Mobile vs. point guards

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Abstract

We study the problem of guarding orthogonal art galleries with horizontal mobile guards (alternatively, vertical) and point guards, using "rectangular vision". We prove a sharp bound on the minimum number of point guards required to cover the gallery in terms of the minimum number of vertical mobile guards and the minimum number of horizontal mobile guards required to cover the gallery. Furthermore, we show that the latter two numbers can be calculated in linear time.

1 Introduction

The number of mobile and point guards required to control the interior of a general or an orthogonal polygon (without holes) has been well-studied as a function of the number of vertices of the polygon (in the introduction we assume the reader is familiar with the concept of mobile guards, point guards, etc., but all these notion is defined precisely in Section 2). Kahn, Klawe, and Kleitman in 1980 [KKK83], and a few years later Győri [Győ86], and O'Rourke [O'R87] proved that $\lfloor n/4 \rfloor$ point guards are sufficient and sometimes necessary to cover the interior of an orthogonal polygon of n vertices. Aggarwal proved in his thesis [Agg84] that any n-vertex orthogonal polygon can be covered by at most $\lfloor \frac{3n+4}{16} \rfloor$ mobile guards, and a strengthening of this result has been shown in [GM16]. These estimates are also shown to be sharp as extremal results. These theorems imply that — from an extremal point of view — only 4/3's as many point guards as mobile guards are needed. However, it was not much studied if we can say something about the ratio of these optima.

The main goal of this paper is to explore the ratio between the numbers of mobile guards and points guards required to control an orthogonal polygon without holes. At first, this appears to be hopeless, as Figure 1 shows a comb, which can

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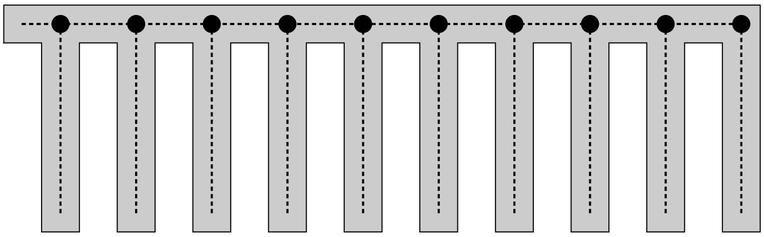


Figure 1: A comb with 10 teeth

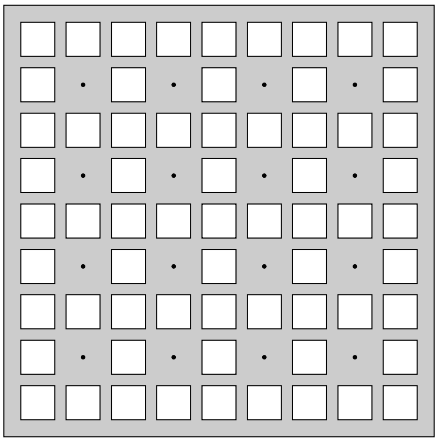


Figure 2: A polygon with holes — unlimited ratio.

be guarded by one mobile guard (whose patrol is shown by a dotted horizontal line). However, to cover the comb using point guards, one has to be placed for each tooth, so ten point guards are needed (marked by solid disks). Combs with arbitrarily high number of teeth clearly demonstrate that the minimum number of points guards required to control an orthogonal polygon cannot be bounded by the minimum size of a mobile guard system covering the comb.

Katz and Morgenstern (in [KM11]) defined and studied the notion of "horizontal sliding cameras". This notion is identical to what we call horizontal mobile r-guard (guard with rectangular vision). The main result of our paper, Theorem 2, shows that a constant factor times the sum of the minimum sizes of a horizontal and a vertical mobile r-guard system can be used to estimate minimum size of a point r-guard system. It is surprising to have such a result after having the comb, but it is similarly unexpected that even this ratio cannot be bounded if the region may contain holes.

Take, for example, Figure 2, which generally contains $3k^2 + 4k + 1$ square holes (in the figure $k = 4$). The regions covered by line of sight vision by the black dots are pairwise disjoint, because the distance between adjacent square holes is less than half of the length of a square hole's side. Therefore no two of the black dots can be covered by one point guard, so at least k^2 point guards are necessary to control gallery. However, $2k + 2$ horizontal mobile guards can easily cover the polygon, and the same holds for vertical mobile guards.

In the last section of the paper, we show that a minimum size horizontal mobile r-guard system can be found in linear time (Theorem 21). This improves the result in [KM11], where it is shown that this problem can be solved in polynomial time.

2 Definitions and the main theorem

A **rectilinear domain** is the closed region of the plane ($\mathbb{R}^2$) whose boundary is an orthogonal polygon, i.e. a closed polygon without self-intersection, so that each segment is parallel to one of the two axes. Consequently, all of its angles are $\pi/2$ (convex) or $3\pi/2$ (reflex).

To avoid confusion, we state that throughout the paper, **vertices** and **sides** refer to subsets of an orthogonal polygon or a rectilinear domain; whereas any **graph** will be defined on a set of **nodes**, of which some pairs are joined by some **edges**. Given a graph G, the edge set $E(G)$ is a subset of the 2-element subsets of the vertices $V(G)$.

Unless otherwise noted, we adhere to the same terminology in the subject of art galleries as O'Rourke [O'R87]. For technical reasons, compared to the definitions in [O'R87] we need to assume extra conditions. In Lemma 1, we prove that we may make the assumptions typeset in *italics* in the following definitions without loss of generality.

Two points x, y in a rectilinear domain D have $\boldsymbol{r}$**-vision** of each other (alternatively, x is r-visible from y) if there exists an axis-aligned *non-degenerate* rectangle in D which contains both x and y. This vision is natural to use in orthogonal art galleries instead of the more powerful line of sight vision. For example, r-vision is invariant on the transformation depicted on Figure 3.

A **point** $\boldsymbol{r}$**-guard** is a point $y \in D$, *such that the two maximal axis-parallel line segments in D containing y do not intersect vertices of D.* A set of points guards $\boldsymbol{r}$**-cover** D if any point $x \in D$ is r-visible from a member of the set. Such a set is called a **point** $\boldsymbol{r}$**-guard system**.

A **vertical mobile** $\boldsymbol{r}$**-guard** is a vertical line segment in D, *such that a maximal axis-parallel line segment in D containing it does not intersect vertices of D.* **Horizontal** mobile guards are defined analogously. A set of (horizontal/vertical/mixed) mobile guards $\boldsymbol{r}$**-cover** D if for any point $x \in D$ there exists a mobile guard and a point y on its line segment such that x is r-visible from y. Such a set is called a **(horizontal/vertical) mobile** $\boldsymbol{r}$**-guard system**.

Lemma 1. *Any rectilinear domain D can be transformed into another rectilinear domain D' so that the classical point guard r-cover, and the classical vertical/horizontal mobile guard r-cover problems in D are equivalent to the respective problems, as per our our definitions, in D'.*

Proof. We can perform the transformation depicted in Figure 3 in D. There is a trivial correspondence between the point and mobile guards of D and D' such that taking this correspondence guard-wise transforms a guarding system of D into a guarding system of D', and vica versa.

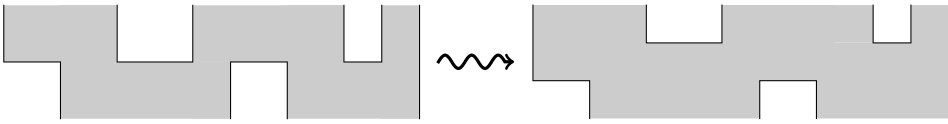


Figure 3: After this transformation, those mobile guards whose maximal containing line segment do not intersect vertices of the rectilinear domain, are just as powerful as mobile guards that are not restricted in such a way.

After performing this operation at every vertical and horizontal occurrence, we get a rectilinear domain D', in which any vertical or horizontal line segment is contained in a non-degenerate rectangle in D'. Therefore degenerate vision between any two points implies non-degenerate vision between the pair. Furthermore, the line segment of any mobile guard can be translated along its normal while staying inside D', and this clearly does not change the set of points r-covered by the guard. Similarly, we can slightly perturb the position of a point guard without changing the set of points of D' it r-covers. $\square$

Theorem 2. *Given a rectilinear domain D let m_V be the minimum size of a vertical mobile r-guard system of D, let m_H be defined analogously for horizontal mobile r-guard systems, and finally let p be the minimum size of a point r-guard system of D. Then*

$$\left\lfloor \frac{4(m_V + m_H - 1)}{3} \right\rfloor \geq p.$$

We omit the prefix "r-" from now on, if it is not deemed confusing. Before moving onto the proof of Theorem 2, we discuss the aspects of its sharpness.

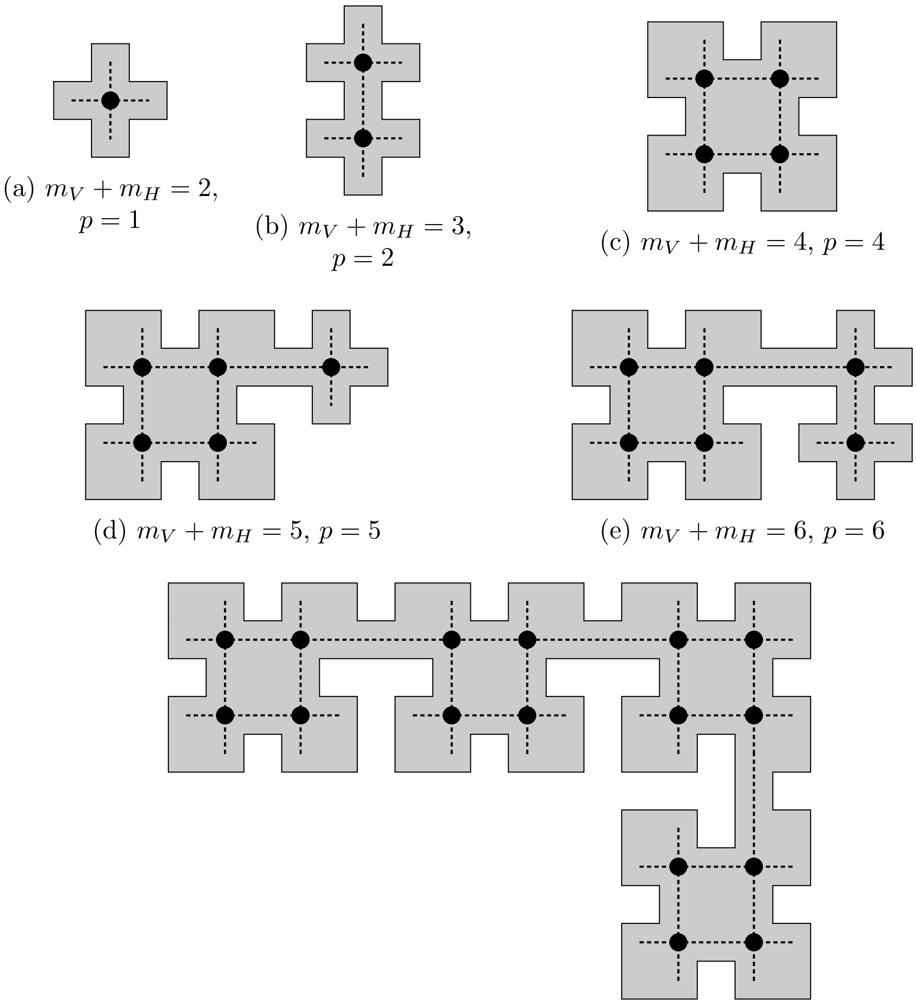
For $m_V + m_H \leq 6$, sharpness of the theorem is shown by the examples in Figure 4. The polygon in Figure 4f can be easily generalized to one satisfying $m_V + m_H = 3k + 1$ and $p = 4k$. For $m_V + m_H = 3k + 2$ and $m_V + m_H = 3k + 3$, we can attach 1 or 2 plus signs to the previously constructed polygons, as shown in Figure 4d and 4e. Thus Theorem 2 is sharp for any fixed value of $m_V + m_H$.

By stringing together a number of copies of the polygons in Figure 4a and 4c in an L-shape (Figure 4f is a special case of this), we can construct rectilinear domains for any (m_H, m_V) pair satisfying $m_V \geq 2(m_H - 1)$ and $m_H \geq 2(m_V - 1)$, such that the polygon satisfies Theorem 2 sharply. The analysis in Section 3 immediately yields that if $m_V = 1$ or $m_H = 1$, then $m_V + m_H - 1$ is an upper bound for the minimum size of a point guard system (see Proposition 9), whose sharpness is shown by combs (Figure 1).

3 Translating the problem into the language of graphs

For graph theoretical notation and theorems used in this paper (say, the block decomposition of graphs), the reader is referred to [Die10].

Definition 3 (Chordal bipartite graph, [GG78])**.** A graph G is chordal bipartite iff any cycle C of ≥ 6 vertices of G has a chord (that is $E(G[C]) \supsetneq E(C)$).



(a) $m_V + m_H = 2$, $p = 1$

(b) $m_V + m_H = 3$, $p = 2$

(c) $m_V + m_H = 4$, $p = 4$

(d) $m_V + m_H = 5$, $p = 5$

(e) $m_V + m_H = 6$, $p = 6$

(f) $m_V + m_H = 13$, $p = 16$

Figure 4: Vertical dotted lines: a minimum size vertical mobile guard system; Horizontal dotted lines: a minimum size horizontal mobile guard system; Solid disks: a minimum size point guard system.

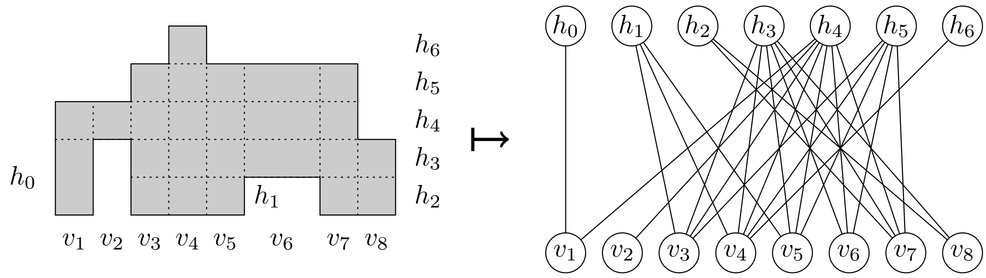


Figure 5: A rectilinear domain and its associated pixelation graph

Let A_V be the set of internally disjoint rectangles we obtain by cutting vertically at each reflex vertex of D. Similarly, let A_H be defined analogously for horizontal cuts of D. We may refer to the elements of these sets as **vertical and horizontal slices**, respectively. Let G be the intersection graph of A_H and A_V, that is $h \in A_H$ and $v \in A_V$ are joined by an edge iff $\text{int}(h) \cap \text{int}(v) \neq \emptyset$; see Figure 5. We may also refer to G as the pixelation graph of D. Clearly, the **set of pixels** $\{\cap e \mid e \in E(G)\}$ is a cover of D. Let us define $c(e)$ as the center of gravity of $\cap e$ (the pixel determined by e).

Let us define the horizontal **R**-tree T_H of D:

$$T_H = \Big(A_H, \Big\{ \{h_1, h_2\} \subseteq A_H \ : \ h_1 \neq h_2,\ h_1 \cap h_2 \neq \emptyset \Big\} \Big),$$

i.e., T_H is the intersection graph of the horizontal slices of D. Similarly, we define T_V as the intersection graph of the vertical slices of D.

Claim 4. *Both T_H and T_V are trees.*

Proof. Connectedness of the trees follows from the connectedness of D. Furthermore, given an edge $e = \{h_1, h_2\} \in E(T)$, the set $D - \cap e = D - h_1 \cap h_2 = D - \partial h_1 \cap \partial h_2$ has two components, therefore $T_H - e$ must have two components as well. □

Lemma 5. *G is a connected chordal bipartite graph.*

Proof. Connectedness of D immediately yields that G is connected too. Suppose C is a cycle of ≥ 6 vertices in G. For each node of the cycle C, connect the centers of gravity of its two incident edges with a line segment. This way we get an orthogonal polygon P in D.

If P is self-intersecting, then the vertices which are represented by the two intersecting line segments are intersecting. This clearly corresponds to a chord of C in G.

If P is simple, then the number of its vertices is $|V(C)|$, thus one of them is a reflex vertex, say $c(v_1 \cap h_1)$ is one. As P lives in D, its interior is a subset of D as well (here we use that D is simply connected). The simpleness of P also implies that the vertical line segment intersecting $c(v_1 \cap h_1)$ after entering the interior of P at $c(v_1 \cap h_1)$, intersects P at least once more when it emerges, say at $c(v_1 \cap h_2)$. As this is not an intersection of the line segments corresponding to two vertices of D, the edge $\{v_1, h_2\}$ is a chord of C. □

It is worth mentioning that even if D is a rectilinear domain with rectilinear hole(s), G may still be chordal bipartite. Take, for example, $[0,3]^2 \setminus (1,2)^2$; the graph associated to it has only one cycle, which is of length 4.

We will use the following technical claim to translate r-vision of points of D into relations in G.

Claim 6. *Let $e_1, e_2 \in E(G)$, where $e_1 = \{v_1, h_1\}$ and $e_2 = \{v_2, h_2\}$, where $v_1, v_2 \in A_V$ and $h_1, h_2 \in A_H$. The points $p_1 \in \text{int}(\cap e_1)$ and $p_2 \in \text{int}(\cap e_2)$ have r-vision of each other in D iff $e_1 \cap e_2 \neq \emptyset$ or $e_1 \cup e_2$ induces a 4-cycle in G.*

Proof. If $v_1 \in e_1 \cap e_2$, then $p_1, p_2 \in v_1$, therefore p_1 and p_2 have r-vision of each other. If $h_1 \in e_1 \cap e_2$, the same holds. If $\{v_1, h_1, v_2, h_2\}$ induces a 4-cycle, then $\text{Conv}((v_1 \cap h_1) \cup (v_1 \cap h_2)) \subseteq v_1 \subseteq D$ by v_1's convexity. Moreover,

$$\begin{aligned} B = &\text{Conv}((v_1 \cap h_1) \cup (v_1 \cap h_2)) \cup \text{Conv}((v_1 \cap h_2) \cup (v_2 \cap h_2)) \cup \\ &\cup \text{Conv}((v_2 \cap h_2) \cup (v_2 \cap h_1)) \cup \text{Conv}((v_2 \cap h_1) \cup (v_1 \cap h_1)) \end{aligned}$$

is contained in D. Since D is simply connected, we have $\text{Conv}(B) \subseteq D$, which is a rectangle containing both p_1 and p_2.

In the other direction, suppose $e_1 \cap e_2 = \emptyset$. If R is an axis-aligned rectangle which contains both p_1 and p_2, then R clearly intersects the interiors of each element of $e_1 \cup e_2$, which implies that $\text{int}(v_2) \cap \text{int}(h_1) \neq \emptyset$ and $\text{int}(v_1) \cap \text{int}(h_2) \neq \emptyset$. Thus $e_1 \cup e_2$ induces a cycle in G. □

This easily implies the following claim.

Claim 7. *Two points $p_1, p_2 \in D$ have r-vision of each other iff $\exists e_1, e_2 \in E(G)$ such that $p_1 \in \cap e_1$, $p_2 \in \cap e_2$, and either $e_1 \cap e_2 \neq \emptyset$ or $e_1 \cup e_2$ induces a 4-cycle in G.*

These claims motivate the following definition.

Definition 8 (r-vision of edges)**.** For any $e_1, e_2 \in E(G)$ we say that e_1 and e_2 have r-vision of each other iff $e_1 \cap e_2 \neq \emptyset$ or there exists a C_4 in G which contains both e_1 and e_2.

Let $Z \subseteq E(G)$ be such that for any $e_0 \in E(G)$ there exists an $e_1 \in Z$ so that e_1 has r-vision of e_0. According Claim 7, if we choose a point from $\text{int}(\cap e_1)$ for each $e_1 \in Z$ then we get a point r-guard system of D.

Observe that any vertical mobile r-guard is contained in $\text{int}(v)$ for some $v \in A_V$ (except ≤ 2 points of the patrol). Extending the line segment the mobile guard patrols increases the area that it covers, therefore we may assume that this line segment intersects each element of $\{\text{int}(\cap e) \mid v \in e \in E(G)\}$, which only depends on some $v \in A_V$. Using Claim 7, we conclude that the set which such a mobile guard covers with r-vision is exactly $\cup\{h \in A_H \mid \{h, v\} \in E(G)\}$. The analogous statement holds for horizontal mobile guards as well.

Thus a vertical mobile guard system of D can be represented by a set $M_V \subseteq A_V$, which dominates each element of A_H in G. Similarly, a horizontal mobile guard system has a representative set $M_H \subseteq A_H$, which dominates A_V in G. Equivalently, $M_H \cup M_V$ is a totally dominating set of G, ie., a subset of $V(G)$ that dominates every vertex in G.

As promised, the following claim has a very short proof using the definitions and claims of this section.

Proposition 9. *If $m_V = 1$ or $m_H = 1$, then $p \leq m_V + m_H - 1$.*

Proof. Let Z be the set of edges of G induced by $M_H \cup M_V$. Clearly, $G[M_H \cup M_V]$ is a star, thus $|Z| = |M_H| + |M_H| - 1$.

We claim that Z covers $E(G)$. There exist two slices, $h_1 \in M_H$ and $v_1 \in M_V$, which are joined by an edge to v_0 and h_0, respectively. Since $G[M_H \cup M_V]$ is a star, $\{v_1, h_1\} \in Z$. This edge has r-vision of e_0, as either $\{v_1, h_1\}$ intersects e_0, or v_0, h_0, v_1, h_1 induces a C_4 in Z. □

Finally, we can state Theorem 2 in a stronger form, conveniently via graph theoretic concepts.

Theorem 2′. *Suppose both A_V and A_H are sets of internally disjoint axis-parallel rectangles of a rectilinear domain D. Also, suppose that for any $v \in A_V$, its top and bottom sides are a subset of ∂D, and for any $h \in A_H$, its left and right sides are a subset of ∂D. Furthermore, suppose that their intersection graph*

$$G = \big(A_H, A_V, \big\{\{h, v\} \subseteq A_V \cup A_H \;:\; \mathrm{int}(v) \cap \mathrm{int}(h) \neq \emptyset\big\}\big)$$

is connected.

If $M_V \subseteq A_V$ dominates A_H in G, and $M_H \subseteq A_H$ dominates A_V in G, then there exists a set of edges $Z \subseteq E(G)$ such that any element of $E(G)$ is r-visible from some element of Z, and

$$|Z| \leq \frac{4}{3} \cdot (|M_V| + |M_H| - 1).$$

Now we are ready to prove the main theorem of this paper.

4 Proof of Theorem 2′

Both A_H and A_V can be extended to a partition of D (while preserving the assumptions on), so G is a subgraph induced by $A_H \cup A_V$ in a chordal bipartite graph (see Lemma 5), thus G is chordal bipartite as well. Let $M = G[M_V \cup M_H]$ be the subgraph induced by the dominating sets. Notice, that the bichordality of G is inherited by M.

Claim 11. *If M is connected, then any edge $e_0 = \{h_0, v_0\} \in E(G)$ is r-visible from some edge of M.*

Proof. As $\Gamma_G(M_V \cup M_H) = V(G)$, there exists two vertices, $v_1 \in M_V$ and $h_1 \in M_H$, such that $\{v_1, h_0\}, \{v_0, h_1\} \in E(G)$.

If $v_0 \in M_V$ or $h_0 \in M_H$, then $\{v_0, h_1\}$ or $\{v_1, h_0\}$ is in $E(M)$.

Otherwise, there exists a path in M, whose endpoints are v_1 and h_1, and this path and the edges $\{v_1, h_0\}$,$\{h_0, v_0\}$,$\{v_0, h_1\}$ form a cycle in G. By the bichordality of G, there exists a C_4 in G which contains an edge of M and e_0. □

We distinguish 3 cases based on the level connectivity of M.

Case 1 M is 2-connected

The 4/3 constant in the statement of Theorem $2'$ is determined by this case. Knowing this, it is not surprising that this is the longest and most complex case of the the proof.

If $E(M)$ consists of a single edge e, then $Z = \{e\}$ is clearly a point guard system of G by Claim 11.

Suppose now, that M has more than two vertices. Any edge of M is contained in a cycle of M, and by the bichordality property, there is such a cycle of length 4. It is easy to see that the convex hull of the pixels determined by the edges of a C_4 is a rectangle. Define

$$D_M = \bigcup_{\{e_1,e_2,e_3,e_4\} \text{ is a } C_4 \text{ in } M} \operatorname{Conv}\left(\bigcup_{i=1}^{4} \cap e_i\right).$$

The simply connectedness of D implies that $D_M \subseteq D$.

Claim 12. *For any slice $s \in V(M)$ the intersection of s and D_M is connected.*

Proof. Suppose that $e_1, e_2 \in E(M)$ are such that $\cap e_1$ and $\cap e_2$ are in two different components of $s \cap D_M$. Since M is 2-connected, there is a path connecting $e_1 \setminus \{s\}$ and $e_2 \setminus \{s\}$ in $M - s$.

Take the shortest cycle in M containing e_1 and e_2. If this cycle contains 4 edges, then the convex hull of their pixels is in D_M, which is a contradiction. Similarly, if the cycle contains more than 4 edges, the chordality of M implies that s is joined to every second node of the cycle, which contradicts our assumption that $s \cap D_M$ is disconnected. □

Claim 13. *For any slice $s \in V(G)$, the intersection of* $\operatorname{int}(s)$ *and D_M is connected.*

Proof. If $s \in V(M)$, we are done by Claim 12. If $s \in V(G) \setminus V(M)$, let $e_1, e_2 \in E(G)$ be the two edges such that $e_1 \cap e_2 = \{s\}$, $\partial(\cap e_1) \bigcap \partial D_M \neq \emptyset$, $\partial(\cap e_2) \bigcap \partial D_M \neq \emptyset$. Then, we must have $(e_1 \bigcup e_2) \setminus \{s\} \subseteq V(M)$. Take the shortest path in M joining $e_1 \setminus \{s\}$ to $e_2 \setminus \{s\}$. The proof can be finished as that of the previous claim. $\square$

Let $B_H \subset M_H$ be the set of those slices whose top and bottom sides both intersect ∂D_M in an uncountable number of points of $\mathbb{R}^2$.

For technical reasons, we split each element of $h \in B_H$ horizontally through $c(h)$ to get two isometric rectangles in $\mathbb{R}^2$; let the set of the resulting refined horizontal slices be B'_H. Let $A'_H = B'_H \bigcup A_H \setminus B_H$ and $M'_H = B'_H \bigcup M_H \setminus B_H$. Let $A'_V = A_V$, $M'_V = M_V$. Let τ be the function which maps $h \in B'_H$ to the $\tau(h) \in A_H$ for which $h \subseteq \tau(h)$ holds, and let τ be the identity function on $A'_V \cup A'_H \setminus B'_H$.

Let G' be the intersection graph of A'_H and A'_V (as in the statement of Theorem 2′). Also, let $M' = G'[M'_H \cup M'_V] = \tau^{-1}(M)$. Observe that τ naturally defines a graph homomorphism $\tau : G' \to G$ (edges are mapped vertex-wise).

Claim 14. *In G', the set M'_H dominates A'_V, and M'_V dominates A'_H. Furthermore, if $Z' \subseteq E(M')$ is a point guard system of G', then $Z = \tau(Z') \subseteq E(M)$ is a point guard system of G.*

Proof. The first statement of this claim holds, since τ maps non-edges to non-edges, and both $M'_H = \tau^{-1}(M_H)$ and $M'_V = \tau^{-1}(M_V)$ by definition. As τ is graph homomorphism, it preserves r-visibility, which implies the second statement of this claim. $\square$

Notice, that M' is 2-connected and $D_M = D_{M'}$. An edge $e \in E(M')$ falls into one of the following 4 categories:

Convex edge: 3 vertices of $\cap e$ fall on ∂D_M, e.g. the edge $\{h_2, v_1\}$ on Figure 6;

Reflex edge: exactly 1 vertex of $\cap e$ falls on ∂D_M, e.g. $\{h''_3, v_3\}$ on Figure 6;

Side edge: two neighboring vertices of $\cap e$ fall on ∂D_M, e.g. $\{h_1, v_4\}$ on Figure 6;

Internal edge: zero vertices of $\cap e$ fall on D_M, e.g. $\{h_2, v_3\}$ on Figure 6.

Notice that on Figure 6, the edge $\{h_3, v_5\}$ fall into neither of the previous categories, as two non-neighboring (diagonally opposite) vertices of its pixel $h_3 \cap v_5$ fall on D_M. This is clearly cannot happen with edges of G', but G may contain edges of this type.

Observe that τ maps convex edges to convex edges, and side edges to side edges. Conversely, the preimages of a convex edge are a convex edge and a side edge (M' is 2-connected), the preimages of a side edge are two side edges, and the preimages of a reflex edge are a reflex edge and an internal edge.

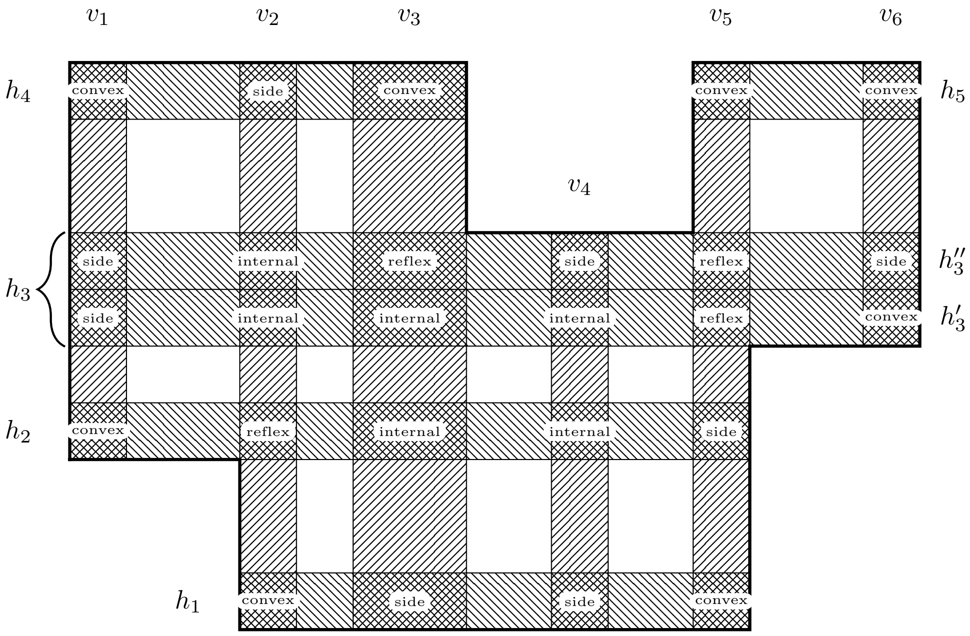


Figure 6: We have $M_H = \{h_1, h_2, h_3, h_4, h_5\}$, $M'_H = M_H - h_3 + h'_3 + h''_3$, and $M_V = M'_V = \{v_1, v_2, v_3, v_4, v_5, v_6\}$. The thick line is the boundary of D_M. Each rectangle pixel is labeled according to the type of its corresponding edge of M'.

Definition 15. For any two edges $e_1, e_2 \in E(M')$, where $e_1 = \{v_1, h_1\}$ and $e_2 = \{v_2, h_2\}$, we write $e_2 \to e_1$ (e_2 dominates e_1) iff either

- $e_1 \cap e_2 \subset A'_H$, and $\exists h_3, h_4 \in M'_H$ such that $\{v_1, v_2, h_3, h_4\}$ induces a C_4 in M', and $h_1 = h_2$ is between h_3 and h_4; or
- $e_1 \cap e_2 \subset A'_V$, and $\exists v_3, v_4 \in M'_V$ such that $\{v_3, v_4, h_1, h_2\}$ induces a C_4 in M', and $v_1 = v_2$ is between v_3 and v_4; or
- $e_1 \cap e_2 = \emptyset$, and $\exists v_3 \in M'_V$ and $h_3 \in M'_H$ such that both $\{v_1, h_2, v_2, h_3\}$ and $\{h_1, v_3, h_2, v_2\}$ induces a C_4 in M'; furthermore, v_1 is between v_2 and v_3, and h_1 is between h_2 and h_3.

We write $e_2 \leftrightarrow e_1$ iff both $e_2 \to e_1$ and $e_1 \to e_2$ hold. Note that $\leftrightarrow$ is a symmetric, but generally intransitive relation.

For example, on Figure 6, $\{h_1, v_3\} \leftrightarrow \{h''_3, v_3\}$, and $\{h_1, v_2\} \to \{h''_3, v_3\}$. Also, $\{h''_3, v_3\} \leftrightarrow \{h''_3, v_1\}$, but $\{h''_3, v_3\} \not\to \{h'_3, v_1\}$.

We will search for a point guard system of M' with very specific properties, which are described by the following definition.

Definition 16. Suppose $Z' \subseteq E(M')$ is such, that

1. Z' contains every convex edge of M,
2. for any non-internal edge $e_1 \in E(M) \setminus Z'$, there exists some $e_2 \in Z'$ for which $e_2 \to e_1$, and

3. for each $h_0 \in A'_H$ for which $\mathrm{int}(h_0) \cap D_M \neq \emptyset$ holds, $\exists\{h_2, v_2\} \in Z'$ such that $\{h_0, v_2\} \in E(M')$ and $N_{M'}(h_2) \supseteq N_{G'}(h_0) \bigcap M'_V$.

If these three properties hold, we call Z' a hyperguard of M'.

Lemma 17. *Any hyperguard Z' of M' is a point guard system of G', i.e., any edge of G' is r-visible from some element of Z'.*

Proof. Let $e_0 = \{v_0, h_0\} \in E(G')$ be an arbitrary edge. By Claim 11, there exists an edge $e_1 \in E(M')$ which has r-vision of e_0, and we also suppose that e_1 is chosen so that $\mathrm{dist}(\cap e_0, \cap e_1)$ is minimal.

Trivially, if $e_1 \in Z'$ (for example, if e_1 is a **convex edge** of M'), then e_0 is r-visible from e_1. Assume now, that $e_1 \notin Z'$.

- If e_1 is a **reflex or side edge** of M', then $\exists e_2 \in Z'$ so that $e_2 \to e_1$. We claim that e_2 has r-vision of e_0 in G'.

  1. If $e_1 \cap e_2 \subset A'_H$: by the choice of e_1 and e_2, v_1 is joined to h_0, h_3, h_4 in G'. The choice of e_1 guarantees that $v_1 \cap h_0$ is between $v_1 \cap h_3$ and $v_1 \cap h_4$. Therefore $\mathrm{int}(v_2 \cap h_0) \neq \emptyset$, so $\{v_0, h_0, v_2, h_1(= h_2)\}$ induces a C_4 in G'.
  2. If $e_1 \cap e_2 \subset A'_V$: the proof proceeds analogously to the previous case.
  3. If $e_1 \cap e_2 = \emptyset$: by the choice of e_1 and e_2, v_1 is joined to h_0, h_3, h_2 in G', and v_1 is joined to v_0, v_3, v_2 in G'. The choice of e_1 guarantees that $v_1 \cap h_0$ is between $v_1 \cap h_3$ and $v_1 \cap h_2$, and that $v_0 \cap h_1$ is between $v_3 \cap h_1$ and $v_2 \cap h_1$. Therefore $\mathrm{int}(v_2 \cap h_0) \neq \emptyset$ and $\mathrm{int}(v_0 \cap h_2) \neq \emptyset$, so $\{v_0, h_0, v_2, h_2\}$ induces a C_4 in G'.

  In any of the three cases, e_0 is r-visible from e_2 in G'.

- If e_1 is an **internal edge** of M', then $\cap e_0 \subset D_M$. By the 3rd property of Z', there $\exists\{h_2, v_2\} \in Z'$ such that $\{h_0, v_2\} \in E(M')$ and $N_{M'}(h_2) \supseteq N_{G'}(h_0) \bigcap M'_V$. An easy argument (use that $D_M \subset D$ are both simply connected) gives that $\{v_0, h_2\} \in E(G')$. Thus $\{v_0, h_2, v_2, h_0\}$ induces a C_4 in G', so e_0 is r-visible from $\{v_2, h_2\} \in Z'$.

We have verified the statement in every case, so the proof of this lemma is complete. □

Notice, that the set of all convex, reflex, and side edges of $E(M')$ form a hyperguard of M'. By Lemma 17, this set is a point guard system of G', and Claim 14 implies that its τ-image is a point guard system of G. The cardinality of the τ-image of this hyperguard is bounded by $2|V(M)| - 4$ (we will see this shortly), which is already a magnitude lower than what the trivial choice of $E(M)$ would give (generally, $|E(M)|$ can be equal to $\Omega(|V(M)|^2)$).

Further analysis of M' allows us to lower the coefficient 2 to 4/3. Readers, who are only interested in a result which is sharp up to a constant factor, may skip to Case 2.

Let the number of convex, side, and reflex edges in M' be c', s', and r', respectively. Claim 12 and 13 allow us to count these objects.

1. The number of reflex vertices of D_M is equal to r': any reflex vertex is a vertex of a reflex edge, and the way M' and D_M is constructed guarantees that exactly one vertex of the pixel of a reflex edge is a reflex vertex of D_M.
2. The number of convex vertices of D_M is equal to c': any convex vertex is a vertex of the pixel of a convex edge, and the way D_M is constructed guarantees that exactly one vertex of the pixel of a convex edge is a convex vertex.
3. The cardinality of $V(M')$ is $c' + s'/2$: the first and last edge incident to any element of $V(M')$ ordered from left-to-right (for elements of M'_H) or from top-to-bottom (for elements of M'_V) is a convex or a side edge. Conversely, any convex edge is the first or last incident edge of exactly one element of M'_H and one element of M'_V. A side edge is the first or last incident edge of exactly one element of $V(M')$.
4. For any reflex edge $e_1 = \{v_1, h_1\} \in E(M')$, there is exactly one reflex or side edge in $E(M')$ which contains v_1 and is in the $\leftrightarrow$ relation with e_1, and the same can be said about h_1.
5. Any side edge $e_1 \in E(M')$ is in $\leftrightarrow$ relation with exactly one reflex or side edge which it intersects. The intersection is the slice in $V(M')$ on which e_1 is a boundary edge.

Define the **auxiliary graph $\boldsymbol{X}$** as follows: let $V(X)$ be the set of reflex and side edges of M', and let

$$E(X) = \Big\{\{e, f\} : \ e \neq f, \ e \cap f \neq \emptyset, \ e \leftrightarrow f\Big\}.$$

By our observations, X is the disjoint union of some cycles and $s'/2$ paths. This structure allows us to select a hyperguard which contains a subset of the reflex and side edges of M', instead of the whole set.

Constructing a hyperguard $\boldsymbol{Z'}$ of $\boldsymbol{M'}$. We will define $(Z'_j)_{j=0}^{\infty}$, a sequence of (set theoretically) increasing sequence of subsets of $E(M')$, and $(X_j)_{j=0}^{\infty}$, a decreasing sequence of induced subgraphs of X.

Additionally, we will define a function $w_j : V(X) \to \{0, 1, 2\}$, and extend its domain to any subgraph $H \subseteq X$ by defining $w_j(H) = \sum_{e \in V(H)} w_j(e)$. The purpose of w_j, very vaguely, is that as Z' will contain every third node of X, we need to keep count of the modulo 3 remainders. Furthermore, w_j serves as buffer in a(n implicitely defined) weight function (see inequality (2)).

For a set $E_0 \subseteq E(X)$, let the indicator function of E_0 be

$$\mathbb{1}_{E_0}(e) = \begin{cases} 1, & \text{if } e \in E_0, \\ 0, & \text{if } e \in E(X) \setminus E_0. \end{cases}$$

Let $Z'_0 = \emptyset$ and $X_0 = X$. By our previous observations, X does not contain isolated nodes. Define $w_0 : V(X) \to \{0,1,2\}$ such that

$$w_0(e) = \begin{cases} 1, & \text{if } d_{X_0}(e) = 1, \\ 0, & \text{if } d_{X_0}(e) = 0. \end{cases}$$

In each step we will define Z'_j, X_j, and w_j so that

- $Z'_{j-1} \subseteq Z'_j$, $X_j \subseteq X_{j-1}$,
- $\{e \in V(X_j) \mid d_{X_j}(e) = 1\} \subseteq w_j^{-1}(1)$,
- $\{e \in V(X_j) \mid d_{X_j}(e) = 0\} = w_j^{-1}(2)$, and
- $\forall e_0 \in V(X) \setminus V(X_j)$, either $e_0 \in Z'_j$, or $\exists e_1 \in Z'_j$ so that $e_1 \to e_0$.

If these hold, then for any path component P_j in X_j, we have $w_j(P_j) \geq 2$.

Phase 1 Let the set of convex edges of M' be C'. Let

$$\begin{aligned}
S' &= \Big\{ e \in V(X) : \ \tau(e) \text{ is a side edge}, \ \exists f \in V(X) \ e \leftrightarrow f, \ e \cap f \subseteq B'_H \Big\}, \\
T' &= \Big\{ f \in V(X) : \ \exists e \in S' \ f \leftrightarrow e, \ \tau^{-1}(\tau(f)) \setminus \{f\} \to N_X(\tau^{-1}(\tau(e))) \setminus \{f\} \Big\}, \\
U' &= \tau^{-1}(\tau(C')) \setminus C', \\
Q' &= \bigcup_{\substack{e_1, e_4 \in S' \cup U' \\ e_2, e_3 \in V(X) \\ e_1 \leftrightarrow e_2, e_2 \leftrightarrow e_3, e_3 \leftrightarrow e_4}} \{e_1, e_2, e_3, e_4\}.
\end{aligned}$$

Take

$$\begin{aligned}
Z'_1 &= \tau^{-1}\left(\tau(C') \bigcup \tau(T')\right), \\
X_1 &= X - T' - N_X(T') - U' - N_X(U'), \\
w_1 &= w_0 - \mathbb{1}_{S'} - \mathbb{1}_{U'} + \sum_{f \in T'} \mathbb{1}_{N_X(N_X(f)) \setminus \{f\} \setminus Q'} + \sum_{e \in U'} \mathbb{1}_{N_X(N_X(e)) \setminus \{e\} \setminus Q'}.
\end{aligned}$$

Phase 2 Take a cycle $e_1, e_2, \ldots, e_{2k_j}$ in X_j ($k_j \geq 2$, $j \geq 1$). This set of nodes of X_j is the edge set of a cycle of length $2k_j$ in M'.

- If $2k_j = 4$, observe that $e_1 \leftrightarrow e_2$, $e_1 \leftrightarrow e_4$, $e_2 \leftrightarrow e_3$, $e_4 \leftrightarrow e_3$ together imply that $e_1 \leftrightarrow e_3$. Take

$$\begin{aligned}
Z'_{j+1} &= \{e_1\} \bigcup Z'_j, \\
X_{j+1} &= X_j - \{e_1, e_2, e_3, e_4\}, \\
w_{j+1} &= w_j.
\end{aligned}$$

- If $2k_j \geq 6$, the chordal bipartiteness of M' implies that without loss of generality there is a chord $f \in E(M')$ which forms a cycle with e_1, e_2, e_3 in M'. Take

$$\begin{aligned} Z'_{j+1} &= \{f\} \bigcup Z'_j, \\ X_{j+1} &= X_j - \{e_{2k_j}, e_1, e_2, e_3, e_4\}, \\ w_{j+1} &= w_j + \mathbb{1}_{e_5} + \mathbb{1}_{e_{2k_j-1}}. \end{aligned}$$

Iterate this step until X_{j_1} is cycle-free.

Phase 3 Take a path $e_1, e_2, \ldots, e_k$ in X_j (for $j \geq j_1$), such that

$$E\left(M'\left[\bigcup_{i=2}^{k-1} e_i\right]\right) \setminus \{e_2, \ldots e_{k-1}\} \neq \emptyset.$$

Using the bipartite chordality of M', there exists a chord $f \in E(M')$ which forms a C_4 with $\{e_{l-1}, e_l, e_{l+1}\}$, where $3 \leq l \leq k-2$. It is easy to see that $e_{l-2} \leftrightarrow e_{l-1}$ implies $f \to e_{l-2}$ and $f \to e_{l-1}$. Similarly, we have that $f \to e_{l+1}$ and $f \to e_{l+2}$. Also, $f \to e_{l-1}$ and $f \to e_{l+1}$ together imply $f \to e_l$. Therefore, we take

$$\begin{aligned} Z'_{j+1} &= \{f\} \bigcup Z'_j, \\ X_{j+1} &= X_j - \{e_{l-2}, e_{l-1}, e_l, e_{l+1}, e_{l+2}\}, \\ w_{j+1} &= w_j + \mathbb{1}_{\{\mathrm{dist}_X(\bullet, e_l)=3\}}. \end{aligned}$$

Iterate this step until X_{j_2} is free of the above defined paths.

Phase 4 The set A'_H is the subset of the nodes of a horizontal $\mathbf{R}$-tree of D. Let $h_{\mathrm{root}} \in A'_H$ be an arbitrarily chosen node serving as the root of the horizontal $\mathbf{R}$-tree. Process the elements of A'_H in decreasing distance (measured in the horizontal $\mathbf{R}$-tree) from h_{root}.

Suppose $h_0 \in A'_H$ is the next horizontal slice to be processed. If $\mathrm{int}(h_0) \cap D_M = \emptyset$ or $h_0 \in M'_H$, then move on to the next slice of A'_H, as the 3rd property of Z' is satisfied by any edge of M' incident to h_0.

Suppose now, that $h_0 \notin M'_H$. It is easy to see that there exists a C_4 in M' whose edge set $\{e_1, e_2, e_3, e_4\}$ satisfies

$$h_0 \cap D_M \subset \mathrm{Conv}\left(\bigcup_{i=1}^{4} \cap e_i\right).$$

Without loss of generality, we may suppose that we chose the C_4 so that the convex hull of the pixels of its edges is minimal. Then e_i (for $i = 1, 2, 3, 4$) is not an internal-edge of M', as this would contradict the choice of the C_4.

If e_i is a convex edge of M', then it is already contained in $Z'_1 \subset Z'$, so it satisfies the 3rd property of Z' for h_0, and we may skip to processing the next slice. If e_i

is a side edge of M', then for any edge $f \in Z'$ which satisfies $f \to e_i$, we have $\cap f \subset \mathrm{Conv}\left(\bigcup_{i=1}^4 \cap e_i\right)$, so f satisfies the 3$^{\text{rd}}$ property of Z' for h_0, and again, we may skip to processing the next slice.

Suppose now, that each e_i (for $i = 1, 2, 3, 4$) is a reflex edge of M'. Let $\{h_1, h_2\} = M'_H \bigcap \bigcup_{i=1}^4 e_i$ and $\{v_1, v_2\} = M'_V \bigcap \bigcup_{i=1}^4 e_i$. The minimality of the chosen C_4 implies that $\{h_1, v_1\} \leftrightarrow \{h_1, v_2\}$ and $\{h_2, v_1\} \leftrightarrow \{h_2, v_2\}$.

If $\{h_1, v_1\}, \{h_1, v_2\}$ were removed in Phase 2 or Phase 3 in one step, then the edge by which Z' is extended in the same step satisfies the 3$^{\text{rd}}$ property of Z' for h_0. The same holds for $\{h_2, v_1\}, \{h_2, v_2\}$. In both cases, we may skip to the next slice to be processed.

Without loss of generality, we may suppose that h_1 is farther away from the root of the horizontal **R**-tree than h_2.

If $\{\{h_1, v_1\}, \{h_1, v_2\}\} \cap V(X_j)$ is non-empty, take the path of P_j of X_j containing this set; otherwise let P_j be the empty graph. Observe, that Claim 13 implies that as a result of Phase 3, for any node $e \in V(P)$, its horizontal slice $e \cap M_H$ is at least as far away from the root as h_1.

Split the path P_j into two components $P_{j,1}$ and $P_{j,2}$ by deleting the edges $\{h_1, v_1\}$ and $\{h_1, v_2\}$ (if one of them is not in $E(X_j)$, then one of the components is empty), so that $\{h_1, v_1\} \notin V(P_{j,2})$ and $\{h_1, v_2\} \notin V(P_{j,1})$.

- If $|V(P_{j,1})| \not\equiv 0 \pmod 3$ or $|V(P_{j,2})| \not\equiv 0 \pmod 3$, then let Y_j be a dominating set of P_j which contains $\{h_1, v_1\}$ or $\{h_1, v_2\}$, and is minimal with respect to these conditions. Set

$$\begin{aligned} Z'_{j+1} &= Y_j \bigcup Z'_j, \\ X_{j+1} &= X_j - P_j, \\ w_{j+1}(e) &= \begin{cases} 0, & \text{if } e \in V(P_j), \\ w_j(e) & \text{if } e \notin V(P_j). \end{cases} \end{aligned}$$

  Clearly, one of $\{h_1, v_1\}$ and $\{h_1, v_2\}$ is contained in $Y_j \subset Z'_{j+1} \subseteq Z'$, and it satisfies the 3$^{\text{rd}}$ property of Z' for h_0.

- If $|V(P_{j,1})| \equiv |V(P_{j,2})| \equiv 0 \pmod 3$, then let Y_j be a minimal dominating set of P_j. Moreover, if $\{\{h_2, v_1\}, \{h_2, v_2\}\} \bigcap (V(X_j) \bigcup Z'_j)$ is non-empty, let f_j be an element of it, otherwise set $f_j = \{h_2, v_1\}$. Take

$$\begin{aligned} Z'_{j+1} &= Y_j \bigcup \{f_j\} \bigcup Z'_j, \\ X_{j+1} &= X_j - P_j - \{f_j\} - N_{X_j}(\{f_j\}), \\ w_{j+1}(e) &= \begin{cases} 0, & \text{if } e \in V(P_j) \bigcup \big\{\{h_2, v_1\}, \{h_2, v_2\}\big\}, \\ w_j(e) + 1, & \text{if } \mathrm{dist}_X(e, f_j) = 2, \\ w_j(e) & \text{otherwise.} \end{cases} \end{aligned}$$

  Observe, that f_j satisfies the 3$^{\text{rd}}$ property of Z' for h_0.

In any case, some element of $Z'_{j+1} \subseteq Z'$ satisfies the 3^{rd} property of Z' for h_0. Furthermore, this holds for any slice of A'_H between h_1 and h_2, so we skip processing these elements.

Phase 5 Lastly, we get X_{j_3} which is the disjoint union of paths and isolated nodes (or it is an empty graph). Take a component P_j of X_j (for some $j \geq j_3$). Let Y_j be a dominating set of P_j (if $|V(P_j)| = 1$, then $Y_j = V(P_j)$). Take

$$\begin{aligned} Z'_{j+1} &= Y_j \bigcup Z'_j, \\ X_{j+1} &= X_j - P_j, \\ w_{j+1}(e) &= \begin{cases} 0, & \text{if } e \in V(P_j), \\ w_j(e) & \text{if } e \notin V(P_j). \end{cases} \end{aligned}$$

By repeating this procedure, eventually X_{j_4} is the empty graph for some $j_4 \geq j_3$. Let $Z' = Z'_{j_4}$. This procedure is orchestrated in a way to guarantee that Z' is a hyperguard of M', so only an upper estimate on the cardinality of $\tau(Z')$ has to be calculated to complete the proof of Case 1.

Estimating the size of $\boldsymbol{Z = \tau(Z')}$. We have

$$|V(X_0)| = r' + s', \; w_0(X) = s', \; |B'_H| = |T'| + |U'|.$$

By definition, $|Z'_1| = c' + |U'| + 2|T'|$ and $|\tau(Z'_1)| = |Z'_1| - |B'_H|$. It is easy to check that

$$|V(X_1)| + w_1(X) + 2|U'| + 5|T'| \leq |V(X_0)| + w_0(X).$$

Therefore, we have

$$\begin{aligned} |Z'_1| + \frac{|V(X_1)| + w_1(X)}{3} &\leq c' + |U'| + 2|T'| + \frac{|V(X_1)| + w_1(X)}{3} \leq \\ &\leq c' + |B'_H| + \frac{|V(X_0)| + w_0(X) - 2|U'| - 2|T'|}{3} \leq \\ &\leq c' + |B'_H| + \frac{r' + 2s' - 2|B'_H|}{3}. \end{aligned} \tag{1}$$

We now show that

$$|Z'_{j+1}| + \frac{|V(X_{j+1})| + w_{j+1}(X)}{3} \leq |Z'_j| + \frac{|V(X_j)| + w_j(X)}{3}. \tag{2}$$

holds for any $j \geq 1$.

In Phase 2, we choose a node from each cycle of X_1. Inequality (2) is preserved, since

$$\begin{aligned} |Z'_{j+1}| &= |Z'_j| + 1, \\ |V(X_{j+1})| &= |V(X_j)| - 5 + \mathbb{1}_{\{4\}}(k_j), \\ w_{j+1}(X) &\leq w_j(X) + 2 - 2 \cdot \mathbb{1}_{\{4\}}(k_j). \end{aligned}$$

In Phase 3, for every $j_2 > j \geq j_1$, we have

$$|Z'_{j+1}| = |Z'_j| + 1,$$
$$|V(X_{j+1})| = |V(X_{j_1})| - 5,$$
$$w_{j+1}(X) \leq w_j(X) + 2.$$

Let $j_3 > j \geq j_2$. If $|V(P_{j,1})| \not\equiv 0 \pmod 3$ and $|V(P_{j,2})| \not\equiv 2 \pmod 3$, then take a dominating set of P_j containing $\{v_1, h_1\}$. We have

$$|Y_j| \leq 1 + \left\lceil \frac{|V(P_{j,1})| - 2}{3} \right\rceil + \left\lceil \frac{|V(P_{j,2})| - 1}{3} \right\rceil \leq$$
$$\leq 1 + \frac{|V(P_{j,1})| - 1}{3} + \frac{|V(P_{j,2})|}{3} = \frac{|V(P_j)| + 2}{3}.$$

Similarly, if $|V(P_{j,1})| \not\equiv 2 \pmod 3$ and $|V(P_{j,2})| \not\equiv 0 \pmod 3$, then there is a small dominating set of P_j containing $\{h_1, v_2\}$. Also, if both $|V(P_{j,1})| \equiv 2 \pmod 3$ and $|V(P_{j,2})| \equiv 2 \pmod 3$, then there is a small dominating set of P_j containing $\{h_1, v_2\}$. Thus, if $|V(P_{j,1})| \not\equiv 0 \pmod 3$ or $|V(P_{j,2})| \not\equiv 0 \pmod 3$, then

$$|Z'_{j+1}| = |Z'_j| + |Y_j| \leq |Z'_j| + \frac{|V(P_j)| + 2}{3},$$
$$|V(X_{j+1})| = |V(X_{j_1})| - |V(P_j)|,$$
$$w_{j+1}(X) \leq w_j(X) - 2.$$

If both $|V(P_{j,1})| \equiv 0 \pmod 3$ and $|V(P_{j,2})| \equiv 0 \pmod 3$, then $|Y_j| = \frac{|V(P_j)|}{3}$. Observe, that

$$\{h_1, v_1\}, \{h_1, v_2\}, \{h_2, v_1\}, \{h_2, v_2\} \notin V(P_k) \text{ for any } k < j.$$

If both $\{h_1, v_1\} \notin Z'_j$ and $\{h_1, v_2\} \notin Z'_j$, but were removed in different steps, then when $\{h_1, v_1\}$ is removed in step k we must have set $w_k(\{h_1, v_2\}) = 1$, which is the consequence of the previous observation. Thus, $w_j(\{h_1, v_2\}) = 1$. Similarly, we must have $w_j(\{h_1, v_1\}) = 1$. This reasoning holds for $\{h_2, v_1\}$ and $\{h_2, v_2\}$, as well.

If P_j is not the empty graph or $f_j \in Z(X_j)$, then inequality (2) trivially holds. If P_j is the empty graph, then $w_j(\{h_1, v_1\}) = w_j(\{h_1, v_2\}) = 1$. If $f_j \in V(X_j)$, these 2 extra weights can be used to compensate for the new degree 1 vertices of X_{j+1}. If $f_j \notin Z(X_j) \bigcup V(X_j)$, then even $w_j(\{h_2, v_1\}) = w_j(\{h_2, v_2\}) = 1$, and in total the 4 extra weights compensate for adding f_j to Z'_{j+1}.

In any case, inequality (2) holds for $j_3 > j \geq j_2$.

For any $j_4 > j \geq j_3$, we have $|Y_j| \leq \left\lceil \frac{|V(P_j)|}{3} \right\rceil \leq \frac{|V(P_j)|+2}{3}$ and $w_j(P_j) = 2$, so inequality (2) holds for j.

Summing it all up. By definition, we have

$$|Z'| = |Z'_{j_4}|,\ X_{j_4} = \emptyset,\ 0 \le w_{j_4}(X).$$

Inequality (2) is preserved by Phase 2 to 5, therefore

$$|Z'| \le |Z'_{j_4}| + \frac{|V(X_{j_4})| + w_{j_4}(X)}{3} \le |Z'_1| + \frac{|V(X_1)| + w_1(X)}{3}.$$

Lastly, using Inequality (1), we get

$$\begin{aligned}
&|Z| = |\tau(Z')| = |\tau(Z' \setminus Z'_1)| + |\tau(Z'_1)| \le |Z' \setminus Z'_1| + |Z'_1| - |B'_H| = \\
&= |Z'| - |B'_H| \le c' + \frac{r' + 2s' - 2|B'_H|}{3} = c' + \frac{(c'-4) + 2s' - 2|B'_H|}{3} = \\
&= \frac{4(c' + s'/2) - 4 - 2|B'_H|}{3} = \frac{4|V(M')| - 4 - 2|B'_H|}{3} = \\
&= \frac{4|M'_H| + 4|M'_V| - 4 - 2|B'_H|}{3} = \frac{4|M_H| + 4|B_H| + 4|M_V| - 4 - 2|B'_H|}{3} = \\
&= \frac{4(|M_H| + |M_V|) - 4}{3},
\end{aligned}$$

as desired.

Case 2 M is connected, but not 2-connected

Let the 2-connected components (or blocks) of M be M_i for $i = 1, \ldots, q$. Since induced graphs of G inherit the chordal bipartite property, by Case 1, there exists a subset $Z_i \subseteq E(M_i)$, such that for any edge $e_0 \in E(G[\Gamma_G(M_i)])$ there exists an edge $e_1 \in Z_i$ which has r-vision of e_0 in $G[\Gamma(M_i)]$, and $|Z_i| \le \frac{4}{3}(|V(M_i)| - 1)$. Let $Z = \cup_{i=1}^{q} Z_i$.

Since the intersection graph of the vertex sets of the 2-connected components is a tree (and any two components intersect in zero or one elements), we have

$$|Z| \le \frac{4}{3}\left(-q + \sum_{i=1}^{q} |V(M_i)|\right) = \frac{4\left(-q + |V(M)| + (q-1)\right)}{3} = \frac{4(|V(M)| - 1)}{3}.$$

Furthermore, given an arbitrary $e_0 = \{v_0, h_0\} \in E(G)$, there exists a $v_1 \in M_V$ and an $h_1 \in M_H$ such that $\{v_1, h_0\}, \{v_0, h_1\} \in E(G)$.

- If $v_0 \in M_V$ or $h_0 \in M_H$ then $\{v_0, h_1\}$ or $\{v_1, h_0\}$ is in $E(M)$.
- Otherwise, there exists a path in M, whose endpoints are v_1 and h_1, and this path and the edges $\{v_1, h_0\}$,$\{h_0, v_0\}$,$\{v_0, h_1\}$ form a cycle in G. By the bichordality of G, there exists a C_4 in G which contains an edge of M and e_0.

In any case, e_0 is r-visible from some $e_1 \in E(M)$. As e_1 is an edge of one of the 2-connected components M_i, we have $e_0 \subset \Gamma_G(M_i)$, therefore $e_0 \in E(G[\Gamma_G(M_i)])$. Thus, some $e_2 \in Z_i$ has r-vision of e_0.

Case 3 M has more than one connected component.

Let us take a decomposition of M into connected components M_i for $i = 1, \dots, t$.

Let $N_i = \Gamma(M_i)$, so we have $M_i \subseteq N_i$ and $\cup_{i=1}^{t} N_i = V(G)$.

For all $i > 1$ let q_i be the number of components of $G[\cup_{k=1}^{i-1} N_k \setminus \cup_{k=i}^{t} N_k]$ to which $N_i \setminus \cup_{k=i+1}^{t} N_k$ is joined in $G[\cup_{k=1}^{i} N_k \setminus \cup_{k=i+1}^{t} N_k]$. Let $F_{i,j}$ be the set of edges joining $N_i \setminus \cup_{k=i+1}^{t} N_k$ to the j^{th} component of $G[\cup_{k=1}^{i-1} N_k \setminus \cup_{k=i}^{t} N_k]$. Furthermore, let $F_{i,j}^V = \{f \in F_{i,j} \mid f \cap A_V \cap N_i \neq \emptyset\}$ and $F_{i,j}^H = \{f \in F_{i,j} \mid f \cap A_H \cap N_i \neq \emptyset\}$.

Claim 18. *For any two edges $f_1, f_2 \in F_{i,j}^V$ either $f_1 \cap f_2 \neq \emptyset$ or $\exists f_3 \in F_{i,j}^V$ such that f_3 intersects both f_1 and f_2. The analogue statement holds for $F_{i,j}^H$.*

Proof. Suppose f_1 and f_2 are disjoint. Since M_i is connected, there is a path in G whose endpoints are $f_1 \cap N_i$ and $f_2 \cap N_i$, while its internal points are in $V(M_i)$; let the shortest such path be Q_1. There is also a path in the j^{th} component of $G[\cup_{k=1}^{i-1} N_k \setminus \cup_{k=i}^{t} N_k]$ whose endpoints are $f_1 \setminus N_1$ and $f_2 \setminus N_i$, let the shortest one be Q_2.

Now Q_1, f_1, Q_2, f_2 form a cycle in $G[\cup_{k=1}^{i} N_k \setminus \cup_{k=i+1}^{t} N_k]$, which is bipartite chordal. Since $V(Q_2) \cap N_i = \emptyset$, there cannot be a chord between $V(M_i) \cap V(Q_1)$ and $V(Q_2)$. This implies that $|V(Q_1)| = 3$ by its choice, and that either $(f_1 \cap N_i) \cup (f_2 \setminus N_i)$ or $(f_2 \cap N_i) \cup (f_1 \setminus N_i)$ is a chord. □

Claim 19. *For any two edges $f^V \in F_{i,j}^V$ and $f^H \in F_{i,j}^H$, the two element set $(f^V \cap N_i) \cup (f^H \cap N_i)$ is an edge of $G[N_i]$.*

Proof. Similar to the proof of Claim 18. □

Let $f_{i,j}^V \in F_{i,j}^V$ which intersects the maximum number of edges from $F_{i,j}$, and choose $f_{i,j}^H \in F_{i,j}^H$ in the same way. If only one of these exist, let $w_{i,j}$ be the existing one, otherwise let $w_{i,j} = (f_{i,j}^V \cap N_i) \cup (f_{i,j}^H \cap N_i)$ (as in Claim 19). Let us finally define $W = \{w_{i,j} \mid i = 2, \dots, t \text{ and } j = 1, \dots, q_i\}$.

Claim 20. $|W| = t - 1$.

Proof. Observe that for every $i = 1, \dots, t$, the subgraph $G[N_i \setminus \cup_{k=i+1}^{t} N_k]$ is connected, since $M_i \subseteq N_i \setminus \cup_{k=i+1}^{t} N_k \subseteq N_i = \Gamma(M_i)$. Moreover, $G[\cup_{k=1}^{t} N_k] = G$ is connected, therefore $t - 1 = \sum_{i=2}^{t} q_i = |W|$. □

By Case 2, there exists a subset $Z_i \subseteq E(M_i)$, such that for any edge $e_0 \in E(G[N_i])$ there exists an edge $e_1 \in Z_i$ which has r-vision of e_0 in $G[N_i]$, and $|Z_i| \leq \frac{4}{3}(|V(M_i)| - 1)$.
Let $Z = W \cup \left(\cup_{i=1}^{t} Z_i\right)$. An easy calculation gives that

$$|Z| \leq (t-1) + \sum_{i=1}^{t} \frac{4|V(M_i)| - 4}{3} \leq \frac{4|V(M)| - 4t + 3(t-1)}{3} \leq$$
$$\leq \frac{4(|M_H| + |M_V| - 1)}{3}.$$

Take an arbitrary edge $e_0 = \{v_0, h_0\} \in E(G)$. We have three cases.

1. If $e_0 \in F^V_{i,j}$ for some i, j, then we claim that $f^V_{i,j} \cap e_0 \neq \emptyset$. Suppose not; by Claim 18 there exists $f \in F^V_{i,j}$ which intersects both e_0 and $f^V_{i,j}$. For any edge $e \in F^V_{i,j}$ intersecting $f^V_{i,j}$ it either intersects f too, or there is an edge intersecting both e and f. Thus f intersects at least as many edges as $f^V_{i,j}$, plus it intersects e_0 too, which contradicts the choice of $f^V_{i,j}$.

   If $w_i = f^V_{i,j}$, then w_i trivially has r-vision of e_0. If both $f^V_{i,j}$ and $f^H_{i,j}$ exist, we have two cases.

   - If $v_0 \in f^V_{i,j}$, then $v_0 \in w_i$ too, so w_i has r-vision of e_0.
   - If $h_0 \in f^V_{i,j}$, then Claim 19 yields that $\{v_0\} \cup (f^H_{i,j} \cap N_i) \in E(G)$. Thus $\Big\{\{v_0, h_0\}, f^V_{i,j}, w_i, \{v_0\} \cup (f^H_{i,j} \cap N_i)\Big\}$ is the edge set of a C_4 in G, so w_i has r-vision of e_0.

2. If $e_0 \in F^H_{i,j}$ for some i, j, the same argument as above gives that $w_{i,j}$ has r-vision of e_0.

3. If neither of the previous two cases holds, then $e_0 \in E(G[N_i])$ for some i, so some element of Z_i has r-vision of it.

Thus Z satisfies Theorem $2'$, and the proof is complete.

5 Algorithmic aspects

Finding a minimum cardinality horizontal mobile r-guard system, which is also known as the *minimum cardinality horizontal sliding cameras* or **MHSC** problem, is known to be polynomial [KM11] in orthogonal polygons without holes. In orthogonal polygons with holes, the problem is NP-hard [BCL$^+$16]. In terms of G, the **MHSC** problem translates to the Total Dominating Set problem, which can be solved in polynomial time for chordal bipartite graphs [DMK90].

Theorem 21. *For a rectilinear domain D given by an ordered list of its vertices, there is an linear time algorithm finding a solution to the optimal horizontal mobile guard problem.*

Proof. First, observe that both the horizontal $\mathbf{R}$-tree T_H and the vertical $\mathbf{R}$-tree T_V of D can be constructed in linear time using linear time triangulation of D ([Cha91], [GHKS96, Section 5]).

Secondly, we implicitly construct the pixelation graph G of D. Observe, that the neighborhood of a vertical slice in G is a path in T_H, and vice versa. Label each horizontal edge of D by the horizontal slice that contains it. Furthermore, label each vertical edge of each horizontal slice by the edge of D containing it; do this for the horizontal edges of vertical slices as well. This step also takes linear time. The endpoints of a path induced by the neighborhood of any node in G can be identified via these labels in $O(1)$ time.

In Section 3, we showed that a horizontal guard system is a subset of $V(T_H)$ which intersects (covers) each element of $\mathcal{F}_H = \{N_G(v) \mid v \in V(T_V)\}$. Dirac's theorem states that ν, the maximum number of disjoint subtrees of the family, is equal to τ, the minimum number of nodes covering each subtree of the family. Obviously, $\nu \leq \tau$. The other direction is proved using a greedy algorithm:

1. Choose an arbitrary node r of T_H to serve as its root. The distance of a vertical slice $v \in V(T_V)$ from r is $\text{dist}_r(v) = \min_{h \in N_G(v)} \text{dist}(h, r)$, and let $h_r(v) = \arg\min_{h \in N_G(v)} \text{dist}(h, r)$.

2. Enumerate the elements of $V(T_V)$ in decreasing order of their distance from r, let $v_1, v_2, \ldots, v_{|V(T_V)|}$ be such an indexing. Let $S_0 = \emptyset$.

3. If $N_G(v_i)$ is disjoint from the elements of $\{N_G(v) \mid v \in S_{i-1}\}$, let $S_i = S_{i-1} \cup \{v_i\}$; otherwise let $S_i = S_{i-1}$.

We claim that $\{h_r(v) \mid v \in S_{|V(T_V)|}\}$ is a cover of $\mathcal{F}_H$. Suppose there exists $v_j \in V(T_V)$ such that $N_G(v_j)$ is not covered. Let i be the smallest index such that $v_i \in S_i$ and $N_G(v_j) \cap N_G(v_i) \neq \emptyset$. Clearly, $i < j$, therefore $\text{dist}_r(v_i) \geq \text{dist}_r(v_j)$. However, this means that $h_r(v_i) \in N_G(v_j)$.

Now $\{h_r(v) \mid v \in S_{|V(T_V)|}\}$ is a cover of the same cardinality as the disjoint set system $\{N_G(v) \mid v \in S_{|V(T_V)|}\}$, proving that $\nu = \tau$.

The first part of the algorithm, including calculating $\text{dist}_r(v)$ and $h_r(v)$ for each v, can be performed in $O(n)$ time, using Gabow & Tarjan's off-line lowest common ancestors algorithm (since each neighborhood $N_G(v)$ for $v \in V(T_V)$ induces a path in T_H).

Calculating the distance decreasing order takes linear time via breadth-first search started from the root. In the i^{th} step of the the third part of the algorithm, we maintain for each node in $V(T_H)$ whether it is under an element of $\{h_r(v) \mid v \in S_i\}$. Summed up for the $|V(T_H)|$ steps, this takes only linear time. $N_G(v_{i+1})$ is disjoint from the elements of $\{N_G(v) \mid v \in S_i\}$ if and only if one of the ends of the path induced by $N_G(v_{i+1})$ is under one of the elements of $\{h_r(v) \mid v \in S_i\}$, which now can be checked in constant time. Thus the algorithm takes in total some constant factor times the size of the input time to run. $\square$

The minimum size r-vision cover of a rectilinear domain by point guards can be computed in $\tilde{O}(n^{17})$ time [WK07]. A linear-time 3-approximation algorithm is described in [LWŻ12].

Corollary 22. *An* $(8/3)$*-approximation of minimum size a point guard system for a given orthogonal polygon can be computed in linear time.*

Proof. Compute m_V and m_H using the previous algorithm. By Theorem 2 and the trivial statement that both $m_H \leq p$ and $m_V \leq p$, we get that $4/3 \cdot (m_H + m_V)$ is an $8/3$-approximation for p. $\square$

Finding a minimum size of mixed vertical and horizontal mobile r-guard system which covers an orthogonal polygon (also known as the *minimum cardinality sliding cameras* or **MSC** problem) has been shown [DM13] to be NP-hard for orthogonal polygons with holes. For orthogonal polygons without holds, the problem translates to the DOMINATING SET problem in G. This is known to be NP-complete in chordal bipartite graphs [MB87]. However, the original problem's NP-hardness is still open. For an orthogonal polygon of n vertices, a covering set of mobile guards of cardinality at most $\lfloor (3n+4)/16 \rfloor$ (which is the extremal bound shown by Aggarwal [Agg84]) can be found in linear time [GM16].

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