

Completing graphs to metric spaces

Andrés Aranda⁴

*Computer Science Institute of Charles University (IUUK)
Charles University
Prague, Czech Republic*

David Bradley-Williams⁵

*Mathematisches Institut
Heinrich-Heine-Universität
Düsseldorf, Germany*

Eng Keat Hng⁶

*Department of Mathematics
London School of Economics and Political Science
London, UK*

Jan Hubička^{1,2,7}

*Computer Science Institute of Charles University (IUUK)
Charles University
Prague, Czech Republic*

Miltiadis Karamanlis⁸

*Department of Mathematics
National Technical University of Athens
Athens, Greece*

Michael Kompatscher^{3,9}

*Institut für Computersprachen, Theory and Logic Group
Technische Universität Wien
Wien, Austria*

Matěj Konečný^{2,10}

*Charles University
Prague, Czech Republic*

Micheal Pawliuk¹¹

*Department of Mathematics and Statistics
University of Calgary
Calgary, Canada*

Abstract

We prove that certain classes of metrically homogeneous graphs omitting triangles of odd short perimeter as well as triangles of long perimeter have the extension property for partial automorphisms and we describe their Ramsey expansions.

This is extended variant of an extended abstract accepted to Eurocomb 2017 which contains proofs of the main statements.

Keywords: Ramsey class, metric space, homogeneous structure, metrically homogeneous graph, extension property for partial automorphisms

¹ Supported by grant ERC-CZ LL-1201 of the Czech Ministry of Education.

² Supported by grant CE-ITI P202/12/G061 of GAČR.

³ Supported by the grant P27600 of the Austrian Science Fund (FWF).

⁴ Email: andres.aranda@gmail.com

⁵ Email: david.bradley-williams@uni-duesseldorf.de

⁶ Email: e.hng@lse.ac.uk

⁷ Email: hubicka@iuuk.mff.cuni.cz

⁸ Email: kararemit@gmail.com

⁹ Email: michael@logic.at

¹⁰ Email: matejkon@gmail.com

¹¹ Email: mpawliuk@ucalgary.ca

1 Introduction

Given integers δ , K and C we consider the class $\mathcal{A}_{K,C}^\delta$ of finite metric spaces $\mathbf{M} = (M, d)$ with integer distances such that

- $d(x, y) \leq \delta$ (the parameter δ is the *diameter* of $\mathcal{A}_{K,C}^\delta$) for every $x, y \in M$;
- the perimeter of every triangle is less than C ; and
- if a triangle has odd perimeter, then it is at least $2K + 1$.

Here a *triangle* is any triple of distinct vertices $u, v, w \in M$ and its *perimeter* is $d(u, v) + d(v, w) + d(w, u)$. We call parameters δ , K and C *acceptable* if it holds that $\delta \geq 2$, $1 \leq K \leq \delta$ and $2\delta + K < C \leq 3\delta + 1$ (this covers acceptable parameters in sense of [Che] with the exception of bipartite graphs). Our main results can be stated as follows:

Theorem 1.1 *The class $\mathcal{A}_{K,C}^\delta$ has the extension property for partial automorphisms (EPPA) for every acceptable choice of δ , K and C .*

Theorem 1.2 *The class $\vec{\mathcal{A}}_{K,C}^\delta$ of free orderings of metric spaces in $\mathcal{A}_{K,C}^\delta$ is a Ramsey class for every acceptable choice of δ , K and C .*

We thus give new examples of classes of metric spaces which are Ramsey when enriched by linear orders and have EPPA, extending the lists of such classes obtained in [Neš07, NVT10, HN16] (for Ramsey classes) and [Sol05, Ver08, Con15] (for the EPPA). While these properties were historically treated independently, we show that both results follow from general constructions (stated as Theorem 1.4 and 1.3 below) and an analysis of an algorithm to fill the gaps in incomplete structures, thus turning them into metric spaces. Before going through the details, we will explain our motivation to consider these rather special-looking classes $\mathcal{A}_{K,C}^\delta$ and all the necessary notions.

It has been observed by Nešetřil [Neš89, Neš05, KPT05] that Ramsey classes have the so-called *amalgamation property*, which in turn implies the existence of a homogeneous Fraïssé limit. Therefore potential candidates for Ramseyness are usually taken from a well known classification programme of homogeneous structures (see [Che] for references). Cherlin recently extended this list by a probably complete classification of classes with metrically homogeneous graphs as limits, where the $\mathcal{A}_{K,C}^\delta$ play an important role. For us it is particularly interesting, because the standard completion algorithm for metric spaces (which we introduce below) generally fails to produce metric spaces satisfying the C bound. More complete results in this direction will appear in [ABWH⁺17].

An *edge-labelled graph* \mathbf{G} is a pair (G, d) where G is the *vertex set* and

d is a partial function from G^2 to \mathbb{N} such that $d(u, v) = 0$ if and only if $u = v$, and either $d(u, v), d(v, u)$ are both undefined or $d(u, v) = d(v, u)$ for each pair of vertices u, v (equivalently, one could think of edge-labelled graph as a relational structure). A pair of vertices u, v such that $d(u, v)$ is defined is called an *edge* of \mathbf{G} . We also call $d(u, v)$ the *length of the edge* u, v . A graph \mathbf{G} is *complete* if every pair of vertices forms an edge and \mathbf{G} is called a *metric space* if the *triangle inequality* holds, that is $d(u, w) \leq d(u, v) + d(v, w)$ for every $u, v, w \in G$. A graph $\mathbf{G} = (G, d)$ is *metric* if there exists a metric space $\mathbf{M} = (G, d')$ such that $d(u, v) = d'(u, v)$ for every edge u, v of \mathbf{G} . Such a metric space \mathbf{M} is also called a *metric completion* of \mathbf{G} .

Given an edge-labelled graph $\mathbf{G} = (G, d)$ the *path distance* $d^+(u, v)$ of u and v is the minimum $\ell = \sum_{1 \leq i \leq n-1} d(u_i, u_{i+1})$ over all possible sequences of vertices for which $u_1 = u$, $u_n = v$ and $d(u_i, u_{i+1})$ is defined for every $i \leq n-1$. If there is no such sequence we put $\ell = \infty$. It is well known that a connected graph $\mathbf{G} = (G, d)$ is metric if and only if $d(u, v) = d^+(u, v)$ for every edge of \mathbf{G} . In this case (G, d^+) is a metric completion of \mathbf{G} which we refer to as the *shortest path completion*. This completion-algorithm also leads to an easy characterisation of metric graphs. The graph \mathbf{G} is metric if and only if it does not contain a *non-metric cycle*. This is an edge-labelled graph cycle such that one distance in the cycle is greater than the sum of the remaining distances. See e.g. [HN16] for details. In this paper we are going to introduce a generalisation of the shortest path completion.

1.1 Extension property for partial automorphisms

Given two edge-labelled graphs $\mathbf{G} = (G, d)$ and $\mathbf{G}' = (G', d')$ a *homomorphism* $\mathbf{G} \rightarrow \mathbf{G}'$ is a function $f : G \rightarrow G'$ such that $d(x, y) = d'(f(x), f(y))$ whenever $d(x, y)$ is defined. A homomorphism f is an *embedding* (or *isometry*) if f is one-to-one and $d(x, y) = d'(f(x), f(y))$ whenever either side makes sense. A surjective embedding is an *isomorphism* and an *automorphism* is an isomorphism $\mathbf{G} \rightarrow \mathbf{G}$. A graph \mathbf{G} is an (*induced*) *subgraph* of \mathbf{H} if the identity mapping is an embedding $\mathbf{G} \rightarrow \mathbf{H}$.

A *partial automorphism* of an edge-labelled graph \mathbf{G} is an isomorphism $f : \mathbf{H} \rightarrow \mathbf{H}'$ where \mathbf{H}, \mathbf{H}' are subgraphs of \mathbf{G} . We say that a class of finite structures \mathcal{K} has the *extension property for partial automorphisms* (EPPA, sometimes called the *Hrushovski extension property*) if whenever $\mathbf{A} \in \mathcal{K}$ there is $\mathbf{B} \in \mathcal{K}$ such that \mathbf{A} is a subgraph of \mathbf{B} and such that every partial automorphism of \mathbf{A} extends to an automorphism of \mathbf{B} .

In addition to being a non-trivial and beautiful combinatorial property,

classes with EPPA have further interesting properties. For example, Kechris and Rosendal [KR07] have shown that the automorphism groups of their Fraïssé limits are amenable.

In 1992, Hrushovski [Hru92] showed that the class \mathcal{G} of all finite graphs has EPPA. A combinatorial argument for Hrushovski’s result was given by Herwig and Lascar [HL00] along with a non-trivial strengthening for certain, more restricted, classes of structures described by forbidden homomorphisms. This result was independently used by Solecki [Sol05] and Vershik [Ver08] to prove EPPA for the class of all finite metric spaces (with integer, rational or real distances – for our presentation we will consider integer distances only). Recently Conant further developed this argument to generalised metric spaces [Con15], where the distances are elements of a distance monoid. As a special case this implies EPPA for classes of metric spaces with distances limited to a given set $S \subseteq \mathbb{N}$ and to metric graphs omitting short triangles of odd perimeter.

1.2 Ramsey classes

For edge-labelled graphs \mathbf{A}, \mathbf{B} denote by $\binom{\mathbf{B}}{\mathbf{A}}$ the set of all subgraphs of \mathbf{B} that are isomorphic to \mathbf{A} . A class \mathcal{C} of structures is a *Ramsey class* if for every two objects \mathbf{A} and \mathbf{B} in \mathcal{C} and for every positive integer k there exists a structure \mathbf{C} in \mathcal{C} such that the following holds: For every partition of $\binom{\mathbf{C}}{\mathbf{A}}$ into k classes there exists an $\tilde{\mathbf{B}} \in \binom{\mathbf{C}}{\mathbf{B}}$ such that $\binom{\tilde{\mathbf{B}}}{\mathbf{A}}$ is contained in a single class of the partition.

The notion of Ramsey classes was isolated in the 1970s and, being a strong combinatorial property, it has found numerous applications, for example, in topological dynamics [KPT05]. It was independently proved by Nešetřil and Rödl [NR76] and Abramson-Harrington [AH78] that the class of all finite linearly ordered hypergraphs is a Ramsey class. Several new classes followed. We briefly outline the results related to Ramsey classes of metric spaces. In 2005 Nešetřil [Neš07] showed that the class of all finite metric spaces is a Ramsey class when enriched by free linear ordering of vertices (see also [Maš16] for alternative proof).

This result was extended to some subclasses \mathcal{A}_S of finite metric spaces where all distances belong to a given set S by Nguyen Van Thé [NVT10]. Recently Hubička and Nešetřil further generalised this result to classes \mathcal{A}_S for all feasible choices of S [HN16] that were earlier identified by Sauer [Sau13], as well as to the class of metric spaces omitting triangles of short odd perimeter.

The list of subclasses of metric spaces which are Ramsey closely corresponds to the list of classes with EPPA. The similarity of the results is not

a coincidence. All of the proofs proceed from a given metric space and, by a non-trivial construction, build an edge-labelled graph with either the desired Ramsey property or EPPA. The actual “amalgamation engines” have been isolated and are based on characterising each class by a set of obstacles in the sense of the definition below. Given a set \mathcal{O} of edge-labelled graphs with edges bounded above by δ , let $\text{Forb}_\delta(\mathcal{O})$ denote the class of all finite or countably infinite edge-labelled graphs \mathbf{G} whose edges are bounded above by δ such that there is no $\mathbf{O} \in \mathcal{O}$ with a homomorphism $\mathbf{O} \rightarrow \mathbf{G}$. Given a class of metric spaces \mathcal{A} , we say that \mathcal{O} is the *set of obstacles* of \mathcal{A} if $\mathcal{A} \subseteq \text{Forb}_\delta(\mathcal{O})$ and moreover every finite $\mathbf{G} \in \text{Forb}_\delta(\mathcal{O})$ has a completion into \mathcal{A} .

The following is a specialisation of main result of [HL00]:

Theorem 1.3 (Herwig-Lascar [HL00]) *Given a finite set \mathcal{O} of edge-labelled graphs, and $\mathbf{G} \in \text{Forb}_\delta(\mathcal{O})$, if there exists some $\mathbf{G}' \in \text{Forb}_\delta(\mathcal{O})$ such that \mathbf{G} is subgraph of \mathbf{G}' and every partial isomorphism between subgraphs of \mathbf{G} extends to an automorphism of \mathbf{G}' then there exists a finite such $\overline{\mathbf{G}} \in \text{Forb}_\delta(\mathcal{O})$.*

The following is a specialisation of Theorem 2.1 of [HN16] (strong amalgamation is defined below):

Theorem 1.4 (Hubička-Nešetřil [HN16]) *Given a strong amalgamation class of finite metric spaces \mathcal{A} , assume that there exists a finite set of obstacles \mathcal{O} of \mathcal{A} . Then the class $\vec{\mathcal{A}}$ of metric spaces from \mathcal{A} along with the free ordering of vertices is Ramsey.*

In the course of both proofs, incomplete edge-labelled graphs are produced. Knowing the characterisation of obstacles and a completion algorithm, it is then possible to turn such a graph into a metric space in the given class.

1.3 The catalogue of metrically homogeneous graphs

It is rather special for a class of structures to have a finite set of obstacles and a successful completion algorithm. Fortunately there is an elaborate list of candidates which can be examined. A weaker notion of completion, known as strong amalgamation, is well studied in the context of Fraïssé theory.

Let \mathbf{A} , \mathbf{B}_1 and \mathbf{B}_2 be edge-labelled graphs and α_1 an embedding of \mathbf{A} into \mathbf{B}_1 , α_2 an embedding of \mathbf{A} into \mathbf{B}_2 , then every edge-labelled graph \mathbf{C} with embeddings $\beta_1 : \mathbf{B}_1 \rightarrow \mathbf{C}$ and $\beta_2 : \mathbf{B}_2 \rightarrow \mathbf{C}$ such that $\beta_1 \circ \alpha_1 = \beta_2 \circ \alpha_2$ is called an *amalgamation* of \mathbf{B}_1 and \mathbf{B}_2 over \mathbf{A} with respect to α_1 and α_2 . See Figure 1.

An *amalgamation class* is a class \mathcal{K} of finite edge-labelled graphs satisfying the following three conditions:

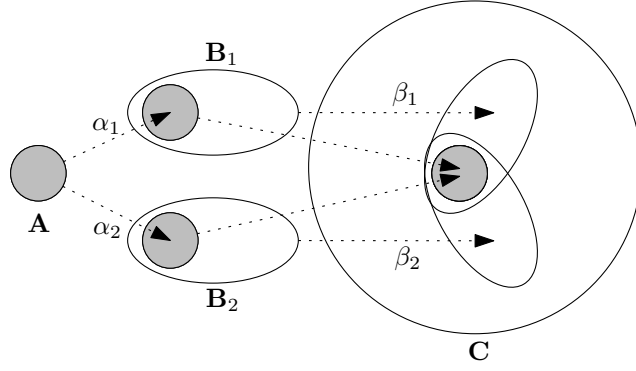


Fig. 1. An amalgamation of \mathbf{B}_1 and \mathbf{B}_2 over \mathbf{A} .

Hereditary property: For every $\mathbf{A} \in \mathcal{K}$ and a subgraph \mathbf{B} of \mathbf{A} we have $\mathbf{B} \in \mathcal{K}$;

Joint embedding property: For every $\mathbf{A}, \mathbf{B} \in \mathcal{K}$ there exists $\mathbf{C} \in \mathcal{K}$ such that \mathbf{C} contains both \mathbf{A} and \mathbf{B} as subgraphs;

Amalgamation property: For $\mathbf{A}, \mathbf{B}_1, \mathbf{B}_2 \in \mathcal{K}$ and α_1 an embedding of \mathbf{A} into \mathbf{B}_1 , α_2 an embedding of \mathbf{A} into \mathbf{B}_2 , there is a $\mathbf{C} \in \mathcal{K}$ which is an amalgamation of \mathbf{B}_1 and \mathbf{B}_2 over \mathbf{A} with respect to α_1 and α_2 .

We say that an amalgamation is *strong* when $\beta_1(x_1) = \beta_2(x_2)$ if and only if $x_1 \in \alpha_1(A)$ and $x_2 \in \alpha_2(A)$. Less formally, a strong amalgamation glues together \mathbf{B}_1 and \mathbf{B}_2 with an overlap no greater than the copy of \mathbf{A} itself.

Classes with the amalgamation property give rise to homogeneous structures. Many examples are provided by a well known classification programme (see [Che] for references). Every such class is a potential candidate to be a Ramsey class, or a class having EPPA. Cherlin recently extended this list by a probably complete classification of classes with metrically homogeneous graphs as limits, where the $\mathcal{A}_{K,C}^\delta$ play an important rôle.

From our perspective they are particularly interesting because they give a condition on the largest perimeter of triangles. The shortest path completion typically violates this axiom and thus a new completion algorithm needs to be given. While an amalgamation procedure is given in [Che] it does not directly generalise to a completion algorithm.

2 Generalised completion algorithm

In this section we will work with fixed acceptable parameters δ , K and C . Put $\mathcal{D} = \{1, 2, \dots, \delta\}^2$. We will refer to elements of \mathcal{D} as *forks*.

Consider a δ -bounded variant of shortest path completion, where in the input graphs there are no distances greater than δ and in the output all edges longer than δ are replaced by an edge of that length. There is an alternative formulation of this completion: for a fork $\mathbf{f} = (a, b)$ define $d^+(\mathbf{f}) = \min(a + b, \delta)$. In the i th step look at all incomplete forks \mathbf{f} (i.e. triples of vertices u, v, w such that exactly two edges are present) such that $d^+(\mathbf{f}) = i$ and define the length of the missing edge to be i .

This algorithm proceeds by first adding edges of length 2, then edges of length 3 and so on up to edges of length δ and has the property that out of all metric completions of a given graph, every edge of the completion yielded by this algorithm is as close to δ as possible.

It makes sense to ask what if, instead of trying to make each edge as close to δ as possible, one would try to make each edge as close to some parameter M as possible. And for M in certain range, it is indeed possible. For each fork $\mathbf{f} = (a, b)$ one can define $d^+(\mathbf{f}) = a + b$ and $d^-(\mathbf{f}) = |a - b|$, i.e. the largest and the smallest possible distance that can metrically complete fork \mathbf{f} . The generalised algorithm will then complete \mathbf{f} by $d^+(\mathbf{f})$ if $d^+(\mathbf{f}) < M$, $d^-(\mathbf{f})$ if $d^-(\mathbf{f}) > M$ and M otherwise. It turns out that there is a good permutation π of $\{1, \dots, \delta\}$, such that if one adds the distances in order given by the permutation, the generalised algorithm will indeed give a correct completion whenever possible. It is easy to check that the choice $M = \delta$ and $\pi = \{1, 2, \dots, \delta\}$ gives exactly the shortest path completion algorithm.

In the following paragraphs we will properly state this idea, introduce some more rules in order to deal with the C bound from the definition of $\mathcal{A}_{K,C}^\delta$ and prove that the algorithm works correctly.

Definition 2.1 (completion algorithm) *Given $c \geq 1$, $\mathcal{F} \subseteq \mathcal{D}$, and an edge-labelled graph $\mathbf{G} = (G, d)$, we say that $\mathbf{G}' = (G, d')$ is the (\mathcal{F}, c) -completion of \mathbf{G} if $d'(u, v) = d(u, v)$ whenever u, v is an edge of \mathbf{G} and $d'(u, v) = c$ if u, v is not an edge of \mathbf{G} and there exist $(a, b) \in \mathcal{F}$, $w \in G$ such that $\{d(u, w), d(v, w)\} = \{a, b\}$. There are no other edges in \mathbf{G}' .*

Given $1 \leq M \leq \delta$, a one-to-one function $t : \{1, 2, \dots, \delta\} \setminus \{M\} \rightarrow \mathbb{N}$ and a function \mathbb{F} from $\{1, 2, \dots, \delta\} \setminus \{M\}$ to the power set of \mathcal{D} , we define the (\mathbb{F}, t, M) -completion of an edge-labelled graph \mathbf{G} as the limit of a sequence of edge-labelled graphs $\mathbf{G}_1, \mathbf{G}_2, \dots$ such that $\mathbf{G}_1 = \mathbf{G}$ and $\mathbf{G}_{k+1} = \mathbf{G}_k$ if $t^{-1}(k)$ is undefined and \mathbf{G}_{k+1} is the $(\mathbb{F}(t^{-1}(k)), t^{-1}(k))$ -completion of \mathbf{G}_k otherwise; for every pair of vertices not forming an edge in this limit, set the distance to M .

We will call a vertex w as in Definition 2.1 a *witness of the edge u, v* . The function t is called the *time function* of the completion because distances a are

added to graph $\mathbf{G}_{t(a)}$ in step $t(a)$ of the completion algorithm. If for a (\mathbb{F}, t, M) -completion and distances a, c there is a distance b such that $(a, b) \in \mathbb{F}(c)$ (i.e. the algorithm might complete a fork (a, b) with distance c), we say that c *expands to* a . In the following paragraphs letters u, v, w will denote vertices and letters a, b, c will denote (lengths of) edges. We will slightly abuse notation and use the term triangle for both triples of vertices u, v, w and for triples of edges a, b, c with $a = d(u, v)$, $b = d(v, w)$, $c = d(u, w)$. The same convention will be used for forks.

Definition 2.2 (magic distances) *Let $M \in \{1, 2, \dots, \delta\}$ be a distance. We say that M is magic (with respect to $\mathcal{A}_{K,C}^\delta$) if $\max(K, \lceil \frac{\delta}{2} \rceil) \leq M \leq \lfloor \frac{C-\delta-1}{2} \rfloor$.*

When the parameters are acceptable, such an M will exist.

Observation 2.3 *The set of magic distances (for a given $\mathcal{A}_{K,C}^\delta$) is exactly the set*

$$S = \{1 \leq a \leq \delta : \text{triangle } aab \text{ is allowed for all } 1 \leq b \leq \delta\}.$$

Proof. If a distance a is in S , then $a \geq K$ (because otherwise the triangle $aa1$ is forbidden by the K bound), $a \geq \lceil \frac{\delta}{2} \rceil$ (because otherwise $aa\delta$ is non-metric) and $a \leq \lfloor \frac{C-\delta-1}{2} \rfloor$ (because otherwise $aa\delta$ is forbidden by the C bound). The other implication follows from the definition of $\mathcal{A}_{K,C}^\delta$. \square

Let M be a magic distance and $1 \leq x \leq \delta$ with $x \neq M$. Define $\mathcal{F}_x^+ = \{(a, b) \in \mathcal{D} : a + b = x\}$, $\mathcal{F}_x^- = \{(a, b) \in \mathcal{D} : |a - b| = x\}$ and $\mathcal{F}_x^C = \{(a, b) \in \mathcal{D} : C - 1 - a - b = x\}$. We further define

$$\mathbb{F}_M(x) = \begin{cases} \mathcal{F}_x^+ \cup \mathcal{F}_x^C & \text{if } x < M \\ \mathcal{F}_x^- & \text{if } x > M. \end{cases}$$

For a magic distance M , we define the function $t_M : \{1, \dots, \delta\} \setminus \{M\} \rightarrow \mathbb{N}$ as

$$t_M(x) = \begin{cases} 2x - 1 & \text{if } x < M \\ 2(\delta - x) & \text{if } x > M. \end{cases}$$

Forks and how they are completed according to \mathbb{F}_M are schematically depicted in Figure 2.

Definition 2.4 (Completion with magic parameter M) *Let M be a magic distance. We then call the (\mathbb{F}_M, t_M, M) -completion (of \mathbf{G}) the completion (of \mathbf{G}) with magic parameter M . We also use the name completion algo-*

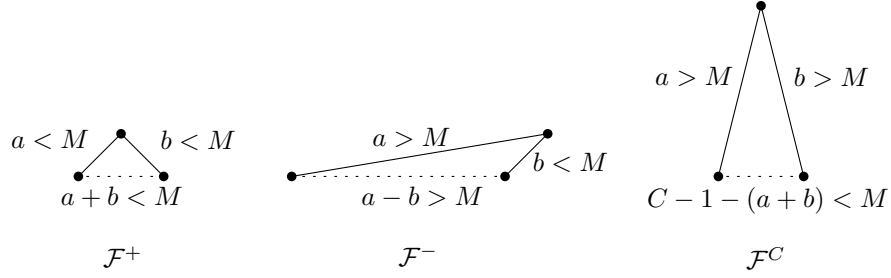


Fig. 2. Forks used by \mathbb{F}_M .

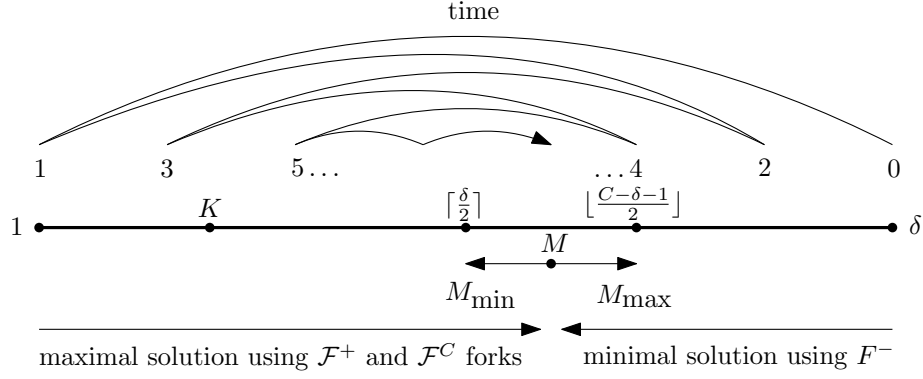


Fig. 3. A sketch of the main parameters of the completion algorithm.

rithm with magic parameter M to emphasise the process of iteratively adding distances.

The interplay of individual parameters of algorithm is schematically depicted in Figure 3.

The class $\mathcal{A}_{K,C}^\delta$ is given by forbidding triangles, that satisfy one of the following conditions: the non-metric condition (i.e. abc is forbidden if $a + b < c$), the K condition ($a + b + c < 2K + 1$ and $a + b + c$ is odd) and the C condition ($a + b + c \geq C$). In our proof we are often going to consider these three classes of forbidden triangles separately. In the following we study how they are related to the magic parameter M . We will use a, b, c for the lengths of the sides of the triangle and without loss of generality assume $a \leq b \leq c$.

If $a + b < c$, then $a < M$, because otherwise $a + b \geq 2M \geq \delta$.

If $a + b + c < 2K + 1$ and $a + b + c$ is odd and the triangle abc is metric, then $a, b, c < K \leq M$, because if $c \geq K$, then from the metric condition $a + b \geq c \geq K$ and hence $a + b + c \geq 2K$, for odd $a + b + c$ this means $a + b + c \geq 2K + 1$.

If $a + b + c \geq C$ then $b, c > M$. Suppose for a contradiction that $a, b \leq M$.

We then have $a + b \geq C - c \geq C - \delta$, but on the other hand $a + b \leq 2M \leq 2 \lfloor \frac{C-\delta-1}{2} \rfloor \leq C - \delta - 1$, which together yield $C - \delta - 1 \geq C - \delta$, a contradiction.

Now we shall precisely state and prove that t_M gives a suitable injection for the algorithm, as stated at the beginning of this section.

Lemma 2.5 (Time Consistency Lemma) *Let a, b be distances different from M . If a expands to b , then $t_M(a) > t_M(b)$.*

Proof. We consider three cases.

If $a < M$ and there is a distance c such that $a = b + c$, then $b < a < M$, hence $t_M(b) < t_M(a)$.

If $a < M$ and there is a distance c such that $a = C - 1 - b - c$, then $b, c > M$ holds (otherwise, if for instance $b \leq M$, we have $C - \delta - 1 \leq C - 1 - c = a + b < 2M \leq 2 \lfloor \frac{C-\delta-1}{2} \rfloor$, a contradiction). By the acceptability conditions on C we obtain the inequality $b = (C - 1) - c - a \geq (2\delta + 1) - \delta - a = \delta + 1 - a$. Hence $t_M(b) \leq 2(a - 1) < 2a - 1 = t_M(a)$.

Finally, we consider the case where $a > M$ and there is a distance c such that $a = |b - c|$. Then either $a = b - c$, which implies $b > a > M$ and thus $t_M(b) < t_M(a)$, or $a = c - b$, which means $b = c - a \leq \delta - a$. Because of $a > M \geq \lceil \frac{\delta}{2} \rceil$, we have $b < M$. So $t_M(b) \leq 2(\delta - a) - 1 < 2(\delta - a) = t_M(a)$. \square

The families \mathbb{F}_M were chosen to include all forks that cannot be completed with M :

Lemma 2.6 *Let \mathbf{G} be an edge-labelled graph and \mathbf{G}' be its completion with magic parameter M . If there is a forbidden triangle (w.r.t. $\mathcal{A}_{K,C}^\delta$) in \mathbf{G}' with an edge of length M , then this edge was already in \mathbf{G} .*

Proof. By Observation 2.3 no triangle of type aMM is forbidden. So suppose that there is a forbidden triangle abM in \mathbf{G}' such that the edge of length M is not in \mathbf{G} . For convenience define $t_M(M) = \infty$, which corresponds to the fact that edges of length M are added in the last step.

If abM is non-metric then either $a + b < M$ or $|a - b| > M$. By Lemma 2.5 we have in both cases that $t_M(a + b)$ resp. $t_M(|a - b|)$ is greater than $t_M(a), t_M(b)$. Therefore the completion algorithm would chose $a + b$ respectively $|a - b|$ as the length of the edge instead of M .

Now that we know that abM is metric, we also know that it is not forbidden by the K bound, because $M \geq K$.

If abM is forbidden by the C bound, then $t_M(C - 1 - a - b) > t_M(a), t_M(b)$ by Lemma 2.5, so the algorithm would set $C - 1 - a - b$ instead of M as the length of the edge. \square

The following lemma generalises the statement that the shortest path completion has all edges of maximum length possible. It will be the key ingredient for proving the correctness of the completion algorithm with magic parameter M .

Lemma 2.7 (Optimality Lemma) *Let $\mathbf{G} = (G, d)$ be an edge-labelled graph such that there is a completion of \mathbf{G} into $\mathcal{A}_{K,C}^\delta$. Denote by $\bar{\mathbf{G}} = (G, \bar{d})$ the completion with magic parameter M of \mathbf{G} and let $\mathbf{G}' = (G, d')$ be an arbitrary completion of \mathbf{G} . Then for every pair of vertices $u, v \in G$ either $d'(u, v) \geq \bar{d}(u, v) \geq M$ or $d'(u, v) \leq \bar{d}(u, v) \leq M$ holds.*

Proof. Suppose that the statement is not true, and take any witness $\mathbf{G}' = (G, d')$. Recall that the completion with magic parameter M is defined as a limit of a sequence $\mathbf{G}_1, \mathbf{G}_2, \dots$ of edge-labelled graphs such that $\mathbf{G}_1 = \mathbf{G}$ and each two subsequent graphs differ at most by adding edges of a single distance.

Take the smallest i such that in the graph $\mathbf{G}_i = (G, d_i)$ there are vertices $u, v \in G$ with $d_i(u, v) > M$ and $d_i(u, v) > d'(u, v)$ or $d_i(u, v) < M$ and $d_i(u, v) < d'(u, v)$. Let $w \in G$ be the witness of the edge $d_i(u, v)$. Then three cases can appear (corresponding to \mathcal{F}^- , \mathcal{F}^+ and \mathcal{F}^C).

In the first case we have $M < d_i(u, v) = |d_{i-1}(u, w) - d_{i-1}(v, w)|$. Without loss of generality $d_{i-1}(u, w) > d_{i-1}(v, w)$, which means that $d_{i-1}(u, w) > M$ and $d_{i-1}(v, w) < M$ (as $M \geq \lceil \frac{\delta}{2} \rceil$). From the minimality of i follows that $d'(u, w) \geq d_{i-1}(u, w)$ and $d'(v, w) \leq d_{i-1}(v, w)$. Since \mathbf{G}' is metric we have $d_i(u, v) = d_{i-1}(u, w) - d_{i-1}(v, w) \leq d'(u, w) - d'(v, w) \leq d'(u, v)$, which is a contradiction.

In the second case we have $M > d_i(u, v) = d_{i-1}(u, w) + d_{i-1}(v, w)$. Analogously to the first case we can show $d_{i-1}(u, w), d_{i-1}(v, w) < M$. By the minimality of i we have $d'(u, w) \leq d_{i-1}(u, w)$ and $d'(v, w) \leq d_{i-1}(v, w)$. Since \mathbf{G}' is metric, we get $d'(u, v) \leq d_i(u, v)$, which contradicts to our assumptions.

In the last case $M > d_i(u, v) = C - 1 - d_{i-1}(u, w) - d_{i-1}(v, w)$. Recall that, by the acceptability of the parameters δ, K and C , we have $C - 1 \geq 2\delta + 1$ and $M \leq \lfloor \frac{C-\delta-1}{2} \rfloor$. Thus we get $d_{i-1}(u, w), d_{i-1}(v, w) > M$ (otherwise, if, say, $d_{i-1}(u, w) \leq M$, we obtain the contradiction $C - \delta - 1 \geq 2M > d_{i-1}(u, w) + d_i(u, v) = C - 1 - d_{i-1}(v, w) \geq C - \delta - 1$). So again $d'(u, w) \geq d_{i-1}(u, w)$ and $d'(v, w) \geq d_{i-1}(v, w)$, which means that the triangle u, v, w in \mathbf{G}' is forbidden by the C bound, which is absurd as \mathbf{G}' is a completion of \mathbf{G} in $\mathcal{A}_{K,C}^\delta$. \square

In the next three lemmas we will use Lemma 2.7 to show that an edge-labelled graph \mathbf{G} has a completion into $\mathcal{A}_{K,C}^\delta$, if and only if the algorithm with magic parameter M gives us such a completion. We will deal with each type

of forbidden triangle separately, starting with the C bound.

Lemma 2.8 *Let $\mathbf{G} = (G, d)$ be an edge-labelled graph such that there is a completion of \mathbf{G} into $\mathcal{A}_{K,C}^\delta$ and let $\mathbf{G}' = (G, d')$ be its completion with magic parameter M . Then there is no triangle forbidden by the C bound in \mathbf{G}' .*

Proof. Suppose for contradiction that there is a triangle with vertices u, v, w in \mathbf{G}' such that $d'(u, v) + d'(v, w) + d'(u, w) \geq C$. For short let $a = d'(u, v)$, $b = d'(v, w)$ and $c = d'(u, w)$. Let a', b', c' be the corresponding edge lengths in an arbitrary completion of \mathbf{G} into $\mathcal{A}_{K,C}^\delta$. Then two cases can appear.

First assume that $a, b, c > M$. Then by Lemma 2.7 we have $a' \geq a$, $b' \geq b$ and $c' \geq c$. Hence $a' + b' + c' \geq C$, which is a contradiction.

In the other case we can assume without loss of generality that $a \leq M$, $b \geq c > M$ and $a + b + c \geq C$. Again by Lemma 2.7 we have that $b' \geq b$ and $c' \geq c$ and $a' \leq a$. If the edge (u, v) was already in \mathbf{G} , then clearly $a' + b' + c' \geq a + b + c \geq C$, which is a contradiction. If (u, v) was not already an edge in \mathbf{G} , then it was added by the completion algorithm with magic parameter M in step $t(a)$. Let $\bar{a} = C - 1 - b - c$. Then clearly $\bar{a} < a$, which means that $t_M(\bar{a}) < t_M(a)$, and as \bar{a} expands to b, c , we have $t_M(b), t_M(c) < t_M(\bar{a})$. But that means that the completion with magic parameter M actually sets the length of the edge u, v to be \bar{a} in step $t_M(\bar{a})$, which is a contradiction. \square

Lemma 2.9 *Let $\mathbf{G} = (G, d)$ be an edge-labelled graph such that there is a completion of \mathbf{G} into $\mathcal{A}_{K,C}^\delta$ and let $\mathbf{G}' = (G, d')$ be its completion with magic parameter M . Then there is no non-metric triangle in \mathbf{G}' .*

Proof. Suppose for a contradiction that there is a triangle with vertices u, v, w in \mathbf{G}' such that $d'(u, v) + d'(v, w) < d'(u, w)$. (Denote $a = d'(u, v)$, $b = d'(v, w)$ and $c = d'(u, w)$ and assume without loss of generality $a \leq b < c$.) Let a', b', c' be the corresponding edge lengths in an arbitrary completion of \mathbf{G} into $\mathcal{A}_{K,C}^\delta$.

If $a, b, c < M$, then $t_M(a) \leq t_M(b) < t_M(a+b) < t_M(c)$, which means that c must be already in \mathbf{G} . But by Lemma 2.7, we have that $a' + b' \leq a + b < c = c'$, which is a contradiction.

If $a, b < M$ and $c \geq M$, then by Lemma 2.7 we have $a' \leq a$, $b' \leq b$ and $c' \geq c$, hence the triangle a', b', c' is again non-metric.

If $a < M$ and $b, c \geq M$, then by Lemma 2.7 we have $a' \leq a$ and $c' \geq c$. If b was already in \mathbf{G} , then \mathbf{G} has no completion – a contradiction. Otherwise clearly $c - a > b \geq M$, so $t_M(c - a) < t_M(b)$. But as $c - a$ expands to c and a , we get $t_M(c - a) > t_M(c), t_M(a)$, which means that the completion algorithm with magic parameter M would complete the edge v, w with the length $c - a$ and not with b . \square

Lemma 2.10 *Let $\mathbf{G} = (G, d)$ be an edge-labelled graph such that there is a completion of \mathbf{G} into $\mathcal{A}_{K,C}^\delta$ and let $\mathbf{G}' = (G, d')$ be its completion with magic parameter M . Then there is no triangle forbidden by the K bound in \mathbf{G}' .*

Proof. Suppose for contradiction that there is a metric triangle with vertices u, v, w in \mathbf{G}' such that $d'(u, v) + d'(v, w) + d'(u, w)$ is odd and less than $2K + 1$. Denote $a = d'(u, v)$, $b = d'(v, w)$ and $c = d'(u, w)$. As we observed above, $a, b, c < K \leq M$. Also assume without loss of generality, that (u, v) was not an edge of \mathbf{G} , but added by the completion algorithm.

Notice that for any two distances e, f it holds that $C - 1 - e - f \geq K$ (simply because $e, f \leq \delta$ and $C > 2\delta + K$), so from the definition of F_x^C we can see that no edge a, b, c was added because of F_x^C , and as they are all small, they either were already in \mathbf{G} or they were added because of F_x^+ .

Let a', b', c' be the corresponding edges in some completion of \mathbf{G} into $\mathcal{A}_{K,C}^\delta$. Then Lemma 2.7 implies that $a' \leq a$, $b' \leq b$ and $c' \leq c$ and thus $a' + b' + c' < 2K + 1$. So if there is a completion in which the triangle u, v, w is not forbidden by the K bound, it is because $a' + b' + c'$ is even. Since $a < M$ there is an x such that $d'(u, x) + d'(v, x) = a$, if a was not in \mathbf{G} . But that means that $d'(u, x) + d'(v, x) + d'(u, v) = 2a < 2K$. Hence by changing the parity of a , the triangle u, v, w becomes forbidden by the K bound. The same argument can be made for the edges b, c and that gives us a contradiction to \mathbf{G} having a completion into $\mathcal{A}_{K,C}^\delta$. \square

From these lemmas we immediately get the following conclusion:

Theorem 2.11 *Let $\mathbf{G} = (G, d)$ be an edge-labelled graph such that there is a completion of \mathbf{G} into $\mathcal{A}_{K,C}^\delta$ and let $\bar{\mathbf{G}} = (G, \bar{d})$ be its completion with magic parameter M . Then $\bar{\mathbf{G}} \in \mathcal{A}_{K,C}^\delta$.*

3 Proofs of the main results

Theorem 2.11 implies the following two crucial lemmas.

Lemma 3.1 *For every acceptable choice of δ, K and C the class $\mathcal{A}_{K,C}^\delta$ has a finite set of obstacles (which are all cycles of diameter at most $2^\delta 3$).*

Proof. Let $\mathbf{G} = (G, d) \in \mathcal{G}^\delta$ be an edge-labelled graph no completion in $\mathcal{A}_{K,C}^\delta$. We seek a subgraph of \mathbf{G} of bounded size which has also no completion into $\mathcal{A}_{K,C}^\delta$.

Consider the sequence of graphs $\mathbf{G}_0, \mathbf{G}_1, \dots, \mathbf{G}_{2M+1}$ as given by Definition 2.1 when completing \mathbf{G} with magic parameter M . Set \mathbf{G}_{2M+2} to be the actual completion.

Because $\mathbf{G}_{2M+2} \notin \mathcal{A}_{K,C}^\delta$ we know it contains a forbidden triangle \mathbf{O} . This triangle always exists, because $\mathcal{A}_{K,C}^\delta$ is 3-constrained. By backward induction on $k = 2M + 1, 2M, \dots, 0$ we obtain cycles \mathbf{O}_k of \mathbf{G}_k such that \mathbf{O}_k has no completion in $\mathcal{A}_{K,C}^\delta$ and there exists a homomorphism $f : \mathbf{O}_k \rightarrow \mathbf{G}_k$.

Put $\mathbf{O}_{2M+1} = \mathbf{O}$. By Lemma 2.6 we know that this triangle is also in \mathbf{G}_{2M+1} . At step k consider every edge u, v of \mathbf{O}_{k+1} which is not an edge of \mathbf{G}_k considering its witness w (i.e. vertex w such that the edges u, w and v, w implied the addition of the edge u, v) and extending \mathbf{O}_k by a new vertex w' and edges $d(u, w') = d(u, w)$ and $d(v, w') = d(v, w)$. One can verify that the completion algorithm will fail to complete \mathbf{O}_k the same way as it failed to complete \mathbf{O}_{k+1} and moreover there is a homomorphism $\mathbf{O}_{k+1} \rightarrow \mathbf{G}_k$.

At the end of this procedure we obtain \mathbf{O}_0 , a subgraph of \mathbf{G} , that has no completion into $\mathcal{A}_{K,C}^\delta$. The bound on the size of the cycle follows from the fact that only δ steps of the algorithm are actually changing the graph and each time every edge may introduce at most one additional vertex.

Let \mathcal{O} consist of all edge-labelled cycles with at most $2^\delta 3$ vertices that are not completable in $\mathcal{A}_{K,C}^\delta$. Clearly \mathcal{O} is finite. To check that \mathcal{O} is a set of obstacles it remains to verify that there is no $\mathbf{O} \in \mathcal{O}$ with a homomorphism to some $\mathbf{M} \in \mathcal{A}_{K,C}^\delta$. Denote by \mathcal{O}' the set of all homomorphic images of structures in \mathcal{O} . Assume, to the contrary, the existence of such an $\mathbf{O} = (O, d) \in \mathcal{O}'$ and $\mathbf{M} = (M, d')$ and a homomorphism $f : \mathbf{O} \rightarrow \mathbf{M}$ and among all those choose one minimising $|\mathbf{O}|$. It follows that $|\mathbf{O}| - |\mathbf{M}| = 1$. Denote by x, y the pair of vertices identified by f . Let $\mathbf{O}' = (O, d'')$ be a metric graph such that $d''(z, z') = d(f(z), f(z'))$ for every pair $\{z, z'\} \neq \{x, x'\}$. It follows that, because $\mathcal{A}_{K,C}^\delta$ has the strong amalgamation property, also $\mathbf{O}' = (O, d'')$ has a completion in $\mathcal{A}_{K,C}^\delta$. \square

Lemma 3.2 *For every metric graph \mathbf{G} and its completion with magic parameter M (which we denote by \mathbf{M}) it holds that every automorphism of \mathbf{G} is also an automorphism of \mathbf{M} .*

Proof. Given \mathbf{G} and its automorphism $f : G \rightarrow G$, it can be verified by induction that for every $k > 0$ f is also an automorphism of graph \mathbf{G}_k as given in Definition 2.1. That for every edge x, y of \mathbf{G}_k which is not an edge of \mathbf{G}_{k-1} it holds that $f(x), f(y)$ is also an edge of \mathbf{G}_k which is not an edge of \mathbf{G}_{k-1} of the same distance. This follows directly from the definition of \mathbf{G}_k . \square

Proof. (of Theorem 1.2) This follows by a combination of Theorem 1.4 and Lemma 3.1. \square

Proof. (of Theorem 1.1) This follows by a combination of Theorem 1.3,

Lemma 3.1 and Lemma 3.2. □

4 Acknowledgement

A significant part of this research was done when the authors were participating in the Ramsey DocCourse programme, in Prague 2016–2017. Similar results were also independently obtained by Rebecca Coulson and will appear in [Cou17].

References

- [ABWH⁺17] Andres Aranda, David Bradley-Williams, Jan Hubička, Miltiadis Karamanlis, Michael Kompatscher, Matěj Konečný, and Micheal Pawliuk. Ramsey expansions of metrically homogeneous graphs. In preparation, 66 pages, 2017.
- [AH78] Fred G. Abramson and Leo A. Harrington. Models without indiscernibles. *Journal of Symbolic Logic*, 43:572–600, 1978.
- [Che] Gregory Cherlin. Homogeneous ordered graphs and metrically homogeneous graphs. In preparation, July–August 2016.
- [Con15] Gabriel Conant. Extending partial isometries of generalized metric spaces. *arXiv:1509.04950*, 2015.
- [Cou17] Rebecca Coulson. *Metrically Homogeneous Graphs: Twisted Isomorphisms, Twisted Automorphisms, and Properties of their Automorphism Groups*. PhD thesis, Rutgers Mathematics Department, 2017.
- [HL00] Bernhard Herwig and Daniel Lascar. Extending partial automorphisms and the profinite topology on free groups. *Transactions of the American Mathematical Society*, 352(5):1985–2021, 2000.
- [HN16] Jan Hubička and Jaroslav Nešetřil. All those Ramsey classes (Ramsey classes with closures and forbidden homomorphisms). Submitted, arXiv:1606.07979, 58 pages, 2016.
- [Hru92] Ehud Hrushovski. Extending partial isomorphisms of graphs. *Combinatorica*, 12(4):411–416, 1992.

- [KPT05] Alexander S. Kechris, Vladimir G. Pestov, and Stevo Todorčević. Fraïssé limits, Ramsey theory, and topological dynamics of automorphism groups. *Geometric and Functional Analysis*, 15(1):106–189, 2005.
- [KR07] Alexander S. Kechris and Christian Rosendal. Turbulence, amalgamation, and generic automorphisms of homogeneous structures. *Proceedings of the London Mathematical Society*, 94(2):302–350, 2007.
- [Maš16] Dragan Mašulović. Pre-adjunctions and the Ramsey property. *arXiv:1609.06832*, 2016.
- [Neš89] Jaroslav Nešetřil. For graphs there are only four types of hereditary Ramsey classes. *Journal of Combinatorial Theory, Series B*, 46(2):127–132, 1989.
- [Neš05] Jaroslav Nešetřil. Ramsey classes and homogeneous structures. *Combinatorics, probability and computing*, 14(1-2):171–189, 2005.
- [Neš07] Jaroslav Nešetřil. Metric spaces are Ramsey. *European Journal of Combinatorics*, 28(1):457–468, 2007.
- [NR76] Jaroslav Nešetřil and Vojtěch Rödl. The Ramsey property for graphs with forbidden complete subgraphs. *Journal of Combinatorial Theory, Series B*, 20(3):243–249, 1976.
- [NVT10] Lionel Nguyen Van Thé. *Structural Ramsey Theory of Metric Spaces and Topological Dynamics of Isometry Groups*. Memoirs of the American Mathematical Society. American Mathematical Society, 2010.
- [Sau13] Norbert W. Sauer. Distance sets of Urysohn metric spaces. *Canadian Journal of Mathematics*, 65(1):222–240, 2013.
- [Sol05] Sławomir Solecki. Extending partial isometries. *Israel Journal of Mathematics*, 150(1):315–331, 2005.
- [Ver08] Anatoly M. Vershik. Globalization of the partial isometries of metric spaces and local approximation of the group of isometries of Urysohn space. *Topology and its Applications*, 155(14):1618–1626, 2008.