# Computing a minimal partition of partial orders into heapable subsets

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#### Abstract

We investigate the partitioning of partial orders into a minimal number of heapable subsets. We prove a characterization result reminiscent of the proof of Dilworth's theorem, which yields as a byproduct a flow-based algorithm for computing such a minimal decomposition. On the other hand, in the particular case of sets and sequences of intervals we prove that this minimal decomposition can be computed by a simple greedy-type algorithm. The paper ends with a couple of open problems related to the analog of the Ulam-Hammersley problem for decompositions of sets and sequences of random intervals into heapable sets.

**Keywords:** partial order; heapable sequence; Dilworth's theorem; greedy algorithm; random intervals.

Mathematics Subject Classifications: 05A18, 68C05, 60K35.

#### 1 Introduction

The *longest increasing subsequence* is a classical problem in combinatorics and algorithmics, with deep connections to statistical shysics and random matrix theory (for a very readable introduction see [20]).

Byers et al. [5] introduced an interesting variant of the concept of increasing sequence: sequence of integers  $A = a_1, a_2, \ldots, a_n$  is *heapable* if numbers  $a_1, a_2, \ldots, a_n$  can be successively inserted as actual leaves into a heap-ordered binary tree (not necessarily complete).

Heapability was further investigated in [13] (and, independently, in [19]). In particular, a subgroup of the authors of the present papers showed that for permutations one can compute in polynomial time a minimal decomposition into heapable subsequences, and connected this problem to a variation on Hammersley's interacting particle process (see also further work in [14, 2, 3], that extended/confirmed some of the conjectures of [13]).

This paper started as a conversation on the heapability of sequences of intervals during a joint Timişoara-Szeged seminar on theoretical computer science in November 2015. Its main purpose is to offer a different perspective on the concept of heapability, by connecting the problem of computing a minimal decomposition of partial ordered sets into "heapable subsequences" (that will be parametrized in this paper by an integer  $k \geq 2$ , and will be called k-chains) to the classical theorems of Dilworth and Kőnig-Egerváry: we show that the number of classes in a minimal decomposition can be obtained as a vertex cover in a bipartite graph. As a byproduct we obtain an efficient algorithm based on network flows for computing such a minimal decomposition. This results and the ones from [13] (where the same parameter was computed, in the case of integer sequences, via a direct greedy algorithm) raise the question whether such greedy algorithms exist for other posets. We answer this question in the affirmative way by adapting the result of [13] to sets and sequences of intervals, ordered by the natural partial order, the setting that motivated the discussion leading to the paper.

The structure of the paper is as follows: in Section 2 we review the main concepts and technical results we will be concerned with. Then, in Section 3 we prove a numerical characterization inspired by (the proof of) Dilworth's theorem for the size of minimal partition of a partial order into k-chains. This result provides, as a byproduct, a flow-based algorithm for the computation of such a minimal partition. We then investigate, in Section 4 a special case of this problem for sequences of intervals, showing that a greedy-type algorithm

computes such a minimal partition in this special case. In Section 5 we show that the result above extends to (unordered) *sets* (rather than sequences) of intervals. We conclude in Section 6 with some open questions raised by our results.

## 2 Preliminaries

We will assume knowledge of standard graph-theoretic notions. In particular, given integer  $k \geq 1$ , rooted tree T is k-ary if every node has at most k descendants. Given a directed graph G = (V, E), we will denote by vc(G) the size of the minimum vertex cover of G.

Let  $k \geq 1$ . A sequence of integers  $A = (a_1, a_2, \ldots, a_n)$  is called k-heapable if there exists a k-ary tree T with n nodes, such that its nodes are labeled by  $a_1, a_2, \ldots, a_n$ , and a bijection between vertices of T and elements of A such that for any two nodes labeled  $a_i, a_j$ , if  $a_j$  is a descendant of  $a_i$  of T then i < j holds for their indices and  $a_i < a_j$ .

We will be concerned with finite partially ordered sets (posets) only. Sequences of elements from a poset naturally embed into this framework by associating, to every sequence  $A = (a_1, \ldots, a_n)$  the poset  $Q_A = \{(i, a_i) : 1 \le i \le n\}$  with partial order  $(i, a_i) \le (j, a_j)$  if and only if  $i \le j$  and  $a_i \le a_j$ . Given poset Q, its subset B is a *chain* if the partial order of Q is a total order on A. Subset C is an *antichain* if no two elements a, b of C are comparable with respect to the patial order relation of Q. Dilworth's theorem [6] states that the minimum number of classes in a chain decomposition of a partial order Q is equal to the size of the largest antichain of Q.

We need to briefly review one classical proof of Dilworth's theorem, due to Fulkerson [9]. This proof employs the Kőnig-Egerváry theorem [16, 7], stating that the size of a maximum matching in a bipartite graph is equal to the minimum vertex cover in the same graph. This result is applied as follows: first, we prove that each matching in  $G_Q$  uniquely corresponds to a chain decomposition of Q. Next we apply Kőnig's theorem to the so-called split graph associated to poset  $Q = (U, \leq)$  [8], i.e. the bipartite graph  $G_Q = (V_1, V_2, E)$ , where  $V_1 = \{x^- : x \in U\}$  and  $V_2 = \{y^+ : y \in U\}$  are independent copies of U, and given  $x \leq y \in U$  we add to E edge  $x^-y^+$ . The conclusion is that the cardinality of a minimum chain decomposition of Q is equal to n, the number of elements in U, minus the size of the smallest vertex cover in  $G_Q$  defined above. Finally, this latter quantity is proved to

coincide to the size of the largest antichain of Q.

We extend the definition of k-heapable sequences to general posets as follows: given integer  $k \geq 1$  and poset  $Q = (U, \leq)$  a subset A of the base set U is called k-heapable (or, equivalently, a k-chain of Q) if there exists a k-ary rooted tree T and a bijection between A and the vertices of T such that for every  $i, j \in A$ , if j is a descendant of i in T then i < j in Q. The k-width of poset Q, denoted by k-wd(Q), was defined in [14] as the smallest number of classes in a partition of Q into k-chains. For k = 1, by Dilworth's theorem, we recover the usual definition of poset width [21]. On the other hand, when  $\pi \in S_n$  is a permutation of n elements, k-wd( $Q_{\pi}$ ) particularizes to the parameter whose scaling properties were investigated in [13].

We deal in the sequel with partial orders on intervals. Without loss of generality all our intervals will be closed subsets of (0,1). We define a partial order of them as follows: Given intervals  $I_1 = [a_1, b_1]$  and  $I_2 = [a_2, b_2]$  with  $a_1 < b_1$  and  $a_2 < b_2$ , we say that  $I_1 \le I_2$  if and only if the entire interval  $I_1$  lies to the left of  $I_2$  on the real numbers axis, that is  $b_1 \le a_2$ . For technical reasons we will also require a total ordering of intervals, denoted by  $\sqsubseteq$  and defined as follows:  $I_1 \sqsubseteq I_2$  if either  $b_1 < b_2$  or  $b_1 = b_2$  and  $a_1 < a_2$ . On the other hand, by the phrase "random intervals" we will mean, similarly to the model in [15], random subintervals of (0,1). A random sample I can be constructed iteratively at each step by choosing two random real numbers  $a, b \in (0,1)$  and taking I = [min(a,b), max(a,b)].

Two important (unpublished) minimax theorems attributed in [11] to Tibor Gallai deal with partitioning sets of intervals. We will only be concerned with the first of them, that states that, given a set of intervals J on the real numbers line the following equality holds:

**Theorem 1.** The minimum number of partition classes of J into pairwise disjoint intervals is equal to the maximum number of pairwise intersecting intervals in J.

Of course, the first quantity in theorem 1 is nothing but the 1-width of the partial order  $\leq$  on intervals. As for random permutations (the well-known Ulam- $Hamersley\ problem\ [17, 22]$ ), the scaling of (the expected value of) this parameter for sets R of n random intervals has been settled and has the form [15]

$$E[1\text{-wd}(\mathbf{R})] \sim \frac{2}{\sqrt{\Pi}} \sqrt{n}.$$

## 3 Main result

Our main result gives a characterization of the k-width of a finite poset that is strongly related to the proof of Dilworth's theorem sketched above:

**Definition 2.** Given poset  $Q = (U, \leq)$  an integer  $k \geq 1$ , the k-split graph associated to poset Q is the bipartite graph  $G_{Q,k} = (V_1, V_2, E)$ , where

- $V_1 = \{x_1^-, \dots, x_k^- : x \in U\}$
- $V_2 = \{y^+ : y \in U\}$
- given  $x, y \in U$ ,  $x \le y$  and  $1 \le i \le k$ , add to E edge  $x_i^- y^+$ .

**Theorem 3.** Let  $Q = (U, \leq)$  be a finite poset with n elements and a fixed integer  $k \geq 1$ . Then

$$k - wd(Q) = n - vc(G_{Q,k}). \tag{1}$$

*Proof.* Define a *left outgoing k-matching* of graph  $G_Q$  to be any set of edges  $A \subseteq E$  such that for every  $x, y \in U$ ,  $deg_A(x^-) \le k$  and  $deg_A(y^+) \le 1$ .

Claim 4. Partitions of Q into k-chains bijectively correspond to left k-matchings of  $G_Q$ . The number of classes of a partition is equal to n minus the number of edges in the associated left k-matching.

Proof. Consider a left k-matching A in  $G_Q$ . Define the partition  $P_A$  as follows: roots of the k-chains consist of those  $x \in U$  for which  $deg_A(x^+) = 0$ . There must be some element  $x \in U$  satisfying this condition, as the minimal elements of P with respect to  $\leq$  satisfy this condition.

Now we recursively add elements of U to the partition P (in parallel) as follows:

1. All elements  $y \in U$  not yet added to any k-chain, and such that  $y^+$  is connected to some  $x^-$  by an edge in A are added to the k-chain containing x, as direct descendants of x. Note that element x (if there exists at least one y with this property) is unique (since  $deg_A(y^+) = 1$ ), so the specification of the k-chain to add y to is well defined. On the other hand, each operation adds at most k successors of any x to its k-chain, since  $deg_A(x^-) \leq k$ .

2. If all direct predecessors of an element  $x \in U$  have been added to some k-chain and are no longer leaves of that k-chain then x becomes the root of a new k-chain.

Conversely, given any partition P of U into k-chains, define set A consisting of edges  $x^-, y^+$  such that x is the father of y in a k-chain. It is immediate that A is a left k-matching.  $\square$ 

**Corollary 5.** There is a bijective mapping between partitions of Q into k-chains and matchings of  $G_{Q,k}$  such that the number of k-chains in a partition is n minus the number of edges in the matching.

Proof. We will actually show how to associate left k-matchings of  $G_Q$  to matchings of  $G_{Q,k}$ . The idea is simple: given a node  $x^-$  of  $G_Q$  with  $l \leq k$  neighbors in  $V_2$ , construct a matching in  $G_{Q,k}$  by giving each of  $x_1^-, \ldots, x_l^-$  a single neighbor from the neighbors of  $x^-$ . In the other direction, if  $e = x_i y$  is an edge in the matching of  $G_{Q,k}$  then consider the edge in appropriate x, y edge in  $G_Q$ . It is easy to check that it gives a k-chain in  $G_Q$ , and the number of k-chains is the number of edges in the left k-matching in  $G_Q$  which is the same as n minus  $vc(G_Q)$ .

We complete the proof of Theorem 3 by applying in a straightforward way (based on Claim 4 and Corollary 5) the Kőnig's theorem to graph  $G_{Q,k}$ .

**Corollary 6.** One can compute parameter k-wd(Q) by creating a flow network  $Z_Q$  and computing the value of the maximum flow of  $Z_Q$  consisting of :

- vertices and edges of  $G_Q$ , with capacity 1.
- a source s, connected to nodes in  $V_1$  by directed edges of capacity k,
- a sink t, that all nodes in V<sub>2</sub> connect to via oriented edges of capacity 1.

computing the maximum s-t network flow value f in network  $Z_Q$  and outputting k-wd(Q)=n-f.

*Proof.* Straightforward, this is simply the maximal flow algorithm for computing the maximal size left k-matching in  $G_Q$ , similar to the construction for maximum matchings in bipartite graphs in the literature.

## 4 Heapability of sequences of intervals: a greedy approach

The result in the previous section naturally raises the following question: for random permutations a greedy approach, extending the well-known patience sorting algorithm [18] works, as it was shown in [13]. On the other hand network flow algorithms are **not** greedy. One is naturally led to inquire whether such an algorithm can be defined/works for any other partial order. In the sequel we provide an affirmative answer to this question, in the case of interval sequences:

**Theorem 7.** For every fixed  $k \geq 1$  there exists a (polynomial time) greedy algorithm that, given the sequence of intervals  $S = (I_1, I_2, ..., I_n)$  as input, computes a minimal partition of S into k-chains.

Before proceeding with the proof of theorem 7, let us remark a potential application of a variant of this result to parallel computing: many algorithms in this area (e.g. algorithms using a parallel prefix-sum design methodology [4]) require the computation of all prefixes of an associative operation  $A_1*A_2*$  $\dots A_n$ . Operations being performed (each corresponding to one computation of a \*-product) are arranged on a binary tree. In the (completely equivalent) max-heap variant of theorem 7, children intervals are required to be less or equal to the parent interval with respect to ordering <. This is quite natural from the standpoint of parallel computing: consider the setting of a parallel-prefix problem where each intermediate \*-computation is a rather costly operation; intervals now represent times when these operations can be scheduled. The requirement that the parent interval be larger than child intervals with respect to  $\leq$  is completely natural, as child computations need to complete in order to feed their results to the parent computation. Thus our result answers the question whether all the time intervals can be scheduled on a single heap-ordered binary tree, and gives such a scheduling, if the answer is affirmative.

#### 4.1 Proof of theorem 7

Without loss of generality, we will present the proof of the theorem for k = 2, but the proof is similar for all k's. The proof employs the concept of slots, adapted for interval sequences from similar concepts for permutations [5, 13]:

**Definition 8.** When a new interval is added to a k-chain it opens k new positions to possibly insert other intervals as direct successors into this node. Each position has an associated integer value that will be called its *slot*. The value of all empty slots created by inserting  $I_1 = [a_1, b_1]$  into a k-chain will be  $b_1$ , the endpoint of  $I_1$ .

**Definition 9.** An interval I is compatible with a slot with value x if all of I lies in  $[x, \infty)$ .

Intuitively, x is the smallest value of the left endpoint of an interval that can be inserted in the k-chain as a child of an interval  $I_1$  with right endpoint x while respecting the heap property. Indeed, as k-chains are (min-)heap ordered, an insertion of an interval I into a k-chain as a child of  $I_1$  is legal if the interval I is greater than  $I_1$  with respect to  $\leq$  relation. This readily translates to the stated condition, that the start point of I must be greater or equal than the slot value of its parent.

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Input: A sequence of intervals I = (I_1, I_2, \dots, I_n).

Output: A partition H of I into k-chains.

for i := 1 to n do:

if I_i = [a_i, b_i] can be inserted into some empty slot

then insert I_i in the highest-valued compatible slot (a child of the node with this slot).

else create new k-chain rooted at I_i
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Figure 1: Our greedy-type algorithm for sequences of intervals.

The proposed greedy algorithm for computing the minimum number of 2-chains a sequence of n intervals can be split into is described in Figure 1. As an example, consider the sequence of intervals  $S_1$  below, with k = 2. The resulting configuration is shown in Figure 2.

$$S_1 = \{[1, 7], [1, 11], [11, 12], [15, 16], [7, 9], [8, 16], [1, 2], [3, 19], [13, 17], [5, 7]\}$$

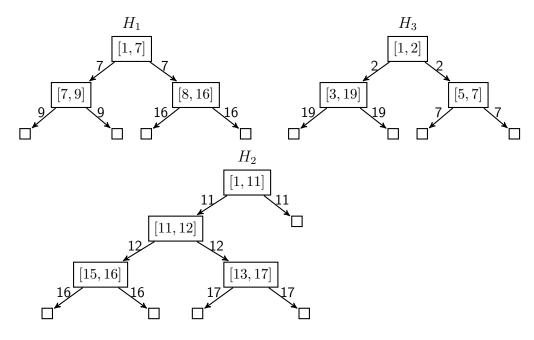


Figure 2: The 2-chain configuration corresponding to  $S_1$ .

By choosing the highest valued slot available for insertion for  $I_k = [a_k, b_k]$ , we make sure that the difference between the slot value s and  $a_k$  is minimal. This is desirable because there may be some interval further down the sequence with starting point value in between the s and  $a_k$  that fits slot s but cannot be inserted there, as the slot is no longer available.

We define the concepts of 2-chain signature and domination between 2-chains in a way to similar to the corresponding concepts for permutations in [5]:

**Definition 10.** Given multiset of slots H, we call the vector of slots, sorted in increasing order, the *signature* of H. We shall denote this by sig(H). By slightly abusing notation, we will employ the previous definition in the obvious way when H is a 2-chain as well.

**Definition 11.** 2-chain  $H_a$  dominates 2-chain  $H_b$  (denoted  $H_a \leq H_b$ ) if  $|sig(H_a)| \leq |sig(H_b)|$  and for all  $1 \leq i \leq |sig(H_a)|$  we have  $sig(H_a)[i] \leq sig(H_b)[i]$ .

For example, for the 2-chains in the Figure 2 their corresponding signatures are, respectively:

$$sig(H_1) = [9, 9, 16, 16];$$
  
 $sig(H_2) = [11, 16, 16, 17, 17];$   
 $sig(H_3) = [7, 7, 19, 19].$ 

Therefore, in our example  $H_1 \leq H_2$ ,  $(H_1 \text{ dominates } H_2)$ , but no other domination relations between  $H_1, H_2, H_3$  are true.

Let  $\sigma = \{H_1, H_2, ..., H_m\}$ , be the set of all 2-chains in the forest at a given moment k,  $sig(\sigma) := \{sig(H_1), sig(H_2), ..., sig(H_m)\}$  be the corresponding (multi)set of all signatures. Denote by [x, y] the interval to be processed next. Let  $H_q$  be the 2-chain selected according to the greedy algorithm described in Figure 1,  $sig(H_q) = \{a_1, a_2, ..., a_{max}\} \in sig(\sigma)$ , and  $H_p$  be an arbitrary 2-chain  $\in \sigma$ ,  $sig(H_p) = \{b_1, b_2, ..., b_{max}\}$ , such that  $H_q \leq H_p$ . Our goal is to compare the 2-chains  $H'_q$  and  $H'_p$  generated by inserting interval [x, y] into the 2-chain  $H_q$  the greedy way (and in an arbitrary compatible slot of  $H_p$ , respectively):

**Lemma 12.** Assume that  $H_q$  dominates  $H_p$ . Then 2-chains  $H'_q$  and  $H'_p$  obtained after inserting a new interval into  $H_q$  the greedy way and into  $H_p$  in an arbitrary way, propagates the domination property, i.e.  $H'_q$  dominates  $H'_p$ .

*Proof.* Proving that  $H'_q \leq H'_p$  is equivalent to proving that  $|sig(H'_q)| \leq |sig(H'_p)|$  and for all indices  $1 \leq l \leq |sig(H'_q)|$ :

$$sig(H'_q)[l] \le sig(H'_p)[l]. \tag{2}$$

The cardinality condition is easy to verify:  $|sig(H'_q)|$  is either  $|sig(H_q)|+1$  or  $|sig(H_q)|+2$ , and similarly for  $|sig(H'_p)|$ . It follows easily from the domination property that when  $|sig(H'_q)|$  increases by 2,  $|sig(H'_p)|$  increases by 2 as well.

As for the second condition, consider the slots from  $H_q$  and  $H_p$  which interfere with the newly arrived interval as follows:

- Let i and i' be the (unique) indices such that  $a_i \leq x < a_{i+1}$  and  $a_{i'} \leq y < a_{i'+1}$  hold in  $H_q$ , with  $i, i' \in 0, \dots, |sig(H_q)|$ .
- Similarly, let j and j' be the unique indices such that  $b_j \leq x < b_{j+1}$  and  $b_{j'} \leq y < b_{j'+1}$  hold in  $H_p$ , with  $i, i' \in 0, \dots, |sig(H_p)|$ .

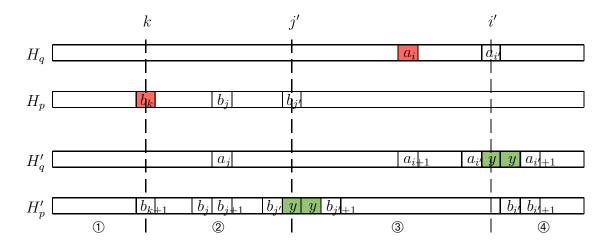


Figure 3: The various cases of inserting a new interval [x, y] into  $H_p$  and  $H_q$ .

(we have employed the conventions  $a_0 = b_0 = -\infty$  and  $a_{sig(H_q)|+1} = b_{|sig(H_p)|+1} = +\infty$ )

Since  $H_q \leq H_p$ , it follows that  $i \geq j$  and  $i' \geq j'$ . Suppose that we insert the interval [x, y] in  $H_p$  in an arbitrary slot  $b_k \leq b_j$  (thus removing one life of slot  $b_k$  and inserting two copies of slot y). The rest of the proof is by a case analysis. The four cases which can be distinguished (Fig. 3) are:

#### Case 1. l < k:

In this case, none of the signatures were affected at position l by the insertion of [x, y], hence:

$$sig(H'_q)[l] = sig(H_q)[l] \le sig(H_p)[l] = sig(H'_p)[l].$$

## Case 2. $l \in [k, j')$ :

In this range all slots from  $H'_p$  have been shifted by one position to the left compared to  $H_p$  due to the removal of  $b_k$ . Consequently:

$$sig(H'_q)[l] = sig(H_q)[l] \le sig(H_p)[l+1] = sig(H'_p)[l].$$

### Case 3. $l \in [j', i')$ :

Knowing that j' is the position in  $H_p$  where we inserted two new slots with value y, i' is the position in  $H_p$  where we inserted these same slots and  $i' \geq j'$ , the following is true:

$$sig(H'_q)[l] \le sig(H_q)[i'] = y = sig(H'_p)[j'] \le sig(H'_p)[l].$$

Case 4.  $l \geq i'$ :

a) For l=i' and l=i'+1:  $sig(H_q')[l]=sig(H_p')[j']=y$ . Since j'< i'< i'+1, then:

$$sig(H'_q)[i'] = y = sig(H'_p)[j'] \le sig(H'_p)[i'].$$
  
 $sig(H'_q)[i'+1] = y = sig(H'_p)[j'] \le sig(H'_p)[i'+1].$ 

b) For l > i' + 1 the two signatures do not have any shifted components compared to the original signatures of  $H_q$  and  $H_p$ , so:

$$sig(H'_q)[i'+k] = sig(H_q)[i'+k-1] \le sig(H_p)[i'+k-1] = sig(H'_p)[i'+k]$$

In conclusion,  $sig(H_q')[l] \leq sig(H_p')[l]$  for any l, and relation  $H_q' \leq H_p'$  follows.  $\square$ 

**Lemma 13.** Given sequence I of intervals, consider the optimal way of intervals to partition I into k-chains. Let r be a stage where our greedy-type algorithm creates a new k-chain. Then the optimal way also creates a new k-chain.

*Proof.* We use the fact that, by Lemma 12, before every step r of the algorithm the multiset  $\Gamma$  of slots created by our greedy algorithm dominates the multiset  $\Omega$  created by the optimal insertion.

Suppose that at stage r the newly inserted interval  $I_r$  element causes a new k-chain to be created. That means that the left endpoint  $a_r$  of  $I_r$  is lower than any of the elements of multiset  $\Gamma$ . By domination, the minimum slot of  $\Omega$  is higher than the minimum slot of  $\Gamma$ . Therefore  $a_r$  is also lower than any slot of  $\Omega$ , which means that the optimal algorithm also creates a new k-chain when inserting  $I_r$ .

Lemma 13 proves that the Greedy algorithm for insertion of new intervals is optimal.

## 4.2 The interval Hammersley (tree) process

One of the most fruitful avenues for the investigation of the scaling properties of the LIS (Longest increasing subsequence) of a random permutation is made via the study of the (so-called hydrodynamic) limit behavior of an interacting particle system known as Hammersley's process [1]. This is a stochastic process that, for the purposes of this paper can be defined (in a somewhat simplified form) as follows: random numbers  $X_0, X_1, \ldots, X_n, \ldots \in (0, 1)$  arrive at integer moments. Each value  $X_j$  eliminates ("kills") the smallest  $X_i > X_j$  that is still alive at moment j. Intuitively, "live" particles represent the top of the stacks in the so-called  $patience\ sorting\ algorithm$  [18] that computes parameter LIS.

The problem of partitioning a random permutation into a minimal set of k-heapable subsequences is similarly connected to a variant of the above process, introduced in [13] and further studied in [2, 3], where it was baptized Hammersley's trees process. Now particles come with k lives, and each particle  $X_j$  merely takes one life of the smallest  $X_i > X_j$ , if any (instead of outright killing it).

The proof of theorem 7 shows that a similar connection holds for sequences of *intervals*. The *Hammersley interval tree process* is defined as follows: "particles" are still numbers in (0,1), that may have up to k lives. However now the sequence  $I_0, I_1, \ldots, I_n, \ldots$  is comprised of random *intervals* in (0,1). When interval  $I_n = [a_n, b_n]$  arrives, it is  $a_n$  that takes a life from the largest live particle  $y < a_n$ . However, it is  $b_n$  that is inserted as a new particle, initially with k lives.

Corollary 14. Live particles in the above Hammersley interval process correspond to slots in our greedy insertion algorithm above. The newly created k-chains correspond to local minima (particle insertions that have a value lower than the value of any particle that is live at that particular moment).

## 5 From sequences to sets of intervals

Theorem 7 dealt with *sequences* of intervals. On the other hand a set of intervals does not come with any particular listing order on the constituent intervals. Nevertheless, the problem can be easily reduced to the sequence case by the following:

**Theorem 15.** Let  $k \geq 1$  and Q be a set of intervals. Then the k-width of Q is equal to the k-width of  $Gr_Q$ , the sequence of intervals obtained by listing the intervals in the increasing order of their right endpoints (with earlier starting intervals being preferred in the case of ties).

*Proof.* Clearly  $k\text{-wd}(Q) \leq k\text{-wd}(G_Q)$ , since a partition of  $Gr_Q$  into k-chains is also a partition of Q.

To prove the opposite direction we need the following:

**Lemma 16.** Let S be a multiset of slots. Let  $I_1 = [a_1, b_1], I_2 = [a_2, b_2]$  be two intervals such that  $b_1 < b_2$  or  $b_1 = b_2$  and  $a_1 < a_2$ . Let  $S_1$  and  $S_2$  be the multisets of slots obtained by inserting the two intervals in the order  $(I_1, I_2)$  and  $(I_2, I_1)$ , respectively. Then  $S_1$  dominates  $S_2$ .

*Proof.* Let x be the largest element in S smaller than  $a_1$ . The effect of inserting  $I_1$  is to remove from S a copy of x (if any) and add instead k copies of  $b_1$ . Then  $I_2$  removes a slot with value  $y \ge b_1$  (because  $b_1 < a_2$ , adding instead k copies of  $b_2$ . If  $y > b_1$  then  $y \in S$ , the two insertions "do not interact", and lead to the same set  $S_1 = S_2$  if performed in the order  $(I_2, I_1)$ .

The only remaining case is when  $y = b_1$ . In this case S cannot contain any element in the range  $(b_1, a_2)$ . Insertion  $(I_1, I_2)$  removes x, adds k - 1 copies of  $b_1$  and k copies of  $b_2$ . On the other hand, insertion  $(I_2, I_1)$  may remove some element x'. It is certainly  $x \le x' \le b_1$ . It then adds k copies of  $b_2$ . Then it removes some  $x'' \le x$  and adds k copies of  $b_1$ . Thus both sets  $S_1, S_2$  can be described as adding k copies of  $b_1$  and k copies of  $b_2$  to S, and then deleting one or two elements,

Two elements

- x and  $b_1$  in the case of order  $(I_1, I_2)$
- x' and x'' in the case of order  $(I_2, I_1)$ .

in the case when some element of S is strictly less than  $a_1$ ,

And one element

- $b_1$  in the case of order  $(I_1, I_2)$
- x'' in the case of order  $(I_2, I_1)$ .

In the second case the result follows by taking into account the fact that  $x'' \leq b_1$  and the following lemma:

**Lemma 17.** Let S be a multiset of slots. Let  $s_2 \leq s_1 \in S$ , and let  $S_1, S_2$  be the sets obtained by deleting from S elements  $s_1$  (or  $s_2$ , respectively). Then  $S_1$  dominates  $S_2$ .

*Proof.* It follows easily from the simple fact that for every  $r \geq 1$  and multiset W the function that maps an integer  $x \in W$  to the r'th smallest element of  $W \setminus x$  is non-increasing (more precisely a function that jump from the r + 1'th down to the r'th smallest element of W)

The first case is only slightly more involved. Since  $x' \leq x$ , by applying Lemma 17 twice we infer:

$$S_1 = S \setminus \{x, b_1\} = (S \setminus \{b_1\}) \setminus \{x\} \preceq (S \setminus \{b_1\}) \setminus \{x'\} = (S \setminus \{x'\}) \setminus \{b_1\}$$
$$\prec (S \setminus \{x'\}) \setminus \{x''\} = S_2.$$

which is what we wanted to prove.

From Lemma 16 we infer the following result

**Lemma 18.** Let  $1 \le r \le n$  and let X, Y be two permutations of intervals  $I_1, I_2, \ldots, I_n$ ,

$$X = (I_1, \dots, I_{r-1}, I_r, I_{r+1}, \dots I_n),$$
  

$$Y = (I_1, \dots, I_{r-1}, I_{r+1}, I_r, \dots I_n).$$

respectively (i.e. X, Y differ by a transposition). If  $I_r \leq I_{r+1}$  then multisets of slots  $S_X, S_Y$  obtained by inserting intervals according to the listing specified by X and Y, respectively, satisfy

$$S_X \leq S_Y$$
.

*Proof.* Without loss of generality one may assume that r = n - 1 (as the result for a general n follows from this special case by repeatedly applying Lemma 12). Let S be the multiset of slots obtained by inserting (in this order) intervals  $I_1, \ldots, I_{r-1}$ . Applying Lemma 16 to intervals  $I_r, I_{r+1}$  we complete the proof of Lemma 18.

Now the opposite direction in the proof of theorem 15 (and the theorem) follows: the multiset of labels obtained by inserting the intervals according to  $Gr_Q$  dominates any multiset of labels arising from a different permutation, since one can "bubble down" smaller intervals (as in bubble sort), until we obtain  $Gr_Q$ . As we do so, at each step, the new multiset of labels dominates

Input: A set of intervals I.

Output: A partition H of I into k-chains.

Sort the intervals w.r.t.  $\sqsubseteq$ :  $I = (I_1, \ldots, I_n)$ .

For i := 1 to n do:

If  $I_i = [a_i, b_i]$  can be inserted into some empty slot

then insert  $I_i$  in the highest-valued compatible slot.

else create new k-chain rooted at  $I_i$ 

Figure 4: The greedy algorithm for sets of intervals.

the old one. Hence  $S_{Gr(Q)}$  dominates all multisets arising from permutations of  $I_1, \ldots, I_n$ , so sequence  $Gr_Q$  minimizes the parameter k-wd among all permutations of Q.

**Corollary 19.** The greedy algorithm in Figure 4 computes the k-width of an arbitrary set of intervals.

**Corollary 20.** Modify the Hammersley interval process to work on sets of intervals by considering them in non-decreasing order according to relation  $\sqsubseteq$ . Then live particles in the modified process correspond to slots obtained using our greedy insertion algorithm for sets of intervals. New k-chains correspond to local minima (such particle insertions that have a value lower than the value of any particle that is live at that particular moment).

## 6 Open questions and future work

Our Theorem 3 is very similar to (the proof of) Dilworth's theorem. It is not yet a proper generalization of this result to the case  $k \ge 1$  because of the lack of a suitable extension of the notion of antichain:

**Open problem 21.** Is there a suitable definition of the concept of k-antichain, that coincides with this concept for k = 1 and leads (via our theorem 3) to an extension of Dilworth's theorem ?

On the other hand, results in Section 4 naturally raise the following:

**Open problem 22.** For which partial orders Q can one compute the parameter k-wd(Q) (and an associate optimal k-chain decomposition) via a greedy algorithm?

Several open problem concerns the limit behavior of the expected value of the k-width of a set of random intervals, for  $k \geq 1$ . As discussed in Section 2, for k = 1 the scaling behavior of this parameter is known [15]. However, in the case of random permutations, the most illuminating description of this scaling behavior is by analyzing the hydrodynamic limit of the Hammersley process [1, 10]. As shown in Section 5, the difference between sequences and sets of intervals is not substantial.

We ask, therefore, whether the success in analyzing this process for random permutations can be replicated in the case of sequences/sets of random intervals:

**Open problem 23.** Analyze the hydrodynamic limit of the Hammersley process for sequences/sets of random intervals.

When  $k \geq 2$  even the scaling behavior is not known, for both sequences and sets of random intervals. The connection with the interval Hammersley process given by Corollaries 14 and 20 provides a convenient, lean way to simulate the dynamics, leading to experimental observations on the scaling constants. A C++ program used to perform these experiments is publicly available at [12]. Based on these experiments we would like to raise the following:

Conjecture 24. For every  $k \ge 2$  there exists a positive constant  $c_k > 0$  such that, if  $R_n$  is a sequence of n random intervals then

$$\lim_{n \to \infty} \frac{E[k \text{-wd}(R_n)]}{n} = c_k \tag{3}$$

Moreover  $c_k = \frac{1}{k+1}$ .

According to this conjecture, just as in the case of random permutations, the scaling behavior of the k-width changes when going from k = 1 to k = 2. Note, though, that the direction of change is different  $(\Theta(\sqrt{n}))$  to  $\Theta(\log n)$  for integer sequences,  $\Theta(\sqrt{n})$  to  $\Theta(n)$  for sequences of intervals). On the other hand, somewhat surprisingly, the scaling behavior of sets of random intervals seems to be similar to that for sequences:

Conjecture 25. For every  $k \geq 2$  there exists a positive constant  $d_k > 0$  such that, if  $W_n$  is a set of n random intervals then

$$\lim_{n \to \infty} \frac{E[k \text{-wd}(W_n)]}{n} = d_k \tag{4}$$

Experiments suggest that  $d_2 = c_2 = \frac{1}{3}$ , and similarly for k = 3, 4  $d_k = c_k = \frac{1}{k+1}$ . Therefore we conjecture that

$$d_k = c_k = \frac{1}{k+1} \text{ for all } k \ge 2.$$
 (5)

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