Strategic Dynamic Pricing with Network Externalities*

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Abstract

We study the optimal pricing policy of a strategic monopolist selling durable goods in a dynamic pricing game with multiple rounds. Customers are forward-looking and experience a (positive) network externality, i.e., each customer's utility depends not only on her valuation of the item and the offered price, but also the weighted sum of the number of other customers who have purchased the item. The monopolist announces and commits to a price sequence and strategic buyers decide on when (if ever) to buy the good in a perfect Bayesian equilibrium. We fully characterize the optimal pricing policy and show that it is linearly increasing in time, where the slope of the price path is described by a single network measure: sum of the entries of the inverse of network externality matrix, termed network effect. Our result shows that increasing the number of rounds and network effect increases both revenue and social welfare. We also study the effect of price discrimination and establish that in the earlier rounds the price offered to more central buyers is lower. This is to encourage them to buy earlier which in turn increases the externality of other buyers and incentivize them to buy in subsequent rounds.

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1 Introduction

The benefits that users derive from various products such as digital products (e.g., computer softwares and smartphone apps) and electronics (e.g., smart phones, hardware devices, and computers) depends, among other things, on the externalities of users who have contributed to improvement of various aspects of the product. These improvements are provided via customers' feedback such as product satisfaction survey resulting in adjusting features and increasing the compatibility of the product. Many online forums provide a platform for exchanging information/opinions regarding the product and customers observe the extent of these externalities over time. Therefore, customer strategies in such settings with "network externalities" will focus, among other things, on not just whether but when to make a purchase. This in turn implies that seller strategies must build up the number of early users to increase these network externalities. Sellers will try to achieve this, among other things, by choosing the right dynamic price path for their products. Despite the ubiquity of these concerns, there is little work on dynamic pricing with network externalities.

In this paper, we study the problem of dynamic pricing in the presence of network externalities. We consider a dynamic game between a seller and a set of buyers. All buyers and seller are forward-looking. The seller announces and *commits* to a price sequence and buyers decide whether and when to buy a single item. The utility of each buyer depends on her valuation of the item, the price, and the (weighted) number of other customers who have already bought the item. Different weights capture differentiation in buyers' preferences regarding various aspects of a product. For example, customers of Microsoft Office products receive positive externalities from previous users. This is because of the improvement of various features of their products by fixing errors and incorporating suggestions provided by their customers. Therefore, by purchasing a certain release of Microsoft office, a buyer enjoys all the features and improvements in the product provided by previous users' feedback.

We consider a Perfect Bayesian Equilibrium (PBE) and show that the optimal price path is non-decreasing. We then show that the equilibrium behavior of buyers (in PBE) can be characterized by a threshold rule in which buyers purchase at different rounds if and only if their valuations exceeds a certain threshold. As the optimal price sequence is non-decreasing, by postponing purchase to future rounds each buyer faces the following tradeoff: on one hand, she has to pay a higher price and on the other hand, her utility from the network externality becomes larger.

Building on this characterization, we first consider a setting with uniform network externalities. We characterize the optimal pricing policy and explicitly find the optimal revenue of the seller. In particular, we show that the optimal pricing policy is linearly increasing. We next consider a block-model with m blocks such that each block (group) i has a fraction of the total number of buyers equal to $\alpha_i n$, where $\alpha_i \in [0, 1]$. The externalities between any pair of buyers depends only on the block they belong to. In particular, we denote the externality between any two buyers from blocks i and j by E_{ij}. This model captures the setting in which different buyers of a product with network externalities, have different preferences for various aspects/features of the product. This can be a result of horizontal product differentiation in which buyers evaluate a product based on a variety of characteristics (see Tirole (1988, Chapter 7) and Talluri and Van Ryzin (2006, chapter 8)). For instance, in evaluating a software quality a group of buyers might care more about user interface of the software while others might care about software functional quality or structural quality. Buyers with similar preferences belong to the same group and have the same externality on buyers from other group. Using techniques from probability theory (namely, Berstein polynomial convergence Lorentz (2012) and Berezin and Zhidkov (2014)), for this general setting, we find the optimal pricing policy as well as the optimal normalized revenue in the limit as the number of buyers goes to infinity. Interestingly, we establish that the optimal pricing policy is again linearly increasing. Note that linear optimal pricing is not a byproduct of uniform valuations for buyers. Surprisingly, we show that even with non-uniform valuations the price sequence is still linear.

Our characterization shows that the properties of both optimal pricing policy and the optimal revenue depends on the quantity $1/\left(\mathbf{1}^T E^{-1} \mathbf{1}\right)$ which we term *network effect*. In particular, the extent of the price difference at two consecutive rounds (slope of the price path) is higher for higher network effect and the optimal revenue is increasing and concave in the total number of rounds and increasing and convex in network effect. The network effect (and hence the optimal revenue) is higher for "asymmetric" networks. More precisely, for a "weakly tied" block-model, for a given sum of network externalities, the revenue is larger for networks in which sum of the products of out-degree and in-degree is lower. For instance, a star network has a higher revenue than a chain network which has a higher revenue than a ring network.

We also consider the effects of price discrimination among different groups of buyers. We show that the optimal pricing policy is linearly increasing with a slope which is in form of a "Bonacich centrality". We establish that in earlier rounds monopolist offers lower prices to groups with higher centralities in order to encourage them to purchase, which in turn further incentivize other customers to purchase in the subsequent rounds.

Finally, we consider two extensions of our model. We study the effects of non-committed seller in a setting with two rounds and uniform externalities. We show that a committed seller obtains a larger revenue than a non-committed seller, hence motivating our focus on committed sellers. We also consider a variation of our model in which buyers obtain utility from the purchase of other buyers in any round of the game. We provide an example which shows that the optimal pricing policy can either be increasing or decreasing. We then provide a full characterization of the optimal non-decreasing pricing policy and highlight its dependence on the network externliaties.

1.1 Related Literature

Our paper relates to two series of works: (i) the study of markets with network externalities and (ii) the study of markets with strategic forward-looking buyer behaviors.

1.1.1 Network Externalities

Markets for products with network externalities has been first studied in Rohlfs (1974), Katz and Shapiro (1985), and Farrell and Saloner (1985). Network externalities can either be direct or indirect. If the utility that a customer derives from the product increases as the number of adopters increases, then a direct externality exist as studied in Katz and Shapiro (1985); Nault and Dexter (1994). Examples include communication services such as mobile phone and instant messaging systems as the value of such services is a function of the number of others who also use the service. Indirect or complementary network externalities arise when there is a positive link between the utility to a customer and the number of other users of the product because of complementary products. This form of network externalities might exist because of interchangeability of complementary products such as computer software, video games, and electric cars because of availability of alternative fueling stations.

Given the importance of network externalities in markets, empirical investigations have examined the implications of direct and indirect network effects in a variety of industries including Gupta et al. (1999); Gandal (1994); Brynjolfsson and Kemerer (1996); Basu et al. (2003); Au and Kauffman (2001); Kauffman and Wang (2002); Matutes and Padilla (1994); Bayus (1987); Gandal et al. (1999, 2000); Cottrell and Koput (1998); Ribstein and Kobayashi (2001); Gallaugher and Wang (2002); Srinivasan et al. (2004). In particular, Au and Kauffman (2001) examine the adoption of electronic bill presentment and payment technology and show the existence of network externalities and its implications, Kauffman and Wang (2002) examine the impact of network externalities in the context of nationally shared electronic banking networks, Matutes and Padilla (1994) demonstrate the presence of network externalities in ATM adoption decisions, Ribstein and Kobayashi (2001) empirically study firms' choice of organizational form and examine network externalities on formations of limited liability partnerships (LLPs) and limited liability companies (LLCs), Gallaugher and Wang (2002) empirically study the market for Web server software and establish the existence of network externalities, Srinivasan et al. (2004) study the effect of network externalities on the pioneer survival in technology market, and Brynjolfsson and Kemerer (1996) empirically study the network externalities in software product market and establish that the network externalities significantly has increased the price of products.

Moreover, on the theory side an extensive study has examined the strategic and welfare implications of network externalities. A consistent finding in the literature is that network externalities alter customer behavior Katz and Shapiro (1985); Farrell and Saloner (1986); Arthur (1989); Economides (1989, 1996); Cabral (2011); Sääskilahti (2015). In particular, Farrell and Saloner (1986) deal with indirect positive network externalities and show that its effect may inhibit innovation. Economides (1996) studies the incentive of an exclusive holder of a technology to share it with competitors in a market with network externalities and shows that the innovator is better off as one of many oligopolists firms rather than as a monopolist (a related question is recently studied in Cai and Raju (2016)). The Marketing literature has examined pricing with "experience" or "network" effects Cabral et al. (1999); Bass (1980); Clarke et al. (1982); Kalish (1983); Besanko and Winston (1990); Candogan et al. (2012), and Ming et al. (2016). In particular, Candogan et al. (2012) and Bloch and Quérou (2013) study the optimal static pricing policy of a seller selling a divisible good (service) to consumers with network externalities. They consider a two-stage game in which a seller decides on the prices and then buyers decide their consumption in an equilibrium. Given a set of prices, their model takes the form of a network game among agents that interact locally, which relates to a series of papers such as Ballester et al. (2006); Bramoullé and Kranton (2007); Corbo et al. (2007); Galeotti and Goyal (2009), and Bramoullé et al. (2014) (a related model is more recently investigated in Cohen and Harsha (2013) and Fainmesser and Galeotti (2016)). More recently, Alizamir et al. (2017) have considered the optimal pricing policy of network goods in a setting with two pricing periods and studied the effect of network structure. Also, closely related are Hartline et al. (2008); Hu et al. (2015) which focus on algorithmic question of finding revenue maximizing strategy.

The most related papers to our work are Dhebar and Oren (1985, 1986); Xie and Sirbu (1995); Cabral et al. (1999), and Goldenberg et al. (2010). Dhebar and Oren (1985) and Dhebar and Oren (1986) use an optimal control methodology to develop a dynamic pricing schedule for a service with expanding network (e.g., communication network). They show that it is optimal for a monopolist to set a lower price when consumers anticipate rapid network growth. Xie and Sirbu (1995) show that an increasing price trajectory is optimal for a duopoly durable goods market with strong network effects. Cabral et al. (1999) study a setting with two buyers in which a monopolist price a product that is subject to network externalities and show that the optimal price increases over time. Finally, Goldenberg et al. (2010) study the effect of network externalities in market growth. They argue that despite the conventional wisdom which suggests that network externalities should derive faster market growth due to bandwagon effect (Economides and Himmelberg (1995); Rohlfs (2003); Varian and Shapiro (1999); Doganoglu and Grzybowski (2007)), there is a chilling effect due to the "wait-and-see" behavior of consumers. Therefore, the growth of network goods has a slow initial phase followed by a fast growth stage.

1.1.2 Strategic Buyers

Settings with strategic buyers are commonly studied in the literature to describe rational and forward-looking buyers, who make inter-temporal purchasing decisions with the goal of maximizing their utility. Many empirical works suggested that assuming myopic customer behavior is no longer a tenable assumption (see Aviv and Pazgal (2008); Li et al. (2014)). The importance of forward-looking customer behavior in shaping firms' pricing decision has been broadly identified by practitioners and firms are largely investing in price optimization algorithms (Schlosser (2004)) and a recent literature has pursued to provide managerial insights for firms to adjust their approach to dynamic pricing Vulcano et al. (2002);

Gallien (2006); Su (2007); Shen and Su (2007); Liu and Van Ryzin (2008); Levin et al. (2009); Board and Skrzypacz (2016); Pai and Vohra (2013); Mersereau and Zhang (2012); Gönsch et al. (2013); Borgs et al. (2014); Cachon and Feldman (2015); Chen and Farias (2015); Yang and Zhang (2015); Chen and Farias (2015); Lobel et al. (2015); Dilme and Li (2016); Bernstein and Martínez-de Albéniz (2016) (see den Boer (2015) for a survey on dynamic pricing literature). In particular, Besbes and Lobel (2015) study the optimal pricing policy of a committed seller that faces customers arriving over time which are strategic in timing their purchases and have heterogeneous valuation and willingness to wait before purchasing. They show that cyclic pricing policies is optimal for this setting. Ajorlou et al. (2016) study the effect of pricing on expanding the installed base through word-of-mouth, and Papanastasiou and Savva (2016) consider a model with two periods and study the effect of social learning on the optimal pricing. They show that the optimal price path can either be increasing or decreasing depending on whether the seller is committed.

The literature with strategic consumer behavior typically assumes firms employ one of two classes of dynamic- pricing policies: (i) with commitment (ii) without commitment (see Stokey (1979); Landsberger and Meilijson (1985); Yin et al. (2009); Whang (2014) for committed pricing and Besanko and Winston (1990); Netessine and Tang (2009); Cachon and Swinney (2009); Dasu and Tong (2010) for non-committed pricing). A question of particular interest in our work is which class of policies (i.e., with or without commitment) is preferred by the firm. In section 5.1, we show that a policy with committed pricing leads to a higher revenue for the seller.[1]

The most related papers to our work are Bulow and Klemperer (1994) and Hörner and Samuelson (2011). In particular, building on the model presented in Bulow and Klemperer (1994), Hörner and Samuelson (2011) study a setting in which a seller who must sell her inventory (single item) before some deadline, facing a group of buyers with independent private values, sets prices in some time periods. In contrast with our problem, they show that the optimal price sequence is decreasing and buyers face the following trade-off: by buying early they pay a higher price with a higher chance of obtaining the item and by waiting they face a lower price with a lower chance of obtaining the item as other buyers also accept the lower price (a related problem is more recently studied in Chen (2012)).

1.2 Notation

For any matrix $M \in \mathbb{R}^m \times \mathbb{R}^n$, we let both $[M]_{ij}$ and M_{ij} to denote the entry at ith row and jth column. We show vectors with bold face letters. The vector of all ones is shown by $\mathbf{1}$, where the dimension of vector is clear from the context. The ith canonical basis is shown by $\mathbf{e}_i$. For any event $\mathcal{E}$, $\mathbf{1}\{\mathcal{E}\} = 1$ if $\mathcal{E}$ holds and 0, otherwise. For any integer n, we let $[n] = \{1, \ldots, n\}$. We denote the transpose of vector $\mathbf{x}$ and matrix M by $\mathbf{x}^T$ and M^T,

[1] Our paper also relates to a vast literature on the "Coase conjecture" Stokey (1981); Bulow (1982); Sobel and Takahashi (1983); Fudenberg et al. (1985); Ausubel and Deneckere (1987); Gul et al. (1986); Kahn (1986); Bagnoli et al. (1989); von der Fehr and Kühn (1995); Kühn and Padilla (1996). Coase (1972) claimed that the price set by a monopolist who is unable to commit to future prices will quickly converge to marginal cost as the time between sales becomes arbitrarily short. The Coase conjecture was confirmed and disconfirmed under a variety of conditions. In all cases the equilibrium solutions obey "Coasian dynamics" (Hart and Tirole (1988)). Coasian dynamics consist of two properties: (i) higher valuation buyers make their purchase no later than lower valuation buyers (skimming property) and (ii) equilibrium price is nonincreasing over time (price monotonicity property). In this paper, we show that the second property does hold when network externalities are present. This is consistent with the findings of Bensaid and Lesne (1996); Mason (2000); Laussel et al. (2015) in a different model with network externalities.

respectively. For two vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^m$, $\mathbf{x} \geq \mathbf{y}$ means entry-wise inequality, i.e. $x_i \geq y_i$, for all $i \in [m]$. We show a weighted directed network by (V, G) where $V = \{1, \ldots, n\}$ represents the set of nodes and G_{ij} represents the weight of the edge from i to j. Out-degree and in-degree of a node i are denoted by $d_i^{(\text{out})} = \sum_{j=1}^{n} G_{ij}$ and $d_i^{(\text{in})} = \sum_{j=1}^{m} G_{ji}$. We let $\mathcal{P}_{[0,1]}(\cdot)$ denote the projection operator onto the interval $[0, 1]$.

2 Model Description

2.1 Environment

We consider a dynamic game between a seller with infinitely many homogeneous items and n buyers in T rounds. The seller announces a price p_t at round t and each buyer decides whether to buy the item or postpone it to future rounds. We denote the sequence of prices by $p_T, p_{T-1}, \ldots, p_1$ where p_t is the price that is announced at round $T + 1 - t$, i.e., when there are t remaining rounds to end of the selling horizon.[2] If a buyer decides to buy an item at price p_t she will exit, i.e., she will not consider any future purchase decisions. Each buyer has a valuation in $[0, 1]$ which is drawn i.i.d. from a distribution with CDF $F(\cdot)$ and pdf $f(\cdot)$ which is known to buyers and seller. We let h^t be the set of buyers that accept the price p_t, i.e., $h^t = \{j \ : \ j \text{ accepts the price } p_t\}$ (we also let $h^{T+1} = \emptyset$). The history at round t which is available to buyers and seller is the set of buyers who have bought the item before the round with price p_t, i.e., $H^t = \{h^{T+1}, h^T, \ldots, h^{t+1}\}$. A behavior strategy for the seller is a finite sequence $\{\sigma_S^t\}_{t=1}^T$, where σ_S^t is a probability transition from H^t to $\mathbb{R}$, mapping the history H^t into a probability distribution over prices. A behavior strategy for buyer i is a finite sequence $\{\sigma_i^t\}_{t=1}^T$, where σ_i^t is a probability transition from $\mathbb{R} \times H^t \times \mathbb{R}$ into $\{0, 1\}$, mapping buyer i's valuation, the history of game, and the current price into a probability of acceptance. We will focus our attention on symmetric perfect Bayesian equilibrium of this game. I.e., each player's strategy specifies optimal actions, given her beliefs and the strategies of the other players (sequential rationality) and each player's belief is consistent with the strategy profile (these beliefs are obtained by using Bayes' rule whenever possible). Moreover, the a buyer's strategy at round t, depends on her valuation as well as her network externality, not on her identity (see Example 1 for illustration of Symmetric versus non-symmetric PBE).[3]

2.2 Utility of Buyers

There exists a strategic complementarity among buyers in which each buyer obtains a higher utility in proportion to the subset of other buyers who have purchased the item. In particular, we suppose that there exist a directed weighted network where the set of nodes is $V = \{1, \ldots, n\}$ representing all buyers and the weight on the edge between i and j, denoted by $g_{ij} \geq 0$, captures the utility that buyer i drives from j's purchase. The weights g_{ij}, $i, j \in V$ are known to buyers and seller. Therefore, the utility of buyer i if she

[2] With this notation p_T is the price announced at the first round and p_1 is the price announced at the last round. This indexing simplifies the notation in the analysis when we use backward induction to find the perfect Bayesian equilibrium.

[3] This is a common assumption in the literature and is used as a selection device among multiple equilibria. For similar arguments see Gul et al. (1986), Chen (2012), Hörner and Samuelson (2011) for dynamic pricing settings, Krishna (2009, Chapter 4) for auction setting, and (Talluri and Van Ryzin , 2006, Chapter 8) for pricing games.

buys the item at round t is

$$v^{(i)} - p_t + \sum_{j \neq i} g_{ij} \mathbf{1}\{j \text{ has bought at round } s > t\}, \tag{2.1}$$

where $v^{(i)} \in [0,1]$ is the valuation of buyer i and p_t is the posted price at time t. The utility of buyer i has a term which depends on the weighted sum of other buyers who has bought the item before buyer i. This utility model captures situations in which a dominant part of the externality comes from the fact that previous users improve the product through their use.[4] Hence, at the time a buyer makes her purchase decision and utilizes the product, what matters is past users which determines how improved the product is. Note that consistent with this interpretation and full rationality (in perfect Bayesian equilibrium), even though buyers derive utility from adoptions in the past, they are *forward-looking*. I.e., they take into account all future behavior and decide to postpone the purchase if it is optimal.

2.3 Utility of Monopolist

We consider a *committed* seller who announces and commits to a price sequence $p_T, p_{T-1}, \dots, p_1$ before the selling season starts. This captures the situation when the seller cannot change the price based on the number of buyers who have decided to buy the product. The revenue of the seller is the summation of the payoff she receives at each round, i.e.,

$$\sum_{t=1}^{T} \sum_{i=1}^{n} p_t \mathbb{E}\left[\mathbf{1}\{i \text{ buys at price } p_t\}\right], \tag{2.2}$$

where we normalized the marginal cost of monopolist to zero. The seller chooses an *optimal price sequence* to maximize revenue. The ability of the seller to commit to a dynamic price path is important for our results. Without such commitment, a Coase conjecture-type reasoning would create a downward pressure on prices and would tend to reduce seller revenue (Coase (1972)). Though such commitment is not possible in some settings, many sellers are able to build a reputation for such commitment, for instance, by creating explicit early discounts which will be lifted later on. In Section 5.1, we consider a seller with no commitment power, i.e., each price must be sequentially rational, given the history of previous plays and anticipations of optimal continuation play. We will show that a seller with no commitment obtains a lower revenue in comparison to a committed seller, hence further motivating our focus on committed prices.

[4] If network externality is

$$\begin{aligned} &\sum_{j \neq i} g_{ij} \mathbf{1}\{j \text{ has bought at round } s > t\} + \rho \sum_{j \neq i} g_{ij} \mathbf{1}\{j \text{ buys at round } s \leq t\} \\ &= (1-\rho) \sum_{j \neq i} g_{ij} \mathbf{1}\{j \text{ has bought at round } s > t\} + \rho \sum_{j \neq i} g_{ij} \mathbf{1}\{j \text{ buyes in any of the rounds}\}, \end{aligned}$$

for some scaling factor $\rho < 1$, then each buyer independent of her purchasing time obtains ρ times the network externality from purchases at all times. This constant term does not alter buyers' strategic behavior and the game is equivalent to the one with utility (2.1).

3 Preliminary Characterizations

Each individual buyer faces a strategic stopping problem, choosing the round (if any) along a sequence of prices at which to accept the offered price and exit the game, taking as given the behavior of other buyers. We next use a well-known single-crossing argument to characterize the buyers behavior. In particular, we show that in the equilibrium, buyer i who accepts the price in a given round t has a valuation which exceeds a threshold $v_t^{(i)}(H^t)$, where this threshold depends on the history of the game at time t and satisfies $v_{t-1}^{(i)}(H^{t-1}) \leq v_t^{(i)}(H^t)$. In the rest of the paper for the sake of readability we refer to $v_t^{(i)}(H^t)$ as $v_t^{(i)}$.

Proposition 1 *(a) The seller's price sequence in any equilibrium is essentially non-decreasing. I.e., any price sequence has a corresponding non-decreasing price sequence in which the equilibrium path (the behavior of buyers) remains the same.*

(b) Given a price sequence, the behavior of buyer $i \in [n]$ in any equilibrium is a thresholding decision. Suppose round t has been reached without buyer i accepting a price. The posterior belief of buyer i's valuation is drawn from a distribution with support $[0, v_{t+1}^{(i)}]$ whose CDF is $F(v)/F(v_{t+1}^{(i)})$, for some $v_{t+1}^{(i)} \in [0,1]$.

Proof The proof is presented in Appendix 7.1. ■

Proposition 1 is a crucial observation on which much of the rest of our analysis builds. Technically, it is simple and relates to previous results in dynamic settings with preferences satisfying single crossing. Its implications in our model are far-reaching, however. Without network externalities, dynamic price paths that are increasing (non-decreasing) would be impossible to sustain because high-valuation buyers will tend to be the first ones to purchase (e.g., in the presence of discounting), and if lower-valuation buyers prefer not to purchase early on, they would also prefer not to purchase later on with higher (or equal) prices. This phenomenon is transformed in the presence of network externalities. Now because the increase in the number of users over time raises the network externality term (regardless of the exact form of network interactions), lower-valuation buyers might be happy to buy later and at higher prices. In fact, it can never be optimal for the seller to have a strictly decreasing price sequence, because this would induce all buyers to delay, while an increasing price sequence would induce high-valuation buyers to purchase early, while lower-valuation ones wait and purchase once the network externality is higher.

In the next lemma we characterize the behavior of the buyers and show that the thresholds are naturally such that the continuation difference in the utility from network externalities equals the continuation difference in the price sequence.

Lemma 1 *Given a price sequence $p_T, \ldots, p_1$, for any equilibrium suppose round t has reached with history H^t. Each buyer i buys the item at round t if her valuation exceeds a threshold $v_t^{(i)}$. These thresholds satisfy the following indifference condition.*

$$\sum_{j \in [n] \setminus H^t \setminus \{i\}} g_{ij} \left(1 - \frac{F(v_t^{(j)})}{F(v_{t+1}^{(j)})}\right) = p_{t-1} - p_t, \qquad i \in [n]. \tag{3.1}$$

Proof The proof is presented in Appendix 7.2. ■

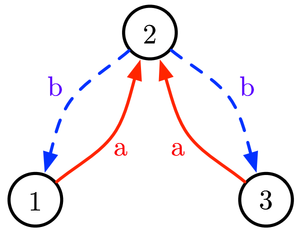


Figure 1: Example 1 illustrates the seller and buyers' behavior.

The sequence $\{v_t^{(i)}\}_{t=1}^{T+1}$, $i \in [n]$ depends on the history of the game and Lemma 1 provides an indifference condition for it, whose boundary conditions (with convention) are $v_{T+1}^{(i)} = 1$. This is because in the first period, the seller and buyers have not yet learned anything about buyer i's valuation.

In the next example we illustrate the main challenges in analyzing the equilibrium of this game which are the possibility of having multiple equilibria as well as having a complicated expected revenue which must capture $(T+1)^n$ many cases corresponding to each of n buyers purchasing at any of T rounds (or not purchasing).

Example 1 We consider a game with 3 players in 2 periods ($n = 3$ and $T = 2$) where the network externalities are $G = \begin{pmatrix} 0 & a & 0 \\ b & 0 & b \\ 0 & a & 0 \end{pmatrix}$, for $a = 4/5$ and $b = 3/5$ (see Figure 1). Using Lemma 1 for the first round, the thresholds determining the strategy of each buyer satisfy the following indifference condition:

$$\begin{aligned}
v_2^{(1)} - p_2 &= v_2^{(1)} - p_1 + a(1 - v_2^{(2)}), \\
v_2^{(2)} - p_2 &= v_2^{(2)} - p_1 + b\left(\left(1 - v_2^{(1)}\right) + \left(1 - v_2^{(3)}\right)\right), \\
v_2^{(3)} - p_2 &= v_2^{(3)} - p_1 + a(1 - v_2^{(2)}).
\end{aligned}$$

This set of equations have multiple solutions which leads to multiple strategies for the first round of the game. In particular, we have

$$v_2^{(2)} = 1 - \frac{p_1 - p_2}{a}, \quad v_2^{(2)} + v_2^{(3)} = 2 - \frac{p_1 - p_2}{a}. \tag{3.2}$$

Using Lemma 1 for the second round, we have

$$\begin{aligned}
v_1^{(1)} &= p_1 - a\mathbf{1}\{2 \text{ bought at price } p_2\}, \\
v_1^{(2)} &= p_1 - b\left(\mathbf{1}\{1 \text{ bought at price } p_2\} + \mathbf{1}\{3 \text{ bought at price } p_2\}\right), \\
v_1^{(3)} &= p_1 - a\mathbf{1}\{2 \text{ bought at price } p_2\}.
\end{aligned} \tag{3.3}$$

Note that these equations for thresholds hold only when they are interior. For instance, if in the first round buyer 2 buys and $a > p_1$, then using (3.3), leads to a negative $v_1^{(1)}$ which is not valid. In this case, $v_1^{(1)} = 0$ is the correct identity (i.e., in the second round, player 1 always buys). However, with our choice of parameters, in the equilibria that we will find all thresholds are interior. In order to characterize the optimal pricing as well as equilibrium behavior of buyers, we first find the expected revenue and then find its maximum. The expected revenue is

$$p_2 \left(\left(1 - v_2^{(1)}\right) + \left(1 - v_2^{(2)}\right) + \left(1 - v_2^{(3)}\right) \right)$$
$$+ p_1 \sum_{S \subseteq \{1,2,3\}} \mathbb{P}\left[H_2 = S\right] \left(\sum_{i \in \{1,2,3\} \setminus S} \mathbb{P}\left[v^{(i)} \geq v_1^{(i)} \mid H_2 = S \right] \right),$$

where the first term is the expected revenue in the first round and the second term is the expected revenue in the second round. Maximizing the expected revenue, leads to the following pairs (among many others) of price sequence and buyers' strategy:

1. Symmetric equilibrium:
   - Price sequence: $p_1 = .6$ and $p_2 = .48$ with expected revenue .38.
   - Buyers' strategy: $v_2^{(2)} = 1 - (p_1 - p_2)/a$, $v_2^{(1)} = v_2^{(3)} = 1 - (p_1 - p_2)/2b$, and the thresholds in the second round are given as in (3.3).
2. Non-symmetric equilibrium:
   - Price sequence: $p_1 = .6$ and $p_2 = .42$ with expected revenue .52.
   - Buyers' strategy: $v_2^{(2)} = 1 - (p_1 - p_2)/a$, $v_2^{(1)} = 1 - (p_1 - p_2)/3b$, $v_2^{(3)} = 1 - 2(p_1 - p_2)/3b$, and the thresholds in the second round are given as in (3.3).

Note that (3.2) does not uniquely define the thresholds as it is an underdetermined system of equation. Any solution to (3.2) results in an equilibrium and the above equilibria are two specific ones obtained by letting $v_2^{(1)} = v_2^{(3)}$ and $v_2^{(3)} = 2v_2^{(1)} - 1$, respectively.

This example motivates our focus on symmetric equilibria. That is, a buyer's strategy at round t, depends on her valuation as well as her network externality, not on her identity (see Chen (2012) and Hörner and Samuelson (2011) for similar arguments). Furthermore, in order to obtain an explicit characterization of the optimal price path we will focus on limiting equilibrium as n goes to infinity. Intuitively, this assumption guarantees that the sample equilibrium path is close to its expectation and enables us to use techniques from probability theory (namely, Berstein polynomial convergence Lorentz (2012) and Berezin and Zhidkov (2014)) to find a closed form formulation for the normalized expected revenue as well as the optimal price sequence.

For ease of exposition, we adopt uniform valuations assumption. However, this assumption is not crucial in deriving our results as we will relax it in Section 4.3 and show how our results generalize beyond uniform valuations.

Assumption 1 The valuations are uniform from $[0, 1]$, i.e., $F(v) = v$ for all $v \in [0, 1]$.

4 Optimal Pricing Policy

In this section, we characterize the optimal pricing sequence of a monopolist seller and highlight the effects of network externalities. We further illustrate the effects of network and price discrimination in the optimal policy and revenue. Recall that the monopolist seller selects a price sequence $p_T, \ldots, p_1$ (and commits to it) that maximizes her expected

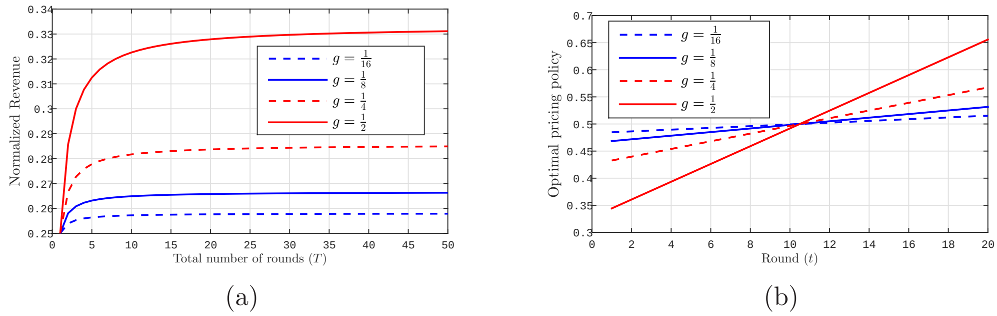


Figure 2: (a) optimal normalized revenue as a function of number of rounds for different network externalities and (b) optimal price path for different network externalities.

revenue, knowing that buyers will follow an equilibrium strategy based on the price sequence. The optimal price sequence is the solution of the following problem:

$$\max_{p_T,\dots,p_1} \sum_{t=1}^{T} \sum_{i=1}^{n} p_t \mathbb{E}\left[\mathbf{1}\{i \text{ buys at price } p_t\}\right]. \tag{4.1}$$

We start by considering uniform network externalities (i.e., same externality between any two buyers) and then extend our characterization to a general block-model network externality.

4.1 Uniform Externalities

We start by considering a setting where the externalities among all buyers are the same, i.e., $g_{ij} = g_{i'j'} = g$ for all $i, j, i', j' \in [n]$ such that $i \neq j$ and $i' \neq j'$. We let $g \in [0, 1]$ denote the normalized externalities. With this notation if k many buyers purchase the item, then the network externality that a buyer obtains is $g\frac{k}{n-1}$. In the next theorem, we provide an explicit characterization of the optimal price path and the optimal revenue.

Theorem 1 *Suppose Assumption 1 holds. The optimal price sequence is linearly increasing given by*

$$p_t = (T-t)\left(\frac{g}{2T - g(T-1)}\right) + \left(\frac{T - g(T-1)}{2T - g(T-1)}\right), \quad t = 1, \dots, T, \tag{4.2}$$

and the optimal normalized limiting revenue (as $n \to \infty$) is

$$\frac{T}{4T - 2g(T-1)}. \tag{4.3}$$

Proof The proof is presented in Appendix 7.3. ∎

Theorem 1 provides the following implications:

- The difference between the optimal price in two consecutive rounds (i.e., slope of the optimal price path) is

$$\frac{g}{2T - g(T-1)}.$$

Figure 2b shows the optimal price path as a function of time. The slope of the optimal price path is increasing in g, showing that larger network externality results in larger price difference. Intuitively, this holds because as the network externality increases, the past purchases contribute more to the utility of buyers, incentivizing them to buy at a higher price. Moreover, the slope is decreasing in T, showing that larger number of rounds makes the price difference smaller. This holds because as the number of rounds increases, the seller encounters less compulsion to increase the price. Figure 2b shows the optimal price path as a function of time.

- The optimal revenue is an increasing concave function in T. Our characterization reveals that the optimal revenue is concave in the number of rounds and approaches the limiting normalized revenue as $T \to \infty$ relatively fast. In particular, in order to reach to q percent of the limiting revenue, the number of periods is

$$\left\lceil \frac{q}{1-q} \frac{g}{(2-g)} \right\rceil,$$

where $\lceil x \rceil$ shows the smallest integer larger than x. For instance, if we want to obtain $q = \%95$ of the optimal revenue (i.e., revenue of infinitely many rounds), then : (i) for $g = \frac{1}{5}$, $T = 3$ many rounds suffices and (ii) for $g = \frac{4}{5}$, $T = 13$ many rounds suffices. Figure 2a shows the optimal normalized revenue as a function of number of rounds.

- The optimal revenue is an increasing convex function in g. Intuitively, this holds because increasing g has two effects. The utility of each buyer depends on the network externality in two different ways, resulting in a convex function: (i) *direct effect*: as g increases, keeping purchase probability of other buyers the same, the network externality part of buyer i' utility increases and (ii) *indirect effect*: as g increases, the purchase probability of other buyers in previous rounds increases, leading to a larger utility.

4.2 General Externalities

Building on the characterization for uniform network externalities, we next consider a general network externality and characterize the price sequence as well as the revenue of the seller. As argued in the Introduction, different buyers of a product with network externalities, have different preferences for various aspects of the product. In order to capture this differentiation in buyers' preferences we consider a block-model (as studied in White et al. (1976); Holland et al. (1983); DiMaggio (1986)). In particular, n buyers are divided into m blocks/groups where buyers who have similar preferences are grouped into one block. Each group $1 \leq i \leq m$ has $n_i = \alpha_i n$ many buyers for some $\alpha_i \in (0, 1]$. We let $A \in \mathbb{R}^m \times \mathbb{R}^m$ be a diagonal matrix such that $[A]_{ii} = \alpha_i$. The normalized externalities among all buyers from group i and group j are equal to E_{ij} (this corresponds to having $g_{ij} = g_{i'j'}$ when i and i' belong to the same group and j and j' also belong to the same group). Hence, we can capture the externalities among all groups by an $m \times m$ matrix denoted by $E \in \mathbb{R}^m \times \mathbb{R}^m$. For instance, if for any $j \in [m]$, $k_j \in [0, \alpha_j n]$ many buyers in group j buy the item, then the externality that a buyer in group i receives is $\sum_{j \in [m]} E_{ij} \frac{k_j}{n}$. Note that the externalities among buyers from the same group $i \in [m]$ is captured by E_{ii}.

In order to guarantee an interior unique solution we adopt the following assumption.

Assumption 2 Matrix E is invertible, $\mathbf{1}^T E^{-1}\mathbf{1} \geq 1$, and $E^{-1}\mathbf{1} \geq \mathbf{0}$.[5]

This assumption is the analogy of assumption $g \in [0,1]$ in Theorem 1. Note that a sufficient condition to guarantee Assumption 2 is to have an invertible E such that $E^{-1}\mathbf{1} \geq \frac{1}{m}\mathbf{1}$. For example, if an invertible E is d-regular ($\sum_j E_{ij} = d$ for any i), then Assumption 2 is equivalent to $d \leq m$. As another example, for $E = I + \delta C$, where δ is small enough (this captures the fact that externalities of buyers within each group is larger than the ones across groups), Assumption 2 always holds.

We next characterize the price sequence as well as the normalized revenue in the limit as $n \to \infty$. We use the following definition and result from Il'inskii and Ostrovska (2002), Lorentz (2012), and Berezin and Zhidkov (2014) in establishing Theorem 3.

Definition 1 (Multivariate Bernstein Polynomials) Let $n_1, \ldots, n_m \in \mathbb{N}$ and f be a function of m variables. The polynomials

$$B_{f,n_1,\ldots,n_m}(x_1,\ldots,x_m) = \sum_{j\in[m]:0\leq k_j \leq n_j} f\left(\frac{k_1}{n_1},\ldots,\frac{k_m}{n_m}\right)\left(\prod_{j=1}^{m}\binom{n_j}{k_j}x_j^{k_j}(1-x_j)^{n_j-k_j}\right)$$

are called the multivariate Bernstein polynomials of f.

Theorem 2 (Pointwise Convergence) *Let $f : [0,1]^m \to \mathbb{R}$ be a continuous function. Then the multivariate Bernstein polynomials $B_{f,n_1,\ldots,n_m}$ converges pointwise to f for $n_1,\ldots,n_m \to \infty$.*

Our key result is that even with a general network structure regulating the externality among users, there is an explicit characterization of optimal prices, highlighting the impact of the form of network externalities on the optimal price path and revenue.

Theorem 3 *Suppose Assumptions 1 and 2 hold. For a general network externality E, the optimal price sequence in the limit (as $n \to \infty$) is linearly increasing given by*

$$p_t = (T-t)\frac{1}{2T\left(\mathbf{1}^T E^{-1}\mathbf{1}\right)-(T-1)} + \frac{T\left(\mathbf{1}^T E^{-1}\mathbf{1}\right)-(T-1)}{2T\left(\mathbf{1}^T E^{-1}\mathbf{1}\right)-(T-1)}, \quad t = 1,\ldots,T, \tag{4.4}$$

with the optimal normalized revenue

$$\frac{\left(\mathbf{1}^T E^{-1}\mathbf{1}\right)T}{4T\left(\mathbf{1}^T E^{-1}\mathbf{1}\right)-2(T-1)}. \tag{4.5}$$

Proof We first prove the theorem for $T = 2$ and then generalize it to $T > 2$ rounds. We characterize the buyers decision given a price sequence and then find the critical thresholds as a function of E and the price sequence. Note that for a symmetric equilibrium, the threshold for all nodes in group i at the first round is the same which we denote by $v_2^{(i)}$. Using Lemma 1, the indifference condition for any $i = 1,\ldots,m$ becomes

$$v_2^{(i)} - p_2 = v_2^{(i)} - p_1 + \sum_j E_{ij}\alpha_j(1-v_2^{(j)}).$$

[5]If E is not invertible, then equilibrium is not unique. In particular, Theorem 3 holds for any pseudo-inverse of E provided that the other two assumptions hold.

By letting $\mathbf{v_2} = (v_2^{(1)}, \ldots, v_2^{(m)})$ and using the definition of A and E, we can write this equation in the following compact form:

$$(EA)\,\mathbf{v_2} = (EA)\,\mathbf{1} - (p_1 - p_2)\mathbf{1}, \quad \mathbf{v_2} = \mathbf{1} - (EA)^{-1}\,\mathbf{1}(p_1 - p_2). \tag{4.6}$$

In the last period, any remaining buyer i' in group i buys the item if $v^{(i')} - p_1 + \sum_{j\in H^2} g_{i'j} \geq 0$, which happens with probability $1 - \frac{p_1 - \sum_{j\in H^2} g_{i'j}}{v_2^{(i)}}$. The seller's expected revenue can be written as

$$\sum_{1\leq k_j \leq n_j} \left(\prod_{j=1}^{m} \binom{n_j}{k_j} (1 - v_2^{(j)})^{k_j} (v_2^{(j)})^{n_j - k_j}\right) \left(p_2 \sum_{j=1}^{m} k_j + p_1 \sum_{j=1}^{m} (n_j - k_j) \mathcal{P}_{[0,1]} \left(1 - \frac{p_1 - \sum_{j'\in H^2} g_{jj'} k_{j'}}{v_2^{(j)}}\right)\right),$$

where the first term of each summand is the probability of a multivariate Binomial random variable capturing the probability that at the first round for each group $j \in [m]$, k_j out of n_j many buyers purchase the item. The second term of each summand is the expected revenue of seller given this event in the two rounds. In particular, for each group j, k_j buyers purchase the product at price p_2 and each of the remaining $(n_j - k_j)$ buyers purchase the product at price p_1 with probability $\left(1 - \frac{p_1 - \sum_{j'\in H^2} g_{jj'} k_{j'}}{v_2^{(j)}}\right)$, provided that this term is in $[0, 1]$. Using Definition 1 and Theorem 2, we have

$$\begin{aligned}
&\lim_{n\to\infty} \frac{1}{n} \sum_{1\leq k_j\leq n_j} \left(\prod_{j=1}^{m} \binom{n_j}{k_j} (1 - v_2^{(j)})^{k_j} (v_2^{(j)})^{n_j - k_j}\right) \\
&\times \left(p_2 \sum_{j=1}^{m} k_j + p_1 \sum_{j} (n_j - k_j) \mathcal{P}_{[0,1]} \left(1 - \frac{p_1 - \sum_{j'} \frac{E_{jj'}}{n} k_{j'}}{v_2^{(j)}}\right)\right) \\
&= p_2 \sum_{j=1}^{m} \alpha_j (1 - v_2^{(j)}) + \sum_{j} p_1 (\alpha_j - (1 - v_2^{(j)})\alpha_j) \mathcal{P}_{[0,1]} \left(1 - \frac{p_1 - \sum_{j'} E_{jj'} (1 - v_2^{(j')}) \alpha_{j'}}{v_2^{(j)}}\right) \\
&= p_2 + (p_1 - p_2)(\alpha^T \mathbf{v}_2) - p_1^2 + p_1 \alpha^T\,(EA)\,(\mathbf{1} - \mathbf{v}_2),
\end{aligned} \tag{4.7}$$

where the first equality follows from Theorem 2 and the second equality follows from Assumption 2. Invoking (4.6) in (4.7) leads to the normalized revenue

$$p_1 - (p_1 - p_2)^2 \alpha^T\,(EA)^{-1}\,\mathbf{1} - p_1 p_2 = p_1 - (p_1 - p_2)^2 \mathbf{1}^T E^{-1} \mathbf{1} - p_1 p_2. \tag{4.8}$$

We next consider an arbitrary $T > 2$ time horizon. The indifference conditions become

$$\begin{aligned}
&EA\,(\mathbf{v}_{t+1} - \mathbf{v}_t) = (p_{t-1} - p_t)\mathbf{1}, \quad t = T, \ldots, 2, \\
&\mathbf{v}_1 - p_1 \mathbf{1} + EA\,(\mathbf{1} - \mathbf{v}_2) = \mathbf{0},
\end{aligned} \tag{4.9}$$

with the convention that $\mathbf{v}_{T+1} = 1$. Similar to (4.8) using (another) generalized Berstein polynomial convergence theorem (see Appendix 7.4 for details), the normalized revenue as $n \to \infty$ becomes

$$\sum_{t=T}^{1} p_t \mathbf{1}^T A\,(\mathbf{v}_{t+1} - \mathbf{v}_t)\,. \tag{4.10}$$

Invoking (4.9) in (4.10) and again using Theorem 2, leads to the following sellers' problem

$$\max_{p_T,\dots,p_1}\left(\left(\sum_{t=T}^{2} p_t \mathbf{1}^T E^{-1}\mathbf{1}\left(p_{t-1}-p_t\right)\right)+p_1\left(1-\mathbf{1}^T E^{-1}\mathbf{1}p_1+p_T\left(\mathbf{1}^T E^{-1}\mathbf{1}-1\right)\right)\right). \tag{4.11}$$

Note that the optimization problem (4.11) becomes identical to optimization problem (7.7) if we replace g by $\frac{1}{\mathbf{1}^T E^{-1}\mathbf{1}}$. Therefore, similar to the argument used in the proof of Theorem 1, the optimal price sequence is

$$p_t=(t-1)\left(\frac{T\left(\mathbf{1}^T E^{-1}\mathbf{1}\right)-1}{2T\left(\mathbf{1}^T E^{-1}\mathbf{1}\right)-(T-1)}\right)-(t-2)\left(\frac{T\left(\mathbf{1}^T E^{-1}\mathbf{1}\right)}{2T\left(\mathbf{1}^T E^{-1}\mathbf{1}\right)-(T-1)}\right),\quad t=1,\dots,T, \tag{4.12}$$

and the optimal normalized revenue is

$$\frac{\left(\mathbf{1}^T E^{-1}\mathbf{1}\right)T}{4T\left(\mathbf{1}^T E^{-1}\mathbf{1}\right)-2(T-1)}. \tag{4.13}$$

Finally, note that this price sequence leads to an interior solution for the thresholds. This is because we have

$$\mathbf{1}-\mathbf{v}_t=\frac{T+1-t}{2T\left(\mathbf{1}^T E^{-1}\mathbf{1}\right)-(T-1)}\left(EA\right)^{-1}\mathbf{1},\quad t=T,\dots,2. \tag{4.14}$$

We also have

$$\mathbf{v}_1=\frac{\left(\mathbf{1}^T E^{-1}\mathbf{1}\right)T-(T-1)}{2T\left(\mathbf{1}^T E^{-1}\mathbf{1}\right)-(T-1)}\mathbf{1}. \tag{4.15}$$

In order to show that all these thresholds are interior since $\mathbf{v}_{T+1}=\mathbf{1}$, it suffices to show that $\mathbf{v}_1\geq\mathbf{0}$ and $\mathbf{v}_{t+1}\geq\mathbf{v}_t$, $t=1,\dots,T$. Using Assumption 2 and in particular $\mathbf{1}^T E^{-1}\mathbf{1}\geq 1$, leads to $\mathbf{v}_1\geq\mathbf{0}$. Again, Using Assumption 2 and in particular $E^{-1}\mathbf{1}\geq\mathbf{0}$ leads to

$$\mathbf{v}_{t+1}-\mathbf{v}_t=\frac{1}{2T\left(\mathbf{1}^T E^{-1}\mathbf{1}\right)-(T-1)}A^{-1}E^{-1}\mathbf{1}\geq\mathbf{0},\quad t=1,\dots,T,$$

completing the proof. ■

Note that the term $\frac{1}{\mathbf{1}^T E^{-1}\mathbf{1}}$ in Theorem 3 plays the role of the term g in Theorem 1. Therefore, the network characteristic that governs the optimal price sequence as well as the optimal revenue is captured by the term $\frac{1}{\mathbf{1}^T E^{-1}\mathbf{1}}$. We refer to $\frac{1}{\mathbf{1}^T E^{-1}\mathbf{1}}$ as *network effect*. All the implications of Theorem 1 hold by replacing g with $\frac{1}{\mathbf{1}^T E^{-1}\mathbf{1}}$. Moreover, Theorem 3 leads to the following implications:

- The optimal revenue is an increasing convex function in $\left(\frac{1}{\mathbf{1}^T E^{-1}\mathbf{1}}\right)$ and also an increasing concave function in T. Moreover, the limiting normalized revenue as $T\to\infty$ is

$$\frac{1}{4-2\left(\frac{1}{\mathbf{1}^T E^{-1}\mathbf{1}}\right)}.$$

- Buyers with valuations below $\mathbf{v}_1$ do not buy the item. Since $\mathbf{v}_1$ as given in (4.15) is decreasing in both $\frac{1}{\mathbf{1}^T E^{-1}\mathbf{1}}$ and T, by increasing $\frac{1}{\mathbf{1}^T E^{-1}\mathbf{1}}$ and T the number of buyers who do not buy the item decreases.

- Using (4.14), the normalized number of buyers in group i who have bought the item after t rounds of pricing is

$$\frac{t}{2T\left(\mathbf{1}^T E^{-1}\mathbf{1}\right) - (T-1)} \left[E^{-1}\mathbf{1}\right]_i .$$

  Therefore, the number of buyers that purchase at time t is increasing in network effects $\frac{1}{\mathbf{1}^T E^{-1}\mathbf{1}}$ and decreasing in T.

4.2.1 Welfare of the Optimal Pricing Policy

Theorem 3 implies that as the number of rounds increases, the revenue becomes larger. One natural question is the impact of the number of rounds on the social welfare of the optimal pricing policy where social welfare is defined as the sum of utilities of buyers and revenue of seller. In the next proposition, we explicitly find the welfare and establish that it is increasing in the number of rounds. Therefore, although social welfare and revenue are not usually aligned with each other, interestingly, increasing the number of rounds improves revenue of the seller as well as the social welfare.

Proposition 2 *Suppose Assumptions 1 and 2 hold. For a general network externality E, for the optimal pricing policy, the welfare is*

$$\frac{T\left(\mathbf{1}^T E^{-1}\mathbf{1}\right)}{\left(2T\left(\mathbf{1}^T E^{-1}\mathbf{1}\right) - (T-1)\right)^2}\left(\left(\frac{3}{2}T\left(\mathbf{1}^T E^{-1}\mathbf{1}\right) - \frac{1}{2}(T-1)\right)\right). \tag{4.16}$$

Furthermore, the welfare is increasing in both T and network effects $\frac{1}{\mathbf{1}^T E^{-1}\mathbf{1}}$.

Proof The proof is presented in Appendix 7.5. ■

We next show the generalization of Theorem 3 for non-uniform valuations. In Section 4.4, we study network effects implications of Theorem 3.

4.3 Non-uniform Valuations

In this subsection, we relax Assumption 1 and show a generalization of our results in Theorem 3. The key result is that even with a general valuation distribution, the optimal pricing policy is still linear in time. Our characterization also highlights the effects of valuation distribution as well as network structure.

Assumption 3 Matrix E is invertible, $\mathbf{1}^T E^{-1}\mathbf{1} \geq f(x)$ for all $x \in [0,1]$, and $E^{-1}\mathbf{1} \geq 0$. We further assume that $\frac{f'(\cdot)}{f(\cdot)}$ is non-increasing and $xf(x)$ is non-decreasing.

Note that the assumptions on $f(\cdot)$ and $F(\cdot)$ are to guarantee that the first order conditions provide the optimal prices. These assumptions are the analogous of the regularity condition for revenue maximization in one round with no externalities.[6] A sufficient condition

[6] We need a stronger assumption than regularity condition because as we see in Theorem (4) we need conditions to guarantee concavity of a multivariate function as opposed to a single variate function $p(1 - F(p))$ which leads to regularity condition (Krishna (2009)).

to guarantee these assumptions is to have non-decreasing $\frac{f'(\cdot)}{f(\cdot)}$ with $\frac{f'(1)}{f(1)} \geq -1$. Also, note that for uniform valuations, we have $f'(\cdot) = 0$ and $f(\cdot) = 1$ and Assumption 3 reduces to Assumption 2.

Theorem 4 *Suppose Assumption 3 holds. For a general network externality E, the optimal price sequence in the limit (as $n \to \infty$) is linearly increasing given by*

$$p_t = (T-t)\frac{1-F(p_T)}{T\left(\mathbf{1}^T E^{-1}\mathbf{1}\right)} + p_T, \quad t = 1, \ldots, T, \tag{4.17}$$

where p_T is the solution of

$$p_T = (1-F(p_T))\left(\frac{1}{f(p_T)} - \frac{T-1}{T\left(\mathbf{1}^T E^{-1}\mathbf{1}\right)}\right).$$

In addition, the optimal normalized revenue is

$$(1-F(p_T))\left(\frac{T-1}{2T}\frac{1}{\mathbf{1}^T E^{-1}\mathbf{1}}(1-F(p_T)) + p_T\right). \tag{4.18}$$

Proof The proof is presented in Appendix 7.6. ∎

4.4 Network Effects

In order to study the effect of network structure in the optimal revenue, we consider a weakly tied block-model, where $E = I + \delta C$, for some small δ. This network externality captures the fact that externalities of buyers within each group is larger than the ones across groups. Using a second order Taylor approximation, the normalized revenue becomes

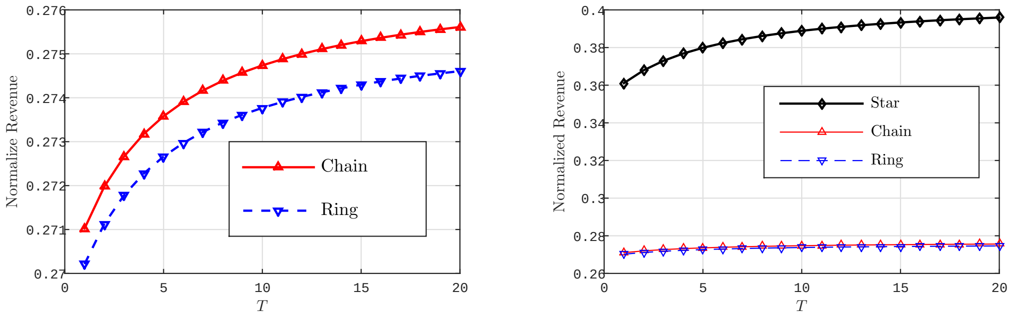
$$\frac{Tm}{4Tm - 2(T-1)} + \delta\frac{T(T-1)\sum_{i,j=1}^m C_{ij}}{2(2mT+1-T)^2}$$
$$+ \delta^2\left(\frac{T(T-1)\left(2T\left(\sum_{i,j=1}^m C_{ij} - (2mT-T+1)\sum_{i,j=1}^m [C^2]_{ij}\right)^2\right)}{2(2mT+1-T)^3}\right). \tag{4.19}$$

We have the following implications:

- The first term is larger than $\frac{1}{4}$, showing that the revenue is larger than the revenue obtained in a market with only one round of pricing.
- The second term shows that larger overall network externality, i.e., larger $\sum_{i,j} C_{ij}$ leads to a larger revenue.
- The third term shows that for a given $\sum_{i,j=1}^m C_{ij}$, the network with the most "asymmetry" has the highest revenue. More specifically, since we have

$$\sum_{i,j=1}^m [C^2]_{ij} = \sum_{i,j=1}^m \sum_{k=1}^m C_{ik}C_{kj} = \sum_{i,k=1}^m C_{ik}d_k^{(\text{out})} = \sum_{k=1}^m d_k^{(\text{out})}d_k^{(\text{in})},$$

  the maximum revenue is obtained for a network with minimum $\sum_k d_k^{(\text{out})}d_k^{(\text{in})}$. As an example, for $C \in \{0,1\}^{m\times m}$ ($C_{ij} = 1$ if there exists an edge between i and j), a bipartite directed graph have the optimal revenue. I.e., each group i cannot both influence others and get influenced by others (i.e., $d_i^{(\text{in})}d_i^{(\text{out})} = 0$ for all $i \in [m]$).



(a) (b)

Figure 3: The normalized revenue as function of the number of periods for $E = I + \delta C$, $m = 10$, $\delta = .29$, $\sum_{i,j} C_{ij} = 30$, and the weight of different edges in each network is the same. The revenue of star is larger than the revenue of chain which is larger than the revenue of ring. This ordering is exactly the ordering of "asymmetry" is these networks.

- For a given $\sum_{i,j=1}^{m} C_{ij}$, among symmetric networks (i.e., $d_i^{(\text{in})} = d_i^{(\text{out})}$, $i \in [m]$) a regular network leads to the largest revenue. I.e., the network for which for any $j \in [m]$, we have $\sum_{j=1}^{m} C_{kj} = \frac{1}{m} \sum_{k,j=1}^{m} C_{kj}$. This follows because we have

$$\sum_{i,j=1}^{m} [C^2]_{ij} = \sum_{i,k=1}^{m} C_{ik} \left(\sum_{j=1}^{m} C_{kj} \right) = \sum_{k=1}^{m} \left(\sum_{j=1}^{m} C_{kj} \right)^2 \geq \frac{1}{m} \left(\sum_{k=1}^{m} \sum_{j=1}^{m} C_{kj} \right)^2,$$

  where we used the symmetry in the last equality and Cauchy-Schwarz in the inequality. Since the inequality is equality if and only if for any j, we have $\sum_{j=1}^{m} C_{kj} = \frac{1}{m} \sum_{k,j=1}^{m} C_{kj}$ (i.e., regular network).

To illustrate the network effects, we next compare the revenue of a chain, a ring, and a star network. Figure 3a shows that the revenue of a chain network is larger than the revenue of a ring network. Moreover, as shown in Figure 3b the revenue of both are dominated by the revenue of a star network. This is because comparing to ring and chain, star network is more asymmetric, i.e. the term $\sum_i d_i^{(\text{out})} d_i^{(\text{in})}$ is minimized for the star network.

4.5 Price Discrimination

We consider a general network externality and study price discrimination. With price discrimination, at each round the price offered to different groups of buyers can be different.

Theorem 5 *Suppose Assumptions 1 and 2 hold. We further suppose $E^{-1} - A$ is positive semidefinite. For a general network externality E, the optimal price sequence is*

$$\mathbf{p}_t = \frac{T-t}{T} \left(I - EA\frac{T-1}{T} \right)^{-1} \mathbf{p}_T + \mathbf{p}_T, \quad t = T, \ldots, 1 \tag{4.20}$$

where $\mathbf{p}_T$ is the solution of

$$EA\mathbf{1} = \left(I + \left(I - \frac{T-1}{T} EA \right)^{-1} \right) EA\mathbf{p}_T. \tag{4.21}$$

Proof The proof is presented in Appendix 7.7. ■

Theorem 5 establishes that the optimal pricing policy is a linear policy and that the slope of this linear policy is given by a "weighted Bonacich centrality" which is $\left(I - \frac{T-1}{T}EA\right)^{-1}$. As an example for $T = 2$, the summation of the offered prices in two rounds is $\mathbf{1}$, i.e., $\mathbf{p}_1 + \mathbf{p}_2 = \mathbf{1}$ and we have the following corollary.

Corollary 1 *Consider the same assumptions as Theorem 5. The optimal price sequence for $T = 2$ is*

$$\mathbf{p}_2 = \mathbf{1} - \frac{1}{2}\left(I - \frac{EA}{4}\right)^{-1}\mathbf{1}, \quad \mathbf{p}_1 = \frac{1}{2}\left(I - \frac{EA}{4}\right)^{-1}\mathbf{1}. \tag{4.22}$$

In particular, if $\alpha_i = \frac{1}{m}$ for all $i \in [m]$, then we have $\mathbf{p}_2 = \mathbf{1} - \frac{1}{2}\mathbf{b}(E, \frac{1}{4m})$, where $\mathbf{b}(E, \frac{1}{4m})$ is the Bonacich *centrality of parameter $\frac{1}{4m}$ in network E.*[7]

Proof The proof is presented in Appendix 7.8. ■

The intuition of this Corollary (as well as Theorem 5) is instructive as it shows that in the first round, seller would offer a lower price to those groups with larger Bonacich centrality. This is to encourage those buyers to purchase in the first rounds which in turn incentivizes more buyers (due to larger centrality) to purchase in the second round. We next compare the revenue in this setting with that of a static pricing which is extensively studied in Candogan et al. (2012).

Remark 1 The buyer's strategies in a setting with one round of pricing is given by a thresholding policy where buyers in group i purchase the item if their valuation exceed $v^{(i)}$. These thresholds satisfy

$$\mathbf{v} - \mathbf{p} + EA\left(\mathbf{1} - \mathbf{v}\right) = \mathbf{0},$$

where $\mathbf{v} = \left(v^{(1)}, \ldots, v^{(m)}\right)$. This leads to $\mathbf{v} = (I - EA)^{-1}(\mathbf{p} - EA\mathbf{1})$. Therefore, the normalized revenue is given by

$$\begin{aligned}\mathbf{p}^T A\left(\mathbf{1} - \mathbf{v}\right) &= \mathbf{p}^T A\left(\mathbf{1} - (I - EA)^{-1}(\mathbf{p} - EA\mathbf{1})\right) \\ &= \mathbf{p}^T A\,(I - EA)^{-1}\mathbf{1} - \mathbf{p}^T A\,(I - EA)^{-1}\mathbf{p}. \end{aligned}\tag{4.23}$$

Maximizing (4.23) over $\mathbf{p}$ leads to

$$\mathbf{p} = \left(A\,(I - EA)^{-1} + \left((I - EA)^{-1}\right)^T A\right)^{-1} A\,(I - EA)^{-1}\mathbf{1}. \tag{4.24}$$

Comparing our results in Corollary 1 and that of (4.24), we should note that the optimal price (4.24) for symmetric networks becomes $\mathbf{p} = \frac{1}{2}\mathbf{1}$. However, in our model, even for symmetric networks, the dependence of optimal price sequence on network structure is present as given in (4.22).

[7] Bonacich centrality of parameter β in network E is defined as $\mathbf{b}(E, \beta) = (I - \beta E)^{-1}\mathbf{1}$ (see Bonacich (1987)).

Remark 2 For a weakly tied network externality, i.e., $E = I + \delta C$, using a first order Taylor approximation, the normalized revenue becomes

$$\sum_{i=1}^{m} \frac{\alpha_i}{4-\alpha_i} + \delta \sum_{i,j=1}^{m} C_{ij} \frac{\alpha_i}{4-\alpha_i} \frac{\alpha_j}{4-\alpha_j}. \tag{4.25}$$

Therefore, unlike the case without price discrimination, for a fixed $\sum_{i,j} C_{ij}$, the revenue of a network that puts more weights among groups with larger size (i.e., larger $\frac{\alpha_i}{4-\alpha_i}\frac{\alpha_j}{4-\alpha_j}$). Also, comparing (4.19) and (4.25), as expected, the revenue with price discrimination is larger than the revenue without price discrimination. This is because from Cauchy-Schwarz inequality we have

$$\left(\sum_{i=1}^{m} \frac{\alpha_i^2}{4-\alpha_i}\right)\left(\sum_{i=1}^{m} 4-\alpha_i\right) \geq 1,$$

which leads to $\sum_{i=1}^{m} \frac{\alpha_i^2}{4-\alpha_i} \geq \frac{1}{4m-1}$. Multiplying by 1/4 and adding 1/4 to both sides results in

$$\sum_{i=1}^{m} \frac{\alpha_i}{4-\alpha_i} = \frac{1}{4}\sum_{i=1}^{m} \frac{\alpha_i^2}{4-\alpha_i} + \frac{1}{4}\sum_{i=1}^{m} \alpha_i \geq \frac{1}{4} + \frac{1}{4}\frac{1}{4m-1} = \frac{m}{4m-1},$$

proving the claim.

5 Extensions

In this section, we provide several extensions of our model and analysis, namely, non-committed monopolist, future-dependent utility of buyers, and price discrimination.

5.1 Equilibrium Prices without Commitment

In this subsection, we consider the problem of a seller without the ability to commit to future prices. It is of course immediate that such lack of commitment will introduce additional constraints on the price choices of the seller and thus reduce equilibrium revenues relative to the commitment benchmark (the reference to equilibrium here in contrast to our earlier reference to optimal prices and revenue reflects the inability of the seller to commit which turns this into a dynamic game between the seller today and its own actions in the future). As is well known from the Coase conjecture literature discussed in the Introduction and Subsection 2.3, the lack of commitment to future prices can lead to a decreasing price path–higher-valuation buyers purchase early on, and then the seller has an incentive to cater to the lower-valuation buyers by reducing prices later. In the limit of our model where externalities disappear, this reasoning would apply and create a decreasing price path. However, in the presence of network externalities, the economic force we identified earlier in the paper is still operational, and the buildup of network externalities will incentivize the seller to have an increasing (non-decreasing) price path. We now provide an explicit characterization of the equilibrium price path with two periods and uniform externalities and find the optimal revenue without commitment.

Proposition 3 *Suppose the same assumptions as Theorem 1 hold and let $T = 2$. If $g \leq \frac{3+\sqrt{13}}{2}$, then in the limit as $n \to \infty$, the optimal price sequence without commitment is*

$$p_2 = \frac{1 + 3g - 2g^2}{2\left(1 + 4g - g^2\right)}, \qquad p_1(H^2) = \frac{g}{2}|H^2| + \frac{2p_2 + g}{2(1+g)}, \tag{5.1}$$

where H^2 is the set of buyers who buy the item in the first round, with the optimal normalized revenue

$$\frac{1 + 4g}{4\left(1 + 4g - g^2\right)}. \tag{5.2}$$

Proof The proof is presented in Appendix 7.9. ■

Remark 3 In this setting, the price sequence without commitment is increasing. In particular, the price in the first round is below $\frac{1}{2}$ and

$$p_1(H^2) = \frac{g}{2}|H^2| + \frac{2p_2 + g}{2(1+g)} \geq \frac{2p_2 + g}{2(1+g)} \geq p_2.$$

In addition, comparing Theorem 1 for $T = 2$ with Proposition 3, we have

$$\frac{1 + 4g}{4\left(1 + 4g - g^2\right)} \leq \frac{1}{4 - g}.$$

Therefore, as expected the optimal revenue without commitment is smaller than the optimal revenue with commitment.

5.2 Model with Utility from all Sales

In this subsection, we consider a variation of our model in which the utility of a buyer depends on the entire subset of buyers who buy the product in T periods. I.e., the utility of buyer i is

$$v^{(i)} - p_t + \sum_{j \neq i} g_{ij} \mathbf{1}\{j \text{ buys at } s \in \{1, \ldots, T\}\},$$

where $v^{(i)}$ is the valuation of buyer i and p_t is the posted price at time t. We assume that customers are individually rational, i.e., if a customer purchases at time t, her utility considering only the purchases already happened, should be non-negative. This assumption is crucial as without this assumption buyers only purchase the item at the round with minimum price. We next show via an example that in this setting the optimal price sequence can either be increasing or decreasing.

Example 2 Let $n = 2$ and $g_{ij} = g$ for $i, j \in \{1, 2\}$. We first find the optimal non-decreasing pricing policy and then the optimal non-increasing pricing policy.

- Non-decreasing policy: suppose $p_2 \leq p_1$. For $g \leq 1$, the optimal pricing policy is $p_2 = p_1 = 1/2$ with revenue $\frac{1+g}{2}$. This is because a buyer with valuation v buys at the second period if and only if $v - p_1 + g(1 - p_2) \geq 0$, and the seller's problem becomes

$$\max_{p_2 \leq p_1} \; p_2(1 - p_2) + p_2 p_1 \left(1 - \frac{p_1 - g(1 - p_2)}{p_2}\right).$$

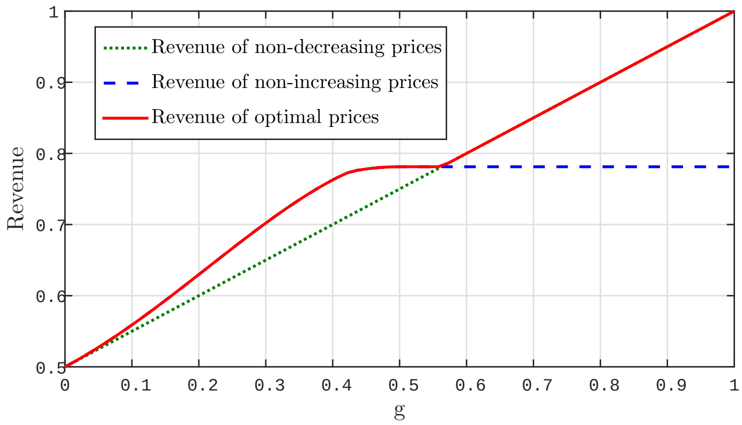


Figure 4: Example 2: the optimal price sequence can either be increasing or decreasing.

- Non-increasing policy: suppose $p_2 \geq p_1$. In this case, even though the price in the first period is higher, a buyer with $v \geq p_2$ buys at the second period if and only if

$$v - p_2 + g(1 - p_1 + g) \geq v - p_1 + g(1 - p_1) \Rightarrow g^2 \geq p_2 - p_1.$$

The seller's problem then becomes

$$\max_{p_2 \geq p_1} \; 2(1-p_2)^2 p_2 + 2p_2(1-p_2)\left(\min\left\{1, 1 - \frac{p_1 - g}{p_2}\right\} p_1 + p_2\right) + p_2^2\left(2p_1\left(1 - \frac{p_1}{p_2}\right)\right),$$

where we have $g^2 \geq p_2 - p_1$. The solution to this problem is

1. $g \geq \frac{1}{2}$: $p_1 = \frac{1}{2}$, $p_2 = \frac{5}{8}$, and revenue is $\frac{25}{32}$.
2. $\frac{\sqrt{13}-1}{6} \leq g < \frac{1}{2}$: $p_1 = g$, $p_2 = \frac{1+g-g^2}{2}$, with revenue $\frac{(1+g-g^2)^2}{2}$.
3. $\sqrt{2} - 1 \leq g < \frac{\sqrt{13}-1}{6}$: $p_1 = g$, $p_2 = \frac{1-g^2}{2}$, with revenue $2g\left(1 + g - 2g^2 - 2g^3\right)$.
4. $g < \sqrt{2} - 1$: $p_1 = \frac{1-g^2}{2}$, $p_2 = \frac{1+g^2}{2}$, with revenue $\frac{1}{2}\left(1 + g + 2g^2 - 2g^3 - 3g^4 + g^5\right)$.

Putting these two cases together the optimal revenue is plotted in Figure 4.

The reason that the optimal price sequence in Example 2 can be decreasing is because we only have 2 buyers and a buyer who observes a lower price in the second round might be willing to purchase at the current (higher) price in order to incentivize the other buyer to purchase at the second round. This effect goes away once we consider a large market in which buyers are non-atomic and hence price taker. In particular, if buyers are price takers and small compared to the industry, they will not exert any market power (see Belleflamme and Peitz (2015, Chapter 2)). Thus, a similar argument to that of Proposition 1 shows that the optimal price path is non-decreasing. We next consider the limiting equilibrium (as $n \to \infty$) and find the optimal non-decreasing price path as well as revenue.

Theorem 6 *Suppose Assumption 1 holds.*

- *For uniform network externalities with $g \in [0, 1)$, in the limit as $n \to \infty$, the optimal pricing is $p_t = \frac{1}{2}$, $t = 1, \ldots, T$ with the normalized revenue*

$$\frac{1}{4}\frac{1 - g^T}{1 - g}.$$

- *For general network externalities with non-increasing $\alpha^T(EA)^t\mathbf{1}$, in the limit as $n \to \infty$, the optimal pricing is $p_t = \frac{1}{2}$, $t = 1, \dots, T$ with the normalized revenue*

$$\frac{1}{4}\mathbf{1}^T A\left(I + (EA) + (EA)^2 + \cdots + (EA)^{T-1}\right)\mathbf{1}.$$

Proof The proof is presented in Appendix 7.10. ■

Theorem 6 leads to the following implications:

- The optimal revenue increases with a weighted network externality, i.e., EA (g for uniform externalities).

- For a general network externality provided that the spectral radius of EA is less than one (Horn and Johnson (2012, Chapter 1)), the limiting revenue as $T \to \infty$ becomes

  $$\frac{1}{4}\mathbf{1}^T A(I - EA)^{-1}\mathbf{1}.$$

  For uniform network externalities this limiting revenue becomes $\frac{1}{4(1-g)}$ and for a weakly tied block-model $E = C + \delta I$ (studied in Section 4.4), first order Taylor series leads to

  $$\frac{1}{4}\sum_{i=1}^{m}\frac{\alpha_i}{1-\alpha_i} + \delta\frac{1}{4}\sum_{i,j=1}^{m}\frac{\alpha_i}{1-\alpha_i}\frac{\alpha_j}{1-\alpha_j}C_{ij},$$

  showing that larger revenue is obtained for a larger weighted summation of network externalities C_{ij} where the weights are $\frac{\alpha_i}{1-\alpha_i}\frac{\alpha_j}{1-\alpha_j}$. This shows that larger externality among groups with larger size leads to a larger revenue.

6 Conclusion

We studied the problem of how to choose a sequence of prices for a product, under price commitment, given a set of customers with heterogeneous valuations who strategically decide their purchase time (if any). The product features network externalities, i.e., the utility of each customer depends on the offered price, her valuation, as well as a weighted number of other buyers who have adopted the item in the past. We established that the problem of finding optimal pricing policy is a tractable one with very few assumptions, and explicitly characterized the optimal policy as a function of the network structure and time horizon. We identified a novel dependence on the network structure termed "network effect". From a structural perspective, the optimal pricing policy is always linearly increasing with a slope that is decreasing in the number of rounds and increasing in "network effects". The optimal revenue is increasing in both the network effects and the number of rounds. Interestingly, the social welfare also increases with both the network effects and the number of rounds. We established that increasing network asymmetry, increases the network effect which in turn increases the revenue. We have also studied the effect of price discrimination and showed that the price of more central groups (defined as the ones with a larger "weighted Bonacich centrality") in the early rounds are smaller which further incentivizes other buyers to purchase in the later rounds. We have further found optimal pricing without commitment for two rounds of pricing and showed that the revenue with commitment is larger than revenue without commitment. The framework and results we present in this paper lay the ground for a potential new approach to the class of dynamic pricing problems with combinatorial structures. Avenues for future research include the expansion of the set of problems that may be tackled through the present approach. For example, the question of strategic dynamic pricing with limited inventory and strategic buyers/seller is a natural extension.

7 Appendix

7.1 Proof of Proposition 1

Proof of part (a): if $p_{s+1} > p_s$ for some $s = T-1, \ldots, 1$, then none of the buyers will buy the item at round $s+1$ (note that the period with price p_s is after the period with price p_{s+1}). This is because if they wait until the next round, the price decreases and the network externality part of their utility does not decrease (it either remains the same or increases). Therefore, the equilibrium path (the behavior of buyers and seller) is the same as a game with $T-1$ rounds in which we eliminate the round with price p_{s+1}. However, since the revenue of the seller is non-decreasing in the number of rounds, it is not optimal for the seller to have $p_{s+1} > p_s$, showing that the price path is non-decreasing.
Part of part (b): for a given equilibrium, suppose that buyer i with valuation v finds it optimal to accept price p_t in round t. Then it must be the case that her utility from purchasing in round t is not smaller than her expected utility from postponing the purchase to future rounds (not purchasing at round t). Therefore, we have

$$
\begin{aligned}
& v - p_t + \sum_j g_{ij} \mathbf{1}\{j \text{ bought at } s > t\} \\
& \geq \mathbb{E}\left[\sum_{s=t-1}^{1} \mathbf{1}\{i \text{ buy at price } p_s\} \left(v - p_s + \sum_j g_{ij} \mathbf{1}\{j \text{ bought at } s' > s\} \right) \mid H_t \right] \\
& = \sum_{s=t-1}^{1} \mathbb{P}\left[i \text{ buys at } s\right] \left(v - p_s + \mathbb{E}\left[\sum_j g_{ij} \mathbf{1}\{j \text{ bought at } s' > s\} \mid H_t \right] \right).
\end{aligned}
$$

Since $\sum_{s=t-1}^{0} \mathbb{P}\left[i \text{ buys at } s\right] \leq 1$ (there is a probability with with i does not buy at all), the derivative in v of the left hand side of this inequality is at least as large as that of the right hand side. Therefore, buyer i with valuations $v' > v$ find it strictly optimal to also accept the offer. This shows that if buyer i at time t does not accept the price, then her valuation is larger than a certain threshold denoted by $v_t^{(i)}$. If buyer i has not yet bought the item at round t, then her valuation is below $v_{t+1}^{(i)}$ with CDF $\mathbb{P}\left[x \leq v \mid x \leq v_{t+1}^{(i)}\right] = \frac{F(v)}{F(v_{t+1}^{(i)})}$.

7.2 Proof of Lemma 1

Using Proposition 1, for a given price sequence $p_T, \ldots, p_1$ and history H^t, the decision of buyer $i \in [n]$ at time t is to buy if and only if her valuation exceeds $v_t^{(i)}$. If the thresholds $v_t^{(i)}$ are interior, then buyer i with valuation $v_t^{(i)}$ must be indifferent between accepting price p_t and waiting until the subsequent round (in a continuation game with $t-1$ periods to go). If buyer i with valuation $v_t^{(i)}$ accepts, her utility (conditional on period t having been reached) is

$$v_t^{(i)} - p_t + \sum_j g_{ij} \mathbf{1}\{j \text{ has bought at } s > t\}. \tag{7.1}$$

By waiting one more period instead, buyer i with valuation $v_t^{(i)}$ obtains utility

$$v_t^{(i)} - p_{t-1} + \sum_j g_{ij} \mathbb{E}\left[\mathbf{1}\{j \text{ has bought at } s > t-1\}\right]. \tag{7.2}$$

Buyer i with threshold $v_t^{(i)}$ at round t must be indifferent between buying at round t and buying at the next round, i.e., round $t-1$. Subtracting (7.1) and (7.2) leads to

$$v_t^{(i)} - p_t = v_t^{(i)} - p_{t-1} + \sum_{j \in [n] \setminus H^t \setminus \{i\}} g_{ij} \mathbb{P}\left[j \text{ buys at round } t\right],$$

where

$$\mathbb{P}\left[j \text{ buys at round } t\right] = \mathbb{P}\left[v^{(j)} \geq v_t^{(j)} \mid v^{(j)} \leq v_{t+1}^{(j)}\right] = 1 - \frac{F(v_t^{(j)})}{F(v_{t+1}^{(j)})},$$

which completes the proof.

7.3 Proof of Theorem 1

We first characterize the buyer's behavior given a price sequence $p_T, \ldots, p_1$ and then find the optimal price sequence for the seller. For a symmetric perfect Bayesian equilibrium, the critical thresholds for all buyers at each round are the same which we denote by $v_T, v_{T-1}, \ldots, v_2, v_1$. Note that this thresholds are random variables and given round t has reached (i.e., a realization of v_{t+1} is observed), v_t is determined by the indifference condition (3.1), resulting in

$$\begin{aligned} &g(v_{t+1} - v_t) = p_{t-1} - p_t, \quad t = T, \ldots, 2, \\ &v_1 = p_1 - g(1 - v_2), \end{aligned} \tag{7.3}$$

with the convention that $v_{T+1} = 1$. Also, the last indifference condition holds because in the last round a buyer with valuation v accepts the price if and only if $v - p_1 + g(1 - v_2) \geq 0$. Again, note that the thresholds $v_T, \ldots, v_1$ are random variables and the seller's problem as given in (4.1) is to maximize

$$\sum_{t=T}^{1} p_t \mathbb{E}\left[\sum_{i=1}^{n} \mathbf{1}\{v^{(i)} \in [v_t, v_{t+1})\}\right]. \tag{7.4}$$

Using the linearity of expectation and invoking (7.3) in (7.4), the normalized expected revenue of seller becomes

$$\left(\sum_{t=T}^{2} p_t \frac{p_{t-1} - p_t}{g}\right) + p_1 \mathbb{E}\left[v_2 - p_1 + g(1 - v_2)\right]. \tag{7.5}$$

Taking summation of the indifference conditions given in (7.3) for $t = T, \ldots, 2$, we obtain a telescopic summation whose expectation leads to

$$g\mathbb{E}[1 - v_2] = p_1 - p_T. \tag{7.6}$$

Plugging (7.6) in (7.5) leads to the following seller's problem

$$\max_{p_T,\ldots,p_1} \left(\sum_{t=T}^{2} p_t \frac{p_{t-1} - p_t}{g} \right) + p_1 \frac{g + p_T(1-g) - p_1}{g}, \tag{7.7}$$

whose solution is the optimal price sequence. We let $h(\mathbf{p})$ denote the objective function, where $\mathbf{p} = (p_T, \ldots, p_1)$. We show $h(\mathbf{p})$ is concave and hence first order condition provides the optimal solution. The Hessian of $h(\mathbf{p})$ is a symmetric matrix $M \in \mathbb{R}^{T\times T}$ such that $M_{i,i+1} = 1$ for $i = 1, \ldots, T-1$, $M_{1,T} = (1-g)$, $M_{ii} = -2$ for $i = 1, \ldots, T$, and $M_{ij} = 0$, otherwise. This matrix is negative semidefinite as for any $\mathbf{x} \in \mathbb{R}^T$, we have

$$\mathbf{x}^T M \mathbf{x} = 2\sum_{i=1}^{T-1} x_i x_{i+1} + 2(1-g)x_1 x_T - 2\sum_{i=1}^{T} x_i^2 \leq 0,$$

where this inequality follows by taking summation of the following inequalities:

$$\begin{aligned} &\frac{1}{2}x_i^2 + \frac{1}{2}x_{i+1}^2 \geq x_i x_{i+1}, \quad i = 1, \ldots, T-1, \\ &\frac{1}{2}x_T^2 + \frac{1}{2}x_1^2 \geq |x_1 x_T| \geq (1-g)\,|x_1 x_T| \geq (1-g)x_1 x_T. \end{aligned} \tag{7.8}$$

Note that we used the assumption $g \in [0, 1]$ in the last two inequalities. Setting the first order conditions equal to zero results in

$$\begin{aligned} \frac{\partial h}{\partial p_T} &= \frac{1}{g}\left(-2p_T + p_{T-1} + p_1(1-g)\right) = 0, \\ \frac{\partial h}{\partial p_t} &= \frac{1}{g}\left(p_{t+1} - 2p_t + p_{t-1}\right) = 0, \quad t = T-1, \ldots, 2, \\ \frac{\partial h}{\partial p_1} &= \frac{1}{g}\left(p_2 + g + p_T(1-g) - 2p_1\right) = 0. \end{aligned}$$

We next find the solution of this set of equations. Starting from the last one, we obtain $p_2 = 2p_1 - (g + p_T(1-g))$. Plugging this into the equation corresponding to $\frac{\partial h}{\partial p_2} = 0$, leads to $p_3 = 3p_1 - 2(g + +p_T(1-g))$. Repeating this argument leads to

$$p_t = tp_1 - (t-1)\left(g + p_T(1-g)\right), \quad t = 1, \ldots, T. \tag{7.9}$$

Using (7.9) for $t = T$ and the equation corresponding to $\frac{\partial h}{\partial p_T} = 0$ yields

$$p_T = Tp_1 - (T-1)\left(g + p_T(1-g)\right),$$

and

$$-2p_T + \left((T-1)p_1 - (T-2)\left(g + p_T(1-g)\right)\right) + p_1(1-g) = 0,$$

which leads to

$$p_T = \frac{T - g(T-1)}{2T - g(T-1)}, \text{ and } \quad p_1 = \frac{T}{2T - g(T-1)}. \tag{7.10}$$

Invoking (7.10) in (7.9) results in

$$p_t = (t-1)\left(\frac{T-g}{2T - g(T-1)}\right) - (t-2)\left(\frac{T}{2T - g(T-1)}\right), \quad t = 1, \ldots, T.$$

Finally, plugging in the obtained price sequence in (7.7) results in the normalize revenue given by (4.3), completing the proof.

7.4 Details of the Proof of Theorem 3

We use the following generalization of Berstein polynomial convergence form Chapter 2.9 of Lorentz (2012).

Theorem 7 (Lorentz (2012)) *Suppose $f(x_1, \ldots, x_k)$ is continuous and defined on k-dimensional simplex $\Delta = \{\mathbf{x} \in \mathbb{R}^k : \ x_i \geq 0, 1 \leq i \leq k, \ \sum_{i=1}^k x_i \leq 1\}$. Then we have $B_{f,n} \to f$, where*

$$B_{f,n}(x_1, \ldots, x_k) = \sum_{\substack{r_i \geq 0, i \in [k] \\ \sum_{i=1}^k r_i \leq n}} f\left(\frac{r_1}{n}, \ldots, \frac{r_k}{n}\right) \binom{n}{r_1, \ldots, r_k} x_1^{r_1} \cdots x_k^{r_k} \left(1 - x_1 - \cdots - x_k\right)^{n - r_1 - \cdots - r_k},$$

and

$$\binom{n}{r_1, \ldots, r_k} = \frac{n!}{r_1! \ldots r_k!(n - r_1 - \cdots - r_k)!}.$$

We now prove Theorem 3 for $T > 2$. The normalized revenue becomes

$$\sum_{\substack{\sum_{s=1}^T k_j^{(s)} \leq n_j \\ j \in [m], s \in [T]}} \left(\prod_{j=1}^m \binom{n_j}{k_j^{(1)}, \ldots, k_j^{(T)}}\right) \left(\left(1 - v_T^{(j)}\right)^{k_j^{(T)}} \times \cdots \times \left(v_2^{(j)} - v_1^{(j)}\right)^{k_j^{(1)}} \left(v_1^{(j)}\right)^{n_j - k_j^{(1)} - \cdots - k_j^{(T)}}\right)\right)$$
$$\frac{1}{n}\left(p_T \sum_{j=1}^m k_j^{(T)} + p_{T-1} \sum_{j=1}^m k_j^{(T-1)} + \cdots + p_1 \sum_{j=1}^m k_j^{(1)}\right), \tag{7.11}$$

where $\binom{n_j}{k_j^{(1)}, \ldots, k_j^{(T)}}$ is the number of possibilities for selecting a partition of $[n_j]$ into $T+1$ subsets of size $k_j^{(1)}, \ldots, k_j^{(T)}$ and $n_j - k_j^{(1)} - \cdots - k_j^{(T)}$. Note that for $s = 1, \ldots, T$, $k_j^{(s)}$ shows the number of buyers in group j who buy at price p_s and the remaining number of buyers in group j (i.e., $n_j - k_j^{(1)} - \cdots - k_j^{(T)}$ many) decide not to buy the product. We can rewrite (7.11) as

$$\lim_{n\to\infty} \sum_{\substack{\sum_{s=1}^{T} k_j^{(s)} \le n_j \\ j\in\{2,\ldots,m-1\}, s\in[T]}} \left(\prod_{j=2}^{m} \binom{n_j}{k_j^{(1)},\ldots,k_j^{(T)}} \right) \left(\left(1-v_T^{(j)}\right)^{k_j^{(T)}} \times\cdots\times \left(v_2^{(j)}-v_1^{(j)}\right)^{k_j^{(1)}} \left(v_1^{(j)}\right)^{n_j-k_j^{(1)}-\cdots-k_j^{(T)}} \right) \right)$$

$$\frac{1}{n}\left(p_T\sum_{j=2}^{m} k_j^{(T)} + p_{T-1}\sum_{j=2}^{m} k_j^{(T-1)} + \cdots + p_1 \sum_{j=2}^{m} k_j^{(1)}\right)$$

$$\times \sum_{\substack{\sum_{s=1}^{T} k_1^{(s)} \le n_1 \\ s\in[T]}} \left(\binom{n_1}{k_1^{(1)},\ldots,k_1^{(T)}} \right) \left(\left(1-v_T^{(1)}\right)^{k_1^{(T)}} \times\cdots\times \left(v_2^{(1)}-v_1^{(1)}\right)^{k_1^{(1)}} \left(v_1^{(1)}\right)^{n_1-k_1^{(1)}-\cdots-k_1^{(T)}} \right) \right)$$

$$\frac{\alpha_1}{n_1}\left(p_T k_1^{(T)} + p_{T-1}k_1^{(T-1)} + \cdots + p_1 k_1^{(1)}\right)$$

$$= \lim_{n\to\infty} \sum_{\substack{\sum_{s=1}^{T} k_j^{(s)} \le n_j \\ j\in\{2,\ldots,m-1\}, s\in[T]}} \left(\prod_{j=2}^{m} \binom{n_j}{k_j^{(1)},\ldots,k_j^{(T)}} \right) \left(\left(1-v_T^{(j)}\right)^{k_j^{(T)}} \times\cdots\times \left(v_2^{(j)}-v_1^{(j)}\right)^{k_j^{(1)}} \left(v_1^{(j)}\right)^{n_j-k_j^{(1)}-\cdots-k_j^{(T)}} \right) \right)$$

$$\frac{1}{n}\left(p_T\sum_{j=2}^{m} k_j^{(T)} + p_{T-1}\sum_{j=2}^{m} k_j^{(T-1)} + \cdots + p_1 \sum_{j=2}^{m} k_j^{(1)}\right)$$

$$\times \alpha_1 \left(p_T(1-v_T^{(1)}) + p_{T-1}(v_T^{(1)}-v_{T-1}^{(1)}) + \cdots + p_1(v_2^{(1)}-v_1^{(1)})\right),$$

where we used Theorem 7 for $n_1 \to \infty$. Again using Theorem 7, $m-1$ times for $j = 2,\ldots,m$ as $n_j \to \infty$, the normalized expected revenue becomes

$$= p_T \sum_{j=1}^{m} \alpha_j \left(1-v_T^{(j)}\right) + p_{T-1}\sum_{j=1}^{m} \alpha_j \left(v_T^{(j)} - v_{T-1}^{(j)}\right) + \cdots + p_1 \sum_{j=1}^{m} \alpha_j \left(v_2^{(j)} - v_1^{(j)}\right)$$

$$= \sum_{t=T}^{1} p_t \mathbf{1}^T A \left(\mathbf{v}_{t+1} - \mathbf{v}_t\right).$$

This expression is given in (4.10) and the rest of the proof is presented after Theorem 3.

7.5 Proof of Proposition 2

The summation of the normalized utility of all buyers in group i and the seller's revenue obtained from these buyers (after canceling out the price as it appears in the utility of buyers and seller with different signs) is

$$\sum_{t=T}^{1} \alpha_i \mathbf{1}\left\{v \in (v_t^{(i)}, v_{t+1}^{(i)}]\right\} \left(v + \mathbf{e}_i^T E A \left(\mathbf{1} - \mathbf{v}_{t+1}\right)\right). \tag{7.12}$$

Taking expectation of (7.12) leads to

$$\sum_{t=T}^{1} \alpha_i \left(\frac{\left(v_{t+1}^{(i)}\right)^2 - \left(v_t^{(i)}\right)^2}{2} + \left(v_{t+1}^{(i)} - v_t^{(i)}\right) \mathbf{e}_i^T E A \left(\mathbf{1} - \mathbf{v}_{t+1}\right) \right), \tag{7.13}$$

where we used $\mathbb{E}\left[v\mathbf{1}\{v \in (a,b]\}\right] = \frac{1}{2}b^2 - \frac{1}{2}a^2$ for v with uniform distribution. Invoking (4.14) and (4.15) in (7.13), the expected normalize welfare for buyers in group i becomes

$$\frac{T}{\left(2T\left(\mathbf{1}^T E^{-1}\mathbf{1}\right) - (T-1)\right)^2} \alpha_i \left(\frac{3}{2}T\left(\mathbf{1}^T E^{-1}\mathbf{1}\right)^2 - \frac{T-1}{2}\left[(EA)^{-1}\mathbf{1}\right]_i\right).$$

Taking summation over all groups $i \in [m]$ and using $\sum_{i=1}^{m} \alpha_i = 1$, the normalized expected welfare can be written as

$$\frac{T}{\left(2T\left(\mathbf{1}^T E^{-1}\mathbf{1}\right) - (T-1)\right)^2}\left(\frac{3}{2}T\left(\mathbf{1}^T E^{-1}\mathbf{1}\right)^2 - \frac{T-1}{2}\left(\mathbf{1}^T E^{-1}\mathbf{1}\right)\right).$$

Finally, note that this expression is increasing in both T and the network effect $\frac{1}{\mathbf{1}^T E^{-1}\mathbf{1}}$.

7.6 Proof of Theorem 4

The proof follows from the same steps as of Theorem 1 and Theorem 3. In particular, analogous to (4.9) and (4.10), the indifference conditions become

$$\begin{aligned}
&EA\left(F\left(\mathbf{v}_{t+1}\right) - F\left(\mathbf{v}_t\right)\right) = (p_{t-1} - p_t)\mathbf{1}, \quad t = T, \ldots, 2, \\
&\mathbf{v}_1 - p_1\mathbf{1} + EA\left(\mathbf{1} - F\left(\mathbf{v}_2\right)\right) = 0,
\end{aligned} \tag{7.14}$$

with the convention that $\mathbf{v}_{T+1} = 1$ where $F(\mathbf{v}) = (F(v^{(1)}), \ldots, F(v^{(m)}))$. The normalized revenue becomes

$$\sum_{t=T}^{1} p_t \mathbf{1}^T A\left(F(\mathbf{v}_{t+1}) - F(\mathbf{v}_t)\right). \tag{7.15}$$

Using (7.14) in (7.15), the normalized revenue can be written as

$$\begin{aligned}
&\left(\sum_{t=T}^{2} p_t \frac{p_{t-1} - p_t}{\left(\frac{1}{\mathbf{1}^T E^{-1}\mathbf{1}}\right)}\right) + p_1\left(F\left(\mathbf{v}_2\right) - F\left(\mathbf{v}_1\right)\right) \\
&= \left(\sum_{t=T}^{2} p_t \frac{p_{t-1} - p_t}{\left(\frac{1}{\mathbf{1}^T E^{-1}\mathbf{1}}\right)}\right) + p_1 \frac{\left(\frac{1}{\mathbf{1}^T E^{-1}\mathbf{1}}\right) + p_T - gF(p_T) - p_1}{\left(\frac{1}{\mathbf{1}^T E^{-1}\mathbf{1}}\right)},
\end{aligned} \tag{7.16}$$

where we used $F(\mathbf{v}_2) = (g - p_1 + p_T) / \left(\frac{1}{\mathbf{1}^T E^{-1}\mathbf{1}}\right)$ in the last equality which follows from telescopic summation of (7.14). Similar to the approach of solving (7.7), letting

$$\begin{aligned}
\mathbf{p} &= (p_T, \ldots, p_1), \\
h(\mathbf{p}) &= \left(\sum_{t=T}^{2} p_t \frac{p_{t-1} - p_t}{\left(\frac{1}{\mathbf{1}^T E^{-1}\mathbf{1}}\right)}\right) + p_1 \frac{\left(\frac{1}{\mathbf{1}^T E^{-1}\mathbf{1}}\right) + p_T - gF(p_T) - p_1}{\left(\frac{1}{\mathbf{1}^T E^{-1}\mathbf{1}}\right)},
\end{aligned}$$

the first order condition results in

$$\begin{aligned}
\frac{\partial h}{\partial p_T} &= -2p_T + p_{T-1} + p_1 - \left(\frac{1}{\mathbf{1}^T E^{-1}\mathbf{1}}\right) p_1 f(p_T) = 0, \\
\frac{\partial h}{\partial p_t} &= p_{t+1} - 2p_t + p_{t-1} = 0, \quad t = T-1, \ldots, 2, \\
\frac{\partial h}{\partial p_1} &= p_2 + \left(\frac{1}{\mathbf{1}^T E^{-1}\mathbf{1}}\right) + p_T - \left(\frac{1}{\mathbf{1}^T E^{-1}\mathbf{1}}\right) F(p_T) - 2p_1 = 0,
\end{aligned}$$

which leads to the price sequence given in (4.17) and revenue given in (4.18). It remains to verify the second order conditions (i.e., the fact that the first order condition gives the optimal price sequence). Similar to the argument used in the proof of Theorem 1, the

Hessian of $h(\cdot)$ is given by a symmetric matrix M where $M_{ii} = -2$ for $i = 1, \dots, T-1$, $M_{TT} = -2 - \left(1/\mathbf{1}^T E^{-1}\mathbf{1}\right) p_1 f'(p_T)$, $M_{i,i+1} = 1$, for $i = 1, \dots, T-1$, and $M_{1T} = 1 - \left(1/\mathbf{1}^T E^{-1}\mathbf{1}\right) f(p_T)$. We next use Gershgorin circle theorem (Horn and Johnson (2012, Chapter 6)) to show that this matrix is negative semidefinite. In particular, for any eigen value λ, we have one of the following cases:

- $|-2-\lambda| \le 2$: this clearly leads to $\lambda \le 0$
- $|-2-\lambda| \le 1 + |1 - \left(\frac{1}{\mathbf{1}^T E^{-1}\mathbf{1}}\right) f(p_T)|$: using Assumption 3 and in particular $f(x)\frac{1}{\mathbf{1}^T E^{-1}\mathbf{1}}$, this leads to

  $$|-2-\lambda| \le 2 - \left(\frac{1}{\mathbf{1}^T E^{-1}\mathbf{1}}\right) f(p_T) \le 2,$$

  resulting in $\lambda \le 0$.
- $|-2 - \left(\frac{1}{\mathbf{1}^T E^{-1}\mathbf{1}}\right) p_1 f'(p_T) - \lambda| \le 1 + |1 - \left(\frac{1}{\mathbf{1}^T E^{-1}\mathbf{1}}\right) f(p_T)|$: first note that using Assumption 3 and in particular non-increasing $\frac{f'(x)}{f(x)}$ and $1 + x\frac{f'(x)}{f(x)} \ge 0$ along with $\frac{1}{\mathbf{1}^T E^{-1}\mathbf{1}} \le 1$ and $p_T \le p_1$ leads to $2 + \left(\frac{1}{\mathbf{1}^T E^{-1}\mathbf{1}}\right) p_1 f'(p_T) \ge 0$. This inequality together with $2 - \left(\frac{1}{\mathbf{1}^T E^{-1}\mathbf{1}}\right) f(p_T) \le 2 + \left(\frac{1}{\mathbf{1}^T E^{-1}\mathbf{1}}\right) p_1 f'(p_T)$ (which again follows from Assumption 3), leads to $\lambda \le 0$.

In all three cases we showed $\lambda \le 0$. Therefore, the objective function in (7.16) is concave and the first order condition provides the optimal solution.

7.7 Proof of Theorem 5

Similar to the proof of Theorem 3 the indifference conditions leads to

$$EA(\mathbf{v}_{t+1} - \mathbf{v}_t) = \mathbf{p}_{t-1} - \mathbf{p}_t, \quad t = T, \dots, 2, \tag{7.17}$$
$$\mathbf{v}_1 - \mathbf{p}_1 + EA(\mathbf{1} - \mathbf{v}_2) = 0, \tag{7.18}$$

and the normalized revenue becomes

$$\sum_{t=T}^{1} \mathbf{p}_t^T A\left(\mathbf{v}_{t+1} - \mathbf{v}_t\right) = \left(\sum_{t=T}^{2} \mathbf{p}_t^T E^{-1}\left(\mathbf{p}_{t-1} - \mathbf{p}_t\right)\right) + \mathbf{p}_1^T A\left(\mathbf{1} - \mathbf{p}_T\right) - \mathbf{p}_1^T E^{-1}\left(\mathbf{p}_1 - \mathbf{p}_T\right). \tag{7.19}$$

Denoting the revenue by $h(\cdot)$ and taking derivative of the revenue leads to

$$\frac{\partial h}{\partial p_T} = \mathbf{p}_{T-1} - 2\mathbf{p}_T - EA\mathbf{p}_1 + \mathbf{p}_1 = 0,$$
$$\frac{\partial h}{\partial p_t} = \mathbf{p}_{t+1} - 2\mathbf{p}_t + \mathbf{p}_{t-1} = 0, \quad t = T-1, \dots, 2,$$
$$\frac{\partial h}{\partial p_1} = \mathbf{p}_2 + E\mathbf{1} - EA\mathbf{p}_T - 2\mathbf{p}_1 + \mathbf{p}_T = 0.$$

Similar to the argument used in the proof of Theorem 1, the optimal price sequence becomes

$$\mathbf{p}_t = (T-t)\mathbf{x} + \mathbf{p}_T, \quad t = T, \dots, 1.$$

Plugging this into the equation corresponding to $\frac{\partial h}{\partial p_1} = 0$, we obtain

$$\mathbf{x} = \frac{1}{T}\left(A^{-1}E^{-1} - (T-1)I\right)^{-1}\mathbf{p}_T.$$

finally using this in the equation corresponding to $\frac{\partial h}{\partial p_T} = 0$, leads to the given price sequence. We next show that the first order condition provides the optimal solution. We show that the Hessian is negative semidefinite.The hessian is given by

$$\begin{pmatrix} -2E^{-1} & E^{-1} & 0 & \dots & E^{-1}-A \\ E^{-1} & -2E^{-1} & E^{-1} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & E^{-1} & -2E^{-1} & E^{-1} \\ E^{-1}-A & 0 & \dots & E^{-1} & -2E^{-1} \end{pmatrix}.$$

We need to show that for any $\mathbf{x}_1, \dots, \mathbf{x}_T \in \mathbb{R}^m$ we have

$$-2\sum_{i=1}^{m} \mathbf{x}_i^T E^{-1}\mathbf{x}_i + \sum_{i \neq j} \mathbf{x}_i^T E^{-1}\mathbf{x}_j - 2\mathbf{x}_1^T A \mathbf{x}_T \leq 0.$$

This follows by taking summation of the following inequalities:

$$\begin{aligned} &\frac{1}{2}\mathbf{x}_i^T E^{-1}\mathbf{x}_i^T + \frac{1}{2}\mathbf{x}_{i+1}^T E^{-1}\mathbf{x}_{i+1}^T \geq \mathbf{x}_i E^{-1}\mathbf{x}_{i+1}, \quad i = 1, \dots, T-1, \\ &\frac{1}{2}\mathbf{x}_T^T(E^{-1}-A)\mathbf{x}_T^T + \frac{1}{2}\mathbf{x}_T^T(E^{-1}-A)\mathbf{x}_T^T \geq \mathbf{x}_1(E^{-1}-A)\mathbf{x}_T \\ &\frac{1}{2}\mathbf{x}_T^T A \mathbf{x}_T^T + \frac{1}{2}\mathbf{x}_T^T A \mathbf{x}_T^T \geq 0, \end{aligned}$$

where the inequalities follows from the assumption that both E^{-1} and $E^{-1} - A$ are positive semidefinite. Note that E^{-1} is positive semidefinite because $E^{-1} - A$ is positive semidefinite and A is diagonal and positive semidefnite. Finally, note that since $E^{-1} - A$ is positive semidefinite, $I - EA\frac{T-1}{T}$ is invertible. This completes the proof.

7.8 Proof of Corollary 1

Using Theorem 5, we have

$$\begin{aligned} \mathbf{p}_2 &= \left(4E^{-1} - A\right)^{-1}\left(2E^{-1} - A\right)\mathbf{1} \\ &= \left(I - 2\left(4E^{-1} - A\right)^{-1}E^{-1}\right)\mathbf{1} = \mathbf{1} - \frac{1}{2}\left(I - \frac{EA}{4}\right)^{-1}\mathbf{1}, \end{aligned}$$

which completes the proof.

7.9 Proof of Proposition 3

For all $i, j \in [n]$, we have $g_{ij} = \tilde{g}$ and that $g = (n-1)\tilde{g}$. Because of symmetry, the thresholds of all buyers are the same. Suppose that $|H^2| = k$, i.e., in the first period k buyers purchase the item. For a given v_2, the probability that a remaining buyer accepts the price p_1 is

$$\mathbb{P}\left[v - p_1 + \tilde{g}k \geq 0 \mid v \geq v_2\right] = 1 - \frac{p_1 - \tilde{g}k}{v_2}. \tag{7.20}$$

Therefore, the revenue in the final round is $p_1\left(1-\frac{p_1-\tilde{g}k}{v_2}\right)$ and the optimal price p_1 in the final round as a function of the remaining buyers is the maximizer of this revenue which is

$$p_1=\frac{v_2+\tilde{g}k}{2}. \tag{7.21}$$

Note that unlike the case with commitment, p_1 can depend on history, i.e., H^2. Using (7.21), the indifference condition becomes

$$v_2-p_2=\sum_{k=0}^{n-1}\binom{n-1}{k}(1-v_2)^k v_2^{n-1-k}\left(v_2-\frac{v_2+\tilde{g}k}{2}+\tilde{g}k\right),$$

which leads to $v_2=\frac{2p_2+g}{1+g}$. The seller's normalized expected revenue in the limit can be written as

$$\begin{aligned}
&\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n}\binom{n}{k}(1-v_2)^k v_2^{n-k}\left(p_2k+p_1(H^2)(n-k)\mathbb{P}\left[v-p_1+\tilde{g}k\geq 0\mid v\geq v_2\right]\right)\\
&=\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n}\binom{n}{k}(1-v_2)^k v_2^{n-k}\left(p_2k+\frac{v_2+\tilde{g}k}{2}(n-k)\frac{v_2+\tilde{g}k}{2v_2}\right)\\
&=p_2(1-v_2)+\frac{1}{4}\left(v_2+g(1-v_2)\right)^2=p_2\frac{1-2p_2}{1+gn}+\frac{1}{4}\left(\frac{2p_2+g}{1+g}+g\frac{1-2p_2}{1+g}\right)^2,
\end{aligned}$$

where the first equality follows from using (7.20) and (7.21), the second equality follows from the closed form equations for the moments of a Binomial distribution and the fact that $\tilde{g}$ is of order $1/n$, and the third equality follows from $v_2=\frac{2p_2+g}{1+g}$. Maximizing this expression over p_2, when $g\leq\frac{3+\sqrt{13}}{2}$ leads to the price sequence and revenue given in (5.1) and (5.2).

7.10 Proof of Theorem 6

We only show the second part as the first part follows from the same line of argument. For $t=1,\dots,T$, we have

$$\mathbf{v}_t-p_t\mathbf{1}+EA(\mathbf{1}-\mathbf{v}_{t+1})=\mathbf{0}, \tag{7.22}$$

with the convention that $\mathbf{v}_{T+1}=1$. (7.22) leads to

$$\begin{aligned}
\mathbf{v}_{T+1}-\mathbf{v}_T&=(1-p_T)\mathbf{1}\\
\mathbf{v}_T-\mathbf{v}_{T-1}&=p_T\mathbf{1}-p_{T-1}\mathbf{1}+(EA)\mathbf{1}(1-p_T)\\
\mathbf{v}_{T-1}-\mathbf{v}_{T-2}&=p_{T-1}\mathbf{1}-p_{T-2}\mathbf{1}+(EA)\mathbf{1}(p_T-p_{T-1})+(EA)^2\mathbf{1}(1-p_T)\\
\mathbf{v}_{T+1-t}-\mathbf{v}_{T-t}&=\sum_{s=0}^{t}\left((EA)^{t-s}\mathbf{1}\right)(p_{T+1-s}-p_{T-s}),\quad t=3,\dots,T-1,
\end{aligned} \tag{7.23}$$

with the convention $p_{T+1}=1$. The normalized revenue can be written as

$$\sum_{t=1}^{T}p_t\alpha^T(\mathbf{v}_{t+1}-\mathbf{v}_t) \tag{7.24}$$

which we need to maximize over non-decreasing price sequences, i.e., $p_T \leq \cdots \leq p_2 \leq p_1$. We show that the optimal solution is $p_t = \frac{1}{2}$, $t = 1, \ldots, T$. We establish this by using the sufficient KKT conditions for optimality (e.g. Bertsekas (1999, Proposition 3.3.2)). We let

$$\begin{aligned}
&\mathbf{p} = (p_T, \ldots, p_1),\\
&f(\mathbf{p}) = -\sum_{t=1}^{T} p_t \alpha^T (\mathbf{v}_{t+1} - \mathbf{v}_t),\\
&g_j(\mathbf{p}) = p_{j+1} - p_j, \quad j = 1, \ldots, T-1,\\
&L(\mathbf{p}, \mu) = f(\mathbf{p}) + \sum_{j=1}^{T-1} \mu_j g_j(\mathbf{p}).
\end{aligned}$$

With this notation, the optimization problem can be rewritten as

$$\begin{aligned}
&\min_{\mathbf{p}} f(\mathbf{p})\\
&\text{s.t. } \mathbf{g}(\mathbf{p}) \leq 0.
\end{aligned}$$

Letting $\mathbf{p}^* = \frac{1}{2}\mathbf{1}$ and

$$\mu_j^* = \frac{1}{2}\sum_{s=1}^{T-j}((\alpha^T (EA)^{s-1}\mathbf{1}) - ((\alpha^T (EA)^{T-s}\mathbf{1})), \quad j = 1, \ldots, T-1,$$

we have

$$\nabla_{\mathbf{p}} L(\mathbf{p}^*, \mu^*) = 0 \tag{7.25}$$
$$g_j(\mathbf{p}^*) \leq 0, \quad j = 1, \ldots, T-1 \tag{7.26}$$
$$\mu_j^*(p_{j+1}^* - p_j^*) = 0, \quad j = 1, \ldots, T-1 \tag{7.27}$$
$$\mu_j^* > 0, \forall j, \text{ s.t. } p_{j+1}^* - p_j^* = 0, \quad j = 1, \ldots, T-1 \tag{7.28}$$
$$\mathbf{y}^T \nabla_{\mathbf{pp}} L(\mathbf{p}^*, \mu^*)\mathbf{y} > 0, \forall \mathbf{y} \neq 0 \text{ s.t. } \nabla g_j(\mathbf{p})^T \mathbf{y} = 0, \forall j, \text{ s.t. } p_{j+1}^* - p_j^* = 0. \tag{7.29}$$

In particular, (7.25) and (7.26) are straightforward to verify, (7.27) holds as all inequalities are active and (7.28) holds because $\alpha^T (EA)^t \mathbf{1}$ is decreasing in t (in particular, we have $\alpha^T (EA)^t \mathbf{1} \leq \alpha^T (EA)^0 \mathbf{1} = 1$). Finally (7.29) holds because all $\mathbf{y} \in \mathbb{R}^{T-1}$ that satisfy the conditions are of the form $y\mathbf{1}$ for some non-zero $y \in \mathbb{R}$ for which

$$\mathbf{y}^T \nabla_{\mathbf{pp}} L(\mathbf{p}^*, \mu^*)\mathbf{y} = y^2 \sum_{i,j=1}^{T} \nabla_{\mathbf{pp}} L(\mathbf{p}^*, \mu^*) = 2y^2 \sum_{t=1}^{T} \alpha^T (EA)^{t-1}\mathbf{1} > 0.$$

Therefore, using Bertsekas (1999, Proposition 3.3.2), $p_t = \frac{1}{2}$, $t = 1, \ldots, T$ is the optimal price sequence which results in revenue $\frac{1}{4}\mathbf{1}^T A \left(I + (EA) + (EA)^2 + \cdots + (EA)^{T-1}\right)\mathbf{1}$.

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