11\rangle$-star. If $\min\{\kappa_2, \kappa_5\} = 8$, then $\mu'(v) \geq 5 - 6 + \frac{3}{8} + \frac{15}{22} \geq 0$, otherwise there is a $\langle 5,8,6,6,10\rangle$-star. If $\min\{\kappa_2, \kappa_5\} \geq 9$, then $\mu'(v) \geq 5 - 6 + \frac{5}{12} + \frac{3}{5} \geq 0$, otherwise there is a $\langle 5,9,6,6,9\rangle$-star.

Subcase 5.3. The 5-vertex v has precisely two 5-neighbors v_1 and v_2.

If $\kappa_4 \geq 12$, then $\mu'(v) \geq 5 - 6 + 1 = 0$ and we are done. If $\kappa_4 \in \{10, 11\}$, then $\mu'(v) \geq 5 - 6 + \frac{4}{5} + \frac{1}{4} \geq 0$, otherwise there is a $\langle 6,6,6,6,11\rangle$-star.

Suppose that v_4 is a 9-vertex. If $\min\{\kappa_3, \kappa_5\} \geq 7$, then $\mu'(v) \geq 5 - 6 + \frac{2}{3} + 2 \cdot \frac{1}{4} \geq 0$. If $\min\{\kappa_3, \kappa_5\} = 6$, then $\mu'(v) \geq 5 - 6 + \frac{2}{3} + \frac{3}{8} \geq 0$, otherwise there is a $\langle 6,6,6,7,9\rangle$-star.

Suppose that v_4 is an 8-vertex. If $\min\{\kappa_3, \kappa_5\} \geq 7$, then $\mu'(v) \geq 5 - 6 + \frac{1}{2} + 2 \cdot \frac{1}{4} = 0$. If $\min\{\kappa_3, \kappa_5\} = 6$, then $\mu'(v) \geq 5 - 6 + \frac{1}{2} + \frac{3}{5} \geq 0$, otherwise there is a $\langle 5,6,6,8,9\rangle$-star.

Suppose that v_4 is a 7-vertex. If $\min\{\kappa_3, \kappa_5\} = 6$, then $\mu'(v) \geq 5 - 6 + \frac{1}{4} + \frac{3}{4} = 0$, otherwise there is a $\langle 5,6,6,7,11\rangle$-star. If $\min\{\kappa_3, \kappa_5\} = 7$, then $\mu'(v) \geq 5 - 6 + 2 \cdot \frac{1}{4} + \frac{3}{5} \geq 0$, otherwise there is a $\langle 5,6,7,7,9\rangle$-star. If $\min\{\kappa_3, \kappa_5\} \geq 8$, then $\mu'(v) \geq 5 - 6 + \frac{1}{4} + 2 \cdot \frac{3}{8} = 0$.

Suppose that v_4 is a 6-vertex. If $\min\{\kappa_3, \kappa_5\} = 6$, then $\mu'(v) \geq 5 - 6 + \frac{3}{2}(\frac{53}{54} - \frac{6}{20}) \geq 0$, otherwise there is a $\langle 5,6,6,6,19\rangle$-star. If $\min\{\kappa_3, \kappa_5\} = 7$, then $\mu'(v) \geq 5 - 6 + \frac{1}{4} + \frac{3}{4} = 0$, otherwise there is a $\langle 5,6,7,6,11\rangle$-star. If $\min\{\kappa_3, \kappa_5\} = 8$, then $\mu'(v) \geq 5 - 6 + \frac{3}{8} + \frac{15}{22} \geq 0$, otherwise there is a $\langle 5,5,8,6,10\rangle$-star. If $\min\{\kappa_3, \kappa_5\} \geq 9$, then $\mu'(v) \geq 5 - 6 + \frac{5}{12} + \frac{3}{5} \geq 0$, otherwise there is a $\langle 5,5,9,6,9\rangle$-star.

Subcase 5.4. The 5-vertex v has precisely two 5-neighbors v_1 and v_3.

As before, we may assume that $\kappa_4 \leq \kappa_5$. If $\kappa_4 \geq 10$, then $\mu'(v) \geq 5 - 6 + 2 \cdot \frac{3}{5} \geq 0$.

Suppose that v_4 is a 6-vertex. By the absence of $\langle 5,6,6,5,\infty\rangle$-stars, we have that $\kappa_5 \geq 7$. If $\kappa_5 = 7$, then $\mu'(v) \geq 5 - 6 + \frac{1}{4} + (\frac{53}{54} - \frac{6}{26}) \geq 0$, otherwise there is a $\langle 5,6,7,5,25\rangle$-star. If $\kappa_5 = 8$, then $\mu'(v) \geq 5 - 6 + \frac{3}{8} + \frac{5}{8} = 0$, otherwise, there is a $\langle 5,6,8,5,15\rangle$-star. If $\kappa_5 = 9$, then $\mu'(v) \geq 5 - 6 + \frac{5}{12} + \frac{3}{5} \geq 0$, otherwise there is a $\langle 5,6,9,5,14\rangle$-star. If $\kappa_5 = 10$, then $\mu'(v) \geq 5 - 6 + \frac{3}{5} + \frac{2}{5} = 0$, otherwise there is a $\langle 5,9,5,6,10\rangle$-star. If $\kappa_5 = 11$, then $\mu'(v) \geq 5 - 6 + \frac{15}{22} + \frac{1}{3} \geq 0$, otherwise there is a $\langle 5,8,5,6,11\rangle$-star. If $12 \leq \kappa_5 \leq 19$, then $\mu'(v) \geq 5 - 6 + \frac{3}{4} + \frac{1}{4} = 0$, otherwise there is a $\langle 5,7,5,6,19\rangle$-star. If $\kappa_5 \geq 20$, then $\mu'(v) \geq 5 - 6 + \frac{3}{2}(\frac{53}{54} - \frac{6}{20}) \geq 0$.

Suppose that v_4 is a 7-vertex. If $\kappa_5 = 7$, then $\mu'(v) \geq 5 - 6 + 2 \cdot \frac{1}{4} + \frac{1}{2} = 0$, otherwise there is a $\langle 5,7,7,5,11\rangle$-star. If $\kappa_5 = 8$, then $\mu'(v) \geq 5 - 6 + \frac{1}{4} + \frac{3}{8} + \frac{2}{5} \geq 0$, otherwise there is a $\langle 5,7,8,5,9\rangle$-star. If $\kappa_5 = 9$, then

$\mu'(v) \geq 5-6+\frac{1}{4}+\frac{5}{12}+\frac{1}{3}=0$, otherwise there is a $\langle 5,8,5,7,9\rangle$-star. If $\kappa_5 \in \{10,11\}$, then $\mu'(v) \geq 5-6+\frac{3}{5}+2\cdot\frac{1}{4} \geq 0$, otherwise there is a $\langle 5,7,5,7,11\rangle$-star. If $\kappa_5 \geq 12$, then $\mu'(v) \geq 5-6+\frac{1}{4}+\frac{3}{4}=0$.

Suppose that v_4 is an 8-vertex. If $\kappa_2 \geq 8$, then $\mu'(v) \geq 5-6+2\cdot\frac{3}{8}+\frac{1}{4}=0$. If $\kappa_2 \leq 7$, then $\mu'(v) \geq 5-6+\frac{3}{8}+\frac{15}{22} \geq 0$, otherwise there is a $\langle 5,7,5,8,10\rangle$-star.

Suppose that v_4 is a 9-vertex. If $\kappa_5 = 9$, then $\mu'(v) \geq 5-6+2\cdot\frac{5}{12}+\frac{1}{4} \geq 0$, otherwise there is a $\langle 5,7,5,9,9\rangle$-star. If $\kappa_5 \geq 10$, then $\mu'(v) \geq 5-6+\frac{5}{12}+\frac{3}{5} \geq 0$.

Subcase 5.5. The 5-vertex v has precisely three 5-neighbors v_1, v_2 and v_3.

If $\min\{\kappa_4,\kappa_5\} = 6$, then $\mu'(v) \geq 5-6+\frac{3}{2}(\frac{53}{54}-\frac{6}{20}) \geq 0$, otherwise there is a $\langle 5,6,6,6,19\rangle$-star. If $\min\{\kappa_4,\kappa_5\} = 7$, then $\mu'(v) \geq 5-6+\frac{1}{4}+\frac{3}{4}=0$, otherwise there is a $\langle 5,6,6,7,11\rangle$-star. If $\min\{\kappa_4,\kappa_5\} = 8$, then $\mu'(v) \geq 5-6+\frac{3}{8}+\frac{15}{22} \geq 0$, otherwise there is a $\langle 5,7,5,8,10\rangle$-star. If $\min\{\kappa_4,\kappa_5\} \geq 9$, then $\mu'(v) \geq 5-6+\frac{5}{12}+\frac{3}{5} \geq 0$, otherwise there is a $\langle 5,7,5,9,9\rangle$-star.

Subcase 5.6. The 5-vertex v has precisely three 5-neighbors v_1, v_2 and v_4.

By the absence of $\langle 5,5,7,5,\infty\rangle$-stars, we have that $\min\{\kappa_3,\kappa_5\} \geq 8$. If $\min\{\kappa_3,\kappa_5\} = 8$, then $\mu'(v) \geq 5-6+\frac{1}{4}+(\frac{53}{54}-\frac{6}{26}) \geq 0$, otherwise there is a $\langle 5,5,8,5,25\rangle$-star. If $\min\{\kappa_3,\kappa_5\} = 10$, then $\mu'(v) \geq 5-6+\frac{2}{5}+\frac{3}{5}=0$, otherwise there is a $\langle 5,5,10,5,14\rangle$-star. If $\min\{\kappa_3,\kappa_5\} = 11$, then $\mu'(v) \geq 5-6+\frac{5}{11}+\frac{4}{7} \geq 0$, otherwise there is a $\langle 5,5,11,5,13\rangle$-star. If $\min\{\kappa_3,\kappa_5\} \geq 12$, then $\mu'(v) \geq 5-6+2\cdot\frac{1}{2}=0$.

It suffices to consider $\min\{\kappa_3,\kappa_5\} = \kappa_3 = 9$. By the absence of $\langle 5,5,9,5,16\rangle$-stars, we have that $\kappa_5 \geq 17$. If v is a wretch, then $\mu'(v) \geq 5-6+\frac{1}{3}+\left(\frac{1}{27}+\frac{17}{27}\right)=0$; otherwise we have that $\mu'(v) \geq 5-6+\left(\frac{1}{3}+\frac{1}{12}\right)+\frac{17}{27} \geq 0$.

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