On the coupling of an arbitrary number of angular momenta: generalized analytic approaches to Clebsch-Gordan decomposition of $SU(2)$ spin representations.

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Abstract

The whole enterprise of Clebsch-Gordan decomposition (CGD) of an arbitrary collection of $SU(2)$ spin representations can be recast as a simple enumerative combinatoric problem. We show here that enumerative combinatorics (EC)[1] is a natural setting for CGD, and easily leads to very general analytic formulae – so far not present in the literature. Based on it, we propose three general methods for computing spin multiplicities; namely, 1) the multi-restricted partition, 2) the generalized binomial and 3) the generating function methods. The last has already been recently discussed in the literature based on statistical mechanical considerations[2], but here we show how it naturally surfaces within the EC approach.

The maximum of the resulting multiset of $SU(2)$ spin representations is easy to establish analytically, but not the minimum. We prove here an analytical equation for computing this minimum. Besides the application of our results to some problems in number theory and statistics, CGD connections to lattice paths and numbers usually encountered in counting problems like Catalan and Riordan's are also briefly discussed.

I. INTRODUCTION

In many areas of physics and chemistry, one is often faced with the daunting task of determining the Clebsch-Gordan decomposition (CGD) series of a given collection of angular momenta. Seemingly unrelated problems like determining how many linearly independent isotropic tensor isomers (particularly useful in determining rotational averages [3] of observables, for example) of rank n there are in a $D-$dimensional space, certain variants of the random walk problem [2], symmetric exclusion processes[4], or even black-hole entropy calculations[5] can all be related to CGD – not to mention its use in the detection and characterization of entangled states[6]. The relevance of CGD is deeply rooted in group theory and manifest in many of its applications.

CGD in the case of two arbitrary $SU(2)$ spin representations is well known and reported in textbooks on the subject [7, 8]. In the general case of N angular momenta, it is possible to repeat the two angular momenta addition scheme over and over, as reported for example in [8]. Needless to say, this approach soon becomes unmanageable as the total number of uncoupled angular momenta N increases, and it gets even worse the higher the magnitude of the momenta involved. An analytical way of determining the Clebsch-Gordan decomposition series is therefore ineluctable. Zachos[9] has provided an analytical expression in the case of N spin-(1/2)s. Curtright, Van Kortryk and Zachos[10] have recently considered the case of a N identical spin$-j$ system. Polychronakos and Sfetsos[2] have also considered the same problem and have shown how to generalize the results – by means of the statistical mechanical partition function – to the case of an arbitrary collection of spins. What is missing though in the literature is a general analytical expression for the computation of the multiplicities of the distinct resulting momenta appearing in the CGD of an arbitrary collection of angular momenta (be it spin or orbital), each of arbitrary magnitude. And with this article we propose material to fill that gap.

The basis of our approach is enumerative combinatorics (EC)[1], and it begins with a mapping of the eigenvalues of the j_z component (henceforth, referred to as "$z-$eigenvalue") of each spin to a subset of $\mathbb{N}_0$ (the set of all natural numbers, zero included). The advantage of the EC approach we discuss below is that it enables one to view the CGD problem from the most general point of view right from the start: thus, making it possible to formulate very general solutions from which one can derive solutions to limit cases like the CGD of

N spin-j (Sec. V) – which is essentially the foremost general case for which one can find in the literature explicit analytic formulae for the spin multiplicities[2, 4, 10]. Inspired by enumerative combinatoric analysis, we propose here three methods for the CGD of an arbitrary collection of spins; namely, 1) the multi-restricted partition, 2) the generating function, and 3) the generalized binomial methods. All three are related to each other: from the first, we derive the second method – from which we derive in turn the third. Though the generating function method has already been discussed in the literature based on quantum statistical mechanics[2], we show here how it emerges within the EC approach. Our generating function and the one discussed in [2] (see also [4]) are related to each other by a proportionality constant which depends on the total spin of the system.

Drawing upon the three methods, we derive general expressions for the dimension of the total $z-$eigenvalue invariant subspaces of the system in Sec. III. This will be followed by derivation of general expressions for the spin multiplicities in Sec. IV – again, based on the three methods. Finally, in Sec. VI we shall illustrate some applications of the general formulae derived in the previous sections as to some problems in quantum physics, number theory and statistics. Connections of CGD to lattice paths and some counting numbers like the Catalan and Riordan's shall be briefly discussed in the same section. But before all these, we shall look into one particular important problem which has not yet been addressed in the literature: this has to do with the minimum of the $SU(2)$ spin representations resulting from the composition of an initial collection of spins. While the maximum is easy to prove, the minimum is not. We prove this minimum in the general case in Sec. II.

The formulae we give here can be easily implemented computationally, thus they could enhance greatly recent theoretical and computational efforts aimed at limiting the computational cost of running simulations on many-body systems involving a considerable number of angular momenta. In a forthcoming article, for example, we shall have the occasion to expound more on the use of these analytical formulae to better characterize the Hilbert space of static multi-spin Hamiltonians and show how they can be employed to run simulations of EPR and NMR spectra which scale polynomially with the number of spins rather than exponentially.

II. COUPLING OF AN ARBITRARY COLLECTION OF ANGULAR MOMENTA

Say we have a finite collection of $SU(2)$ spin representations: $\mathcal{A} \equiv \{j_1, j_2, \ldots, j_N\}$, and $j_i \in \{\frac{1}{2}, 1, \frac{3}{2}, 2, \ldots\}$. The coupling of these angular momenta will give rise to a multiset[1, 11] $\mathcal{E}$ of angular momenta which we indicate as

$$\mathcal{E} \equiv \{\underbrace{J_0, \ldots, J_0}_{\lambda_0}, \underbrace{J_1, \ldots, J_1}_{\lambda_1}, \ldots, \underbrace{J_m, \ldots, J_m}_{\lambda_m}\} \equiv \left\{J_0^{\lambda_0}, J_1^{\lambda_1}, \ldots, J_m^{\lambda_m}\right\} \tag{1}$$

where λ_κ is the multiplicity of J_κ.

In group theoretical terms, we may argue as follows: The angular momentum j_i is associated with the Hilbert space $\mathcal{H}^{(j_i)}$ of dimension $2j_i + 1$. The Hilbert space $\mathcal{H}$ of the collection is given by the tensor product

$$\mathcal{H} = \mathcal{H}^{(j_1)} \otimes \mathcal{H}^{(j_2)} \otimes \cdots \otimes \mathcal{H}^{(j_N)} \ . \tag{2}$$

This representation of the Hilbert space $\mathcal{H}$ is reducible into distinct irreducible components $\mathcal{H}^{(J_\kappa)}$, of dimension $2J_\kappa + 1$. $\mathcal{H}$ in terms of its irreducible components is therefore

$$\mathcal{H} = \underbrace{\mathcal{H}^{(J_0)} \oplus \ldots \oplus \mathcal{H}^{(J_0)}}_{\lambda_0} \oplus \underbrace{\mathcal{H}^{(J_1)} \oplus \ldots \oplus \mathcal{H}^{(J_1)}}_{\lambda_1} \oplus \cdots \oplus \underbrace{\mathcal{H}^{(J_m)} \oplus \ldots \oplus \mathcal{H}^{(J_m)}}_{\lambda_m} \ . \tag{3}$$

We now determine the distinct elements of $\mathcal{E}$. To begin with, we shall assume henceforth the following ordering of the distinct elements of $\mathcal{E}$: $J_0 > J_1 > \ldots > J_m$.

Theorem 1. *Given the finite multiset $\mathcal{A} = \{j_1, j_2, \ldots, j_N\}$ of $SU(2)$ spin representations, then the maximum (J_0) and minimum (J_m) of the related coupled representation $\mathcal{E}$ are given by the expressions:*

$$J_0 = \max \mathcal{E} = \sum_i j_i \tag{4}$$

$$J_m = \min \mathcal{E} = \upsilon_{\mathcal{A}} \cdot H(\upsilon_{\mathcal{A}}) + (1 - H(\upsilon_{\mathcal{A}})) \cdot \frac{(2J_0 \bmod 2)}{2} \tag{5}$$

respectively, where

$$\upsilon_{\mathcal{A}} \equiv 2 \cdot \max \mathcal{A} - J_0 \ , \tag{6}$$

and where $H(x)$ is the Heaviside step function, defined here to be

$$H(x) \equiv \begin{cases} 0, & \textit{if } x < 0 \\ 1, & \textit{if } x \geq 0 \ . \end{cases} \tag{7}$$

Proof. Eq. (4) is obvious and we shall not consider it further. The same cannot be said about Eq. (5).

To prove Eq. (5), it is important to keep in mind that $J_m = \min \mathcal{E}$. Now, from the multiset $\mathcal{A} = \{j_1, j_2, \ldots, j_N\}$, we consider the submultiset $\widetilde{\mathcal{A}}$, defined as

$$\widetilde{\mathcal{A}} \equiv \mathcal{A} \setminus \{\max \mathcal{A}\} \ , \tag{8}$$

i.e. $\widetilde{\mathcal{A}}$ is the submultiset obtained after removing the largest element (or one of the largest elements, if its multiplicity is greater than 1) of $\mathcal{A}$ from the latter. The coupled representation of $\widetilde{\mathcal{A}}$ will yield the multiset $\widetilde{\mathcal{E}}$ where – in analogy to Eq. (1) –

$$\widetilde{\mathcal{E}} \equiv \{\underbrace{\widetilde{J}_0, \ldots, \widetilde{J}_0}_{\widetilde{\lambda}_0}, \underbrace{\widetilde{J}_1, \ldots, \widetilde{J}_1}_{\widetilde{\lambda}_1}, \ldots, \underbrace{\widetilde{J}_m, \ldots, \widetilde{J}_m}_{\widetilde{\lambda}_m}\} \ . \tag{9}$$

with $\widetilde{J}_l > \widetilde{J}_{l'}$ if $l < l'$, by convention. We now couple each element of $\widetilde{\mathcal{E}}$ with $\max \mathcal{A}$ to generate the multiset $\mathcal{E}$. Since the coupling of $\widetilde{J}_l$ ($\in \widetilde{\mathcal{E}}$) with $\max \mathcal{A}$ gives rise to the Clebsch-Gordan decomposition series

$$\max \mathcal{A} + \widetilde{J}_l, \ \max \mathcal{A} + \widetilde{J}_l - 1, \ldots, \left|\max \mathcal{A} - \widetilde{J}_l\right| \ , \tag{10}$$

– with all resulting momenta ending up as elements of $\mathcal{E}$ – we conclude that

$$\min \mathcal{E} = \min \left\{ \left|\max \mathcal{A} - \widetilde{J}_l\right| \ \middle| \ \widetilde{J}_l \in \widetilde{\mathcal{E}} \right\} \tag{11}$$

It is important to notice at this point that

$$\max \mathcal{A} \geq \widetilde{J}_m \qquad \text{and} \qquad \widetilde{J}_0 \geq \widetilde{J}_m \tag{12}$$

where the equality in the former *may* hold when $\dim \widetilde{\mathcal{A}} = 1$, while the equality in the latter holds when $\dim \widetilde{\mathcal{A}} = 1$. Eq. (12) leads us to consider two possible scenarios:

A. $\max \mathcal{A} \geq \widetilde{J}_0$

In this case, it follows from Eq. (12) that

$$\max \mathcal{A} \geq \widetilde{J}_0 \geq \widetilde{J}_m \ . \tag{13}$$

Therefore, given Eq. (11) and the fact that $J_l > J_{l'}$ if $l < l'$, we conclude that

$$\begin{aligned} \min \mathcal{E} = J_m &= \max \mathcal{A} - \widetilde{J}_0 \\ &= 2 \cdot \max \mathcal{A} - J_0 \ , \end{aligned} \tag{14}$$

since $\max \mathcal{A} + \widetilde{J}_0 = J_0$.

B. $\max \mathcal{A} < \widetilde{J}_0$

When this relation is satisfied, then according to Eq. (12),

$$\widetilde{J}_0 > \max \mathcal{A} \geq \widetilde{J}_m \tag{15}$$

which means that there certainly exists a $\widetilde{J}_l^* \in \widetilde{\mathcal{E}}$ which satisfies Eq. (11). Our aim here though, is not to determine $\widetilde{J}_l^*$ but the difference $\left|\max \mathcal{A} - \widetilde{J}_l^*\right|$. In passing, we draw the reader's attention to the fact that the elements of $\widetilde{\mathcal{E}}$ (and also $\mathcal{E}$) are either all integers or half-integers, so we can talk of the nature of any of the elements as being an integer or half-integer by just referring to any other element. With this last observation in mind, one can easily show that,

$$\left|\max \mathcal{A} - \widetilde{J}_l\right| \geq \begin{cases} 0, \text{ if both } \widetilde{J}_0 \text{ and } \max \mathcal{A} \text{ are integers or half-integers} \\ \frac{1}{2}, \text{ if either } \widetilde{J}_0 \text{ or } \max \mathcal{A} \text{ is half-integer } \end{cases}, \tag{16}$$

since all the elements of $\widetilde{\mathcal{A}}$ are less or equal to $\max \mathcal{A}$. Thus, when $\max \mathcal{A} < \widetilde{J}_0$,

$$\min \mathcal{E} = J_m = \begin{cases} 0, \text{ if both } \widetilde{J}_0 \text{ and } \max \mathcal{A} \text{ are integers or half-integers} \\ \frac{1}{2}, \text{ if either } \widetilde{J}_0 \text{ or } \max \mathcal{A} \text{ is half-integer } \end{cases}, \tag{17}$$

which can be written more succinctly as,

$$\begin{aligned} J_m &= \frac{1}{2}\left[2\left(\max \mathcal{A} + \widetilde{J}_0\right) \mod 2\right] \\ &= \frac{1}{2}\left[2J_0 \mod 2\right] . \end{aligned} \tag{18}$$

Putting together Eqs. (14) and (18) yields Eq. (5). □

Going back to Eq. (5), we observe that

$$J_m = \begin{cases} \upsilon_{\mathcal{A}}, \text{ if } \upsilon_{\mathcal{A}} \geq 0 , \\ 0, \text{ if } \upsilon_{\mathcal{A}} < 0 \text{ and } \upsilon_{\mathcal{A}} \text{ is an integer} , \\ \frac{1}{2}, \text{ if } \upsilon_{\mathcal{A}} < 0 \text{ and } \upsilon_{\mathcal{A}} \text{ is half-integer} . \end{cases} \tag{19}$$

Since $J_0 - J_\kappa = \kappa, \ \forall\ \kappa \in \{0, \ldots, m\}$, it follows that the number of distinct elements of $\mathcal{E}$, $N'_{\mathcal{E}}$, is certainly

$$N'_{\mathcal{E}} = J_0 - J_m + 1 , \tag{20}$$

and so $m = N'_{\mathcal{E}} - 1$. As already noted above, if J_0 is half-integer, so are all the elements of $\mathcal{E}$, and the same applies when J_0 is an integer.

III. THE TOTAL $z-$EIGENVALUE INVARIANT SUBSPACES AND THEIR DIMENSIONS

Each angular momentum j_i of $\mathcal{A} = \{j_1, j_2, \ldots, j_N\}$ has its own set of possible $z-$eigenvalues, which we indicate as $\mathcal{M}^{[i]}$, *i.e.*

$$\mathcal{M}^{[i]} \equiv \{j_i, j_i - 1, \ldots, -j_i\} \ , \qquad i \in \{1, 2, \ldots, N\} \ . \tag{21}$$

The cardinality of $\mathcal{M}^{[i]}$, $\dim \mathcal{M}^{[i]}$, is obviously $(2j_i + 1)$. To make contact with EC, we note that elements of the set $\mathcal{M}^{[i]}$ can also be represented as

$$\mathcal{M}^{[i]} \equiv \{j_i - n_i \mid \ n_i \in \mathbb{N}_0 \ \wedge \ 0 \leq n_i \leq 2j_i\} \ . \tag{22}$$

For a given j_i, we denote the set of all possible n_i according to Eq. (22) as $\mathcal{N}^{[i]}$; *i.e.* $\mathcal{N}^{[i]} \equiv \{0, 1, 2, \ldots, 2j_i\}$. Evidently, there is a one-to-one correspondence between the set $\mathcal{N}^{[i]}$ and the set $\mathcal{M}^{[i]}$. Due to this bijective relation between the two sets, we can perform operations with $\mathcal{N}^{[i]}$ instead of $\mathcal{M}^{[i]}$. This is very convenient because – unlike $\mathcal{M}^{[i]}$ – the set $\mathcal{N}^{[i]}$ is always a subset of $\mathbb{N}_0$, independent of whether j_i is half-integer or not.

Each element J_l of the coupled representation of $\mathcal{A}$, *i.e.* $\mathcal{E}$, also has its own set $\mathcal{M}'^{[l]}$ of possible $z-$eigenvalues, and – in analogy to Eq. (22) – we can write

$$\mathcal{M}'^{[l]} \equiv \left\{J_l - n_l \mid \ n_l \in \mathcal{N}^{[l]}\right\} \ , \tag{23}$$

where $\mathcal{N}^{[l]} \equiv \{0, 1, 2, \ldots, 2J_l\}$. The multiset sum of the various $\mathcal{M}'^{[l]}$ yields the multiset $\mathcal{M}'$, that is,

$$\mathcal{M}' \equiv \biguplus_{l=0}^{m} \ \mathcal{M}'^{[l]} \ . \tag{24}$$

$\mathcal{M}'$ contains as elements all possible total $z-$eigenvalues in the coupled representation, each repeated a certain number of times Ω. The multiplicity Ω of each reflects the number of possible $J_l \in \mathcal{E}$ from which it can originate.

To determine these multiplicities, what we do is to first make use of the fact that the $z-$eigenvalues in the coupled representation are simply given by the sum of their uncoupled counterparts. On that account, it follows from Eq. (22) that

$$\begin{aligned} \mathcal{M}' &= \left\{ \sum_i (j_i - n_i) \middle| \, j_i \in \mathcal{A} \ , \ n_i \in \mathcal{N}^{[i]} \right\} \\ &= \left\{ J_0 - \sum_i n_i \middle| \ n_i \in \mathcal{N}^{[i]} \right\} \end{aligned} \tag{25}$$

where we have made use of Eq. (4). Given that the sum of a finite number of natural numbers is also obviously a natural number, we can rewrite Eq. (25) as

$$\mathcal{M}' = \{\, J_0 - n |\, n \in \mathcal{N} \} \tag{26}$$

where,

$$n \equiv \sum_i n_i \ , \qquad n_i \in \mathcal{N}^{[i]} \tag{27a}$$

$$\mathcal{N} \equiv \left\{ 0, 1, \ldots, \sum_i \ \max \mathcal{N}^{[i]} \right\} = \{0, 1, \ldots, 2J_0\} \ . \tag{27b}$$

It is therefore straightforward to see that the multiplicity of the element $(J_0 - n)$ of the multiset $\mathcal{M}'$ is equivalent to the number of possible ways of getting the number n according to Eq. (27a). In fact, Eq. (27a) defines a "multi-restricted partition" of the integer n, whereby in addition to the number of parts restricted to N, each part n_i is also restricted to the set $\mathcal{N}^{[i]}$. Contrary to common partitions, the number zero is an admissible part here for the sake of mathematical consistency; the results remain unchanged if one ignores them in the following. We shall come back to this point later on.

We shall assume the notation $M_n \equiv J_0 - n$. And so there is a one-to-one correspondence between $\{n\}$ and the set of distinct M_n.

Theorem 2. *Let $\Omega_{\mathcal{A},n}$ represent the number of elements of the multiset $\mathcal{E}$ which admit M_n as a possible eigenvalue of its $z-$component. Then, $\Omega_{\mathcal{A},n}$ is given by the following expression:*

$$\boxed{\Omega_{\mathcal{A},n} = \sum_{A(n)\ \in\ P(\mathcal{A};n)} \ \prod_{\nu=0}^{\dim \widetilde{A}(n)-1} \binom{\omega_n(\nu) - (1-\delta_{\nu,0}) \sum_{l=0}^{\nu-1} s_n(l)}{s_n(\nu)}} \tag{28}$$

where,

- $P(\mathcal{A};n) \equiv$ *the set of all (unordered) multi-restricted partition of the integer n into at most* $\dim \mathcal{A} = N$ *parts, with the value of each part being at most only one of the N values of $\{2j_1, \ldots, 2j_N\}$, where $j_i \in \mathcal{A}$;*

- $A(n) \equiv$ *an element of $P(\mathcal{A};n)$; the sum in Eq. (28) is over all $A(n)$;*

- $\widetilde{A}(n) \equiv$ *the set of distinct elements of* $A(n)$. ν *runs over all elements* $\{a_n(\nu)\}$ *of* $\widetilde{A}(n)$*: with* $a_n(0)$ *as the largest integer,* $a_n(1)$ *the second largest integer, and so forth;*

- $\omega_n(\nu) \equiv$ *the number of* $2j_i$ *greater or equal to* $a_n(\nu)$*;*

- $s_n(\nu) \equiv$ *the number of times* $a_n(\nu)$ *appears in* $A(n)$.

Proof. It is clear that for any given finite multiset $\mathcal{A}$ we can define the multiset $\mathcal{N}_{max}$, where

$$\mathcal{N}_{max}(\mathcal{A}) \equiv \left\{\max \mathcal{N}^{[i]}\right\}_{\mathcal{A}} = \{2j_1, 2j_2, \ldots, 2j_N\} \ . \tag{29}$$

Say $P(\mathcal{A}; n)$ the set of all (unordered) multi-restricted partitions of the integer n into $\dim \mathcal{A}(= N)$ parts, with the restriction that the $i-$th part, n_i, belongs to the set $\mathcal{N}^{[i]}$. That is,

$$P(\mathcal{A}; n) \equiv \{A(n; 1), A(n; 2), \ldots, A(n; q_n)\} \tag{30}$$

where $A(n; k)$ is the $k-$th multi-restricted partition of n in reference to $\mathcal{A}$ according to the description given above; $q_n \equiv \dim P(\mathcal{A}; n)$. For convenience, we choose to write the elements of $P(\mathcal{A}; n)$ as multisets, *i.e.*

$$A(n; k) \equiv \{a_{k,0}^{s_n(k,0)}, a_{k,1}^{s_n(k,1)}, \ldots, a_{k,\mu_{k,n}}^{s_n(k,\mu_{k,n})}\} \tag{31}$$

where the $a_{k,\nu}$s are positive integers (in other words, parts of the restricted partition of n); $s(k,\nu)$ is the multiplicity of the part $a_{k,\nu}$ and $\mu_{k,n} \equiv \dim A(n; k) - 1$. Evidently,

$$n = \sum_{\nu=0}^{\mu_{k,n}} s_n(k,\nu) \cdot a_{k,\nu} \ , \qquad N = \sum_{\nu=0}^{\mu_{k,n}} s_n(k,\nu) \ . \tag{32}$$

Finally, let us denote the set of all distinct elements of $A(n; k)$ as $\widetilde{A}(n; k)$:

$$\widetilde{A}(n; k) \equiv \{a_{k,0}, a_{k,1}, \ldots, a_{k,\mu_{k,n}}\} \ . \tag{33}$$

To determine the restricted partition $\Omega_{\mathcal{A},n}$ of n as explained above according to (27a), we may consider the elements of $\mathcal{A}$ as unconnected channels (N in total) among which we need to distribute *successfully* the elements of $A(n,k)$. The caveats here are: 1) the positive integers the channel i can accommodate cannot exceed $\max \mathcal{N}^{[i]}$; 2) given the multiset $A(n,k)$, each channel can accommodate only one of $A(n,k)$; 3) by "successfully" we mean the elements of $A(n,k)$ must be distributed among the N channels without leaving none behind, according to the preceding caveats. Given $A(n;k)$, we denote the number of ways the distribution can be successfully achieved as $\varrho[A(n;k)]$. It is evident then that

$$\Omega_{\mathcal{A},n} = \sum_k \varrho[A(n;k)] \ , \qquad A(n;k) \in P(\mathcal{A};n). \tag{34}$$

How do we then determine $\varrho[A(n;k)]$? In order to ensure a successful distribution of the elements of $A(n;k)$ among the N channels, we must distribute first $\max \widetilde{A}(n;k)$, followed by the second largest integer of $\widetilde{A}(n;k)$, and so on. To simplify matters, we assume that $a_{k,\nu} > a_{k,\nu'}$ if $\nu < \nu'$. Thus, $\max \widetilde{A}(n;k) = a_{k,0}$. If we denote the number of ways of distributing the number $a_{k,\nu} \in \widetilde{A}(n;k)$ as $D_n(k,\nu)$, then it follows that

$$D_n(k,0) = \begin{pmatrix} \omega_n(0) \\ s_n(k,0) \end{pmatrix} \ , \tag{35}$$

where $\omega_n(\nu)$ is the number of channels which can successfully take in $a_{k,\nu}$. In general, $\omega_n(\nu)$ is the number of elements of the multiset $\mathcal{N}_{max}(\mathcal{A})$ which are not less than $a_{k,\nu}$:

$$\omega_n(\nu) = \sum_{i=1}^{N} \delta_{1,H(2j_i - a_{k,\nu})} \qquad 2j_i \in \mathcal{N}_{max}(\mathcal{A}) \ , \tag{36}$$

where $H(x)$ is the Heaviside step function defined in Eq. (7).

We now move on to the second largest integer of $\widetilde{A}(n;k)$: $a_{k,1}$. The number of channels which can take in this integer is $\omega_n(1)$, which includes the $\omega_n(0)$ channels since $a_{k,0} > a_{k,1}$. But of the $\omega_n(0)$ channels, $s_n(k,0)$ of them have already been occupied by $a_{k,0}$. We are therefore left effectively with $\omega_n(1) - s_n(k,0)$ channels available to accommodate the $s_n(k,1)$ integers of value $a_{k,1}$. Hence,

$$D_n(k,1) = \begin{pmatrix} \omega_n(1) - s_n(k,0) \\ s_n(k,1) \end{pmatrix} \ . \tag{37}$$

Following similar arguments, for $a_{k,2}$ we shall have at our disposal $\omega_n(2) - s_n(k,0) - s_n(k,1)$ channels to distribute $s_n(k,2)$ of it. And so,

$$D_n(k,2) = \begin{pmatrix} \omega_n(1) - s_n(k,0) - s_n(k,1) \\ s_n(k,2) \end{pmatrix} \ . \tag{38}$$

We easily infer from the above arguments that, for a given $\mathcal{A}$,

$$D_n(k,\nu) = \begin{pmatrix} \omega_n(\nu) - (1 - \delta_{0,\nu}) \sum_{l=0}^{\nu-1} s_n(k,l) \\ s_n(k,\nu) \end{pmatrix} \ . \tag{39}$$

It is easy to realize that for a given $A(n;k)$, $\varrho[A(n;k)]$ is the product of the various $D_n(k,\nu)$, *i.e.*

$$\varrho[A(n;k)] = \prod_{\nu=0}^{\dim \widetilde{A}(n;k)-1} D_n(k,\nu) \ . \tag{40}$$

Putting together Eqs. (34), (40) and (39), and simply writing $s_n(k,\nu) \to s_n(\nu)$, $A(n;k) \to A(n)$, $\widetilde{A}(n;k) \to \widetilde{A}(n)$, we get Eq. (28). $\square$

We shall refer to Eq. (28) as the *multi-restricted partition method* for determining $\{\Omega_{\mathcal{A},n}\}$.

One can now understand why the parts with value $n_i = 0$ do not contribute to $\Omega_{\mathcal{A},n}$: the effective number of ways of distributing any allowed multiplicity of the number zero among the channels, according to the above criteria, is always 1. Furthermore, it is easy to realize that when $\mathcal{A} = \left\{\frac{1}{2}^N\right\}$ – *i.e.* in the case of N spin-$\frac{1}{2}$s, Eq. (28) reduces to

$$\Omega_{\mathcal{A},n} = \binom{N}{n} \ . \tag{41}$$

An important property of the integers $\{\Omega_n\}$ (in the following, we shall on some occasions simply write Ω_n instead of $\Omega_{\mathcal{A},n}$) is that

$$\Omega_n = \Omega_{2J_0-n} \ , \qquad \forall n \in \mathcal{N} \ , \tag{42}$$

which implies a reciprocal distribution of these integers. The relation in Eq. (42) holds because Ω_n and Ω_{2J_0-n} are the dimensions of the subspaces of total $z-$eigenvalues M_n and M_{2J_0-n}, respectively. But from Eq. (26), we have that $|M_n| = |M_{2J_0-n}|$, *i.e.* the two subspaces are related by T-symmetry – which necessarily implies that they must have the same dimension. Moreover,

$$\max\ \{\Omega_n\} = \Omega_{\lfloor J_0 \rfloor} \ . \tag{43}$$

where $\lfloor \bullet \rfloor$ is the floor function. We add that $\{\Omega_n\}$ is a multiset and so the subspace characterized by $n = \lfloor J_0 \rfloor$ may not be the only subspace of dimension max $\{\Omega_n\}$.

A. The generating function $G_{\mathcal{A},\Omega}(x)$ of $\{\Omega_n\}$

Theorem 3. *The multiset of integers $\{\Omega_n\}$ of a given $\mathcal{A}$ are the coefficients of the polynomial function $G_{\mathcal{A},\Omega}(x)$,* i.e.

$$G_{\mathcal{A},\Omega}(x) = \sum_{n=0}^{2J_0} \Omega_n \ x^n \tag{44a}$$

$$G_{\mathcal{A},\Omega}(x) \equiv \prod_{\alpha} \left[\Lambda_\alpha(x)\right]^{N_\alpha} \tag{44b}$$

where the index α runs over distinct elements $\{j_\alpha\}$ of $\mathcal{A}$, N_α is the multiplicity of j_α in $\mathcal{A}$, and $\Lambda_\alpha(x)$ is the polynomial function

$$\Lambda_\alpha(x) \equiv 1 + x + x^2 + \ldots + x^{2j_\alpha} \ . \tag{45}$$

Proof. To proof Eq. (44), it is advantageous to recall that each Ω_n characterizes the dimension of the invariant subspace of constant $M_n \in \mathcal{M}'$. The sum total of the dimension of these subspaces must necessarily be the same as the tensor space of the spaces of $\mathcal{A}$ elements. This yields us the identity

$$\prod_{\alpha} (2j_\alpha + 1)^{N_\alpha} = \sum_{n=0}^{2J_0} \Omega_n \ . \tag{46}$$

The LHS of Eq. (46) can be seen as the result of the limit

$$\lim_{q\to 1} \sum_{n=0}^{2J_0} \Omega_n \ q^n = \sum_{n=0}^{2J_0} \Omega_n \ . \tag{47}$$

We may thus seek to find the polynomial $G_{\mathcal{A},\Omega}(q)$ such that

$$\lim_{q\to 1} \ G_{\mathcal{A},\Omega}(q) = \prod_{\alpha} (2j_\alpha + 1)^{N_\alpha} \ . \tag{48}$$

It is manifest from Eq. (48) that the latter is satisfied if there exists a polynomial in q, $g_{\mathcal{A},\Omega,\alpha}(q)$, such that

$$\lim_{q\to 1} \ g_{\mathcal{A},\Omega,\alpha}(q) = 2j_\alpha + 1 \ . \tag{49}$$

$g_{\mathcal{A},\Omega,\alpha}(q)$ must have positive coefficients and be of degree $2j_\alpha$ so as to ensure that $\deg\left[\ G_{\mathcal{A},\Omega}(q)\right] = \deg\left[\ \sum_{n=0}^{2J_0} \Omega_n \ q^n\right] = 2J_0$. From these conditions, we conclude that all the coefficients of the polynomial $g_{\mathcal{A},\Omega,\alpha}(q)$ must be of value 1. But this implies that $g_{\mathcal{A},\Omega,\alpha}(q)$ is none other but the $q-$analog of the integer $2j_\alpha + 1$, *i.e.*

$$g_{\mathcal{A},\Omega,\alpha}(q) = \left[2j_\alpha + 1\right]_q \ , \tag{50}$$

where by definition[12],

$$[n]_q \equiv \frac{q^n - 1}{q - 1} \ . \tag{51}$$

Given that $(2j_\alpha + 1)$ is a positive integer, we have that

$$[2j_\alpha + 1]_q = \frac{q^{2j_\alpha+1} - 1}{q - 1} = 1 + q + q^2 + \ldots + q^{2j_\alpha} \ . \tag{52}$$

Accordingly, it follows from Eqs. (48), (49) and (50) that

$$G_{\mathcal{A},\Omega}(q) = \prod_\alpha \left([2j_\alpha + 1]_q \right)^{N_\alpha} \tag{53}$$

which gives Eq. (44b) after substituting q with x and making use of the definition in Eq. (45). As should be expected, $\deg [\, G_{\mathcal{A},\Omega}(q)] = \deg \left[\, \sum_{n=0}^{2J_0} \Omega_n \, q^n \right] = 2J_0$. □

Going back to the identity in Eq. (42), we conclude that $G_{\mathcal{A},\Omega}(q)$ is a *reciprocal* polynomial (see [13, Def. 3.6]). But that is not all: it is also *unimodal* (see [13, Def. 3.7]), given Eq. (43) and the fact that $\Omega_n \leq \Omega_{n'}$, for $n < n'$ and $0 \leq n, n' \leq \lfloor J_0 \rfloor$.

The generating function $G_{\mathcal{A},\Omega}(q)$ allows an alternative and more analytic expression for the integers Ω_n. As it happens, the following theorem holds:

Theorem 4. *Given the multiset $\mathcal{A}$, the integer $\Omega_{\mathcal{A},n}$ – as defined above – satisfies the relation,*

$$\boxed{\begin{aligned} \Omega_{\mathcal{A},n} &= \sum_{s_1=0}^{N_1} \cdots \sum_{s_\sigma=0}^{N_\sigma} (-1)^{s_1+s_2+\ldots+s_\sigma} \binom{N + n - 1 - \sum_{\alpha=1}^{\sigma}(2j_\alpha+1)s_\alpha}{N-1} \binom{N_1}{s_1} \cdots \binom{N_\sigma}{s_\sigma} \\ &= \sum_{\substack{\sum_{\alpha=1}^{\sigma}(2j_\alpha+1)s_\alpha \leq n \\ 0 \leq s_\alpha \leq N_\alpha}} (-1)^{s_1+s_2+\ldots+s_\sigma} \binom{N + n - 1 - \sum_{\alpha=1}^{\sigma}(2j_\alpha+1)s_\alpha}{N-1} \binom{N_1}{s_1} \cdots \binom{N_\sigma}{s_\sigma} \end{aligned}} \tag{54}$$

where σ is the number of distinct angular momenta present in the multiset $\mathcal{A}$; the index α runs over all distinct elements of $\mathcal{A}$; j_α is the $\alpha-th$ distinct element of $\mathcal{A}$ with multiplicity N_α and $0 \leq n \leq \sum_\alpha 2N_\alpha j_\alpha = 2J_0$.

Proof. The prove is very simple. Surely, from Eqs. (53) and (52) we may rewrite $G_{\mathcal{A},\Omega}(q)$ as

$$G_{\mathcal{A},\Omega}(q) = (1-q)^{-N} \prod_\alpha \left(1 - q^{2j_\alpha+1}\right)^{N_\alpha} \ . \tag{55}$$

After expanding the factor with negative exponent on the RHS as a negative binomial series (see for example [14]) and the other factors by the normal binomial theorem, and collecting terms with the same q^n, we get the result in Eq. (54). □

Eq. (54) is a generalization of a result in [15], and it is readily implementable computationally as to Eq. (28). As far as we know, Eq. (54) is also the most concise general analytical formula for calculating $\Omega_{\mathcal{A},n}$. The summation reduces to very few terms, given the condition that $\sum_{\alpha=1}^{\sigma}(2j_\alpha+1)s_\alpha \leq n$. We shall term Eq. (54) as the *generalized binomial method* for determining $\Omega_{\mathcal{A},n}$.

We mention that the generating function $G_{\mathcal{A},\Omega}(x)$ is similar to that provided in [2] – the difference being a constant of proportionality which depends exponentially on J_0. In particular, in [2] the generating function was arrived at by considering the statistical mechanical partition function $\mathcal{Z}_{\mathcal{A}}$ of a system of non-interacting spins coupled with a magnetic field. Indeed, if we let $x \to e^{-\beta}$, then

$$\mathcal{Z}_{\mathcal{A}} = e^{-\beta J_0}\; G_{\mathcal{A},\Omega}\left(e^{-\beta}\right) \ . \tag{56}$$

IV. THE MULTIPLICITIES $\{\lambda_\kappa\}$ AND CORRESPONDING GENERATING FUNCTION $G_{\mathcal{A},\lambda}(x)$

Given that the multiset $\mathcal{M}'$ (see Eq. (24)) is generated by $\mathcal{E}$ (see Eq. (1)) and $\Omega_{\mathcal{A},n}$ is the dimension of the invariant subspace of constant $M_n \in \mathcal{M}'$, then

$$\Omega_{\mathcal{A},n} = \sum_{\kappa=0}^{n} \lambda_{\mathcal{A},\kappa} \ , \qquad n \in \{0,1,\ldots,m\} \tag{57}$$

since if $J_l < |M_n|$, then $M_n \notin \mathcal{M}'^{[l]}$ (recall also that according to the convention employed in this article, $J_l > J_{l'}$ if $l < l'$). From Eq. (57), we derive that the multiplicity $\lambda_{\mathcal{A},\kappa}$ of the distinct element J_κ of $\mathcal{E}$ is therefore given by the relation

$$\lambda_{\mathcal{A},\kappa} = \Omega_{\mathcal{A},\kappa} - (1-\delta_{\kappa,0})\; \Omega_{\mathcal{A},\kappa-1} \ , \qquad \kappa \in \{0,1,\ldots,m\} \ . \tag{58}$$

This is the multi-restricted partition way of determining $\lambda_{\mathcal{A},\kappa}$. An elementary analysis of the recursive relation in Eq. (58) shows that the generating function $G_{\mathcal{A},\lambda}(x)$ for $\{\lambda_{\mathcal{A},\kappa}\}$ must be related to $G_{\mathcal{A},\Omega}(x)$ through the equation

$$\boxed{G_{\mathcal{A},\lambda}(x) = (1-x)\; G_{\mathcal{A},\Omega}(x) = \sum_{\kappa=0}^{2J_0+1} \lambda_{\mathcal{A},\kappa}\; x^\kappa} \ . \tag{59}$$

The positive coefficients of the generating function $G_{\mathcal{A},\lambda}(x)$ give the multiplicities $\{\lambda_{\mathcal{A},\kappa}\}$ in Eq. 1. Nevertheless, a few observations are due here: Unlike $G_{\mathcal{A},\Omega}(x)$, $\deg\left[G_{\mathcal{A},\lambda}(x)\right] = 2J_0+1$.

Moreover, exploiting the fact that $G_{\mathcal{A},\Omega}(x)$ is reciprocal and unimodal, it can be easily proved that

$$\lambda_\kappa = -\lambda_{2J_0+1-\kappa} \ , \tag{60}$$

which – together with the fact that $G_{\mathcal{A},\lambda}(x=1)=0, \forall \mathcal{A}$ – implies that out of the $2(J_0+1)$ terms of $G_{\mathcal{A},\lambda}(x)$, only an even number of them has nonzero coefficients. This number is precisely $2(m+1)$. We call those terms with zero coefficients the "sinking terms". We see that when $m = J_0$, there are no sinking terms in $G_{\mathcal{A},\lambda}(x)$. In general, the number of sinking terms one should expect is $2J_m$.

Similar to Eq. (54) of theorem 4, the following theorem can be stated:

Theorem 5. *Let $\mathcal{A}$ be a multiset of angular momenta and $\mathcal{E}$ its corresponding coupled representation multiset (see Eq. (1)). Then, the multiplicity $\lambda_{\mathcal{A},\kappa}$ of $J_\kappa \in \mathcal{E}$ is given by the following expression:*

$$\boxed{\lambda_{\mathcal{A},\kappa} = \sum_{\substack{\sum_{\alpha=1}^{\sigma}(2j_\alpha+1)s_\alpha \le \kappa \\ 0\le s_\alpha \le N_\alpha}} (-1)^{s_1+s_2+\ldots+s_\sigma} \begin{pmatrix} N+\kappa-2-\sum_{\alpha=1}^{\sigma}(2j_\alpha+1)s_\alpha \\ N-2 \end{pmatrix} \begin{pmatrix} N_1 \\ s_1 \end{pmatrix} \cdots \begin{pmatrix} N_\sigma \\ s_\sigma \end{pmatrix}} \tag{61}$$

where $0 \le \kappa \le m$.

The proof follows the same line of reasoning as that of theorem 4, so we leave it as an exercise for the reader. Eq. (61) is the generalized binomial approach in determining $\lambda_{\mathcal{A},n}$.

V. LIMIT CASE: UNIVARIATE SPIN SYSTEMS.

The limit case of "univariate spin" system, *i.e.* when $\mathcal{A} = \{j^N\}$, has been extensively discussed in the literature [2, 4, 10] lately. We show here that the same results can be easily deduced from the generalized equations (28) and (61). For convenience, we shall sometimes represent the integers $\{\Omega_{\mathcal{A},n}\}$ and $\{\lambda_{\mathcal{A},n}\}$ of $\mathcal{A} = \{j^N\}$ simply as $\Omega_{(2j+1)^{\otimes N},n}$ and $\lambda_{(2j+1)^{\otimes N},n}$, respectively.

Beginning with the integers $\{\Omega_{\mathcal{A},n}\}$, we see that in the case of a univariate spin system of N spin-j (or in general N angular momenta $j_1 = j_2 = \ldots = j_N = j$), Eq. (54) reduces to

$$\boxed{\Omega_{(2j+1)^{\otimes N},n} = \sum_{s=0}^{\lfloor \frac{n}{2j+1} \rfloor} (-1)^s \begin{pmatrix} N+n-1-(2j+1)s \\ N-1 \end{pmatrix} \begin{pmatrix} N \\ s \end{pmatrix}} \ . \tag{62}$$

That is, in the case of a univariate spin system, the integers $\{\Omega_{\mathcal{A},n}\}$ are none but the entries of an extended Pascal triangle (see [15] and compare Eq. (63) with [4, Eq. A.6]).

Proposition 1. *In terms of generalized hypergeomentric functions[16], it can be easily shown that Eq.* (62) *yields*

$$\boxed{\Omega_{(2j+1)^{\otimes N},n} = \begin{pmatrix} N+n-1 \\ n \end{pmatrix} {}_{2j+2}F_{2j+1}\left(\left. \begin{matrix} -N, -\frac{n}{2j+1}, \ldots, -\frac{n-i}{2j+1}, \ldots -\frac{n-2j}{2j+1} \\ -\frac{N+n-1}{2j+1}, \ldots, -\frac{N+n-1-i'}{2j+1}, \ldots -\frac{N+n-1-2j}{2j+1} \end{matrix} \right| 1 \right)} \ . \tag{63}$$

Proof. First, we note that we can rewrite the sum in Eq. (63) as $\sum_{s=-\infty}^{\infty} t_s$. Moreover, the term ratio t_{s+1}/t_s is

$$\frac{t_{s+1}}{t_s} = \frac{\prod_{i=0}^{2j} \left(s - \frac{n-i}{2j+1}\right)}{\prod_{i'=0}^{2j} \left(s - \frac{N+n-1-i'}{2j+1}\right)} \frac{s-N}{s+1} \ , \tag{64}$$

and

$$t_0 = \begin{pmatrix} N+n-1 \\ n \end{pmatrix} \ , \tag{65}$$

from which we infer Eq. (63). □

We draw the reader's attention to the fact that both the number of upper and lower parameters of ${}_{2j+2}F_{2j+1}$ can be reduced by one each any time the following condition is satisfied:

$$i' - i = N - 1 \ . \tag{66}$$

Those pair of upper and lower parameters characterized by i and i' satisfying Eq. (66) do not contribute to the generalized hypergeometric function in Eq. (63). Thus, in general, one may make the transformation ${}_{2j+2}F_{2j+1} \to {}_{2j+2-\epsilon}F_{2j+1-\epsilon}$, where

$$\epsilon = \dim R_{j,N,1} \ , \text{ where } R_{j,N,1} \equiv \{ (i',i) | \, i' - i = N-1 \ \wedge \ 0 \ \leq i, i' \in \mathbb{N}_0 \leq 2j \} \tag{67}$$

and where ${}_{2j+2-\epsilon}F_{2j+1-\epsilon}$ excludes all indexed pair of upper and lower parameters (i', i) of ${}_{2j+2-\epsilon}F_{2j+1-\epsilon}$ which are elements of $R_{j,N,1}$. Obviously, when $2j < N-1$ neither the number of upper nor lower parameters in Eq. (63) can be reduced.

When it comes to the multiplicities $\{\lambda_{\mathcal{A},n}\}$, we deduce from Eq. (61) that for univariate spin systems

$$\boxed{\lambda_{(2j+1)^{\otimes N},\kappa} = \sum_{s=0}^{\lfloor \frac{\kappa}{2j+1} \rfloor} (-1)^s \begin{pmatrix} N+\kappa-2-(2j+1)s \\ N-2 \end{pmatrix} \begin{pmatrix} N \\ s \end{pmatrix}} \ . \tag{68}$$

This result is identically the same as Polychronakos and Sfetsos'[2] (who arrived at their result employing common composition rules of $SU(2)$ representations), after one performs the transformation according to Eq. (22).

Proposition 2. *The summation in Eq.* (68) *yields*

$$\boxed{\lambda_{(2j+1)^{\otimes N},\kappa} = \begin{pmatrix} N+\kappa-2 \\ \kappa \end{pmatrix} {}_{2j+2}F_{2j+1}\left(\begin{matrix} -N, -\frac{\kappa}{2j+1}, \ldots, -\frac{\kappa-i}{2j+1}, \ldots - \frac{\kappa-2j}{2j+1} \\ -\frac{N+\kappa-2}{2j+1}, \ldots, -\frac{N+\kappa-2-i}{2j+1}, \ldots - \frac{N+\kappa-2-2j}{2j+1} \end{matrix} \middle| 1 \right)} \ . \tag{69}$$

We leave the proof to the reader. Just as in the case of $\Omega_{(2j+1)^{\otimes N},\kappa}$, Eq. (63), the hypergeometric function in Eq. (69) can be reduced to ${}_{2j+2-\epsilon'}F_{2j+1-\epsilon'}$ where

$$\epsilon' = \dim R_{j,N,2} \ , \ \text{where } R_{j,N,2} \equiv \{ (i',i) | \, i'-i = N-2 \ \wedge \ 0 \ \le i, i' \in \mathbb{N}_0 \le 2j \} \ , \tag{70}$$

on the condition that $2j \ge N-2$.

VI. APPLICATIONS AND FURTHER CONSIDERATIONS

A. A simple illustration

Let us consider a system of 4 spin-1 and 2 spin-1/2 particles. Thus, $\mathcal{A} = \left\{\frac{1}{2}, \frac{1}{2}, 1, 1, 1, 1\right\}$. This could be, for instance, the case of a monoradical ion with 6 nonzero spin nuclei: four of spin-1 and two of spin-1/2. The addition of these six angular momenta will yield a series of angular momenta (*i.e.* the Clebsch-Gordan series), which will constitute the set $\mathcal{E}$ (see Eq. (1)). The maximum of $\mathcal{E}$, J_0, according to Eq. (4) is $J_0 = 5$. In this particular case, $\nu_{\mathcal{A}} = -3$ and so the minimum of the Clebsch-Gordan series (*i.e.* of $\mathcal{E}$) is $J_m = 0$ (see Eqs. (5) and (6)). So, the distinct spin angular momenta we get from the addition of the six initial spin angular momenta are

$$J_0 = 5, \quad J_1 = 4, \quad J_2 = 3, \quad J_3 = 2, \quad J_4 = 1, \quad J_5 = 0 \ . \tag{71}$$

We now determine the multiplicity of the various distinct J_κ employing the three methods described above.

The multi-restricted partition method.

The multiplicity of the total angular momenta $\{J_\kappa\}$ can be determined by first determining the first 6 values of $\{\Omega_n\}$ (see Eqs. (28) and (44)). Employing Eq. (28), we may calculate $\Omega_0, \Omega_1, \ldots, \Omega_5$. For the sake of brevity, we illustrate here the calculation of Ω_4 and λ_4. First of all, we need to determine the set $P(\mathcal{A}; n = 4)$. In other words, we need to determine all the possible ways of writing the integer 4 as the sum of at most 6 nonzero integers, with the restriction that no more than 4 parts can be greater than 2. With this prescription, we find that

$$\begin{aligned} P(\mathcal{A}; n = 4) = \{A(4)\} &= \{(1,1,1,1,0,0), (2,1,1,0,0,0), (2,2,0,0,0,0)\} \\ &= \{(1,1,1,1), (2,1,1), (2,2)\} \; . \end{aligned} \tag{72}$$

Let us call $(1,1,1,1) \equiv A_x(4)$, $(2,1,1) \equiv A_y(4)$ and $(2,2) \equiv A_z(4)$. Then, $\widetilde{A}_x(4) = \{1\}$, $\widetilde{A}_y(4) = \{1,2\}$ and $\widetilde{A}_z(4) = \{2\}$ and so from Eq. (28) it follows that

$$\Omega_4 = \begin{pmatrix} 6 \\ 4 \end{pmatrix} + \begin{pmatrix} 4 \\ 1 \end{pmatrix} \begin{pmatrix} 5 \\ 2 \end{pmatrix} + \begin{pmatrix} 4 \\ 2 \end{pmatrix} = 61 \tag{73}$$

where the first, second and third terms are the contributions from $A_x(4), A_y(4)$ and $A_z(4)$, respectively.

According to Eq. (58), in order to determine λ_4 we need Ω_3. For $n = 3$ here,

$$P(\mathcal{A}; n = 3) = \{(1,1,1), (1,2)\} \; . \tag{74}$$

And so from Eq. (28) we find that

$$\Omega_3 = \begin{pmatrix} 6 \\ 3 \end{pmatrix} + \begin{pmatrix} 4 \\ 1 \end{pmatrix} \begin{pmatrix} 5 \\ 1 \end{pmatrix} = 40 \; , \tag{75}$$

which means $\lambda_4 = 21$, according to Eq. (58).

The generalized binomial approach.

For the same reason as above, we only illustrate here how to calculate Ω_4 and λ_4 using the generalized binomial method.

With $\mathcal{A} = \{\frac{1}{2}^2, 1^4\}$, we only have two distinct momenta. Let us associate the dummy variable s_1 with $j_\alpha = 1/2$ and s_2 with $j_\alpha = 1$. Then, from Eq. (54) it follows that

$$\Omega_4 = \sum_{\substack{2s_1+3s_2\leq 4\\ 0\leq s_1\leq 2,\ 0\leq s_2\leq 4}} (-1)^{s_1+s_2} \binom{9-2s_1-3s_2}{5} \binom{2}{s_1} \binom{4}{s_2} . \tag{76}$$

The only pairs of s_1 and s_2 which satisfy the condition $2s_1 + 3s_2 \leq 4$ are $(s_1, s_2) \in \{(0,0), (1,0), (2,0), (0,1)\}$, which correspond to four summands:

$$\Omega_4 = \binom{9}{5} - \binom{7}{5}\binom{2}{1} + \binom{5}{5}\binom{2}{2} - \binom{6}{5}\binom{4}{1} = 61 . \tag{77}$$

Similarly, in accord with Eq. (61) we find that

$$\begin{aligned} \lambda_4 &= \sum_{\substack{2s_1+3s_2\leq 4\\ 0\leq s_1\leq 2,\ 0\leq s_2\leq 4}} (-1)^{s_1+s_2} \binom{8-2s_1-3s_2}{4} \binom{2}{s_1} \binom{4}{s_2} , \\ &= \binom{8}{4} - \binom{6}{4}\binom{2}{1} + \binom{4}{4}\binom{2}{2} - \binom{5}{4}\binom{4}{1} = 21 . \end{aligned} \tag{78}$$

The generating function method.

The numbers Ω_n are easily determined using the generating function given in Eq. (44). In this case, we have

$$G_{\mathcal{A},\Omega}(x) = (1+x)^2(1+x+x^2)^4 = \sum_{n=0}^{2J_0} \Omega_n\ x^n . \tag{79}$$

But,

$$(1+x)^2(1+x+x^2)^4 = x^{10}+6x^9+19x^8+40x^7+61x^6+70x^5+61x^4+40x^3+19x^2+6x+1 . \tag{80}$$

From Eq. (80) we see that $\Omega_4 = 61$, as calculated earlier according to both the multi-restricted partition and generalized binomial methods.

We are now ready to calculate the multiplicity of $J_0, J_1, \ldots, J_m$. From Eqs. (58) and (80) we have that

$$\lambda_0 = 1\ ,\ \lambda_1 = 5\ ,\ \lambda_2 = 13\ ,\ \lambda_3 = 21\ ,\ \lambda_4 = 21\ ,\ \lambda_{m=5} = 9. \tag{81}$$

The same result can be obtained if we make use of the generating function $G_{\mathcal{A},\lambda}(x)$ in Eq. (59). Indeed, in this case,

$$G_{\mathcal{A},\lambda}(x) = 1 + 5x + 13x^2 + 21x^3 + 21x^4 + 9x^5 - 9x^6 - 21x^7 - 21x^8 - 13x^9 - 5x^{10} - x^{11} \ , \quad (82)$$

from which we observe that $\lambda_4 = 21$ as previously determined. Note the absence of sinking terms in Eq. (82).

Adopting a notation similar to that in [9], we may write the Clebsch-Jordan decomposition series from the coupling of 2 spin-1/2 and 4 spin-1 angular momenta as

$$\mathbf{2}^{\otimes 2} \otimes \mathbf{3}^{\otimes 4} = \mathbf{11} \oplus 5 \cdot \mathbf{9} \oplus 13 \cdot \mathbf{7} \oplus 21 \cdot \mathbf{5} \oplus 21 \cdot \mathbf{3} \oplus 9 \cdot \mathbf{1} \ , \quad (83)$$

where the integers in boldface represent the dimension of a representation; the integers multiplying the boldface integers on the right-hand side are the multiplicities. The Hilbert space $\mathcal{H}$ of the whole system is of dimension 324. With the above calculated multiplicities, one can verify that

$$\sum_{\kappa=0}^{m} \lambda_\kappa \ (2J_\kappa + 1) = 324 \ , \quad (84)$$

as should be expected.

In passing, we note that term symbols in atomic physics can be readily determined in similar manner.

B. Isotropic tensors, Riordan numbers, Catalan numbers and lattice paths

A rank N tensor in a D dimensional Euclidean space, $T_{D,N}$, has the same irreducible representations as the composition of N spin-$\left(\frac{D-1}{2}\right)$ representations in $SU(2)$. The dimension of the basis set of its irreducible components are given by the coefficients of $G_{\mathcal{A},\lambda}(x)$ – here, $\mathcal{A} = \{(\frac{D-1}{2})^N\}$. For example, the irreducible components of a rank 10 tensor in 3−dimensional space have the same dimensions as the irreducible components of the coupling of 10 spin-1 particles:

$$\mathbf{3}^{\otimes 10} = 603 \cdot \mathbf{1} \oplus 1585 \cdot \mathbf{3} \oplus 2025 \cdot \mathbf{5} \oplus 1890 \cdot \mathbf{7} \oplus 1398 \cdot \mathbf{9} \oplus 837 \cdot \mathbf{11} \oplus 405 \cdot \mathbf{13} \oplus 155 \cdot \mathbf{15} \oplus 45 \cdot \mathbf{17} \oplus 9 \cdot \mathbf{19} \oplus \mathbf{21} \ . \quad (85)$$

The irreducible component $\mathbf{1}$ is the S state, thus it represents the totally symmetric part of the tensor. The multiplicities of the various representations are also the characters of the rotation group in that representation; which means that the totally symmetric representation of a rank 10 tensor in $D = 3$ has a basis set of dimension 603. These are also the number of independent isotropic tensor isomers present in $T_{3,10}$.

Isotropic tensors are crucial when determining rotational averages of observables, and they are particularly useful in essentially all studies related to matter-radiation interaction[3]. To correctly perform rotational averages, it is of fundamental importance to know beforehand how many linearly independent isotropic isomers there are. As we have shown above, one can use CGD to determine this number by making use of the relations derived above. We also mention that the number of independent isotropic isomers in $D = 3$ of rank $n = 0, 1, 2, \ldots$ are collectively called Motzkin sum [17] or Riordan[18] numbers and their generating function is[17]

$$\begin{aligned} G_{3D}(x) &= \frac{1}{2x}\left(1 - \sqrt{\frac{1-3x}{1+x}}\right) \\ &= 1 + x^2 + x^3 + 3x^4 + 6x^5 + 15x^6 + 36x^7 + 91x^8 + 232x^9 + \ldots \end{aligned} \tag{86}$$

Following similar arguments, a rank N tensor in $D = 2$ has the same irreducible representations as a multispinor of rank N. We thus understand from Eq. (5) that there will be no isotropic tensors when N is odd since in that case $J_m = 1/2$. In fact, any tensor of rank odd (even) N in $D = 2$ is fermionic (bosonic). In particular, the number of independent isotropic tensors in $T_{2,N=2n}$, $n \in \mathbb{N}_0$ is given by $\lambda_{2^{\otimes N},N/2}$, *i.e.* the multiplicity of the $N/2-$th distinct element of the associated multiset $\mathcal{E}$ obtained after coupling N spin-(1/2)s. We thus infer from Eqs. (28) or (41) and (58) that

$$\begin{aligned} \lambda_{2^{\otimes N},N/2} &= \binom{N}{N/2} - \binom{N}{N/2-1} = \frac{1}{N/2+1}\binom{N}{N/2} \\ &\equiv C_{N/2} \end{aligned} \tag{87}$$

where we have recognized the integers $\{\lambda_{2^{\otimes N},N/2}\}$ as being the well-known Catalan numbers, $\{C_{N/2}\}$. More appropriately, the $\{C_{N/2}\}$ are the so-called aerated Catalan numbers[19] since $C_{N/2} = \lambda_{2^{\otimes N},N/2} = 0$ for odd N. The generating function $G_{2D}(x)$ for the integers $\{\lambda_{2^{\otimes N},N/2}\}$

is of the form[19]

$$
\begin{aligned}
G_{2D}(x) &= \frac{1}{2x^2}\left(1-\sqrt{1-4x^2}\right) \\
&= \sum_{N=0}^{\infty} \lambda_{2^{\otimes N},N/2}\; x^N \\
&= 1 + x^2 + 2x^4 + 5x^6 + 14x^8 + 42x^{10} + 132x^{12} + \ldots
\end{aligned}
\tag{88}
$$

where we notice the appearance of only even terms (corresponding to bosonic states). From Eqs. (58) and (41) we have that

$$
\lambda_{2^{\otimes N},\kappa} = \frac{N+1-2\kappa}{N+1-\kappa}\binom{N}{\kappa}\;, \qquad \kappa \in \{0,1,\ldots,m=N/2\}\;. \tag{89}
$$

Drawing on the results obtained with the generalized binomial method (Eq. (69)), we end up with the following identities:

$$
\lambda_{2^{\otimes N},\kappa} = \binom{N+\kappa-2}{\kappa}\; {}_3F_2\left(\left.\begin{matrix} -N, -\frac{\kappa}{2}, -\frac{\kappa-1}{2} \\ -\frac{N+\kappa-2}{2}, -\frac{N+\kappa-3}{2} \end{matrix}\right| 1\right) \tag{90a}
$$

$$
C_\nu = \binom{3\nu-2}{\nu}\; {}_3F_2\left(\left.\begin{matrix} -2\nu, -\frac{\nu}{2}, -\frac{\nu-1}{2} \\ -\frac{3\nu-2}{2}, -\frac{3\nu-3}{2} \end{matrix}\right| 1\right) \tag{90b}
$$

$$
R_\nu = \binom{2\nu-2}{\nu}\; {}_4F_3\left(\left.\begin{matrix} -\nu, -\frac{\nu}{3}, -\frac{\nu-1}{3}, -\frac{\nu-2}{3} \\ -\frac{2\nu-2}{3}, -\frac{2\nu-3}{3}, -\frac{2\nu-4}{3} \end{matrix}\right| 1\right)\;. \tag{90c}
$$

where $\nu \in \{0,1,2,3,\ldots\}$, and $\{C_\nu\}$ and $\{R_\nu\}$ are the Catalan and Riordan numbers, respectively. The easiness with which these identities were derived from Eq. (69) is worth noting.

Moreover, the popping up of Catalan and Riordan numbers in these limit cases is very telling. They strike at a deeper connection between enumerative combinatorics and CGD. In particular, given that the Catalan and Riordan numbers are related to some counting problems in lattice paths (or random walks), it is only fair to ask if every CGD problem can be re-interpreted as a lattice path problem. We call this the "CGD-lattice path duality" problem. A quick analysis of the issue seems to weigh in for the affirmative, but a more formal prove need to be given. For example, a very important characteristic of the lattice paths counted by Catalan triangle is that at each step one can move up ($\Delta = +1$) or down ($\Delta = -1$), while with paths counted by the Motzkin numbers one can stay put ($\Delta = 0$)

besides the up and down moves[19]. These restricted variations of one's position ($\Delta = 0, \pm 1$) are reminiscent of transitions between quantum states due to the system's interaction with an environment. And the steps may be interpreted as multiples of the transition timescale. So far, the random walk connection has been discussed in the literature only in the case of N spin-js [2] (see also [10, sec. 3.2]). Extending this interpretation to the generalized multivariate $\mathscr{A}$ may set the ground for a formal way of mapping lattice gas models to spin dynamics, which might prove to be computationally advantageous (especially in EPR and NMR simulations). Movassagh and Shor's[20] recent application of lattice paths, among other mathematical techniques, to prove the violation of the area law in some $D = 1$ models is particularly interesting and could offer some insights as to how to achieve the lattice gas-spin dynamics mapping in general.

C. CGD and exclusion processes on graphs.

An even more interesting application of our results has to do with exclusion processes (see [4] and references therein). In [4], Mendonça considering a simple symmetric exclusion process (SSEP) on a graph (with characteristics defined in the article), showed that its infinitesimal generator $\mathcal{H}$ is equivalent to the Hamiltonian of the isotropic Heisenberg spin-1/2 quantum ferromagnet on the same graph. The conservation of particles in the process implied that $\mathcal{H}$ can be blocked-diagonalized, with each invariant subspace conserving n particles. The peculiar characteristic of the SSEP model is that each vertex of the graph can accommodate not more than a particle at a time. We can thus assign to each vertex the number 0 (when it is empty) or 1 (when occupied). Here, these values are literally counting the number of particles in that vertex, and the fact that we can associate or 0 or 1 to each vertex is an indication that each vertex in this model is acting like a spin-$1/2$ in $SU(2)$. The degeneracy of the eigenstate characterized by a total number of particles n is simply $\Omega_{2^{\otimes N},n}$ – which makes perfect sense if we recall how the expression for $\Omega_{\mathscr{A},n}$ was derived under the multi-restricted partition method in Sec. III.

Our results can be used to analyze the degeneracy of more complicated graphs. One may consider for example a graph of N vertices, whereby the vertex i can accommodate at most a given finite number n_i of particles (or assume a finite number of states which can be ordered). Say we represent the vertices as the multiset $\{n_1^{d_1}, \ldots, n_\sigma^{d_\sigma}\}$, where $\sum_{\alpha=1}^{\sigma} d_\alpha = N$.

Then the degeneracy of the invariant subspace characterized by n particles is $\Omega_{\mathcal{A},n}$ (Eq. (54)), where $\mathcal{A} = \{(\frac{n_1}{2})^{d_1}, \ldots, (\frac{n_\sigma}{2})^{d_\sigma}\}$. Here, we should expect the generator of the process to be equivalent to the Hamiltonian of some phenomenon related to some paraparticle.

In the special case whereby each vertex can be occupied by an infinite number of particles, we lose exclusion and $\mathcal{A} = \{\infty^N\}$ (*i.e.* the process becomes equivalent to the composition of N spin-∞ representations of $SU(2)$). We thus end up with vertices which act like bosonic eigenstates. And the generating function for $\Omega_{\infty^{\otimes N},n}$ is

$$(1 + x + x^2 + x^3 + \ldots)^N = \sum_{n=0}^{\infty} \Omega_{\infty^{\otimes N},n}\ x^n\ . \tag{91}$$

But $(1 + x + x^2 + x^3 + \ldots)^N = (1 - x)^{-N}$, from which we derive that

$$\Omega_{\infty^{\otimes N},n} = \begin{pmatrix} N + n - 1 \\ n \end{pmatrix} \tag{92}$$

and,

$$\lambda_{\infty^{\otimes N},\kappa} = \begin{pmatrix} N + \kappa - 2 \\ \kappa \end{pmatrix}\ . \tag{93}$$

Comparing Eqs. (92) and (93) with Eqs. (63) and (69), respectively, we see that both hypergeometric functions in the latter pair satisfy the limit: $\lim_{j\to\infty}\ {}_{2j+2}F_{2j+1}\left(\begin{smallmatrix}\cdots\\ \cdots\end{smallmatrix}\middle|\, 1\right) = 1$.

D. Number theory: Multi-restricted partition

The main results of this paper can also be applied to some very interesting problems in enumerative combinatorics, and this should come as no surprise. We briefly consider below the example of multi-restricted partition of a given integer.

Say $p(n_1^{d_1}, \ldots, n_\sigma^{d_\sigma}; n)$ the number of partitions of the integer n into at most $N = \sum_{\alpha=1}^{\sigma} d_\alpha$ parts, with d_α parts being at most n_α ($\alpha \in \{1, \ldots, \sigma\}$). For example, $p(2^5, 4^3, 5^4; 16)$ is the number of partitions of the integer 16 into at most 12 parts, with 5 of them being at most of value 2, 3 being at most 4 and 4 being at most 5.

To determine $p(n_1^{d_1}, \ldots, n_\sigma^{d_\sigma}; n)$, we need to make the following distinction:

The number zero is an admissible part. When this is so, we shall represent $p(n_1^{d_1}, \ldots, n_\sigma^{d_\sigma}; n)$ as $p_0(n_1^{d_1}, \ldots, n_\sigma^{d_\sigma}; n)$ (and reserve the former for the case whereby zero is not an admissible part). In this case, $p_0(n_1^{d_1}, \ldots, n_\sigma^{d_\sigma}; n)$ is exactly $\Omega_{\mathcal{A},n}$ of the CGD of

N spins: d_1 being spin-$\frac{n_1}{2}$ representations of $SU(2)$, d_2 being spin-$\frac{n_2}{2}$, and so on. Thus, $\mathcal{A} = \{ \left(\frac{n_1}{2}\right)^{d_1}, \ldots, \left(\frac{n_\sigma}{2}\right)^{d_\sigma} \}$, and given that $p_0(n_1^{d_1}, \ldots, n_\sigma^{d_\sigma}; n) = \Omega_{\mathcal{A},n}$ we can use any of the three methods discussed above to determine $p_0(n_1^{d_1}, \ldots, n_\sigma^{d_\sigma}; n)$. Resorting to the generating function approach for example, we can – following Eq. (53) – state that

$$\boxed{\prod_{\alpha=1}^{\sigma} \left(\sum_{i=0}^{n_\alpha} x^i \right)^{d_\alpha} = \sum_{n=0}^{\sum_{\alpha=1}^{\sigma} d_\alpha n_\alpha} p_0(n_1^{d_1}, \ldots, n_\sigma^{d_\sigma}; n)\ x^n} \ . \tag{94}$$

For instance,

$$(1+x+x^2)^5(1+x+x^2+x^3+x^4)^3(1+x+x^2+x^3+x^4+x^5)^4 = \sum_{n=0}^{42} p_0(2^5, 4^3, 5^4; n)\ x^n \ . \tag{95}$$

Certainly, it also follows from Eq. (42) that

$$p_0(n_1^{d_1}, \ldots, n_\sigma^{d_\sigma}; n) = p_0\left(n_1^{d_1}, \ldots, n_\sigma^{d_\sigma}; \sum_{\alpha=1}^{\sigma} d_\alpha n_\alpha - n \right) \ . \tag{96}$$

The limit case $p(n_1^{d_1}; n)$ is what is termed *restricted partition* in the literature, and is usually indicated as $p(n_1, d_1; n)$ (see for example [13, Chap. 3]), with the significant difference that the integer zero is not an admissible part according to the latter. To differentiate between the two, we shall write $p_0(n_1, d_1; n)$ to indicate that zero is an admissible part, and simply $p(n_1, d_1; n)$ when it is not. It is not hard to see that $p_0(c, d; n) = \Omega_{(c+1)^{\otimes d}, n}$, thus we may directly employ Eq. (62) or (63). Indeed, based on the latter, we have that

$$\boxed{p_0(c, d; n) = \binom{d+n-1}{n} {}_{c+2}F_{c+1}\left(\left. \begin{matrix} -d, -\frac{n}{c+1}, \ldots, -\frac{n-i}{c+1}, \ldots - \frac{n-c}{c+1} \\ -\frac{d+n-1}{c+1}, \ldots, -\frac{d+n-1-i'}{c+1}, \ldots - \frac{d+n-1-c}{c+1} \end{matrix} \right| 1 \right)} \ . \tag{97}$$

Needless to say, the generating function for $\{p_0(c, d; n)\}$ is

$$\left(1 + x + x^2 + \ldots + x^c\right)^d = \sum_{n=0}^{dc} p_0(c, d; n)\ x^n \ . \tag{98}$$

Zero is not an admissible part We can still map this case to the case in which zero is an admissible part. All we need to do is to make the following transformations: $n_\alpha \longrightarrow n_\alpha - 1$, $n \longrightarrow n - N$. Therefore,

$$p(n_1^{d_1}, \ldots, n_\sigma^{d_\sigma}; n) = p_0\left((n_1 - 1)^{d_1}, \ldots, (n_\sigma - 1)^{d_\sigma}; n - N\right) \ . \tag{99}$$

All we have done is just a rescaling of the integers. Moreover, for the common restricted partition we can write

$$p(c, d; n) = p_0(c - 1, d; n - d) \tag{100}$$

Note that n in Eqs. (99) and (100) is $N \leq n \leq \sum_{\alpha=1}^{\sigma} d_\alpha n_\alpha$.

E. A simple statistical application: the rolling of N dices

The equations derived in the last section may find application in a number of statistical problems. We illustrate here just one. Suppose we throw N dices. We ask: what is the probability $P_6(N,n)$ that the sum of the outcomes is n? Certainly,

$$P_6(N,n) = \frac{1}{6^N} \times p(6,N;n) = \frac{1}{6^N} \times p_0(5,N;n') \tag{101a}$$

$$= \frac{1}{6^N} \times \begin{pmatrix} n-1 \\ n' \end{pmatrix} {}_7F_6\left(\left. \begin{matrix} -N, -\frac{n'}{6}, -\frac{n'-1}{6}, -\frac{n'-2}{6}, -\frac{n'-3}{6}, -\frac{n'-4}{6}, -\frac{n'-5}{6} \\ -\frac{n-1}{6}, -\frac{n-2}{6}, -\frac{n-3}{6}, -\frac{n-4}{6}, -\frac{n-5}{6}, -\frac{n-6}{6} \end{matrix} \right| 1 \right) . \tag{101b}$$

where $n' \equiv n - N$, and obviously $N \leq n \leq 6N$, where use has been made of Eq. (97).

Consider now the coupling of N spin-5/2s. $P_6(N,n)$ can be re-interpreted here as the probability that a randomly chosen $z-$eigenvalue in the coupled representation is of value M_n.

VII. CONCLUSION

The seemingly unavailing effort of mapping the $z-$eigenvalues of $SU(2)$ spins to the set of natural numbers allows one to place the composition of an arbitrary multiset of $SU(2)$ spins into the context of enumerative combinatorics, as we have shown above. The striking gain here is a very general re-interpretation of the CGD conundrum which allows one to solve the problem in diverse ways and in general terms. This is a feat hardly achievable without the $z-$eigenvalues-positive integers mapping. That this simplifies analytic CGD is remarkable in its own right. But in retrospect, we note that the hallmark of quantum mechanics is the quantization of observables – to which we can always associate a countable set or multiset, in general. And this fact constitutes the very first instance of connection between quantum mechanics and enumerative combinatorics. Our work shows how and why EC can be a valid mathematical toolkit to be included in the arsenal of mathematical techniques employed in quantum mechanics. The proving of the minimum J_m (sec. II), and the three methods outlined in secs. III and IV for analytic CGD are all indicators of its relevance to quantum mechanics. The connection between EC and lattice paths may also offer interesting approaches to the study of spin dynamics, and bridge the gap between the latter and lattice gas models. We believe this to be an important effort worth pursuing.

Moreover, besides the applications discussed above, the analytical expressions presented here may find very broad applications in diverse fields – from quantum algebra, quantum information, quantum statistical mechanics to group theory, not to mention quantum chemistry, and in particular quantum magnetic resonance (as we shall show in an upcoming article). The analytical method presented here may also be employed to study spin-orbit coupling in many-body systems.

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[1] R. Stanley, *Enumerative Combinatorics*, Cambridge Studies in Advanced Mathematics, Vol. 1 (Cambridge University Press, 2011).

[2] A. P. Polychronakos and K. Sfetsos, Nuclear Physics B **913**, 664 (2016).

[3] D. L. Andrews and T. Thirunamachandran, The Journal of Chemical Physics **67**, 5026 (1977).

[4] J. R. G. Mendonça, Journal of Physics A: Mathematical and Theoretical **46**, 295001 (2013).

[5] E. R. Livine and D. R. Terno, Phys. Rev. A **72**, 022307 (2005).

[6] E. Cohen, T. Hansen, and N. Itzhaki, Scientific Reports **6** (2016).

[7] R. N. Zare, *Angular momentum: understanding spatial aspects in chemistry and physics* (Wiley, 1988).

[8] P. W. Atkins and R. S. Friedman, *Molecular Quantum Mechanics* (OUP Oxford, 2011).

[9] C. K. Zachos, Modern Physics Letters A **07**, 1595 (1992).

[10] T. L. Curtright, T. S. Van Kortryk, and C. K. Zachos, Physics Letters A **381**, 422 (2017).

[11] D. Singh, A. Ibrahim, T. Yohanna, and J. Singh, Novi Sad Journal of Mathematics **37**, 73 (2007).

[12] V. Kac and P. Cheung, *Quantum Calculus*, Universitext (Springer New York, 2001).

[13] G. E. Andrews, *The theory of partitions*, Encyclopedia of Mathematics and its Applications (Cambridge University Press, 1976).

[14] J. Riordan, *Introduction to Combinatorial Analysis*, Dover Books on Mathematics (Dover Publications, 2002).

[15] R. C. Bollinger, Mathematics Magazine **66**, 87 (1993).

[16] W. Koepf, *Hypergeometric Summation*, Universitext (Springer London, 2014).

[17] S. Paolucci, *Continuum Mechanics and Thermodynamics of Matter* (Cambridge University Press, 2016); E. W. Weisstein, "Isotropic tensor," From *MathWorld*–A Wolfram Web Resource. `http://mathworld.wolfram.com/IsotropicTensor.html`, [Online; accessed 22-May-2017].

[18] F. R. Bernhart, Discrete Mathematics **204**, 73 (1999).

[19] R. Donaghey and L. W. Shapiro, Journal of Combinatorial Theory, Series A **23**, 291 (1977).

[20] R. Movassagh and P. W. Shor, arXiv preprint arXiv:1408.1657 (2014), 10.1073/pnas.1605716113.