A Complexity Trichotomy for the Six-Vertex Model

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Abstract

We prove a complexity classification theorem that divides the six-vertex model into exactly three types. For every setting of the parameters of the model, the computation of the partition function is precisely: (1) Solvable in polynomial time for every graph, or (2) #P-hard for general graphs but solvable in polynomial time for planar graphs, or (3) #P-hard even for planar graphs. The classification has an explicit criterion. In addition to matchgates and matchgates-transformable signatures, we discover previously unknown families of planar-tractable partition functions by a non-local connection to #CSP, defined in terms of a "loop space". For the proof of #P-hardness, we introduce the use of Möbius transformations as a powerful new tool to prove that certain complexity reductions succeed in polynomial time.

1 Introduction

Partition functions are Sum-of-Product computations. In physics, one considers a set of particles connected by some bonds. Then physical considerations impose various local constraints, each with a suitable weight. Given a configuration satisfying all the local constraints, the product of local weights is the weight of the configuration, and its sum over all configurations is the value of the partition function. It encodes much information about the physical system.

This is essentially the same set-up as counting constraint satisfaction problems (#CSP). Take a set of constraint functions $\mathcal{F}$, the problem #CSP($\mathcal{F}$) is as follows: The input is a bipartite graph $G = (U, V, E)$, where U are the variables (spins), V is labeled by constraint functions from $\mathcal{F}$, and E describes how the constraints are applied on the variables. The output is the sum, over all assignments to variables in U, of the product of constraint function evaluations in V. Note that each function in $\mathcal{F}$ has a fixed arity, and in general takes values in $\mathbb{C}$ (not just $\{0, 1\}$). A spin system is the special case of #CSP where the constraints are binary functions (in which case each $v \in V$ has degree 2 and can be replaced by an edge).

By definition, a partition function is an exponential sized sum. But in some cases, clever algorithms exist that can compute it in polynomial time. Well-known examples of partition functions from physics that have been investigated intensively in complexity theory include the Ising model, Potts model, hardcore gas and Ice model [14, 10, 11, 18, 25]. Most of these are spin systems. If particles take (+) or (−) spins, each can be modeled by a Boolean variable, and local constraints

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are expressed by edge (binary) constraint functions. These are nicely modeled by the #CSP framework. Some physical systems are more naturally described as orientation problems, and these can be modeled by Holant problems, of which #CSP is a special case. Roughly speaking, Holant problems [7] (see Section 2 for definitions) are tensor networks where edges of a graph are variables while vertices are local constraint functions. Spin systems can be simulated easily as Holant problems, but Freedman, Lovász and Schrijver proved that simulation in the reverse direction is generally not possible [8]. In this paper we study a family of partition functions that fit the Holant problems naturally, but *not* as a spin system. This is the *six-vertex model*.

In physics, the six-vertex model concerns crystal lattices with hydrogen bonds. Remarkably it can be expressed perfectly as a family of Holant problems with 6 parameters, although in physics people are more focused on regular planar structures such as lattice graphs, and asymptotic limit. Previously, without being able to account for the planar restriction, it has been proved [5] that there is a complexity dichotomy where computing the partition function Z_{Six} is either in P or #P-hard. However the more interesting problem is what happens on planar structures where physicists had discovered some remarkable algorithms, such as Kasteleyn's algorithm for planar perfect matchings [21, 15, 16]. Concomitantly, and also probably because of that, to achieve a complete complexity classification in the planar case is more challenging. It must isolate precisely those problems that are #P-hard in general graphs but P-time computable on planar graphs.

In this paper we prove a complexity trichotomy theorem for the six-vertex models: According to the 6 parameters from $\mathbb{C}$, the partition function Z_{Six} is either computable in P-time, or #P-hard on general graphs but computable in P-time on planar graphs, or remains #P-hard on planar graphs. The classification has an explicit criterion. In addition to matchgates and matchgates-transformable signatures, we discover previously unknown families of planar-tractable Z_{Six} by a non-local connection to #CSP, defined in terms of a "loop space".

Linus Pauling in 1935 first introduced the six-vertex models to account for the residual entropy of water ice [20]. We have a large number of oxygen and hydrogen atoms in a 1 to 2 ratio. Each oxygen atom (O) is connected by a bond to four other neighboring oxygen atoms (O), and each bond is occupied by one hydrogen atom (H). Physical constraint requires that each (H) is closer to either one or the other of the two neighboring (O), but never in the middle of the bond. Pauling argued [20] that, furthermore, the allowed configurations are such that at each oxygen (O) site, exactly two hydrogen (H) are closer to it, and the other two are farther away. The placement of oxygen and hydrogen atoms can be naturally represented by vertices and edges of a 4-regular graph. The constraint on the placement of hydrogen atoms (H) can be represented by an orientation of the edges of the graph, such that at every vertex (O), exactly two edges are oriented toward the vertex, and exactly two edges are oriented away from it. In other words, this is an *Eulerian orientation*. Since there are $\binom{4}{2} = 6$ local valid configurations, this is called the six-vertex model. In addition to water ice, potassium dihydrogen phosphate KH_2PO_4 (KDP) also satisfies this model.

The valid local configurations of the six-vertex model are illustrated in Figure 1. The energy E of the system is determined by six parameters $\epsilon_1, \epsilon_2, \ldots, \epsilon_6$ associated with each type of the local configuration. If there are n_i sites in local configurations of type i, then $E = n_1\epsilon_1 + n_2\epsilon_2 + \ldots + n_6\epsilon_6$. Then the partition function is $Z_{\text{Six}} = \sum e^{-E/k_BT}$, where the sum is over all valid configurations, k_B is Boltzmann's constant, and T is the system's temperature. Mathematically, this is a *sum-of-product* computation where the sum is over all Eulerian orientations of the graph, and the product is over all vertices where each vertex contributes a factor $c_i = c^{\epsilon_i}$ if it is in configuration i ($1 \le i \le 6$) for some constant c.

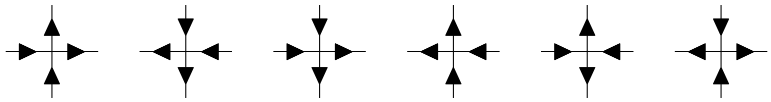


Figure 1: Valid configurations of the six-vertex model

Some choices of the parameters are well-studied. On the square lattice graph, when modeling ice one takes $\epsilon_1 = \epsilon_2 = \ldots = \epsilon_6 = 0$. In 1967, Elliott Lieb [19] famously showed that, as the number N of vertices approaches ∞, the value of the "partition function per vertex" $W = Z^{1/N}$ approaches $\left(\frac{4}{3}\right)^{3/2} \approx 1.5396007\ldots$ (Lieb's square ice constant). This matched experimental data 1.540 ± 0.001 so well that it is considered a triumph. Other well-known six-vertex models include: the KDP model of a ferroelectric ($\epsilon_1 = \epsilon_2 = 0$, and $\epsilon_3 = \epsilon_4 = \epsilon_5 = \epsilon_6 > 0$), the Rys F model of an antiferroelectric ($\epsilon_1 = \epsilon_2 = \epsilon_3 = \epsilon_4 > 0$, and $\epsilon_5 = \epsilon_6 = 0$). Historically these are widely considered among the most significant applications ever made of statistical mechanics to real substances. In classical statistical mechanics the parameters are all real numbers while in quantum theory the parameters are complex numbers in general.

Disregarding the planarity restriction, [5] proved that computing the partition function Z_{Six} is either in P or #P-hard. However known cases of planar P-time computable Z_{Six} (but #P-hard on general graphs) are all #P-hard in this classification. In this paper we tackle the more difficult planar case, and prove a complexity trichotomy theorem. The most interesting part is the classification of those Z_{Six} which are #P-hard in general but P-time computable on planar structures. The classification is valid for all parameter values $c_1, c_2, \ldots, c_6 \in \mathbb{C}$. (To state our theorem in strict Turing machine model, we take $c_1, c_2, \ldots, c_6$ to be *algebraic* numbers.) The dependence of this trichotomy on the values $c_1, c_2, \ldots, c_6$ is explicit.

We show that constraints that are expressible as matchgates (denoted by $\mathscr{M}$) or those that are transformable by a holographic transformation to matchgates (denoted by $\widehat{\mathscr{M}}$) do constitute a family of Z_{Six} which are #P-hard in general but P-time computable on planar structures. This is as expected and is known before (Kasteleyn's algorithm for planar perfect matchings, and Valiant's holographic algorithms based on matchgates [23, 24]). However we also discover an additional family of such Z_{Six} which are not transformable to matchgates. The P-time tractability on planar graphs is via a non-local transformation to #CSP, where the variables in #CSP correspond to certain loops in the six-vertex model graph. The fact that the #CSP instance is P-time tractable depends heavily on the global topological constraint imposed by the planar structure.

After carving out this last tractable family, we set about to prove that everything else is #P-hard, even for the planar case. A powerful tool in such proofs is the interpolation technique [22]. Typically an interpolation proof can succeed when certain quantities (such as eigenvalues) are not roots of unity, lest the iteration repeat after a bounded number of steps. A sufficient condition is that these quantities have complex norm $\neq 1$. However for some constraint functions, we can show that all constructions necessarily produce only relevant quantities of unit norm. In this case we introduce Möbius transformations $\mathfrak{z} \mapsto \frac{a\mathfrak{z}+b}{c\mathfrak{z}+d}$. It turns out that in this case the constraint function defines a natural Möbius transformation that maps unit circle to unit circle on $\mathbb{C}$. By exploiting the conformal mapping property we can obtain a suitable Möbius transformation which generates a group of infinite order. This allows us to show that our interpolation proof succeeds.

2 Preliminaries and Notations

In this paper, $\mathfrak{i}$ denotes $\sqrt{-1}$, a square root of -1.

2.1 Definitions and Notations

A constraint function f of arity k is a map $\{0,1\}^k \to \mathbb{C}$. Fix a set $\mathcal{F}$ of constraint functions. A signature grid $\Omega = (G, \pi)$ is a tuple, where $G = (V, E)$ is a graph, π labels each $v \in V$ with a function $f_v \in \mathcal{F}$ of arity $\deg(v)$, and the incident edges $E(v)$ at v with input variables of f_v. We consider all 0-1 edge assignments σ, each gives an evaluation $\prod_{v \in V} f_v(\sigma|_{E(v)})$, where $\sigma|_{E(v)}$ denotes the restriction of σ to $E(v)$. The counting problem on the instance Ω is to compute

$$\operatorname{Holant}(\Omega; \mathcal{F}) = \sum_{\sigma: E \to \{0,1\}} \prod_{v \in V} f_v(\sigma|_{E(v)}).$$

The Holant problem parameterized by the set $\mathcal{F}$ is denoted by Holant($\mathcal{F}$). If $\mathcal{F} = \{f\}$ is a single set, for simplicity, we write $\{f\}$ as f directly, and also we write $\{f, g\}$ as f, g. When G is a planar graph, the corresponding signature grid is called a planer grid. We use Holant $(\mathcal{F} \mid \mathcal{G})$ to denote the Holant problem over signature grids with a bipartite graph $H = (U, V, E)$, where each vertex in U or V is assigned a signature in $\mathcal{F}$ or $\mathcal{G}$ respectively. Signatures in $\mathcal{F}$ are considered as row vectors (or covariant tensors); signatures in $\mathcal{G}$ are considered as column vectors (or contravariant tensors). Similarly, Pl-Holant $(\mathcal{F} \mid \mathcal{G})$ denotes the Holant problem over signature grids with a planar bipartite graph.

A constraint function is also called a signature. A signature f of arity 4 has the signature matrix $M(f) = M_{x_1x_2,x_4x_3}(f) = \begin{bmatrix} f_{0000} & f_{0010} & f_{0001} & f_{0011} \\ f_{0100} & f_{0110} & f_{0101} & f_{0111} \\ f_{1000} & f_{1010} & f_{1001} & f_{1011} \\ f_{1100} & f_{1110} & f_{1101} & f_{1111} \end{bmatrix}$. If (i, j, k, ℓ) is a permutation of $(1, 2, 3, 4)$, then the 4×4 matrix $M_{x_ix_j,x_\ell x_k}(f)$ lists the 16 values with row index $x_ix_j \in \{0,1\}^2$ and column index $x_\ell x_k \in \{0,1\}^2$ in lexicographic order. Without other specification, $M(f)$ denotes $M_{x_1x_2,x_4x_3}(f)$. The planar six-vertex model is Pl-Holant($\neq_2|\ f$), where $M(f) = \begin{bmatrix} 0 & 0 & 0 & a \\ 0 & b & c & 0 \\ 0 & z & y & 0 \\ x & 0 & 0 & 0 \end{bmatrix}$. We call the submatrix $\begin{bmatrix} M(f)_{2,2} & M(f)_{2,3} \\ M(f)_{3,2} & M(f)_{3,3} \end{bmatrix} = \begin{bmatrix} b & c \\ z & y \end{bmatrix}$ denoted by $M_{\text{In}}(f)$ the inner matrix of $M(f)$, and the submatrix $\begin{bmatrix} M(f)_{1,1} & M(f)_{1,4} \\ M(f)_{4,1} & M(f)_{4,4} \end{bmatrix} = \begin{bmatrix} 0 & a \\ x & 0 \end{bmatrix}$ denoted by $M_{\text{Out}}(f)$ the outer matrix of $M(f)$. A binary signature g has the signature matrix $M(g) = M_{x_1,x_2}(g) = \begin{bmatrix} g_{00} & g_{01} \\ g_{10} & g_{11} \end{bmatrix}$. Without other specification, $M(g)$ denotes $M_{x_1,x_2}(g)$. We use $(\neq_2)$ to denote binary DISEQUALITY signature $(0, 1, 1, 0)^T$. It has the signature matrix $M(\neq_2) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Let $N = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$. Note that N is the double DISEQUALITY, which is the function of connecting two pairs of edges by $(\neq_2)$. A function is symmetric if its value depends only on the Hamming weight of its input. A symmetric function f on k Boolean variables can be expressed as $[f_0, f_1, \ldots, f_k]$, where f_w is the value of f on inputs of Hamming weight w. For example, $(=_k)$ is the EQUALITY signature $[1, 0, \ldots, 0, 1]$ (with $k - 1$ many 0's) of arity k. The support of a function f is the set of inputs on which f is nonzero.

Counting constraint satisfaction problems (#CSP) can be defined as a special case of Holant problems. An instance of #CSP($\mathcal{F}$) is presented as a bipartite graph. There is one node for each variable and for each occurrence of constraint functions respectively. Connect a constraint node to a variable node if the variable appears in that occurrence of constraint, with a labeling on the edges for the order of these variables. This bipartite graph is also known as the *constraint graph*. If we attach each variable node with an EQUALITY function, and consider every edge as a variable, then the #CSP is just the Holant problem on this bipartite graph. Thus $\#\mathrm{CSP}(\mathcal{F}) \equiv_T \mathrm{Holant}\,(\mathcal{EQ} \mid \mathcal{F})$, where $\mathcal{EQ} = \{=_1, =_2, =_3, \dots\}$ is the set of EQUALITY signatures of all arities. By restricting to planar constraint graphs, we have the planar #CSP framework, which we denote by Pl-#CSP. The construction above also shows that $\text{Pl-}\#\mathrm{CSP}(\mathcal{F}) \equiv_T \text{Pl-Holant}\,(\mathcal{EQ} \mid \mathcal{F})$.

2.2 Gadget Construction

One basic notion used throughout the paper is gadget construction. We say a signature f is *constructible* or *realizable* from a signature set $\mathcal{F}$ if there is a gadget with some dangling edges such that each vertex is assigned a signature from $\mathcal{F}$, and the resulting graph, when viewed as a black-box signature with inputs on the dangling edges, is exactly f. If f is realizable from a set $\mathcal{F}$, then we can freely add f into $\mathcal{F}$ while preserving the complexity.

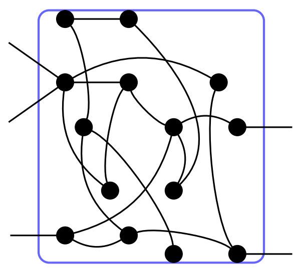


Figure 2: An $\mathcal{F}$-gate with 5 dangling edges.

Formally, this notion is defined by an $\mathcal{F}$-gate. An $\mathcal{F}$-gate is similar to a signature grid (G, π) for $\mathrm{Holant}(\mathcal{F})$ except that $G = (V, E, D)$ is a graph with some dangling edges D. The dangling edges define external variables for the $\mathcal{F}$-gate (See Figure 2 for an example). We denote the regular edges in E by $1, 2, \dots, m$ and the dangling edges in D by $m+1, \dots, m+n$. Then we can define a function f for this $\mathcal{F}$-gate as

$$f(y_1, \dots, y_n) = \sum_{x_1, \dots, x_m \in \{0,1\}} H(x_1, \dots, x_m, y_1, \dots, y_n),$$

where $(y_1, \dots, y_n) \in \{0,1\}^n$ is an assignment on the dangling edges and $H(x_1, \dots, x_m, y_1, \dots, y_n)$ is the value of the signature grid on an assignment of all edges in G, which is the product of evaluations at all vertices in V. We also call this function f the signature of the $\mathcal{F}$-gate.

An $\mathcal{F}$-gate is planar if the underlying graph G is a planar graph, and the dangling edges, ordered counterclockwise corresponding to the order of the input variables, are in the outer face in a planar embedding. A planar $\mathcal{F}$-gate can be used in a planar signature grid as if it is just a single vertex with the particular signature.

Using planar $\mathcal{F}$-gates, we can reduce one planar Holant problem to another. Suppose g is the signature of some planar $\mathcal{F}$-gate. Then $\text{Pl-Holant}(\mathcal{F}, g) \leq_T \text{Pl-Holant}(\mathcal{F})$. The reduction is simple. Given an instance of $\text{Pl-Holant}(\mathcal{F}, g)$, by replacing every occurrence of g by the $\mathcal{F}$-gate, we get an instance of $\text{Pl-Holant}(\mathcal{F})$. Since the signature of the $\mathcal{F}$-gate is g, the Holant values for these two signature grids are identical. There are three common gadgets we will use in this paper.

Suppose f_1 and f_2 have signature matrices $M_{x_ix_j,x_\ell x_k}(f_1)$ and $M_{x_sx_t,x_vx_u}(f_2)$, where (i,j,k,ℓ) and (s,t,u,v) are permutations of $(1,2,3,4)$. By connecting x_ℓ with x_s, x_k with x_t, both using DISEQUALITY $(\neq_2)$, we get a signature of arity 4 with the signature matrix $M_{x_ix_j,x_\ell x_k}(f_1)NM_{x_sx_t,x_vx_u}(f_2)$ by matrix product with row index x_ix_j and column index x_vx_u (See Figure 3). In this paper, we

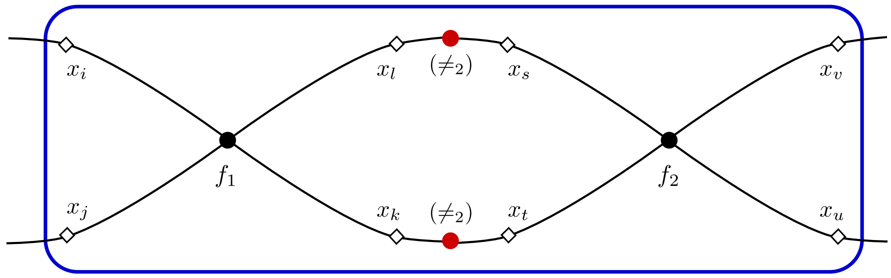


Figure 3: Connect variables x_ℓ, x_k of f_1 with variables x_s, x_t of f_2 both using $(\neq_2)$.

focus on planar graphs, and we assume the edges incident to a vertex are ordered counterclockwise. When connecting two signatures, we need to keep the counterclockwise order of the edges incident to each vertex. In order to satisfy this planar property, (i,j,k,ℓ) and (s,t,u,v) both have to be cyclic permutations of $(1,2,3,4)$. In this paper, given a signature f of arity 4 with the signature matrix $M_{x_ix_j,x_\ell x_k}(f)$, we always assume (i,j,k,ℓ) is a cyclic permutation of $(1,2,3,4)$. There are four cyclic permutations of $(1,2,3,4)$, so correspondingly, a signature f has four 4×4 signature matrices $M_{x_1x_2,x_4x_3}(f) = \begin{bmatrix} 0&0&0&a\\ 0&b&c&0\\ 0&z&y&0\\ x&0&0&0 \end{bmatrix}$, $M_{x_2x_3,x_1x_4}(f) = \begin{bmatrix} 0&0&0&y\\ 0&a&z&0\\ 0&c&x&0\\ b&0&0&0 \end{bmatrix}$, $M_{x_3x_4,x_1x_2}(f) = \begin{bmatrix} 0&0&0&x\\ 0&y&c&0\\ 0&z&b&0\\ a&0&0&0 \end{bmatrix}$, and $M_{x_4x_1,x_3x_2}(f) = \begin{bmatrix} 0&0&0&b\\ 0&x&z&0\\ 0&c&a&0\\ y&0&0&0 \end{bmatrix}$. Note that no matter in which signature matrix, only the pair (c,z) is always in the inner matrix. We call (c,z) the inner pair, and (a,x), (b,y) outer pairs. On the other hand, given a signature f' of arity 4 with the signature matrix $M_{x_1x_2,x_4x_3}(f') = M_{x_ix_j,x_kx_\ell}(f)$, we may relabel the variables x_1,x_2,x_3,x_4 of f' by x_i,x_j,x_ℓ,x_k. Then, the signature f' is exactly the signature f. In fact, f' can be viewed as a rotation form of f. The four rotation forms of f are denoted by f, $f^{\frac{\pi}{2}}$, f^{π} and $f^{\frac{3\pi}{2}}$. Once we get one form, all the four rotation forms can be freely used. In the proof, after one construction, we may use this property to get a similar construction and conclusion by quoting this rotational symmetry.

A binary signature g has the signature vector $g(x_1,x_2) = (g_{00}, g_{01}, g_{10}, g_{11})^T$, and also $g(x_2,x_1) = (g_{00}, g_{10}, g_{01}, g_{11})^T$. Without other specification, g denotes $g(x_1,x_2)$. Let f be a signature of arity 4 with the signature matrix $M_{x_ix_j,x_\ell x_k}(f)$ and (s,t) be a permutation of $(1,2)$. By connecting x_ℓ with x_s and x_k with x_t, both using DISEQUALITY $(\neq_2)$, we get a binary signature with the signature matrix $M_{x_ix_j,x_kx_\ell}Ng(x_s,x_t)$ by matrix product with index x_ix_j (See Figure 4). If $g_{00} = g_{11}$, then $N(g_{00}, g_{01}, g_{10}, g_{11})^T = (g_{11}, g_{10}, g_{01}, g_{00})^T = (g_{00}, g_{10}, g_{01}, g_{11})^T$ and similarly, $N(g_{00}, g_{10}, g_{01}, g_{11})^T = (g_{00}, g_{01}, g_{10}, g_{11})^T$. Therefore, $M_{x_ix_j,x_\ell x_k}Ng(x_s,x_t) = M_{x_ix_j,x_\ell x_k}g(x_t,x_s)$, which means connecting variables x_ℓ, x_k of the signature f with variables x_s, x_t of the signature g using double DISEQUALITY N are equivalent to connecting variables x_ℓ, x_k of the signature f with variables x_t, x_s of the signature g directly. In this paper, we use $M_{x_ix_j,x_\ell x_k}g(x_t,x_s)$ instead of $M_{x_ix_j,x_\ell x_k}Ng(x_s,x_t)$ to represent connecting f and g using double DISEQUALITY N when $g_{00} = g_{11}$. Since g is a binary signature, we can rotate it by 180 degree. Then the variables x_s and x_t change their positions with each other, and that rotation do not destroy the planar graph. That is, both $g(x_s,x_t)$ and $g(x_t,x_s)$ can be freely used once we get one of them.

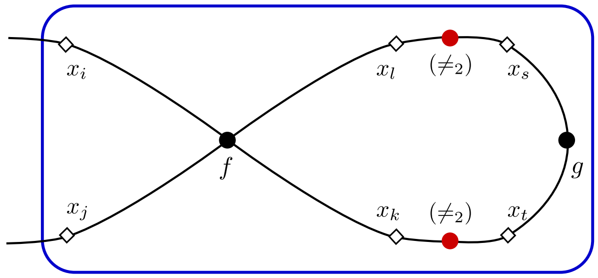


Figure 4: Connect variables x_ℓ, x_k of f with variables x_s, x_t of g both using $(\neq_2)$.

A binary signature g also has the 2×2 signature matrix $M(g) = M_{x_1,x_2}(g) = \left[\begin{smallmatrix} g_{00} & g_{01} \\ g_{10} & g_{11}\end{smallmatrix}\right]$. Without other specification, $M(g)$ denotes $M_{x_1,x_2}(g)$. A signature f of arity 4 also has the 2×8 signature matrix $M_{x_1,x_2x_4x_3}(f) = \left[\begin{smallmatrix} f_{0000} & f_{0010} & f_{0001} & f_{0011} & f_{0100} & f_{0110} & f_{0101} & f_{0111} \\ f_{1000} & f_{1010} & f_{1001} & f_{1011} & f_{1100} & f_{1110} & f_{1101} & f_{1111}\end{smallmatrix}\right]$. Suppose the signature matrix of g is $M_{x_s,x_t}(g)$ and the signature matrix of f is $M_{x_i,x_jx_\ell x_k}(f)$. By connecting x_t with x_i using Disequality $(\neq_2)$, we get a signature h of arity 4 with the signature matrix $M_{x_s,x_t}(g)Z^TZM_{x_i,x_jx_\ell x_k}(f)$ by matrix product with row index x_s and column index $x_jx_\ell x_k$ (See Figure 5). We may change

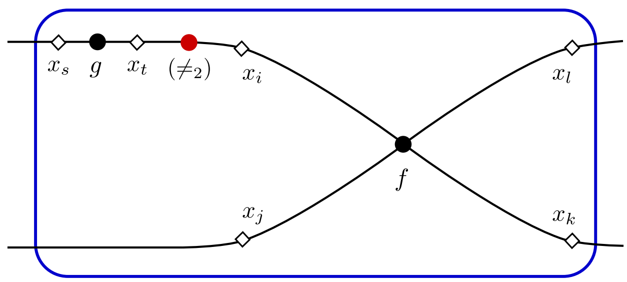


Figure 5: Connect variable x_t of g with variable x_i of f using $(\neq_2)$.

this form to a signature matrix with row index x_sx_j and column index $x_\ell x_k$. In particular, if $M_{x_s,x_t}(g) = \left[\begin{smallmatrix} 0 & 1 \\ t & 0\end{smallmatrix}\right]$, then

$$\begin{aligned} M_{x_s,x_jx_\ell x_k}(h) &= M_{x_s,x_t}(g)M(\neq_2)M_{x_i,x_jx_\ell x_k}(f) \\ &= \begin{bmatrix} 0 & 1 \\ t & 0\end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0\end{bmatrix}\begin{bmatrix} f_{0000} & f_{0010} & f_{0001} & f_{0011} & f_{0100} & f_{0110} & f_{0101} & f_{0111} \\ f_{1000} & f_{1010} & f_{1001} & f_{1011} & f_{1100} & f_{1110} & f_{1101} & f_{1111}\end{bmatrix} \\ &= \begin{bmatrix} f_{0000} & f_{0010} & f_{0001} & f_{0011} & f_{0100} & f_{0110} & f_{0101} & f_{0111} \\ tf_{1000} & tf_{1010} & tf_{1001} & tf_{1011} & tf_{1100} & tf_{1110} & tf_{1101} & tf_{1111}\end{bmatrix}. \end{aligned}$$

We change the name of the index x_s in $M_{x_s,x_jx_\ell x_k}(h)$ by x_i, then $M_{x_ix_j,x_\ell x_k}(h) = \left[\begin{smallmatrix} f_{0000} & f_{0010} & f_{0001} & f_{0011} \\ f_{0100} & f_{0110} & f_{0101} & f_{0111} \\ tf_{1000} & tf_{1010} & tf_{1001} & tf_{1011} \\ tf_{1100} & tf_{1110} & tf_{1101} & tf_{1111}\end{smallmatrix}\right]$. That is, when connecting the variable x_t of signature g with the variable x_i of signature f, let the entries of $M_{x_ix_j,x_\ell x_k}(f)$ with index $x_i = 1$ multiply by t and that will give the signature matrix of the constructed signature h. Similarly, when connecting the variable x_s of g with the variable x_i of f, let the entries of $M_{x_ix_j,x_\ell x_k}(f)$ with index $x_i = 0$ multiply by t and that will give $M_{x_ix_j,x_\ell x_k}(h)$.

A constant scalar $C \neq 0$ does not change the complexity of a Holant problem. That is, $\operatorname{Holant}(\mathcal{F}\cup\{f\}) \equiv_T \operatorname{Holant}(\mathcal{F}\cup\{Cf\})$ for any signature set $\mathcal{F}$, signature f and constant scalar

$C \neq 0$. In planar instances, that is Pl-Holant$(\mathcal{F} \cup \{f\}) \equiv_T$ Pl-Holant$(\mathcal{F} \cup \{Cf\})$. In other words, Cf is realizable by f for any $C \neq 0$. Hence, we can pick a nonzero entry of the signature matrix $M(f)$ and divide all entries in $M(f)$ by the nonzero entry we picked. We call this operation normalization. By normalization, we may assume the nonzero entry we picked is of value 1.

2.3 Holographic Transformation

To introduce the idea of holographic transformation, it is convenient to consider bipartite graphs. For a general graph, we can always transform it into a bipartite graph while preserving the Holant value, as follows. For each edge in the graph, we replace it by a path of length two. (This operation is called the *2-stretch* of the graph and yields the edge-vertex incidence graph.) Each new vertex is assigned the binary Equality signature $(=_2) = [1, 0, 1]$.

For an invertible 2-by-2 matrix $T \in \mathrm{GL}_2(\mathbb{C})$ and a signature f of arity n, written as a column vector (contravariant tensor) $f \in \mathbb{C}^{2^n}$, we denote by $T^{-1}f = (T^{-1})^{\otimes n} f$ the transformed signature. For a signature set $\mathcal{F}$, define $T^{-1}\mathcal{F} = \{T^{-1}f \mid f \in \mathcal{F}\}$ the set of transformed signatures. For signatures written as row vectors (covariant tensors) we define $\mathcal{F}T$ similarly. Whenever we write $T^{-1}f$ or $T^{-1}\mathcal{F}$, we view the signatures as column vectors; similarly for fT or $\mathcal{F}T$ as row vectors. In the special case of the Hadamard matrix $H_2 = \frac{1}{\sqrt{2}} \left[\begin{smallmatrix} 1 & 1 \\ 1 & -1 \end{smallmatrix}\right]$, we also define $\widehat{\mathcal{F}} = H_2\mathcal{F}$. Note that H_2 is orthogonal. Since constant factors are immaterial, for convenience we sometime drop the factor $\frac{1}{\sqrt{2}}$ when using H_2.

Let $T \in \mathrm{GL}_2(\mathbb{C})$. The holographic transformation defined by T is the following operation: given a signature grid $\Omega = (H, \pi)$ of Holant $(\mathcal{F} \mid \mathcal{G})$, for the same bipartite graph H, we get a new grid $\Omega' = (H, \pi')$ of Holant $\left(\mathcal{F}T \mid T^{-1}\mathcal{G}\right)$ by replacing each signature in $\mathcal{F}$ or $\mathcal{G}$ with the corresponding signature in $\mathcal{F}T$ or $T^{-1}\mathcal{G}$.

Theorem 2.1 (Valiant's Holant Theorem [25])**.** *For any $T \in \mathrm{GL}_2(\mathbb{C})$,*

$$\mathrm{Holant}(\Omega; \mathcal{F} \mid \mathcal{G}) = \mathrm{Holant}(\Omega'; \mathcal{F}T \mid T^{-1}\mathcal{G}).$$

Therefore, a holographic transformation does not change the complexity of the Holant problem in the bipartite setting. Clearly, this theorem holds for planar instances.

Definition 2.2. *We say a signature set $\mathcal{F}$ is $\mathscr{C}$-transformable if there exists a $T \in \mathrm{GL}_2(\mathbb{C})$ such that $(0, 1, 1, 0)T^{\otimes 2} \in \mathscr{C}$ and $T^{-1}\mathcal{F} \subseteq \mathscr{C}$.*

This definition is important because if Pl-Holant$(\mathscr{C})$ is tractable, then Pl-Holant$(\neq_2 \mid \mathcal{F})$ is tractable for any $\mathscr{C}$-transformable set $\mathcal{F}$.

2.4 Polynomial Interpolation

Polynomial interpolation is a powerful technique to prove #P-hardness for counting problems. We introduce this technique with the following lemmas.

Lemma 2.3. *Let f be a 4-ary signature with the signature matrix $M(f) = \left[\begin{smallmatrix} 0 & 0 & 0 & 1 \\ 0 & b & 0 & 0 \\ 0 & 0 & b & 0 \\ 1 & 0 & 0 & 0 \end{smallmatrix}\right]$, where $b \neq 0$ is not a root of unity. Let χ_1 be a 4-ary signature with the signature matrix $M(\chi_1) = \left[\begin{smallmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{smallmatrix}\right]$. Then for any signature set $\mathcal{F}$ containing f, we have*

$$\text{Pl-Holant}(\neq_2 \mid \mathcal{F} \cup \{\chi_1\}) \leqslant_T \text{Pl-Holant}(\neq_2 \mid \mathcal{F}).$$

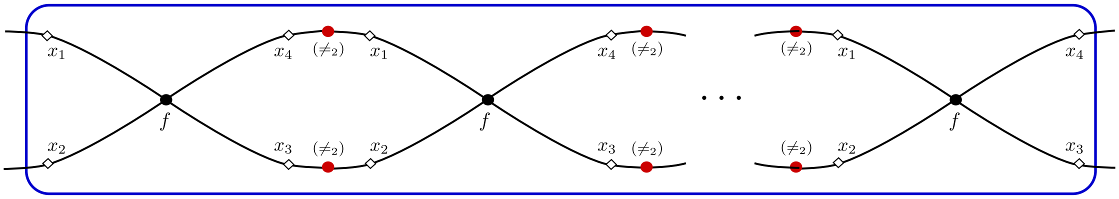


Figure 6: A chain of $2s+1$ many copies of f linked by double Disequality N

Proof. We construct a series of gadgets f_{2s+1} by a chain of $2s+1$ many copies of f linked by double Disequality N (See Figure 6). We know f_{2s+1} has the signature matrix

$$M(f_{2s+1}) = M(f)(NM(f))^{2s} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & b^{2s+1} & 0 & 0 \\ 0 & 0 & b^{2s+1} & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

The matrix $M(f_{2s+1})$ has a good form for polynomial interpolation. Suppose χ_1 appears m times in an instance Ω of Pl-Holant($\neq_2|\ \mathcal{F} \cup \{\chi_1\}$). We replace each appearance of χ_1 by a copy of the gadget f_{2s+1} to get an instance Ω_{2s+1} of Pl-Holant($\neq_2|\ \mathcal{F} \cup \{f_{2s+1}\}$), which is also an instance of Pl-Holant($\neq_2|\ \mathcal{F}$). We divide Ω_{2s+1} into two parts. One part consists of m signatures f_{2s+1} and its signature is represented by $(M(f_{2s+1}))^{\otimes m}$. Here we rewrite $(M(f_{2s+1}))^{\otimes m}$ as a column vector. The other part is the rest of Ω_{2s+1} and its signature is represented by A which is a tensor expressed as a row vector. Then, the Holant value of Ω_{2s+1} is the dot product $\langle A, (M(f_{2s+1}))^{\otimes m}\rangle$, which is a summation over $4m$ bits. That is, the value of the $4m$ edges connecting the two parts. We can stratify all $0,1$ assignments of these $4m$ bits having a nonzero evaluation of a term in Pl-Holant$_{\Omega_{2s+1}}$ into the following categories:

- There are i many copies of f_{2s+1} receiving inputs 0011 or 1100;
- There are j many copies of f_{2s+1} receiving inputs 0110 or 1001;

where $i + j = m$.

For any assignment in the category with parameter (i, j), the evaluation of $(M(f_{2s+1}))^{\otimes m}$ is clearly $b^{(2s+1)j}$. Let a_{ij} be the summation of values of the part A over all assignments in the category (i, j). Note that a_{ij} is independent from the value of s since we view the gadget f_{2s+1} as a block. Since $i + j = m$, we can denote a_{ij} by a_j. Then, we rewrite the dot product summation and get

$$\text{Pl-Holant}_{\Omega_{2s+1}} = \langle A, (M(f_{2s+1}))^{\otimes m}\rangle = \sum_{0 \leqslant j \leqslant m} a_j b^{(2s+1)j}.$$

Under this stratification, the Holant value of Pl-Holant($\Omega, \neq_2|\ \mathcal{F} \cup \{\chi_1\}$) can be represented as

$$\text{Pl-Holant}_{\Omega} = \langle A, (M(\chi_1))^{\otimes m}\rangle = \sum_{0 \leqslant j \leqslant m} a_j.$$

Since $b \neq 0$ is not a root of unity, the Vandermonde coefficients matrix

$$\begin{bmatrix} b^0 & b^1 & \cdots & b^m \\ (b^3)^0 & (b^3)^1 & \cdots & (b^3)^m \\ \vdots & \vdots & \vdots & \vdots \\ (b^{2m+1})^0 & (b^{2m+1})^1 & \cdots & (b^{2m+1})^m \end{bmatrix}$$

has full rank. By oracle querying the values of Pl-Holant$_{\Omega_{2s+1}}$, we can solve the coefficients a_j in polynomial time and obtain the value of $p(x) = \sum_{0 \leqslant j \leqslant m} a_j x^j$ for any x. Let $x = 1$, we get Pl-Holant$_\Omega$. Therefore, we have Pl-Holant$(\neq_2|\ \mathcal{F} \cup \{\chi_1\}) \leqslant_T$ Pl-Holant$(\neq_2|\ \mathcal{F})$. $\square$

Corollary 2.4. *Let f be a 4-ary signature with the signature matrix $M(f) = \left[\begin{smallmatrix} 0 & 0 & 0 & 1 \\ 0 & b & 0 & 0 \\ 0 & 0 & b & 0 \\ -1 & 0 & 0 & 0 \end{smallmatrix}\right]$, where $b \neq 0$ is not a root of unity. Let χ_2 be a 4-ary signature with the signature matrix $M(\chi_2) = \left[\begin{smallmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \end{smallmatrix}\right]$. Then for any signature set $\mathcal{F}$ containing f, we have*

$$\text{Pl-Holant}(\neq_2|\ \mathcal{F} \cup \{\chi_2\}) \leqslant_T \text{Pl-Holant}(\neq_2|\ \mathcal{F}).$$

Proof. We still construct a series of gadgets f_{2s+1} by a chain of odd copies of f linked by double DISEQUALITY N. We know f_{2s+1} has the signature matrix

$$M(f_{2s+1}) = M(f)(NM(f))^{2s} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & b^{2s+1} & 0 & 0 \\ 0 & 0 & b^{2s+1} & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}.$$

Suppose χ_2 appears m times in an instance Ω of Pl-Holant$(\neq_2|\ f \cup \chi_2)$. We replace each appearance of χ_2 by a copy of the gadget f_{2s+1} to get an instance Ω_{2s+1} of Pl-Holant$(\neq_2|\ \mathcal{F} \cup \{f_{2s+1}\})$. Same as the proof of Lemma 2.3, we divide Ω_{2s+1} into two parts. One part is represented by $(M(f_{2s+1}))^{\otimes m}$ and the other part is represented by A. Then, the Holant value of Ω_{2s+1} is the dot product $\langle A, (M(f_{2s+1}))^{\otimes m}\rangle$. We can stratify all $0, 1$ assignments of these $4m$ bits having a nonzero evaluation of a term in Pl-Holant$_{\Omega_{2s+1}}$ into the following categories:

- There are i many copies of f_{2s+1} receiving inputs 0011;
- There are j many copies of f_{2s+1} receiving inputs 0110 or 1001;
- There are k many copies of f_{2s+1} receiving inputs 1100;

where $i + j + k = m$.

For any assignment in those categories with parameters (i, j, k) where $k \equiv 0 (\text{mod } 2)$, the evaluation of $(M(f_{2s+1}))^{\otimes m}$ is clearly $(-1)^k b^{(2s+1)j} = b^{(2s+1)j}$. And for any assignment in those categories with parameters (i, j, k) where $k \equiv 1 (\text{mod } 2)$, the evaluation of $(M(f_{2s+1}))^{\otimes m}$ is clearly $(-1)^k b^{(2s+1)j} = -b^{(2s+1)j}$. Since $i + j + k = m$, the index i is determined by j and k. Let a_{j0} be the summation of values of the part A over all assignments in those categories (i, j, k) where $k \equiv 0 (\text{mod } 2)$, and a_{j1} be the summation of values of the part A over all assignments in those categories (i, j, k) where $k \equiv 1 (\text{mod } 2)$. Note that a_{j0} and a_{j1} are independent from the value of s. Let $a_j = a_{j0} - a_{j1}$. Then, we rewrite the dot product summation and get

$$\text{Pl-Holant}_{\Omega_{2s+1}} = \langle A, (M(f_{2s+1}))^{\otimes m}\rangle = \sum_{0 \leqslant j \leqslant m} (a_{j0} b^{(2s+1)j} - a_{j1} b^{(2s+1)j}) = \sum_{0 \leqslant j \leqslant m} a_j b^{(2s+1)j}.$$

Under this stratification, the Holant value of Pl-Holant $(\Omega; \neq_2|\ f \cup \chi_2)$ can be represented as

$$\text{Pl-Holant}_\Omega = \langle A, (M(\chi_2))^{\otimes m}\rangle = \sum_{0 \leqslant j \leqslant m} (a_{j0} - a_{j1}) = \sum_{0 \leqslant j \leqslant m} a_j.$$

Since $b \neq 0$ is not a root of unity, the Vandermonde coefficients matrix has full rank. Hence we can solve the coefficients in polynomial time and obtain the value of $p(x) = \sum_{0 \leqslant j \leqslant m} a_j x^j$ for any x. Let $x = 1$, we get Pl-Holant$_\Omega$. Therefore, we have Pl-Holant$(\neq_2|\ \mathcal{F} \cup \{\chi_2\}) \leqslant_T$ Pl-Holant$(\neq_2|\ \mathcal{F})$. $\square$

Lemma 2.5. *Let $g = (0, 1, t, 0)^T$ be a binary signature, where $t \neq 0$ is not a root of unity. Then for any binary signature g' of the form $(0, 1, t', 0)^T$ and any signature set $\mathcal{F}$ containing g, we have*

$$\text{Pl-Holant}\left(\neq_2 \mid \mathcal{F} \cup \{g'\}\right) \leqslant_T \text{Pl-Holant}\left(\neq_2 \mid \mathcal{F}\right).$$

Inductively, for any finite signature set $\mathcal{B}$ consisting of binary signatures of the form $(0, 1, t', 0)^T$ and any signature set $\mathcal{F}$ containing g, we have

$$\text{Pl-Holant}\left(\neq_2 \mid \mathcal{F} \cup \mathcal{B}\right) \leqslant_T \text{Pl-Holant}\left(\neq_2 \mid \mathcal{F}\right).$$

Proof. Note that $M(g) = \left[\begin{smallmatrix} 0 & 1 \\ t & 0 \end{smallmatrix}\right]$. Connect the variable x_2 of a copy of g with the variable x_1 of another copy of g using $(\neq_2)$. We get a signature g_2 with the signature matrix

$$M(g_2) = M_{x_1,x_2}(g) M(\neq_2) M_{x_1,x_2}(g) = \begin{bmatrix} 0 & 1 \\ t & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ t & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ t^2 & 0 \end{bmatrix}.$$

That is, $g_2 = (0, 1, t^2, 0)^T$. Recursively, we can construct $g_s = (0, 1, t^s, 0)^T$ for $s \in \mathbb{N}$. Here, g_1 denotes g. Given an instance Ω' of $\text{Pl-Holant}\left(\neq_2 \mid \mathcal{F} \cup \{g'\}\right)$, same as the proof in Lemma 2.3, we can replace each appearance of g' by g_s and get an instance Ω_s of $\text{Pl-Holant}\left(\neq_2 \mid \mathcal{F}\right)$. Similarly, the Holant value of Ω_s can be represented as

$$\text{Pl-Holant}_{\Omega_s} = \sum_{0 \leqslant j \leqslant m} a_j (t^s)^j,$$

while the Holant value of Ω' can be represented as

$$\text{Pl-Holant}_{\Omega'} = \sum_{0 \leqslant j \leqslant m} a_j (t')^j.$$

Since $t \neq 0$ is not a root of unity, we know all t^s are distinct, which means the Vandermonde coefficients matrix has full rank. Hence, we can solve the coefficients in polynomial time and obtain the value of $p(x) = \sum\limits_{0 \leqslant j \leqslant m} a_j x^j$ for any x. Let $x = t'$, we get $\text{Pl-Holant}_{\Omega'}$. Therefore, we have $\text{Pl-Holant}(\neq_2 \mid \mathcal{F} \cup \{g'\}) \leqslant_T \text{Pl-Holant}(\neq_2 \mid \mathcal{F})$. The second part of this lemma follows directly by the first part. □

Remark: Note that the reason why the interpolation can succeed is that we can construct polynomially many binary signatures g_s of the form $(0, 1, t_s, 0)^T$, where all t_s are distinct such that the Vandermonde coefficients matrix has full rank. According to this, we have the following corollary.

Corollary 2.6. *Given a signature set $\mathcal{F}$, if we can use $\mathcal{F}$ to construct polynomially many distinct binary signatures $g_s = (0, 1, t_s, 0)^T$, then for any finite signature set $\mathcal{B}$ consisting of binary signatures of the form $(0, 1, t', 0)^T$, we have*

$$\text{Pl-Holant}\left(\neq_2 \mid \mathcal{F} \cup \mathcal{B}\right) \leqslant_T \text{Pl-Holant}\left(\neq_2 \mid \mathcal{F}\right).$$

In Lemma 6.4, we will show how to construct polynomially many distinct binary signatures $g_s = (0, 1, t_s, 0)^T$ using Möbius transformation[1]. A Möbius transformation of the extended complex plane $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, the complex plane plus a point at infinity, is a rational function of the form $\mathfrak{z} \to \frac{a\mathfrak{z}+b}{c\mathfrak{z}+d}$ of complex variable $\mathfrak{z}$, where the coefficients a, b, c, d are complex numbers satisfying det $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] = ad - bc \neq 0$. It is a bijective conformal map. In particular, a Möbius transformation mapping the unit circle $S^1 = \{z \mid |z| = 1\}$ to itself is of the form $\varphi(\mathfrak{z}) = e^{\mathfrak{i}\theta}\frac{(\mathfrak{z}+\alpha)}{1+\bar{\alpha}\mathfrak{z}}$ denoted by $\mathcal{M}(\alpha, e^{\mathfrak{i}\theta})$, where $|\alpha| \neq 1$. When $|\alpha| < 1$, it maps the interior of S^1 to the interior, while when $|\alpha| > 1$, it maps the interior of S^1 to the exterior. A Möbius transformation is completely determined by its values on any 3 distinct points.

An interpolation proof based on a lattice structure will be given in Lemma 6.1, where the following lemma is used.

Lemma 2.7. *[5] Suppose $\alpha, \beta \in \mathbb{C} - \{0\}$, and the lattice $L = \{(j,k) \in \mathbb{Z}^2 \mid \alpha^j \beta^k = 1\}$ has the form $L = \{(ns, nt) \mid n \in \mathbb{Z}\}$, where $s, t \in \mathbb{Z}$ and $(s,t) \neq (0,0)$. Let ϕ and ψ be any numbers satisfying $\phi^s \psi^t = 1$. If we are given the values $N_\ell = \sum_{j,k \geq 0,\ j+k \leq m} (\alpha^j \beta^k)^\ell x_{j,k}$ for $\ell = 1, 2, \ldots \binom{m+2}{2}$, then we can compute $\sum_{j,k\geq 0,\ j+k\leq m} \phi^j \psi^k x_{j,k}$ in polynomial time.*

2.5 Tractable Signature Sets

We give some signatures that are known to be computable in polynomial time, called tractable. There are three families: affine signatures, product-type signatures, and matchgate signatures [4].

Affine Signatures

Definition 2.8. *For a signature f of arity n, the support of f is*

$$\text{supp}(f) = \{(x_1, x_2, \ldots, x_n) \in \mathbb{Z}_2^n \mid f(x_1, x_2, \ldots, x_n) \neq 0\}.$$

Definition 2.9. *A signature $f(x_1, \ldots, x_n)$ of arity n is* affine *if it has the form*

$$\lambda \cdot \chi_{AX=0} \cdot \mathfrak{i}^{Q(X)},$$

where $\lambda \in \mathbb{C}$, $X = (x_1, x_2, \ldots, x_n, 1)$, A is a matrix over $\mathbb{Z}_2$, $Q(x_1, x_2, \ldots, x_n) \in \mathbb{Z}_4[x_1, x_2, \ldots, x_n]$ is a quadratic (total degree at most 2) multilinear polynomial with the additional requirement that the coefficients of all cross terms are even, i.e., Q has the form

$$Q(x_1, x_2, \ldots, x_n) = a_0 + \sum_{k=1}^{n} a_k x_k + \sum_{1 \leq i < j \leq n} 2b_{ij} x_i x_j,$$

and χ is a 0-1 indicator function such that $\chi_{AX=0}$ is 1 iff $AX = 0$. We use $\mathscr{A}$ to denote the set of all affine signatures.

Follows by the definition directly, one can check the following two lemmas.

Lemma 2.10. *Let g be a binary signature with support of size 4. Then, $g \in \mathscr{A}$ iff g has the signature matrix $M(g) = a \left[\begin{smallmatrix} \mathfrak{i}^{q_{00}} & \mathfrak{i}^{q_{01}} \\ \mathfrak{i}^{q_{10}} & \mathfrak{i}^{q_{11}} \end{smallmatrix}\right]$, where $q_{00}, q_{01}, q_{10}, q_{11} \in \mathbb{N}$ and $q_{00} + q_{01} + q_{10} + q_{11} \equiv 0 (\text{mod } 2)$.*

Lemma 2.11. *Let h be a unary signature with support of size 2. Then, $h \in \mathscr{A}$ iff h is of the form $M(h) = a \begin{bmatrix} \mathfrak{i}^{q_0} & \mathfrak{i}^{q_1} \end{bmatrix}$, where $q_0, q_1 \in \mathbb{N}$.*

Product-Type Signatures

Definition 2.12. *A signature on a set of variables X is of* product type *if it can be expressed as a product of unary functions, binary equality functions* $([1,0,1])$*, and binary disequality functions* $([0,1,0])$*, each on one or two variables of X. We use $\mathscr{P}$ to denote the set of product-type functions.*

Theorem 2.13. *Let $\mathcal{F}$ be any set of complex-valued signatures in Boolean variables. If $\mathcal{F} \subseteq \mathscr{A}$ or $\mathscr{P}$, then* $\operatorname{Holant}(\neq_2 \mid \mathcal{F})$ *is tractable.*

Problems defined by $\mathscr{A}$ are tractable essentially by Gauss sums [2]. Problems defined by $\mathscr{P}$ are tractable by a propagation algorithm.

Matchgate Signatures Matchgates were introduced by Valiant [23, 24] to give polynomial-time algorithms for a collection of counting problems over planar graphs. As the name suggests, problems expressible by matchgates can be reduced to computing a weighted sum of perfect matchings. The latter problem is tractable over planar graphs by Kasteleyn's algorithm [16], a.k.a. the FKT algorithm [21, 15]. These counting problems are naturally expressed in the Holant framework using *matchgate signatures*. We use $\mathscr{M}$ to denote the set of all matchgate signatures; thus Pl-Holant($\mathscr{M}$) is tractable.

The parity of a signature is even (resp. odd) if its support is on entries of even (resp. odd) Hamming weight. We say a signature satisfies the even (resp. odd) Parity Condition if all entries of odd (resp. even) weight are zero. For signatures of arity no more than 4, the matchgate signatures are characterized by the following lemma.

Lemma 2.14. *(cf. Lemma 2.3, Lemma 2.4 in [3]) If f has arity $\leqslant 3$, then $f \in \mathscr{M}$ iff f satisfies the Parity Condition.*

If f has arity 4 and f satisfies even Parity Condition, i.e.,

$$M_{x_1x_2,x_4x_3}(f) = \begin{bmatrix} f_{0000} & 0 & 0 & f_{0011} \\ 0 & f_{0110} & f_{0101} & 0 \\ 0 & f_{1010} & f_{1001} & 0 \\ f_{1100} & 0 & 0 & f_{1111} \end{bmatrix},$$

then $f \in \mathscr{M}$ iff

$$\det M_{\text{Out}}(f) = f_{0000}f_{1111} - f_{1100}f_{0011} = f_{0110}f_{1001} - f_{1010}f_{0101} = \det M_{\text{In}}(f)$$

Holographic transformations extend the reach of the FKT algorithm even further, as stated below.

Theorem 2.15. *Let $\mathcal{F}$ be any set of complex-valued signatures in Boolean variables. If $\mathcal{F}$ is $\mathscr{M}$-transformable, then* Pl-$\operatorname{Holant}(\neq_2 \mid \mathcal{F})$ *is tractable.*

Recall the signature class $\widehat{\mathscr{M}} = H_2\mathscr{M}$, where $H_2 = \frac{1}{\sqrt{2}}\left[\begin{smallmatrix} 1 & 1 \\ 1 & -1 \end{smallmatrix}\right]$. Follows by the definition directly, One can check the following lemmas:

Lemma 2.16. *A signature f with the signature matrix $M(f) = \left[\begin{smallmatrix} 0 & 0 & 0 & a \\ 0 & b & c & 0 \\ 0 & z & y & 0 \\ x & 0 & 0 & 0 \end{smallmatrix}\right]$ is $\mathscr{M}$-transformable iff $f \in \widehat{\mathscr{M}}$.*

Lemma 2.17. *A signature g with the signature matrix $M(g) = \left[\begin{smallmatrix} g_{00} & g_{01} \\ g_{10} & g_{11} \end{smallmatrix}\right]$ is in $\widehat{\mathscr{M}}$ iff $g_{00} = \epsilon g_{11}$ and $g_{01} = \epsilon g_{10}$, where $\epsilon = \pm 1$.*

Lemma 2.18. *A signature f with the signature matrix $M(f) = \left[\begin{smallmatrix} 0 & 0 & 0 & 0 \\ 0 & b & c & 0 \\ 0 & z & y & 0 \\ 0 & 0 & 0 & 0 \end{smallmatrix}\right]$ is in $\widehat{\mathscr{M}}$ iff $b = \epsilon y$ and $c = \epsilon z$, where $\epsilon = \pm 1$.*

Lemma 2.19. *If f has the signature matrix $M(f) = \left[\begin{smallmatrix} 0 & 0 & 0 & a \\ 0 & b & 0 & 0 \\ 0 & 0 & y & 0 \\ x & 0 & 0 & 0 \end{smallmatrix}\right]$, where $abxy \neq 0$, then $f \notin \widehat{\mathscr{M}}$.*

2.6 Known Dichotomies and Hardness Results

Definition 2.20. *A 4-ary signature is non-singular redundant iff in one of its four 4×4 signature matrices, the middle two rows are identical and the middle two columns are identical, and the determinant*

$$\det \begin{bmatrix} f_{0000} & f_{0010} & f_{0011} \\ f_{0100} & f_{0110} & f_{0111} \\ f_{1100} & f_{1110} & f_{1111} \end{bmatrix} \neq 0.$$

Theorem 2.21. *[6] If f is a non-singular redundant signature, then* Pl-Holant$(\neq_2 | f)$ *is* #P*-hard.*

Theorem 2.22. *[17] Let G be a connected planar graph and $\mathcal{EO}(H)$ be the set of all Eulerian Orientation of the medial graph $H = H(G)$. Then*

$$\sum_{O \in \mathcal{EO}(H)} 2^{\beta(O)} = 2T(G; 3, 3),$$

where $\beta(O)$ is the number of saddle vertices in orientation O, i.e., vertices in which the edges are oriented "in, out, in, out" in cyclic order.

Remark: Note that $\sum_{O \in \mathcal{EO}(H)} 2^{\beta(O)}$ can be expressed as Pl-Holant$(\neq_2 | f)$, where f has the signature matrix $M(f) = \left[\begin{smallmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 2 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{smallmatrix}\right]$. Therefore, we have Pl-Holant$(\neq_2 | f)$ is #P-hard.

Theorem 2.23. *[4] Let $\mathcal{F}$ be any set of complex-valued signatures in Boolean variables. Then* Pl-#CSP$(\mathcal{F})$ *is* #P*-hard unless $\mathcal{F} \subseteq \mathscr{A}$, $\mathcal{F} \subseteq \mathscr{P}$, or $\mathcal{F} \subseteq \widehat{\mathscr{M}}$, in which case the problem is computable in polynomial time. If $\mathcal{F} \subseteq \mathscr{A}$ or $\mathcal{F} \subseteq \mathscr{P}$, then* #CSP$(\mathcal{F})$ *is computable in polynomial time without planarity; otherwise* #CSP$(\mathcal{F})$ *is* #P*-hard.*

Theorem 2.24. *[5] Let f be a 4-ary signature with the signature matrix $M(f) = \left[\begin{smallmatrix} 0 & 0 & 0 & a \\ 0 & b & c & 0 \\ 0 & z & y & 0 \\ x & 0 & 0 & 0 \end{smallmatrix}\right]$, then* Holant $(\neq_2 | f)$ *is #P-hard except for the following cases:*

- *$f \in \mathscr{P}$;*
- *$f \in \mathscr{A}$;*
- *there is a zero in each pair $(a, x), (b, y), (c, z)$;*

in which cases Holant $(\neq_2 | f)$ *is computable in polynomial time.*

3 Main Theorem and Proof Outline

Theorem 3.1. *Let f be a signature with the signature matrix $M(f) = \left[\begin{smallmatrix} 0 & 0 & 0 & a \\ 0 & b & c & 0 \\ 0 & z & y & 0 \\ x & 0 & 0 & 0 \end{smallmatrix}\right]$, where $a, b, c, x, y, z \in \mathbb{C}$. Then* Pl-Holant$(\neq_2 | f)$ *is polynomial time computable in the following cases, and #P-hard otherwise:*

1. *$f \in \mathscr{P}$ or $\mathscr{A}$;*
2. *There is a zero in each pair (a, x), (b, y), (c, z);*
3. *$f \in \mathscr{M}$ or $\widehat{\mathscr{M}}$;*
4. *$c = z = 0$ and*
   (i). *$(ax)^2 = (by)^2$, or*
   (ii). *$x = a\mathfrak{i}^{\alpha}, b = a\sqrt{\mathfrak{i}}^{\beta}$, and $y = a\sqrt{\mathfrak{i}}^{\gamma}$, where $\alpha, \beta, \gamma \in \mathbb{N}$, and $\beta \equiv \gamma \pmod 2$;*

If f satisfies condition 1 or 2, then Holant$(\neq_2 | f)$ *is computable in polynomial time without the planarity restriction; otherwise (the non-planar)* Holant$(\neq_2 | f)$ *is #P-hard.*

Let N be the number of zeros in $\{a, b, c, x, y, z\}$. If $N \geqslant 3$, then either there is a zero pair, or we have $N = 3$ and each pair has exactly one zero. We define Case I as $N = 3$ with each pair having exactly one zero. We define Case II as having a zero pair, which also includes some cases of $N = 2$. For the remaining cases of $N = 2$ the two zeros are in different pairs. In particular there is an outer pair that has a single zero. We define Case III to be $N = 2$ and having no zero pair, or $N = 1$ and the zero is in an outer pair. In Case III an outer pair has exactly one zero, and the other two pairs together have at most one zero. Then we define Case IV as having $N = 1$ and the zero is in an inner pair, or $N = 0$.

Case I is tractable, even for non-planar Holant$(\neq_2 | f)$ (see [5]).

In Case II, depending on whether the zero pair is inner or outer we have two different connections to #CSP. A previously established connection to #CSP [5] can be adapted in the planar setting to the case with a zero outer pair. This connection is a local transformation, and we observe that it preserves planarity. A significantly more involved non-local connection to #CSP is discovered in this paper when the inner pair is zero (and no outer pair is zero). We show that by the support structure of the signature we can define a set of circuits, which forms a partition of the edge set. There are exactly two valid configurations along such a circuit, corresponding to its two cyclic orientations. These circuits may intersect in complicated ways, including self-intersections. But we can define a #CSP problem, where the variables are these circuits, and their edge functions exactly account for the intersections. We show that Pl-Holant$(\neq_2 | f)$ is equivalent to these #CSP problems, which are non-planar in general. However, crucially, because Pl-Holant$(\neq_2 | f)$ is planar, every two such circuits must intersect even times. Due to the planarity of Pl-Holant$(\neq_2 | f)$ we can exactly carve out a new class of tractable problems via this non-local #CSP connection.

For #P-hardness proofs in this paper, one particularly difficult case is in Lemma 6.4. In this case, all constructable binary signatures correspond to points on the unit circle S^1, and any iteration of the construction amounts to mapping this point by a Möbius transformation which preserves S^1.

The 4 Cases are defined formally as follows:

I. There is exactly one zero in each pair. In this case, Holant $(\neq_2 | f)$ is tractable, proved in [5].

II. There is a zero pair:
   1. An outer pair (a, x) or (b, y) is a zero pair. We prove that Pl-Holant$(\neq_2 | f)$ is tractable if $f \in \mathscr{P}, \mathscr{A}, \mathscr{M}$ or $\widehat{\mathscr{M}}$, and is #P-hard otherwise.

      In this Case II.1, we can rotate the signature f such that the matrix $M_{\mathrm{Out}}(f)$ is the zero matrix. We reduce Pl-#CSP($M_{\text{In}}(f)$) to Pl-Holant $(\neq_2|\ f)$ via a local replacement (Lemma 4.2). We apply the dichotomy of Pl-#CSP to get #P-hardness (Theorem 4.3). Tractability of Pl-Holant $(\neq_2|\ f)$ follows from known tractable signatures.

2. The inner pair (c, z) is a zero pair and no outer pair is a zero pair. We prove that Pl-Holant($\neq_2|\ f$) is #P-hard unless f satisfies condition 4, in which it is tractable. This is the non-local reduction described above. The tractable condition 4 is previously unknown. (Curiously, in Case II.2, condition 4 subsumes $f \in \mathscr{M}$.)

III. 1. There are exactly two zeros and they are in different pairs;

2. There is exactly one zero and it is in an outer pair.

We prove that Pl-Holant($\neq_2|\ f$) is #P-hard unless $f \in \mathscr{M}$, in which case it is tractable. In this case, there is a single zero in an outer pair. By connecting two copies of the signature f, we can construct a 4-ary signature f_1 such that one outer pair is a zero pair. When $f \notin \mathscr{M}$, we can realize a signature $M(g) = \left[\begin{smallmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{smallmatrix}\right]$ by interpolation using f_1 (Lemma 5.1). This g can help us "extract" $M_{\text{In}}(f)$. By connecting f and g, we can construct a signature that belongs to Case II. We then prove #P-hardness using the result of Case II (Theorem 5.2).

IV. 1. There is exactly one zero and it is in the inner pair;

2. All values in $\{a, x, b, y, c, z\}$ are nonzero.

We prove that Pl-Holant($\neq_2|\ f$) is #P-hard unless $f \in \mathscr{M}$, in which it is tractable. Assume $f \notin \mathscr{M}$. The main idea is to use Möbius transformations. However, there are some settings where we cannot do so, either because we don't have the initial signature to start the process, or the matrix that would define the Möbius transformation is singular. So we first treat the following two special cases.

- If $a = \epsilon x$, $b = \epsilon y$ and $c = \epsilon z$, where $\epsilon = \pm 1$, by interpolation based on a lattice structure, either we can realize a non-singular redundant signature or reduce from the evaluation of the Tutte polynomial at $(3, 3)$, both of which are #P-hard (Lemma 6.1).
- If $\det \left[\begin{smallmatrix} b & c \\ z & y \end{smallmatrix}\right] = 0$ or $\det \left[\begin{smallmatrix} a & z \\ c & x \end{smallmatrix}\right] = 0$, then either we can realize a non-singular redundant signature or a signature that is #P-hard by Lemma 6.1 (Lemma 6.2).

If f does not belong to the above two cases, we want to realize binary signatures of the form $(0, 1, t, 0)^T$, for arbitrary values of t. If this can be done, by carefully choosing the values of t, we can construct a signature that belongs to Case III and it is #P-hard when $f \notin \mathscr{M}$ (Lemma 6.3). We realize binary signatures by connecting f with $(\neq_2)$. This corresponds naturally to a Möbius transformation. By discussing the following different forms of binary signatures we get, we can either realize arbitrary $(0, 1, t, 0)^T$ or a signature belonging to Case II.2 that does not satisfy condition 4, therefore is #P-hard (Theorem 6.8).

- If we can get a signature of the form $g = (0, 1, t, 0)^T$ where $t \neq 0$ is not a root of unity, then by connecting a chain of g, we can get polynomially many distinct binary signatures $g_i = (0, 1, t^i, 0)^T$. Then, by interpolation, we can realize arbitrary binary signatures of the form $(0, 1, t', 0)^T$.
- Suppose we can get a signature of the form $(0, 1, t, 0)^T$, where $t \neq 0$ is an n-th primitive root of unity ($n \geqslant 5$). Now, we only have n many distinct signatures $g_i = (0, 1, t^i, 0)^T$. But we can relate f to two Möbius transformations due to $\det \left[\begin{smallmatrix} b & c \\ z & y \end{smallmatrix}\right] \neq 0$ and $\det \left[\begin{smallmatrix} a & z \\ c & x \end{smallmatrix}\right] \neq 0$. For each Möbius transformation φ, we can realize the signatures $g = (0, 1, \varphi(t^i), 0)^T$. If $|\varphi(t^i)| \neq 0, 1$ or ∞ for some i, then this is treated above. Otherwise, since φ is a bijection on the extended complex plane $\widehat{\mathbb{C}}$, it can map at most two points of S^1 to 0 or ∞. Hence,

$|\varphi(t^i)| = 1$ for at least three t^i. But a Möbius transformation is determined by any three distinct points. This implies that φ maps S^1 to itself. Such φ have a known special form $e^{i\theta}\dfrac{\mathfrak{z}+\alpha}{1+\bar{\alpha}\mathfrak{z}}$. By exploiting its property we can construct a signature f' such that its corresponding Möbius transformation φ' defines an infinite group. This implies that $\varphi'^k(t)$ are all distinct. Then, we can get polynomially many distinct binary signatures $(0, 1, \varphi'^k(t), 0)$, and realize arbitrary binary signatures of the form $(0, 1, t', 0)^T$ (Lemma 6.4).

- Suppose we can get a signature of the form $(0, 1, t, 0)^T$ where $t \neq 0$ is an n-th primitive root of unity ($n = 3, 4$). Then we can either relate it to two Möbius transformations mapping the unit circle to itself, or realize a double pinning $(0, 1, 0, 0)^T = (1, 0)^T \otimes (0, 1)^T$ (Corollary 6.5).
- Suppose we can get a signature of the form $(0, 1, 0, 0)^T$. By connecting f with it, we can get new signatures of the form $(0, 1, t, 0)^T$. Similarly, by analyzing the value of t, we can either realize arbitrary binary signatures of the form $(0, 1, s, 0)^T$, or realize a signature that belongs to Case II.2, which is #P-hard (Lemma 6.6).
- Suppose we can only get signatures of the form $(0, 1, \pm 1, 0)$. That implies $a = \epsilon x$, $b = \epsilon y$ and $c = \epsilon z$, where $\epsilon = \pm 1$. This has been treated before.

These 4 Cases above cover all possibilities. If $N \geq 3$, then there is a zero pair or there is exactly one zero in each pair. This is either in Case I or Case II. If $N = 2$, it is either in Case II or Case III.1. If $N = 1$, it is either in Case III.2 or Case IV.1. If $N = 0$, it is in Case IV.2.

4 Case II: One Zero Pair

If an outer pair is a zero pair, by rotational symmetry, we may assume (a, x) is a zero pair.

Definition 4.1. *Given a 4-ary signature f with the signature matrix*

$$M(f) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & b & c & 0 \\ 0 & z & y & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{4.1}$$

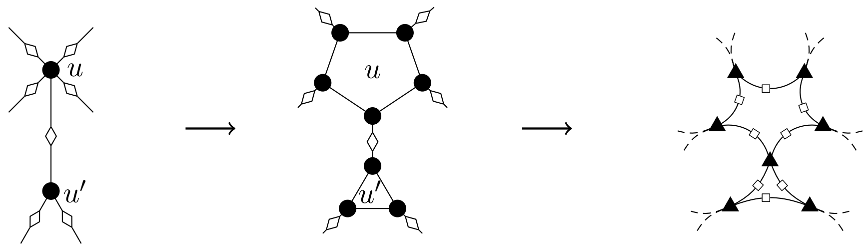
let f_{In} denote the binary signature of the form $M(f_{\text{In}}) = M_{\text{In}}(f) = \left[\begin{smallmatrix} b & c \\ z & y \end{smallmatrix}\right]$. Given a set $\mathcal{F}$ consisting of signatures of the form (4.1), let $\mathcal{F}_{\text{In}}$ denote the signature set $\{f_{\text{In}} \mid f \in \mathcal{F}\}$.

Lemma 4.2. *Let $\mathcal{F}$ be a set consisting of signatures of the form (4.1). Then,*

$$\text{Pl-\#CSP}(\mathcal{F}_{\text{In}}) \leqslant_T \text{Pl-Holant}\,(\neq_2 \mid \mathcal{F})\,.$$

Proof.[5] We prove this reduction in two steps. In each step, we begin with a signature grid and end with a new signature grid such that the Holant values of both signature grids are the same.

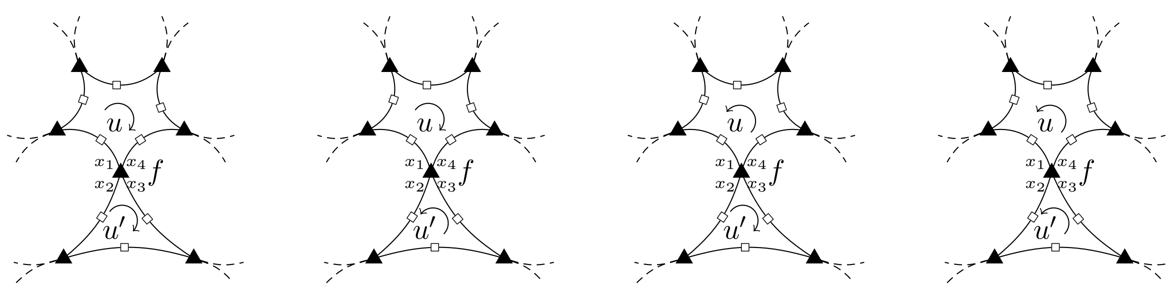
For step one, let $G = (U, V, E)$ be a planar bipartite graph representing an instance of Pl-#CSP$(\mathcal{F}_{\text{In}})$, where each $u \in U$ is a variable, and each $v \in V$ has degree two and is labeled by some $f_{\text{In}} \in \mathcal{F}_{\text{In}}$. We define a cyclic order of the edges incident to each vertex $u \in U$, and decompose u into $k = \deg(u)$ vertices. Then we connect the k edges originally incident to u to these k new vertices so that each vertex is incident to exactly one edge. We also connect these k new vertices in a cycle according to



(a) (b) (c)

Figure 7: The reduction from #Pl-CSP(f_{In}) to Pl-Holant($\neq_2|\ f$). The circle vertices are labeled by $(=_d)$, where d is the degree of the corresponding vertex, the diamond vertices are labeled by f_{In}, the triangle vertices are labeled by the corresponding f, and the square vertices are labeled by $(\neq_2)$.

the cyclic order (see Figure 7b). Thus, in effect we have replaced u by a cycle of length $k = \deg(u)$. (If $k = 1$ there is a self-loop.) Each of k vertices has degree 3, and we label them by $(=_3)$. Clearly this preserves planarity and does not change the value of the partition function. The resulting graph has the following properties: (1) every vertex has either degree 2 or degree 3; (2) each degree 2 vertex is connected to degree 3 vertices; (3) each degree 3 vertex is connected to exactly one degree 2 vertex.



(a) $f_{\text{In}00} \leftrightarrow f_{0110}$ (b) $f_{\text{In}01} \leftrightarrow f_{0101}$ (c) $f_{\text{In}10} \leftrightarrow f_{1010}$ (d) $f_{\text{In}11} \leftrightarrow f_{1001}$

Figure 8: Assign input variables of f: Suppose the binary signature f_{In} is applied to (the ordered pair) (u, u'). The variables u and u' have been replaced by cycles of length $\deg(u)$ and $\deg(u')$ respectively. For the cycle C_u representing a variable u, we associate the value $u = 0$ with a clockwise orientation, and $u = 1$ with a counter-clockwise orientation. Then by the support of f, (x_1, x_4) can only take assignment $(0, 1)$ or $(1, 0)$, and similarly (x_2, x_3) can only take assignment $(0, 1)$ or $(1, 0)$.

Now step two. For every $v \in V$, v has degree 2 and is labeled by some f_{In}. We contract the two edges incident to v. The resulting graph $G' = (V', E')$ is 4-regular and planar. We put a node on every edge of G' and labet it by $(\neq_2)$ (see Figure 7c). Next, we assign the corresponding f to every $v' \in V'$ after this contraction. The input variables x_1, x_2, x_3, x_4 are carefully assigned at each copy of f (as illustrated in Figure 8) such that there are exactly two configurations to each original cycle, which correspond to cyclic orientations, due to the $(\neq_2)$ on it and the support set of f. These cyclic orientations correspond to the $\{0,1\}$ assignments at the original variable $u \in U$. Under this one-to-one correspondence, the value of f_{In} is perfectly mirrored by the value of f. Therefore, we have Pl-#CSP$(\mathcal{F}_{\text{In}}) \leqslant_T$ Pl-Holant $(\neq_2 |\, \mathcal{F})$. $\square$

Theorem 4.3. *Let $\mathcal{F}$ be a set consisting of signatures of the form (4.1). Then* Pl-Holant$(\neq_2 |\, \mathcal{F})$ *is #P-hard unless $\mathcal{F} \subseteq \mathscr{P}$, $\mathcal{F} \subseteq \mathscr{A}$, or $\mathcal{F} \subseteq \widehat{\mathscr{M}}$, in which cases the problem is tractable.*

Proof. Tractability follows by Theorems 2.13 and 2.15. For any $f \in \mathcal{F}$, notice that the support of f is on $\chi_{x_1 \neq x_2}$ and $\chi_{x_3 \neq x_4}$, where χ is the indicator function. We have

$$f(x_1, x_2, x_3, x_4) = f_{\text{In}}(x_1, x_4) \cdot \chi_{x_1 \neq x_2} \cdot \chi_{x_3 \neq x_4}.$$

Thus, $\mathcal{F}_{\text{In}} \subseteq \mathscr{P}$ or $\mathscr{A}$ is equivalent to $\mathcal{F} \subseteq \mathscr{P}$ or $\mathscr{A}$. In addition, by Lemmas 2.17 and 2.18, $\mathcal{F}_{\text{In}} \subseteq \widehat{\mathscr{M}}$ is equivalent to $\mathcal{F} \subseteq \widehat{\mathscr{M}}$. Therefore, if $\mathcal{F} \nsubseteq \mathscr{P}, \mathscr{A}$ or $\widehat{\mathscr{M}}$, then $\mathcal{F}_{\text{In}} \nsubseteq \mathscr{P}, \mathscr{A}$ or $\widehat{\mathscr{M}}$. By Theorem 2.23, Pl-#CSP$(\mathcal{F}_{\text{In}})$ is #P-hard, and then by Lemma 4.2, Pl-Holant $(\neq_2 |\, \mathcal{F})$ is #P-hard. $\square$

Remark: One may observe that if $f \in \mathscr{M}$, then Pl-Holant $(\neq_2 |\, \mathcal{F})$ is also tractable as $\mathcal{F}$ and $(=_2)$ are both realized by matchgates. However, Theorem 4.3 already accounted for this case because for $\mathcal{F}$ consisting of signatures of the form (4.1), $\mathcal{F} \subseteq \mathscr{M}$ implies $\mathcal{F} \subseteq \mathscr{P}$.

Now, we consider the case that the inner pair is a zero pair and no outer pair is a zero pair.

Definition 4.4. *Given a 4-ary signature f with the signature matrix*

$$M(f) = \begin{bmatrix} 0 & 0 & 0 & a \\ 0 & b & 0 & 0 \\ 0 & 0 & y & 0 \\ x & 0 & 0 & 0 \end{bmatrix}, \tag{4.2}$$

where $(a, x) \neq (0, 0)$ and $(b, y) \neq (0, 0)$, let $\mathcal{G}_f$ denote the set of all binary signatures g_f of the form

$$M(g_f) = \begin{bmatrix} a^{k_1+\ell_1} y^{k_2+\ell_2} x^{k_3+\ell_3} b^{k_4+\ell_4} & a^{k_2+\ell_4} y^{k_3+\ell_1} x^{k_4+\ell_2} b^{k_1+\ell_3} \\ a^{k_4+\ell_2} y^{k_1+\ell_3} x^{k_2+\ell_4} b^{k_3+\ell_1} & a^{k_3+\ell_3} y^{k_4+\ell_4} x^{k_1+\ell_1} b^{k_2+\ell_2} \end{bmatrix},$$

where $k_1, k_2, k_3, k_4, \ell_1, \ell_2, \ell_3, \ell_4 \in \mathbb{N}$, and $k = k_1 + k_2 + k_3 + k_4 = \ell_1 + \ell_2 + \ell_3 + \ell_4 = \ell$, and let $\mathcal{H}_f$ denote the set of all unary signatures h_f of the form

$$M(h_f) = \begin{bmatrix} a^{m_1} y^{m_2} x^{m_3} b^{m_4} & a^{m_3} y^{m_4} x^{m_1} b^{m_2} \end{bmatrix},$$

where $m_1, m_2, m_3, m_4 \in \mathbb{N}$.

Let $k = k_1 = \ell_1 = \ell = 1$, we get a specific signature in $\mathcal{G}_f$ denoted by g_{1_f}, where $M(g_{1_f}) = \left[\begin{smallmatrix} a^2 & by \\ by & x^2 \end{smallmatrix}\right]$. Let $k = k_1 = \ell_3 = \ell = 1$, we get another specific signature in $\mathcal{G}_f$ denoted by g_{2_f}, where $M(g_{2_f}) = \left[\begin{smallmatrix} ax & b^2 \\ y^2 & ax \end{smallmatrix}\right]$.

Remark: For any $i, j \in \{1, 2, 3, 4\}$, let $k = k_i = \ell_j = \ell = 1$, we can get 16 signatures in $\mathcal{G}_f$ that have similar signature matrices to $M(g_{1_f})$ and $M(g_{2_f})$. In fact, $\mathcal{G}_f$ is the closure of the Hadamard product of these 16 basic signature matrices.

Lemma 4.5. *Let f be a signature of the form (4.2). Then,*

$$\text{Pl-Holant}(\neq_2|\ f) \leqslant_T \#\text{CSP}(\mathcal{G}_f \cup \mathcal{H}_f), \tag{4.3}$$

If $a^2 = x^2 \neq 0$, $b^2 = y^2 \neq 0$ and $\left(\frac{b}{a}\right)^8 \neq 1$, then

$$\#\text{CSP}(g_{1_f}, g_{2_f}) \leqslant_T \text{Pl-Holant}(\neq_2|\ f). \tag{4.4}$$

Proof. We divide the proof into two parts: We show the reduction (4.3) in Part I, and the reduction (4.4) in Part II.

Part I: Since we are considering planar graph, we can view it in a plane. Given a vertex of arity 4, list the four edges incident to it in counterclockwise order. We say two edges are not adjacent if there is exactly one other edge between them. Given an instance $\Omega = (G, \pi)$ of Pl-Holant($\neq_2|\ f$), two edges in G are called 2-ary edge twins if they are incident to a vertex of degree 2, and 4-ary edge twins if they are incident to a vertex of degree 4 but they are not adjacent. Both 2-ary edge twins and 4-ary edge twins are called edge twins.

Each edge has an unique 2-ary edge twin at its endpoint of degree 2 and an unique 4-ary edge twin at its endpoint of degree 4. That is, edge twins induce a binary relation and the transitive closure of this relation on an edge forms a circuit. Therefore, graph G can be divided into some circuits $C_1, C_2, \cdots, C_k$. Note that C_i may include repeated vertices called self-intersection vertices, but no repeated edges. We pick an edge e_i of C_i to be the leader edge of C_i. Given the leader edge e_i, the direction from its endpoint of degree 2 to its endpoint of degree 4 gives an orientation of the circuit C_i. In edge twins, depending on the orientation, we can say one edge is the successor of the other edge. When we list the assignment of edges in a circuit, we start with its leader edge and follow with the successor of the leader edge, and so forth.

For any nonzero term in the sum

$$\text{Pl-Holant}_\Omega = \sum_{\sigma: E \mapsto \{0,1\}} \prod_{v \in V} f_v(\sigma\ |_{E_{(v)}}),$$

the assignment of all edges $\sigma : E \mapsto \{0, 1\}$ can be uniquely extended by the assignment of all leader edges $\sigma' : \{e_1, e_2, \cdots e_k\} \mapsto \{0, 1\}$. This is because at each vertex v, $f_v(\sigma\ |_{E_{(v)}}) \neq 0$ if and only if each pair of edge twins in $E_{(v)}$ is assigned value $(0, 1)$ or $(1, 0)$. More specifically, all edges in C_i take assignment $(0, 1, 0, 1, \cdots, 0, 1)$ when $e_i = 0$ and $(1, 0, 1, 0, \cdots, 1, 0)$ when $e_i = 1$. In other words, all pairs of 4-ary edge twins in C_i take assignment $(0, 1)$ when $e_i = 0$ and $(1, 0)$ when $e_i = 1$. Then, we have

$$\text{Pl-Holant}_\Omega = \sum_{\sigma': \{e_1, \cdots, e_k\} \mapsto \{0,1\}} \prod_{v \in V} f_v(\sigma'\ |_{E_{(v)}}).$$

Let $V_{i,j} = C_i \cap C_j$ $(i < j)$ be the set of all intersection vertices in C_i and C_j. Since G is a planar graph, $|V_{i,j}| \equiv 0 \pmod 2$. Let $\sigma'_{(e_i,e_j)}$ denote the restriction of σ' on edges e_i and e_j. Define binary function $g_{i,j}$ on e_i and e_j: Given an assignment $\sigma'_{(e_i,e_j)} : \{e_i, e_j\} \mapsto \{0, 1\}$, we have

$$g_{i,j}(e_i, e_j) = \prod_{v \in V_{i,j}} f_v(\sigma'_{(e_i,e_j)}\ |_{E_{(v)}}).$$

Since all edges incident to vertices in $V_{i,j}$ are either in C_i or C_j, the assignment of these edges can be extended by the assignment $\sigma'_{(e_i,e_j)}$. Hence, $g_{i,j}$ is well-defined. We show $g_{i,j} \in \mathcal{G}_f$ by induction on the number n of self-intersection vertices in C_i.

- First, $n = 0$. That is, C_i is a simple cycle without self-intersection. By Jordan Curve Theorem, C_i divides the plane into an interior region and an exterior region. According to the orientation of C_i, we denote the left side of C_i to be the interior region and the right side to be the exterior region. At a half of intersection vertices in $V_{i,j}$, C_j enters the interior of C_i, and at the other half of intersection vertices, C_j exits. We call those vertices "entry-vertices" and "exit-vertices" separately (See Figure 9).

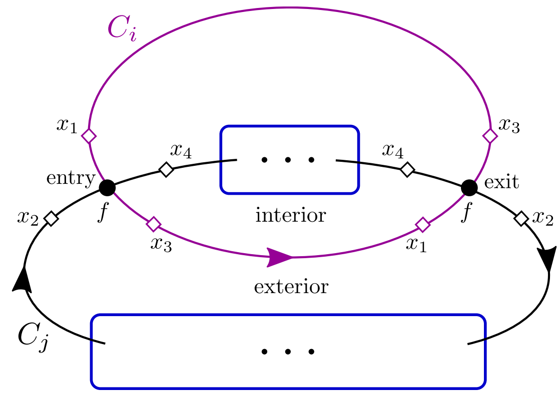


Figure 9: Intersection vertices between C_i and C_j

For each vertex in $V_{i,j}$, consider the two pairs of edge twins incident to it. We label the edge twins in C_i by variables (x_1, x_3) obeying the orientation of C_i. That is, x_3 is always the successor of x_1. Hence, all variables (x_1, x_3) take the same assignment $(0, 1)$ when $e_i = 0$ and $(1, 0)$ when $e_i = 1$. Then, the labeling (x_2, x_4) of edge twins in C_j is uniquely determined by the counterclockwise order of (x_1, x_2, x_3, x_4). Moreover, at any vertex in $V_{i,j}$, the variable x_2 is always in the exterior of C_i (See Figure 9), which means at entry-vertices, x_4 is the successor of x_2, while at exit-vertices, x_2 is the successor of x_4. Therefore, at entry-vertices, variables (x_2, x_4) take assignment $(0, 1)$ when $e_j = 0$ and $(1, 0)$ when $e_j = 1$, while at exit-vertices, they take assignment $(1, 0)$ and $(0, 1)$ correspondingly.

(e_i, e_j)	entry-vertices					exit-vertices				
	(x_1, x_2, x_3, x_4)	f	$f^{\frac{\pi}{2}}$	f^{π}	$f^{\frac{3\pi}{2}}$	(x_1, x_2, x_3, x_4)	f	$f^{\frac{\pi}{2}}$	f^{π}	$f^{\frac{3\pi}{2}}$
$(0, 0)$	$(0, 0, 1, 1)$	a	y	x	b	$(0, 1, 1, 0)$	b	a	y	x
$(0, 1)$	$(0, 1, 1, 0)$	b	a	y	x	$(0, 0, 1, 1)$	a	y	x	b
$(1, 1)$	$(1, 1, 0, 0)$	x	b	a	y	$(1, 0, 0, 1)$	y	x	b	a
$(1, 0)$	$(1, 0, 0, 1)$	y	x	b	a	$(1, 1, 0, 0)$	x	b	a	y

Table 1: The values of f at intersection vertices

According to the different assignments of (e_i, e_j) as listed in Column 1 of Table 1, Column 2 and Column 7 (indexed by (x_1, x_2, x_3, x_4)) list the assignments of (x_1, x_2, x_3, x_4) at entry-vertices and exit-vertices separately. Note that under this labeling, signature f has four rotation forms:

$$M(f) = \begin{bmatrix} 0 & 0 & 0 & a \\ 0 & b & 0 & 0 \\ 0 & 0 & y & 0 \\ x & 0 & 0 & 0 \end{bmatrix}, M(f^{\frac{\pi}{2}}) = \begin{bmatrix} 0 & 0 & 0 & y \\ 0 & a & 0 & 0 \\ 0 & 0 & x & 0 \\ b & 0 & 0 & 0 \end{bmatrix}, M(f^{\pi}) = \begin{bmatrix} 0 & 0 & 0 & x \\ 0 & y & 0 & 0 \\ 0 & 0 & b & 0 \\ a & 0 & 0 & 0 \end{bmatrix} \text{ and } M(f^{\frac{3\pi}{2}}) = \begin{bmatrix} 0 & 0 & 0 & b \\ 0 & x & 0 & 0 \\ 0 & 0 & a & 0 \\ y & 0 & 0 & 0 \end{bmatrix}.$$

Columns 3, 4, 5, 6 and Columns 8, 9, 10, 11 list the corresponding values of signature f in four forms f, $f^{\frac{\pi}{2}}$, f^{π} and $f^{\frac{3\pi}{2}}$ separately.

Suppose there are k_1 many entry-vertices assigned f, k_2 many entry-vertices assigned $f^{\frac{\pi}{2}}$, k_3 many entry-vertices assigned f^{π}, and k_4 many entry-vertices assigned $f^{\frac{3\pi}{2}}$, while there are ℓ_4 many exit-vertices assigned f, ℓ_1 many exit-vertices assigned $f^{\frac{\pi}{2}}$, ℓ_2 many exit-vertices assigned f^{π}, and ℓ_3 many exit-vertices assigned $f^{\frac{3\pi}{2}}$. Then, according to the assignments of (e_i, e_j), the values of $g_{i,j}$ are listed in Table 2 :

(e_i, e_j)	$g_{i,j}(e_i, e_j) = f^{k_1}(f^{\frac{\pi}{2}})^{k_2}(f^{\pi})^{k_3}(f^{\frac{3\pi}{2}})^{k_4}(f^{\frac{\pi}{2}})^{\ell_1}(f^{\pi})^{\ell_2}(f^{\frac{3\pi}{2}})^{\ell_3}f^{\ell_4}$
$(0,0)$	$a^{k_1}y^{k_2}x^{k_3}b^{k_4}a^{\ell_1}y^{\ell_2}x^{\ell_3}b^{\ell_4}$
$(0,1)$	$b^{k_1}a^{k_2}y^{k_3}x^{k_4}y^{\ell_1}x^{\ell_2}b^{\ell_3}a^{\ell_4}$
$(1,1)$	$x^{k_1}b^{k_2}a^{k_3}y^{k_4}x^{\ell_1}b^{\ell_2}a^{\ell_3}y^{\ell_4}$
$(1,0)$	$y^{k_1}x^{k_2}b^{k_3}a^{k_4}b^{\ell_1}a^{\ell_2}y^{\ell_3}x^{\ell_4}$

Table 2: The values of $g_{i,j}$

That is,

$$M(g_{i,j}) = \begin{bmatrix} a^{k_1+\ell_1}y^{k_2+\ell_2}x^{k_3+\ell_3}b^{k_4+\ell_4} & a^{k_2+\ell_4}y^{k_3+\ell_1}x^{k_4+\ell_2}b^{k_1+\ell_3} \\ a^{k_4+\ell_2}y^{k_1+\ell_3}x^{k_2+\ell_4}b^{k_3+\ell_1} & a^{k_3+\ell_3}y^{k_4+\ell_4}x^{k_1+\ell_1}b^{k_2+\ell_2} \end{bmatrix}.$$

Since the number of entry-vertices is equal to the number of exit-vertices, $k_1+k_2+k_3+k_4 = \ell_1+\ell_2+\ell_3+\ell_4$. Hence, we have $g_{i,j} \in \mathcal{G}_f$.

- Then, suppose $g_{i,j} \in \mathcal{G}_f$ holds for any circuit C_i with at most n many self-intersection vertices. C_i can be decomposed into two edge-disjoint circuits, each of which has at most n many self-intersection vertices (See Figure 10).

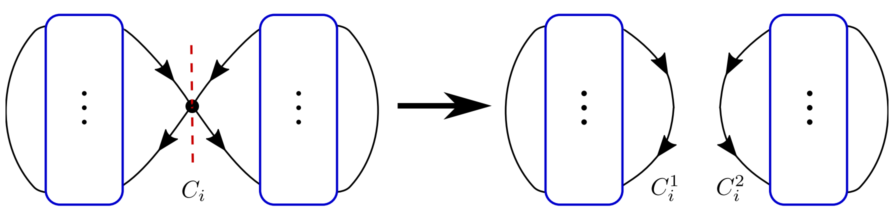


Figure 10: Decompose C_i into C_i^1 and C_i^2.

Now, we have $C_i = C_i^1 \cup C_i^2$. Each C_i^s $(s = 1, 2)$ is a circuit with at most n many self-intersection vertices. The orientation of C_i induces the orientations of C_i^1 and C_i^2. Clearly, the assignment of edges in C_i^s can be uniquely extended by the assignment of e_i. We can still view e_i as the leader edge of both C_i^1 and C_i^2. Let $V_{i,j}^s = C_i^s \cap C_j$ be the set of all intersection vertices in C_i^s and C_j. Then $V_{i,j} = V_{i,j}^1 \cup V_{i,j}^2$, where $V_{i,j}^1 \cap V_{i,j}^2 = \emptyset$. Same as the definition of $g_{i,j}$, we define binary function:

$$g_{i,j}^{(s)}(e_i, e_j) = \prod_{v \in V_{i,j}^s} f_v(\sigma'_{(e_i,e_j)}\,|_{E_{(v)}}).$$

As we have showed above, $g_{i,j}^{(s)}$ is well-defined and by induction hypothesis, it is in $\mathcal{G}_f$. Also, we have

$$g_{i,j} = \prod_{v \in V_{i,j}} f_v(\sigma'_{(e_i,e_j)} \mid_{E_{(v)}}) = \prod_{v \in V^1_{i,j}} \prod_{v \in V^2_{i,j}} f_v(\sigma'_{(e_i,e_j)} \mid_{E_{(v)}}) = g^{(1)}_{i,j} g^{(2)}_{i,j}.$$

Note that $\mathcal{G}_f$ is closed under function multiplication. That is, $g_{i,j} \in \mathcal{G}_f$.

Let V_i be the set of all self-intersection vertices in C_i. Let $\sigma'_{(e_i)}$ denote the restriction of σ' on e_i. Define unary function h_i on e_i: Given an assignment $\sigma'_{(e_i)} : e_i \mapsto \{0,1\}$, we have

$$h_i(e_i) = \prod_{v \in V_i} f_v(\sigma'_{(e_i)} \mid_{E_{(v)}}).$$

The assignment of those edges incident to vertices in V_i can be uniquely extended by the assignment $\sigma'_{(e_i)}$. Hence, h_i is well-defined. We show $h_i \in \mathcal{H}_f$.

For each vertex in V_i, since it is a self-intersection vertex, the two pairs of edge twins incident to it are both in C_i. We still label each pair of edge twins by a pair of variables (x_1, x_3) obeying the orientation of C_i. That is, x_3 is always the successor of x_1. Now, at each vertex in V_i, the four edges incident to it are labeled by (x_1, x_1, x_3, x_3) listed in counterclockwise order. We pick the proper pair of variables (x_1, x_3) and change it to (x_2, x_4) such that the label of four edges is (x_1, x_2, x_3, x_4) in counterclockwise order. Clearly, (x_2, x_4) and (x_1, x_3) take the same assignment. That is, at each vertex in V_i, the assignment of (x_1, x_2, x_3, x_4) is $(0,0,1,1)$ when $e_i = 0$, and $(1,1,0,0)$ when $e_i = 1$. Under this labeling, signature f still has four rotation forms. The values of signature f in four forms are listed in Table 3.

e_i	(x_1, x_2, x_3, x_4)	f	$f^{\frac{\pi}{2}}$	f^{π}	$f^{\frac{3\pi}{2}}$
0	$(0,0,1,1)$	a	y	x	b
1	$(1,1,0,0)$	x	b	a	y

Table 3: The values of f at self-intersection vertices

Suppose there are m_1 many vertices assigned f, m_2 many vertices assigned $f^{\frac{\pi}{2}}$, m_3 many vertices assigned f^{π} and m_4 many vertices assigned $f^{\frac{3\pi}{2}}$. Then, we have

$$M(h_i) = [a^{m_1} y^{m_2} x^{m_3} b^{m_4} \quad a^{m_3} y^{m_4} x^{m_1} b^{m_2}].$$

That is, $h_i \in \mathcal{H}_f$.

For any vertex $v \in V$, it is either in some $V_{i,j}$ or some V_i. Thus,

$$\begin{aligned} \text{Pl-Holant}_{\Omega} &= \sum_{\sigma' : \{e_1, \cdots, e_k\} \mapsto \{0,1\}} \left(\prod_{\substack{v \in V_{i,j} \\ 1 \leqslant i < j \leqslant k}} f_v(\sigma' | E_{(v)}) \right) \left(\prod_{\substack{v \in V_i \\ 1 \leqslant i \leqslant k}} f_v(\sigma' | E_{(v)}) \right) \\ &= \sum_{\sigma' : \{e_1, \cdots, e_k\} \mapsto \{0,1\}} \left(\prod_{1 \leqslant i < j \leqslant k} g_{i,j}(e_i, e_j) \right) \left(\prod_{1 \leqslant i \leqslant k} h_i(e_i) \right), \end{aligned}$$

where $g_{i,j} \in \mathcal{G}_f$ and $h_i \in \mathcal{H}_f$. Therefore, Pl-Holant$(\neq_2 \mid f) \leqslant_T \#\text{CSP}(\mathcal{G}_f \cup \mathcal{H}_f)$.

Here, we give an example for the reduction (4.3).

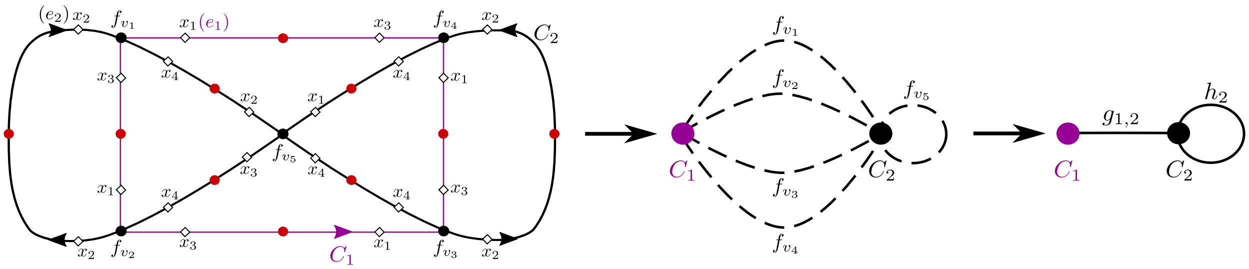


Figure 11: An example for the reduction (4.3)

Example. Given an instance $\Omega = (G, \pi)$ of Pl-Holant $(\neq_2 |\ f)$, there are two circuits C_1 (The Square) and C_2 (The Horizontal Eight) in G (See Figure 11). Each circuit C_i has a leader edge e_i. Given the leader, the direction from its endpoint of degree 2 to the endpoint of degree 4 gives a default orientation of the circuit. Given a nonzero term in the sum Pl-Holant$_\Omega$, as a consequence of the support of f, the assignment of edges in each circuit is uniquely determined by the assignment of its leader. That is, any assignment of the leaders $\sigma' : \{e_1, e_2\} \mapsto \{0, 1\}$ can be uniquely extended to an assignment of all edges $\sigma : E \mapsto \{0, 1\}$ such that on each circuit, the $\{0, 1\}$ assignments alternate.

Consider the signatures f on the intersection vertices between C_1 and C_2, ($f_{v_1}, f_{v_2}, f_{v_3}$ and f_{v_4}). Due to planarity, there are an even number of intersection vertices. Assume C_1 does not have self-intersection (as is The Square). Otherwise, we will decompose C_1 further and reason inductively. Without self-intersection, C_1 has an interior and exterior region (Jordan Curve Theorem) depending on its default orientation. With respect to C_1, The circuit C_2 enters and exits the interior of C_1 alternately. Thus, we can divide the intersection vertices into an equal number of "entry-vertices" and "exit-vertices". In this example, f_{v_1} and f_{v_4} are on "entry-vertices", while f_{v_2} and f_{v_3} are on "exit-vertices". By analyzing the values of each f when e_1 and e_2 take assignment 0 or 1, we can view each f as a binary constraint on C_1 and C_2. Depending on the 4 different rotation forms of f and whether f is on "entry-vertices" or "exit-vertices", the resulting binary constraint has 8 different forms (See Table 1). By multiplying these constraints, we get the binary constraint $g_{1,2}$. This can be viewed as a binary edge function on the circuits C_1 and C_2. The property of $g_{1,2}$ crucially depends on there are an even number of intersection vertices. Given a particular assignment $\sigma'_{(e_1,e_2)} : \{e_1, e_2\} \mapsto \{0, 1\}^2$, we have

$$g_{1,2}(e_1, e_2) = \prod_{1 \leqslant i \leqslant 4} f_{v_i}(\sigma'_{(e_1,e_2)} \mid_{E(v)}).$$

If the placement of f_{v_1} were to be rotated clockwise $\frac{\pi}{2}$, then f_{v_1} will be changed to $f_{v_1}^{\frac{\pi}{2}}$ in the above formula, where $M_{x_1x_2,x_4x_3}(f_{v_1}^{\frac{\pi}{2}}) = M_{x_2x_3,x_1x_4}(f_{v_1})$.

For the self-intersection vertex f_{v_5}, the notions of "entry-vertex" and "exit-vertex" do not apply. f_{v_5} gives rise to an unary constraint h_2 on e_2. Depending on the 4 different rotation forms of f, h_2 has 4 different forms (see Table 3 in full version). Given a particular assignment $\sigma'_{(e_2)} : e_2 \mapsto \{0, 1\}$, we have

$$h_2(e_2) = f_{v_5}(\sigma'_{(e_2)} \mid_{E(v)}).$$

Therefore, we have

$$\begin{aligned}
\text{Pl-Holant}_\Omega &= \sum_{\sigma:E\mapsto\{0,1\}} \prod_{v\in V} f_v(\sigma\mid_{E_{(v)}}) \\
&= \sum_{\sigma':\{e_1,e_2\}\mapsto\{0,1\}} \left(\prod_{1\leqslant i\leqslant 4} f_{v_i}(\sigma'|E_{(v)})\right) f_{v_5}(\sigma'|E_{(v)}) \\
&= \sum_{\sigma':\{e_1,e_2\}\mapsto\{0,1\}} g_{1,2}(e_1,e_2)h_2(e_2).
\end{aligned}$$

Part II: Given an instance I of $\#\text{CSP}(g_{1_f}, g_{2_f})$. Consider binary constraints on variables x_i and x_j $(i<j)$. Note that g_{1_f} is symmetric, that is, $g_{1_f}(x_i,x_j) = g_{1_f}(x_j,x_i)$. We assume there are $s_{i,j}$ many constraints $g_{1_f}(x_i,x_j)$, $t_{i,j}$ many constraints $g_{2_f}(x_i,x_j)$ and $t'_{i,j}$ many constraints $g_{2_f}(x_j,x_i)$. These are all constraints between x_i and x_j. Let $g_{i,j}(x_i,x_j)$ be the function product of these constraints. That is,

$$g_{i,j}(x_i,x_j) = g_{1_f}^{s_{i,j}}(x_i,x_j)g_{2_f}^{t_{i,j}}(x_i,x_j)g_{2_f}^{t'_{i,j}}(x_j,x_i).$$

Then, we have

$$\#\text{CSP}(I) = \sum_{\sigma':\{x_1,\cdots,x_k\}\mapsto\{0,1\}} \prod_{1\leqslant i<j\leqslant n} g_{i,j}(x_i,x_j).$$

We prove the reduction (4.4) in two steps. We first reduce $\#\text{CSP}(I)$ to an instance $\Omega_i (i=1,2)$ of Pl-Holant $(\neq_2|\ f,\chi_i)$ respectively, where $\chi_1 = \begin{bmatrix} 0&0&0&1\\0&1&0&0\\0&0&1&0\\1&0&0&0 \end{bmatrix}$ and $\chi_2 = \begin{bmatrix} 0&0&0&1\\0&1&0&0\\0&0&1&0\\-1&0&0&0 \end{bmatrix}$. Ω_i is constructed as follows:

1. In a plane, draw cycle D_1. Then draw cycle D_2, and let D_2 intersect with D_1 at least $2(s_{1,2}+t_{1,2}+t'_{1,2})$ many times. This can be done since we can let D_2 enter and exit the interior of D_1 alternately and there will be an even number of intersection vertices. Successively draw cycles D_j until D_k, and let D_j intersect with D_i at least $2(s_{i,j}+t_{i,j}+t'_{i,j})$ many times for $1\leqslant i<j\leqslant k$. This can be done since we can let D_j enter and exit the interiors of $D_1,\cdots,D_{j-1}$ successively. Note that, when letting D_j intersect with other cycles, it may intersect with D_i again. This is why we say D_j intersects with D_i at least $2(s_{i,j}+t_{i,j}+t'_{i,j})$ many times. We will deal with those extra intersection vertices later. Moreover, when drawing these cycles, we can easily make them satisfy the following conditions:
   a. There is no self-intersection.
   b. There is no more than two cycles that intersect with others at the same vertex. That is, each intersection vertex is of degree 4.
2. Replace each edge by a path of length two. We finally get a planar bipartite graph $G=(V,E)$. On one side, all vertices are of degree 2, and on the other side, all vertices are of degree 4. We can still define edge twins as defined in Part I. Moreover, we still divide the graph into some circuits $C_1,\cdots,C_k$. In fact, C_i is just the cycle D_i after the replacement of each edge by a path of length two.
   Let $V_{i,j} = C_i\cap C_j$ $(i<j)$ be the intersection vertices in C_i and C_j. Clearly, $|V_{i,j}|$ is even and no less than $2(s_{i,j}+t_{i,j}+t'_{i,j})$. Since there is no self-intersection, each circuit is a simple cycle. We can define "entry-vertices" and "exit-vertices" as in Part I. Among $V_{i,j}$, a half of them are entry-vertices and the other half are exit-vertices. As we did in Part I, we pick an edge e_i

as the leader edge of C_i and this gives an orientation of C_i. List the edges in C_i according to the orientation of C_i. Then, all edges in C_i take assignment $(0,1,0,1,\cdots,0,1)$ when $e_i=0$ and $(1,0,1,0,\cdots,1,0)$ when $e_i=1$.

3. Label the vertex of degree 2 by $(\neq_2)$. For any vertex in $V_{i,j}$, as we showed in Part I, we can label the four edges incident to it by variables (x_1,x_2,x_3,x_4) in a way such that when $(e_i,e_j)=(s,t)$, we have $(x_1,x_2,x_3,x_4)=(s,t,1-s,1-t)$ at entry-vertex, and $(x_1,x_2,x_3,x_4)=(s,1-t,1-s,t)$ at exit-vertex (See Table 1). Note that f has four rotation forms under this labeling. Label $s_{i,j}$ many entry-vertices by f and $s_{i,j}$ many exit-vertices by $f^{\frac{\pi}{2}}$, $t_{i,j}$ many entry-vertices by f^{π} and $t_{i,j}$ many exit-vertices by $f^{\frac{3\pi}{2}}$, and $t'_{i,j}$ many entry-vertices by f^{π} and $s_{i,j}$ many exit-vertices by $f^{\frac{\pi}{2}}$. Let $V'_{i,j}$ be the set of these $2(s_{i,j}+t_{i,j}+t'_{i,j})$ vertices. Refer to Table 2, we have

$$\prod_{v\in V'_{ij}} f_v(\sigma'_{(e_i,e_j)}\mid_{E_{(v)}}) = g_{1_f}^{s_{i,j}}(e_i,e_j)g_{2_f}^{t_{i,j}}(e_i,e_j)g_{2_f}^{t'_{i,j}}(e_j,e_i)=g_{i,j}(e_i,e_j).$$

For any vertex in $V_{i,j}\backslash V'_{i,j}$, if label it by an auxiliary signature $\chi_1=\begin{bmatrix}0&0&0&1\\0&1&0&0\\0&0&1&0\\1&0&0&0\end{bmatrix}$, then refer to Table 2 (Here $a=x=b=y=1$), we have

$$\prod_{v\in V_{i,j}\backslash V'_{ij}} \chi_1(\sigma'_{(e_i,e_j)}\mid_{E_{(v)}})\equiv 1.$$

We can also label the vertex in $V_{i,j}\backslash V'_{i,j}$ by auxiliary signature $\chi_2=\begin{bmatrix}0&0&0&1\\0&1&0&0\\0&0&1&0\\-1&0&0&0\end{bmatrix}$. Note that in $V_{i,j}\backslash V'_{i,j}$, the number of entry-vertices are equal to the number of exit-vertices. We label all entry-vertices by χ_2 and label all exit-vertices by its rotation form $\chi_2^{\frac{\pi}{2}}=\begin{bmatrix}0&0&0&1\\0&1&0&0\\0&0&-1&0\\1&0&0&0\end{bmatrix}$. Refer to Table 2 (Here $a=b=y=1, x=-1$, and $k=k_1=\ell_1=\ell$), we have

$$\prod_{v\in V_{i,j}\backslash V'_{ij}} \chi_2(\sigma'_{(e_i,e_j)}\mid_{E_{(v)}})\equiv 1.$$

Now, we get an instance Ω_i $(i=1,2)$ for each problem Pl-Holant $(\neq_2\mid f,\chi_i)$ respectively. Note that χ_i has the same support as f. As we have showed in Part I, for any nonzero term in the sum Pl-Holant$_{\Omega_i}$, the assignment of all edges $\sigma: E\mapsto\{0,1\}$ can be uniquely extended by the assignment of all leader edges $\sigma':\{e_1,e_2,\cdots,e_k\}\mapsto\{0,1\}$. Therefore, we have

$$\begin{aligned}\#\mathrm{CSP}(I) &= \sum_{\sigma':\{e_1,\cdots,e_k\}\mapsto\{0,1\}}\prod_{1\leqslant i<j\leqslant n} g_{i,j}(e_i,e_j)\\ &= \sum_{\sigma':\{e_1,\cdots,e_k\}\mapsto\{0,1\}}\left(\prod_{\substack{v\in V'_{i,j}\\1\leqslant i<j\leqslant n}} f_v(\sigma'|E_{(v)})\right)\left(\prod_{\substack{v\in V_{i,j}\backslash V'_{i,j}\\1\leqslant i<j\leqslant n}}\chi_{i_v}(\sigma'|E_{(v)})\right)\\ &= \text{Pl-Holant}_{\Omega_i}\end{aligned}$$

That is, $\#\mathrm{CSP}(g_{1_f},g_{2_f})\leqslant_T$ Pl-Holant $(\neq_2\mid f,\chi_i)$.

Then, we show

$$\text{Pl-Holant}(\neq_2|\ f, \chi_1) \leqslant_T \text{Pl-Holant}(\neq_2|\ f)$$

when $a = \epsilon x, b = \epsilon y$, where $\epsilon = \pm 1$ and

$$\text{Pl-Holant}(\neq_2|\ f, \chi_2) \leqslant_T \text{Pl-Holant}(\neq_2|\ f)$$

when $a = \epsilon x, b = -\epsilon y$, where $\epsilon = \pm 1$ by interpolation.

- If $a = x$ and $b = y$, since they are all no zeros and $(\frac{b}{a})^8 \neq 1$, by normalization, we may assume $M(f) = \left[\begin{smallmatrix} 0 & 0 & 0 & 1 \\ 0 & b & 0 & 0 \\ 0 & 0 & b & 0 \\ 1 & 0 & 0 & 0 \end{smallmatrix}\right]$, where $b \neq 0$ and $b^8 \neq 1$.

  If b is not a root of unity, by Lemma 2.3, we have $\text{Pl-Holant}(\neq_2|\ f, \chi_1) \leqslant_T \text{Pl-Holant}(\neq_2|\ f)$. Otherwise, b is a root of unity. Construct gadget $f_\boxtimes$ as showed in Figure 12. Given an

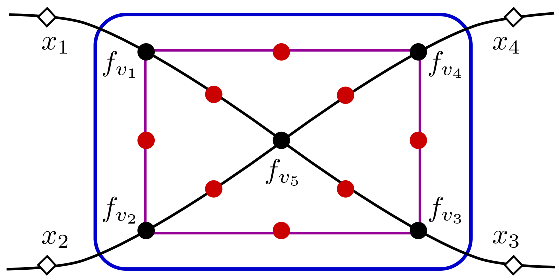


Figure 12: The SQUARE gadget

  assignment (x_1, x_2, x_3, x_4) of $f_\boxtimes$, $f_\boxtimes(x_1, x_2, x_3, x_4) \neq 0$ if and only if (x_1, x_2, x_3, x_4) is equal to the assignment of f_{v_5}. That is, $f_\boxtimes$ has support $(0, 0, 1, 1)$, $(1, 1, 0, 0)$, $(0, 1, 1, 0)$ and $(1, 0, 0, 1)$. In fact, each DIAGONAL LINE in this gadget is a part of some circuit, which means the edges in it can only take assignment $(0, 1, 0, 1, 0, 1)$ or $(1, 0, 1, 0, 1, 0)$, otherwise the term is zero. Also, the SQUARE cycle in this gadget is a circuit itself, which means the edges in it can only take assignment $(0, 1, 0, 1, 0, 1, 0, 1)$ or $(1, 0, 1, 0, 1, 0, 1, 0)$. We simplify them by $(0, 1)$ and $(1, 0)$.

  For the signature f, if one pair of its edge twins flips its assignment between $(0, 1)$ and $(1, 0)$, then the value of f changes from 1 to b, or b to 1. If two pairs of edge twins both flip their assignments, then the value of f does not change. According to this property, we give the following Table 4. Here, we label vertices v_1, v_2, v_3, v_4 and v_5 in a way such that the values of f on these vertices are all 1 under the assignment $(x_1, x_2, x_3, x_4) = (0, 0, 1, 1)$ and SQUARE$= (0, 1)$ (Row 2). When the assignment of SQUARE flips from $(0, 1)$ to $(1, 0)$, one pair of edge twins of each vertex except v_5 flips its assignment. So the values of f on these vertices except v_5 change from 1 to b (Row 3). When (x_1, x_3) flips its assignment, one pair of edge twins of v_1, v_3 and v_5 flip their assignments. When (x_2, x_4) flips its assignment, one pair of edge twins of v_2, v_4 and v_5 flip their assignments. Using this fact, we get other rows correspondingly.

  Hence, $f_\boxtimes$ has the signature matrix $M(f_\boxtimes) = \left[\begin{smallmatrix} 0 & 0 & 0 & 1+b^4 \\ 0 & 2b^3 & 0 & 0 \\ 0 & 0 & 2b^3 & 0 \\ 1+b^4 & 0 & 0 & 0 \end{smallmatrix}\right]$. Since $b^4 \neq -1$, $1 + b^4 \neq 0$, by normalization, we have $M(f_\boxtimes) = \left[\begin{smallmatrix} 0 & 0 & 0 & 1 \\ 0 & \frac{2b^3}{1+b^4} & 0 & 0 \\ 0 & 0 & \frac{2b^3}{1+b^4} & 0 \\ 1 & 0 & 0 & 0 \end{smallmatrix}\right]$. Since $|b| = 1$ and $b^4 \neq 1$, we have

(x_1,x_2,x_3,x_4)	SQUARE	f_{v_1}	f_{v_2}	f_{v_3}	f_{v_4}	f_{v_5}	$f_{\boxtimes}$
$(0,0,1,1)$	$(0,1)$	1	1	1	1	1	$1+b^4$
	$(1,0)$	b	b	b	b	1	
$(1,1,0,0)$	$(0,1)$	b	b	b	b	1	$1+b^4$
	$(1,0)$	1	1	1	1	1	
$(0,1,1,0)$	$(0,1)$	1	b	1	b	b	$2b^3$
	$(1,0)$	b	1	b	1	b	
$(1,0,0,1)$	$(0,1)$	b	1	b	1	b	$2b^3$
	$(1,0)$	1	b	1	b	b	

Table 4: The values of gadget $f_{\boxtimes}$ when $a = x = 1$ and $b = y$

$|1+b^4| < 2$. Then $|\frac{2b^3}{1+b^4}| > |b^3| = 1$, which means $\frac{2b^3}{1+b^4}$ is not a root of unity. By Lemma 2.3, we have Pl-Holant$(\neq_2|\ f, \chi_1) \leqslant_T$ Pl-Holant$(\neq_2|\ f, f_{\boxtimes})$. Since $f_{\boxtimes}$ is constructed by f, we have Pl-Holant$(\neq_2|\ f, \chi_1) \leqslant_T$ Pl-Holant$(\neq_2|\ f)$.

- If $a = -x$ and $b = -y$, then $M(f) = \begin{bmatrix} 0 & 0 & 0 & a \\ 0 & b & 0 & 0 \\ 0 & 0 & -b & 0 \\ -a & 0 & 0 & 0 \end{bmatrix}$. Connect the variable x_4 with x_3 of f using $(\neq_2)$, and we get a binary signature g', where
$$g' = M_{x_1x_2,x_4x_3}(0,1,1,0)^T = (0,b,-b,0)^T.$$
Since $b \neq 0$, g' can be normalized as $(0,1,-1,0)^T$. Connect the variable x_2 of g' with the variable x_1 of f, and we get a signature f' with the signature matrix $M(f') = \begin{bmatrix} 0 & 0 & 0 & a \\ 0 & b & 0 & 0 \\ 0 & 0 & b & 0 \\ a & 0 & 0 & 0 \end{bmatrix}$. As we have proved above, Pl-Holant$(\neq_2|\ f, \chi_1) \leqslant_T$ Pl-Holant$(\neq_2|\ f, f')$. Since f' is constructed by f, we have Pl-Holant$(\neq_2|\ f, \chi_1) \leqslant_T$ Pl-Holant$(\neq_2|\ f)$.
- If $a = -x$, $b = y$ or $a = x$, $b = -y$, by normalization and rotational symmetry, we may assume $M(f) = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & b & 0 & 0 \\ 0 & 0 & b & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}$, where $b \neq 0$ and $b^8 \neq 1$.
If b is not a root of unity, by Corollary 2.4, we have Pl-Holant$(\neq_2|\ f, \chi_2) \leqslant_T$ Pl-Holant$(\neq_2|\ f)$. Otherwise, b is a root of unity. Construct gadget $f_{\boxtimes}$ in the same way as showed above. Similarly, given an assignment (x_1,x_2,x_3,x_4) of $f_{\boxtimes}$, $f_{\boxtimes}(x_1,x_2,x_3,x_4) \neq 0$ if and only if (x_1,x_2,x_3,x_4) is equal to the assignment of f_{v_5}. Also, the edges in SQUARE can only take assignment $(0,1)$ or $(1,0)$.
For the signature f, if one pair of its edge twins flips its assignment between $(0,1)$ and $(1,0)$, then the value of f changes from ± 1 to b, or b to ∓ 1. If two pairs of edge twins both flip their assignments, then the value of f does not change if the value is b, or change its sign if the value is ± 1. According to this property, we have the following Table 5. Here, we label vertices v_1, v_2, v_3, v_4 and v_5 in a way such that the values of f on these vertices are all 1 under the assignment $(x_1,x_2,x_3,x_4) = (0,0,1,1)$ and SQUARE$=(0,1)$ (Row 2). When the assignment of SQUARE flips from $(0,1)$ to $(1,0)$, one pair of edge twins of each vertex except v_5 flips its assignment. So the values of f on these vertices except v_5 change from 1 to b (Row 3). When (x_1,x_3) flips its assignment, one pair of edge twins of v_1, v_3 and v_5 flip their assignments. When (x_2,x_4) flips its assignment, one pair of edge twins of v_2, v_4 and v_5 flip their assignments. Using this fact, we get other rows correspondingly.

(x_1,x_2,x_3,x_4)	Square	f_{v_1}	f_{v_2}	f_{v_3}	f_{v_4}	f_{v_5}	$f_{\boxtimes}$
$(0,0,1,1)$	$(0,1)$	1	1	1	1	1	$1+b^4$
	$(1,0)$	b	b	b	b	1	
$(1,1,0,0)$	$(0,1)$	b	b	b	b	-1	$-(1+b^4)$
	$(1,0)$	-1	-1	-1	-1	-1	
$(0,1,1,0)$	$(0,1)$	1	b	1	b	b	$2b^3$
	$(1,0)$	b	-1	b	-1	b	
$(1,0,0,1)$	$(0,1)$	b	1	b	1	b	$2b^3$
	$(1,0)$	-1	b	-1	b	b	

Table 5: The values of gadget $f_{\boxtimes}$ when $a=-x=1$ and $b=y$

Hence, $f_{\boxtimes}$ has the signature matrix $\begin{bmatrix} 0 & 0 & 0 & 1+b^4 \\ 0 & 2b^3 & 0 & 0 \\ 0 & 0 & 2b^3 & 0 \\ -(1+b^4) & 0 & 0 & 0 \end{bmatrix}$. Since $|b|=1$ and $b^8\neq 1$, we have $\frac{2b^3}{1+b^4}$ is not a root of unity. By Corollary 2.4, $\text{Pl-Holant}(\neq_2|\, f,\chi_2) \leqslant_T \text{Pl-Holant}(\neq_2|\, f, f_{\boxtimes})$, and hence $\text{Pl-Holant}(\neq_2|\, f,\chi_2) \leqslant_T \text{Pl-Holant}(\neq_2|\, f)$.

In summary, we have

$$\begin{tikzcd}
 & \text{Pl-Holant}\,(\neq_2|\, f,\chi_1) \arrow[dr, "a=\epsilon x,\, b=\epsilon y\ (\epsilon=\pm 1)"] & \\
\#\text{CSP}(g_{1_f},g_{2_f}) \arrow[ur] \arrow[dr] & \leqslant_T \qquad\qquad \leqslant_T & \text{Pl-Holant}\,(\neq_2|\, f) \\
 & \text{Pl-Holant}\,(\neq_2|\, f,\chi_2) \arrow[ur, "a=\epsilon x,\, b=-\epsilon y\ (\epsilon=\pm 1)"'] &
\end{tikzcd}$$

Therefore, we have $\#\text{CSP}(g_{1_f},g_{2_f}) \leqslant_T \text{Pl-Holant}\,(\neq_2|\, f)$ when $a^2=x^2\neq 0$, $b^2=y^2\neq 0$ and $(\frac{b}{a})^8\neq 1$. $\square$

Theorem 4.6. *Let f be a 4-ary signature of the form (4.2). Then* $\text{Pl-Holant}(\neq_2|\, f)$ *is #P-hard unless*

(i). $(ax)^2=(by)^2$, *or*

(ii). $x=a\mathfrak{i}^{\alpha}, b=a\sqrt{\mathfrak{i}}^{\beta}, y=a\sqrt{\mathfrak{i}}^{\gamma}$, *where* $\alpha,\beta,\gamma\in\mathbb{N}$, *and* $\beta\equiv\gamma \pmod 2$,

in which cases, the problem is tractable.

Proof of Tractability:

- In case (i), if $ax=by=0$, then f has support of size at most 2. So we have $f\in\mathscr{P}$, and hence $\text{Pl-Holant}(\neq_2|\, f)$ is tractable by Theorem 2.13. Otherwise, $(ax)^2=(by)^2\neq 0$. For any signature g in $\mathcal{G}_f$, we have $g_{00}\cdot g_{11}=(ax)^{k_1+\ell_1+k_3+\ell_3}(by)^{k_2+\ell_2+k_4+\ell_4}$ and $g_{01}\cdot g_{10}=(ax)^{k_2+\ell_2+k_4+\ell_4}(by)^{k_1+\ell_1+k_3+\ell_3}$. Since $(k_1+\ell_1+k_3+\ell_3)-(k_2+\ell_2+k_4+\ell_4)\equiv k+\ell\equiv 0 \pmod 2$, we have

$$\frac{g_{00}\cdot g_{11}}{g_{01}\cdot g_{10}}=\left(\frac{ax}{by}\right)^{(k_1+\ell_1+k_3+\ell_3)-(k_2+\ell_2+k_4+\ell_4)}=\left(\frac{(ax)^2}{(by)^2}\right)^{\frac{(k_1+\ell_1+k_3+\ell_3)-(k_2+\ell_2+k_4+\ell_4)}{2}}=1.$$

That is, $g \in \mathscr{P}$. Since any signature h in $\mathcal{H}_f$ is unary, $h \in \mathscr{P}$. Hence, we have $\mathcal{G}_f \cup \mathcal{H}_f \subseteq \mathscr{P}$. By Theorem 2.23, #CSP($\mathcal{G}_f \cup \mathcal{H}_f$) is tractable. By reduction (4.3) of Lemma 4.5, we have Pl-Holant($\neq_2|\ f$) is tractable.

- In case (ii), for any signature g in $\mathcal{G}_f$, $M(g)$ is of the form

$$a^{k+l}\begin{bmatrix}\sqrt{\mathfrak{i}}^{\beta(k_4+\ell_4)+\gamma(k_2+\ell_2)+2\alpha(k_3+\ell_3)} & \sqrt{\mathfrak{i}}^{\beta(k_1+\ell_3)+\gamma(k_3+\ell_1)+2\alpha(k_4+\ell_2)}\\ \sqrt{\mathfrak{i}}^{\beta(k_3+\ell_1)+\gamma(k_1+\ell_3)+2\alpha(k_2+\ell_4)} & \sqrt{\mathfrak{i}}^{\beta(k_2+\ell_2)+\gamma(k_4+\ell_4)+2\alpha(k_1+\ell_1)}\end{bmatrix} = a^{k+\ell}\begin{bmatrix}\sqrt{\mathfrak{i}}^{p_{00}} & \sqrt{\mathfrak{i}}^{p_{01}}\\ \sqrt{\mathfrak{i}}^{p_{10}} & \sqrt{\mathfrak{i}}^{p_{11}}\end{bmatrix},$$

where p_{00}, p_{01}, p_{10} and p_{11} denote the exponents of $\sqrt{\mathfrak{i}}$ in g correspondingly. Since $\beta \equiv \gamma \pmod 2$, if they are both even, then $p_{00} \equiv p_{01} \equiv p_{10} \equiv p_{11} \equiv 0 \pmod 2$; if they are both odd, then $p_{00} \equiv p_{11} \equiv k_2 + \ell_2 + k_4 + \ell_4 \equiv k_1 + \ell_1 + k_3 + \ell_3 \equiv p_{01} \equiv p_{10} \pmod 2$. If these exponents are all odd, we can take out a $\sqrt{\mathfrak{i}}$. Hence, g is of the form $a'(\mathfrak{i}^{q_{00}}, \mathfrak{i}^{q_{01}}, \mathfrak{i}^{q_{10}}, \mathfrak{i}^{q_{11}})^T$, where $a' = a^{k+l}$ or $a^{k+l}\sqrt{\mathfrak{i}}$, and for all $i, j \in \{0, 1\}$, $q_{ij} = \frac{p_{ij}}{2}$ or $\frac{p_{ij}-1}{2}$. Thus,

$$q_{00} + q_{01} + q_{10} + q_{11} \equiv (p_{00} + p_{01} + p_{10} + p_{11})/2 \pmod 2.$$

Moreover, since $p_{00} + p_{01} + p_{10} + p_{11} = (k+\ell)(\beta + \gamma + 2\alpha) \equiv 0 \pmod 4$, we have $q_{00} + q_{01} + q_{10} + q_{11} \equiv 0 \pmod 2$. Therefore, $g \in \mathscr{A}$ by Lemma 2.10.
In this case, for any signature h in $\mathcal{H}_f$, $M(h)$ is of the form

$$a^m\begin{bmatrix}\sqrt{\mathfrak{i}}^{\beta m_2+\gamma m_4+2\alpha m_3} & \sqrt{\mathfrak{i}}^{\beta m_4+\gamma m_2+2\alpha m_1}\end{bmatrix}.$$

Since $\beta \equiv \gamma \pmod 2$, we have $\beta m_2 + \gamma m_4 \equiv \beta m_4 + \gamma m_2 \pmod 2$. Hence, h is of the form $a''(i^{q_0}, i^{q_1})$, where $a'' = a^m$ or $a^m\sqrt{\mathfrak{i}}$. That is, $h \in \mathscr{A}$ by Lemma 2.11. Hence, $\mathcal{G}_f \cup \mathcal{H}_f \subseteq \mathscr{A}$. By Theorem 2.23, #CSP($\mathcal{G}_f \cup \mathcal{H}_f$) is tractable. By reduction (4.3) of Lemma 4.5, we have Pl-Holant($\neq_2|\ f$) is tractable.

Proof of Hardness: Note that $M_{x_1x_2,x_4x_3}(f) = \begin{bmatrix}0&0&0&a\\0&b&0&0\\0&0&y&0\\x&0&0&0\end{bmatrix}$ and $M_{x_3x_4,x_2x_1}(f) = \begin{bmatrix}0&0&0&x\\0&y&0&0\\0&0&b&0\\a&0&0&0\end{bmatrix}$. Connect variables x_4, x_3 of a copy of signature f with variables x_1, x_2 of another copy of signature f both using $(\neq_2)$. We get a signature f_1 with the signature matrix

$$M(f_1) = M_{x_1x_2,x_4x_3}(f) N M_{x_1x_2,x_4x_3}(f) = \begin{bmatrix}0&0&0&a^2\\0&0&by&0\\0&by&0&0\\x^2&0&0&0\end{bmatrix}.$$

Similarly, connect x_4, x_3 of a copy of signature f with x_3, x_4 of another copy of signature f both using $(\neq_2)$. We get a signature f_2 with the signature matrix

$$M(f_2) = M_{x_1x_2,x_4x_3}(f) N M_{x_3x_4,x_2x_1}(f) = \begin{bmatrix}0&0&0&ax\\0&0&b^2&0\\0&y^2&0&0\\ax&0&0&0\end{bmatrix}.$$

Notice that $M(f_1^{\frac{\pi}{2}}) = \begin{bmatrix}0&0&0&0\\0&a^2&by&0\\0&by&x^2&0\\0&0&0&0\end{bmatrix}$, $M(f_2^{\frac{\pi}{2}}) = \begin{bmatrix}0&0&0&0\\0&ax&b^2&0\\0&y^2&ax&0\\0&0&0&0\end{bmatrix}$, $M(g_{1_f}) = \begin{bmatrix}a^2&by\\by&x^2\end{bmatrix}$ and $M(g_{2_f}) = \begin{bmatrix}ax&b^2\\y^2&ax\end{bmatrix}$. That is, $g_{i_f} = f_i^{\frac{\pi}{2}}{}_{\text{In}}$. Thus, we have $f_i(x_1, x_2, x_3, x_4) = g_i(x_1, x_2) \cdot \chi_{x_2 \neq x_3} \cdot \chi_{x_1 \neq x_4}$. Now, we analyze g_{1_f} and g_{2_f}.

- If $\{g_{1_f}, g_{2_f}\} \subseteq \mathscr{P}$, then either $(ax)^2 = (by)^2$ due to degeneracy, or g_{1_f} and g_{2_f} are generalized Equality or generalized Disequality respectively. In the later case, since $(a, x) \neq (0, 0)$ and $(b, y) \neq (0, 0)$, it forces that $ax = by = 0$. So we still have $(ax)^2 = (by)^2$. That is, $\{a, b, x, y\}$ belongs to case (i).
- If $\{g_{1_f}, g_{2_f}\} \subseteq \mathscr{A}$, there are two subcases.
  - If both g_{1_f} and g_{2_f} have support of size at most 2, then we have $ax = by = 0$ due to $(a, x) \neq (0, 0)$ and $(b, y) \neq (0, 0)$. This belongs to case (i).
  - Otherwise, both g_{1_f} and g_{2_f} have support of size 4, which means $abxy \neq 0$. Let $x' = \frac{x}{a}, b' = \frac{b}{a}$ and $y' = \frac{y}{a}$. By normalization, we have

$$M(g_{1_f}) = a^2 \begin{bmatrix} 1 & b'y' \\ b'y' & x'^2 \end{bmatrix}.$$

Since $g_{1_f} \in \mathscr{A}$, by Lemma 2.10, x'^2 and $b'y'$ are both powers of $\mathfrak{i}$, and the sum of all exponents is even. It forces that $x'^2 = \mathfrak{i}^{2\alpha}$ for some $\alpha \in \mathbb{N}$. Then, we can choose α such $x' = \mathfrak{i}^{\alpha}$. Also, we have

$$M(g_{2_f}) = a^2 \begin{bmatrix} x' & b'^2 \\ y'^2 & x' \end{bmatrix}.$$

Since $g_{2_f} \in \mathscr{A}$ and x' is already a power of $\mathfrak{i}$, y'^2 and b'^2 are both powers of $\mathfrak{i}$. That is, $b' = \sqrt{\mathfrak{i}}^{\beta}$ and $y' = \sqrt{\mathfrak{i}}^{\gamma}$. Also, since $g_{1_f} \in \mathscr{A}$, $b'y' = \sqrt{\mathfrak{i}}^{\beta+\gamma}$ is a power of $\mathfrak{i}$, which means $\beta \equiv \gamma \pmod 2$. That is, $\{a, b, x, y\}$ belongs to case (ii).
- If $\{g_{1_f}, g_{2_f}\} \subseteq \widehat{\mathscr{M}}$, then by Lemma 2.17, we have $a^2 = x^2$ and $b^2 = y^2$, denoted by case (iii).

Therefore, if $\{a, b, x, y\}$ does not belong to case (i), case (ii) or case (iii), then $\{g_{1_f}, g_{2_f}\} \nsubseteq \mathscr{P}, \mathscr{A}$ or $\widehat{\mathscr{M}}$. By Theorem 2.23, we have Pl-#CSP(g_{1_f}, g_{2_f}) is #P-hard. Recall $g_{1_f} = f_{1\ \mathrm{In}}^{\frac{\pi}{2}}$ and $g_{2_f} = f_{2\ \mathrm{In}}^{\frac{\pi}{2}}$. By Lemma 4.2, Pl-Holant$(\neq_2 |\ f_1^{\frac{\pi}{2}}, f_2^{\frac{\pi}{2}})$ is #P-hard, and hence Pl-Holant$(\neq_2 |\ f)$ is #P-hard.

Otherwise, $\{a, b, x, y\}$ does not belong to case (i) or case (ii), but belongs to case (iii). Then $a^2 = x^2 \neq 0$, $b^2 = y^2 \neq 0$ and $\frac{b}{a}$ is not a power of $\sqrt{\mathfrak{i}}$, that is $(\frac{b}{a})^8 \neq 1$. By reduction (4.4) of lemma 4.6, we have #CSP$(g_{1_f}, g_{2_f}) \leqslant_T$ Pl-Holant$(\neq_2 |\ f)$. Moreover, since $\{a, b, x, y\}$ does not belong to case (i) or case (ii), we have $\{g_{1_f}, g_{2_f}\} \nsubseteq \mathscr{P}$ or $\mathscr{A}$. By Theorem 2.23, #CSP(g_{1_f}, g_{2_f}) is #P-hard. Therefore, we have Pl-Holant$(\neq_2 |\ f)$ is #P-hard. $\square$

5 Case III: Exactly Two Zeros and They Are in Different Pairs or Exactly One Zero and It Is in an Outer Pair

If there are exactly two zeros and they are in different pairs, there must be a zero in an outer pair. By rotational symmetry, we may assume a is zero and we prove this case in Theorem 5.2. We first give the following lemma.

Lemma 5.1. *Let f be a 4-ary signature with the signature matrix $M(f) = \left[\begin{smallmatrix} 0&0&0&0 \\ 0&b&c&0 \\ 0&z&y&0 \\ 0&0&0&0 \end{smallmatrix}\right]$, where $\det M_{\mathrm{In}}(f) = by - cz \neq 0$. Let g be a 4-ary signature with the signature matrix $M(g) = \left[\begin{smallmatrix} 0&0&0&0 \\ 0&0&1&0 \\ 0&1&0&0 \\ 0&0&0&0 \end{smallmatrix}\right]$. Then for any signature set $\mathcal{F}$ containing f, we have*

$$\text{Pl-Holant}(\neq_2 |\ \mathcal{F} \cup \{g\}) \leqslant_T \text{Pl-Holant}(\neq_2 |\ \mathcal{F}).$$

Proof. We construct a series of gadgets f_s by a chain of s copies of f linked by double DISEQUALITY N. f_s has the signature matrix

$$M(f_s) = M(f)(NM(f))^{s-1} = N(NM(f))^s = N \begin{bmatrix} 0 & \mathbf{0} & 0 \\ \mathbf{0} & \begin{bmatrix} z & y \\ b & c \end{bmatrix}^s & \mathbf{0} \\ 0 & \mathbf{0} & 0 \end{bmatrix}.$$

Consider the inner matrix $N_{\text{In}}M_{\text{In}}(f)$ of $NM(f)$. Suppose $N_{\text{In}}M_{\text{In}}(f) = Q^{-1}\Lambda Q$, where $\Lambda = \left[\begin{smallmatrix} \lambda_1 & \mu \\ 0 & \lambda_2 \end{smallmatrix}\right]$ is the Jordan Canonical Form. Note that $\lambda_1\lambda_2 = \det \Lambda = \det(N_{\text{In}}M_{\text{In}}(f)) \neq 0$. We have $M(f_s) = NP^{-1}\Lambda_s P$, where

$$P = \begin{bmatrix} 1 & \mathbf{0} & 0 \\ \mathbf{0} & Q & \mathbf{0} \\ 0 & \mathbf{0} & 1 \end{bmatrix} \quad \text{and} \quad \Lambda_s = \begin{bmatrix} 0 & \mathbf{0} & 0 \\ \mathbf{0} & \begin{bmatrix} \lambda_1 & \mu \\ 0 & \lambda_2 \end{bmatrix}^s & \mathbf{0} \\ 0 & \mathbf{0} & 0 \end{bmatrix}.$$

1. If $\mu = 0$, and $\frac{\lambda_2}{\lambda_1}$ is a root of unity. Suppose $(\frac{\lambda_2}{\lambda_1})^n = 1$. Then $\Lambda_n = \left[\begin{smallmatrix} 0 & 0 & 0 & 0 \\ 0 & \lambda_1^n & 0 & 0 \\ 0 & 0 & \lambda_2^n & 0 \\ 0 & 0 & 0 & 0 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 0 & 0 & 0 & 0 \\ 0 & \lambda_1^n & 0 & 0 \\ 0 & 0 & \lambda_1^n & 0 \\ 0 & 0 & 0 & 0 \end{smallmatrix}\right]$, and $M(f_n) = \left[\begin{smallmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda_1^n & 0 \\ 0 & \lambda_1^n & 0 & 0 \\ 0 & 0 & 0 & 0 \end{smallmatrix}\right] = \lambda_1^n \left[\begin{smallmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{smallmatrix}\right]$. After normalization, we can realize the signature g.
2. If $\mu = 0$, and $\frac{\lambda_2}{\lambda_1}$ is not a root of unity. The matrix $\Lambda_s = \left[\begin{smallmatrix} 0 & 0 & 0 & 0 \\ 0 & \lambda_1^s & 0 & 0 \\ 0 & 0 & \lambda_2^s & 0 \\ 0 & 0 & 0 & 0 \end{smallmatrix}\right]$ has a good form for interpolation. Suppose g appears m times in an instance Ω of Pl-Holant($\neq_2|\ \mathcal{F} \cup \{g\}$). Replace each appearance of g by a copy of gadget f_s to get an instance Ω_s of Pl-Holant($\neq_2|\ \mathcal{F} \cup \{f_s\}$), which is also an instance of Pl-Holant($\neq_2|\ \mathcal{F}$). We can treat each of the m appearances of f_s as a new gadget composed of four functions in sequence N, P^{-1}, Λ_s and P, and denote this new instance by Ω'_s. We divide Ω'_s into two parts. One part consists of m signatures $\Lambda_s^{\otimes m}$. Here $\Lambda_s^{\otimes m}$ is expressed as a column vector. The other part is the rest of Ω'_s and its signature is represented by A which is a tensor expressed as a row vector. Then the Holant value of Ω'_s is the dot product $\langle A, \Lambda_s^{\otimes m} \rangle$, which is a summation over $4m$ bits. That is, the value of the $4m$ edges connecting the two parts. We can stratify all $0,1$ assignments of these $4m$ bits having a nonzero evaluation of a term in Pl-Holant$_{\Omega'_s}$ into the following categories:
   - There are i many copies of Λ_s receiving inputs 0110;
   - There are j many copies of Λ_s receiving inputs 1001;

   where $i + j = m$.
   For any assignment in the category with parameter (i,j), the evaluation of $\Lambda_s^{\otimes m}$ is clearly $\lambda_1^{si}\lambda_2^{sj} = \lambda^{sm}\left(\frac{\lambda_2}{\lambda_1}\right)^{sj}$. Let a_{ij} be the summation of values of the part A over all assignments in the category (i,j). Note that a_{ij} is independent from the value of s since we view the gadget Λ_s as a block. Since $i + j = m$, we can denote a_{ij} by a_j. Then we rewrite the dot product summation and get

$$\text{Pl-Holant}_{\Omega_s} = \text{Pl-Holant}_{\Omega'_s} = \langle A, \Lambda_s^{\otimes m} \rangle = \sum_{0 \leqslant j \leqslant m} \lambda^{sm} a_j (\frac{\lambda_2}{\lambda_1})^{sj}.$$

Note that $M(g) = NP^{-1}(NM(g))P$, where $NM(g) = \begin{bmatrix} 0&0&0&0 \\ 0&1&0&0 \\ 0&0&1&0 \\ 0&0&0&0 \end{bmatrix}$. Similarly, divide Ω into two parts. Under this stratification, we have

$$\text{Pl-Holant}_\Omega = \langle A, (NM(g))^{\otimes m} \rangle = \sum_{0 \leqslant j \leqslant m} a_j.$$

Since $\frac{\lambda_2}{\lambda_1}$ is not a root of unity, the Vandermonde coefficients matrix

$$\begin{bmatrix} \lambda^m \left(\frac{\lambda_2}{\lambda_1}\right)^0 & \lambda^m \left(\frac{\lambda_2}{\lambda_1}\right)^1 & \cdots & \lambda^m \left(\frac{\lambda_2}{\lambda_1}\right)^m \\ \lambda^{2m} \left(\frac{\lambda_2}{\lambda_1}\right)^{2\cdot 0} & \lambda^{2m} \left(\frac{\lambda_2}{\lambda_1}\right)^{2\cdot 1} & \cdots & \lambda^{2m} \left(\frac{\lambda_2}{\lambda_1}\right)^{2\cdot m} \\ \vdots & \vdots & \vdots & \vdots \\ \lambda^{(m+1)m} \left(\frac{\lambda_2}{\lambda_1}\right)^{(m+1)\cdot 0} & \lambda^{(m+1)m} \left(\frac{\lambda_2}{\lambda_1}\right)^{(m+1)\cdot 1} & \cdots & \lambda^{(m+1)m} \left(\frac{\lambda_2}{\lambda_1}\right)^{(m+1)\cdot m} \end{bmatrix}$$

has full rank. Hence, by oracle querying the values of $\text{Pl-Holant}_{\Omega_s}$, we can solve coefficients a_j and obtain the value of Pl-Holant_Ω in polynomial time.

3. If $\mu = 1$, and $\lambda_1 = \lambda_2$ denoted by λ. Then $\Lambda_s = \begin{bmatrix} 0&0&0&0 \\ 0&\lambda^s&s\lambda^{s-1}&0 \\ 0&0&\lambda^s&0 \\ 0&0&0&0 \end{bmatrix}$. We use this form to give a polynomial interpolation. As in the case above, we can stratify the assignments of $\Lambda_s^{\otimes m}$ of these $4m$ bits having a nonzero evaluation of a term in $\text{Pl-Holant}_{\Omega'_s}$ into the following categories:
   - There are i many copies of Λ_s receiving inputs 0110 or 1001;
   - There are j many copies of Λ_s receiving inputs 0101;

   where $i + j = m$.
   For any assignment in the category with parameter (i, j), the evaluation of $\Lambda_s^{\otimes m}$ is clearly $\lambda^{si}(s\lambda^{s-1})^j = \lambda^{sm}(\frac{s}{\lambda})^j$. Let a_{ij} be the summation of values of the part A over all assignments in the category (i, j). a_{ij} is independent from s. Since $i + j = m$, we can denote a_{ij} by a_j. Then, we rewrite the dot product summation and get

$$\text{Pl-Holant}_{\Omega_s} = \text{Pl-Holant}_{\Omega'_s} = \langle A, \Lambda_s^{\otimes m} \rangle = \lambda^{sm} \sum_{0 \leqslant j \leqslant m} a_j \left(\frac{s}{\lambda}\right)^j.$$

   Similarly, divide Ω into two parts. Under this stratification, we have

$$\text{Pl-Holant}_\Omega = \langle A, (NM(g))^{\otimes m} \rangle = a_0.$$

   The Vandermonde coefficients matrix

$$\begin{bmatrix} \lambda^m \left(\frac{1}{\lambda}\right)^0 & \lambda^m \left(\frac{1}{\lambda}\right)^1 & \cdots & \lambda^m \left(\frac{1}{\lambda}\right)^m \\ \lambda^{2m} \left(\frac{2}{\lambda}\right)^0 & \lambda^{2m} \left(\frac{2}{\lambda}\right)^1 & \cdots & \lambda^{2m} \left(\frac{2}{\lambda}\right)^m \\ \vdots & \vdots & \vdots & \vdots \\ \lambda^{(m+1)m} \left(\frac{m+1}{\lambda}\right)^0 & \lambda^{(m+1)m} \left(\frac{m+1}{\lambda}\right)^1 & \cdots & \lambda^{(m+1)m} \left(\frac{m+1}{\lambda}\right)^m \end{bmatrix}$$

   has full rank. Hence, we can solve a_0 in polynomial time and it is the value of Pl-Holant_Ω.

Therefore, we have Pl-Holant($\neq_2|\ \mathcal{F}\cup\{g\}$) $\leqslant_T$ Pl-Holant($\neq_2|\ \mathcal{F}$). $\square$

Theorem 5.2. *Let f be a 4-ary signature with the signature matrix*

$$M(f)=\begin{bmatrix}0&0&0&0\\0&b&c&0\\0&z&y&0\\x&0&0&0\end{bmatrix},$$

where $x\neq 0$ and there is at most one number in $\{b,c,y,z\}$ that is 0. Then Pl-Holant($\neq_2|\ f$) *is #P-hard unless $f\in\mathscr{M}$, in which case the problem is tractable.*

Proof. Tractability follows by Theorem 2.15.

Suppose $f\notin\mathscr{M}$. By Lemma 2.14, $\det M_{\text{In}}(f)\neq\det M_{\text{Out}}(f)$, that is $\det\left[\begin{smallmatrix}b&c\\z&y\end{smallmatrix}\right]=by-cz\neq 0$. Note that $M_{x_1x_2,x_4x_3}(f)=\left[\begin{smallmatrix}0&0&0&0\\0&b&c&0\\0&z&y&0\\x&0&0&0\end{smallmatrix}\right]$, $M_{x_3x_4,x_2x_1}(f)=\left[\begin{smallmatrix}0&0&0&x\\0&y&c&0\\0&z&b&0\\0&0&0&0\end{smallmatrix}\right]$, and $M_{x_2x_3,x_1x_4}(f)=\left[\begin{smallmatrix}0&0&0&y\\0&0&z&0\\0&c&x&0\\b&0&0&0\end{smallmatrix}\right]$. Connect variables x_4,x_3 of a copy of signature f with variables x_3,x_4 of another copy of signature f both using ($\neq_2$). We get a signature f_1 with the signature matrix

$$M(f_1)=M_{x_1x_2,x_4x_3}(f)NM_{x_3x_4,x_2x_1}(f)=\begin{bmatrix}0&0&0&0\\0&b_1&c_1&0\\0&z_1&y_1&0\\0&0&0&0\end{bmatrix},$$

where $\left[\begin{smallmatrix}b_1&c_1\\z_1&y_1\end{smallmatrix}\right]=\left[\begin{smallmatrix}b&c\\z&y\end{smallmatrix}\right]\cdot\left[\begin{smallmatrix}z&b\\y&c\end{smallmatrix}\right]$. Here, $\det\left[\begin{smallmatrix}b_1&c_1\\z_1&y_1\end{smallmatrix}\right]=-(by-cz)^2\neq 0$. By Lemma 5.1, we have

$$\text{Pl-Holant}(\neq_2|\ f,g)\leqslant_T\text{Pl-Holant}(\neq_2|\ f,f_1),$$

where g has the signature matrix $M(g)=\left[\begin{smallmatrix}0&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&0\end{smallmatrix}\right]$.

- If $bcyz\neq 0$, connect variables x_1,x_4 of signature f with variables x_1,x_2 of signature g both using ($\neq_2$). We get a signature f_2 with the signature matrix

$$M(f_2)=M_{x_2x_3,x_1x_4}(f)NM_{x_1x_2,x_4x_3}(g)=\begin{bmatrix}0&0&0&0\\0&0&z&0\\0&c&x&0\\0&0&0&0\end{bmatrix}.$$

- Otherwise, connect variables x_4,x_3 of signature f with variables x_1,x_2 of signature g both using ($\neq_2$). We get a signature f_2 with the signature matrix

$$M(f_2)=M_{x_1x_2,x_4x_3}(f)NM_{x_1x_2,x_4x_3}(g)=\begin{bmatrix}0&0&0&0\\0&b&c&0\\0&z&y&0\\0&0&0&0\end{bmatrix},$$

  and there is exactly one in $\{b,c,y,z\}$ that is zero.

In both cases, the support of f_2 has size 3, which means $f_2\notin\mathscr{P},\mathscr{A}$ or $\widehat{\mathscr{M}}$. By Theorem 4.3, Pl-Holant($\neq_2|\ f_2$) is #P-hard. Since

$$\text{Pl-Holant}(\neq_2|\ f_2)\leqslant_T\text{Pl-Holant}(\neq_2|\ f,g)\leqslant_T\text{Pl-Holant}(\neq_2|\ f,f_1)\leqslant_T\text{Pl-Holant}(\neq_2|\ f),$$

we have Pl-Holant($\neq_2|\ f$) is #P-hard. $\square$

6 Case IV: Exactly One Zero and It Is in the Inner Pair or All Values Are Nonzero

By rotational symmetry, if there is one zero in the inner pair, we may assume $c = 0$. We first consider the case that $a = \epsilon x, b = \epsilon y$ and $c = \epsilon z$, where $\epsilon = \pm 1$.

Lemma 6.1. *Let f be a 4-ary signature with the signature matrix*

$$M(f) = \begin{bmatrix} 0 & 0 & 0 & a \\ 0 & b & c & 0 \\ 0 & \epsilon c & \epsilon b & 0 \\ \epsilon a & 0 & 0 & 0 \end{bmatrix}, \ \epsilon = \pm 1 \textit{ and } abc \neq 0.$$

Then Pl-Holant$(\neq_2 | \, f)$ *is #P-hard if $f \notin \mathscr{M}$.*

Proof. If $\epsilon = -1$. Connect the variable x_4 with x_3 of f using $(\neq_2)$, and we get a binary signature g_1, where

$$g_1 = M_{x_1x_2,x_4x_3}(f)(0,1,1,0)^T = (0, b+c, -(b+c), 0)^T.$$

Also connect the variable x_1 with x_2 of f using $(\neq_2)$, and we get a binary signature g_2, where

$$g_2 = ((0,1,1,0)M_{x_1x_2,x_4x_3}(f))^T = (0, b-c, -(b-c), 0)^T.$$

Since $bc \neq 0$, $b+c$ and $b-c$ can not be both zero. Without loss of generality, assume $b+c \neq 0$. By normalization, we have $g_1 = (0,1,-1,0)^T$. Then, connect the variable x_2 of g_1 with the variable x_1 of f using $(\neq_2)$, and we get a signature with the signature matrix $\left[\begin{smallmatrix} 0 & 0 & 0 & a \\ 0 & b & c & 0 \\ 0 & c & b & 0 \\ a & 0 & 0 & 0 \end{smallmatrix}\right]$. Therefore, it suffices to show #P-hardness for the case that $\epsilon = 1$.

Since $f \notin \mathscr{M}$, by Lemma 2.14, $c^2 - b^2 \neq a^2$. We prove #P-hardness in three cases depending on the values of a, b and c.

Case 1: If $c^2 - b^2 \neq 0$ and $|c+b| \neq |c-b|$, or $c^2 - a^2 \neq 0$ and $|c+a| \neq |c-a|$. By rotational symmetry, we may assume $c^2 - b^2 \neq 0$ and $|c+b| \neq |c-b|$. Normalizing f by assuming $a = 1$, we have $M(f) = \left[\begin{smallmatrix} 0 & 0 & 0 & 1 \\ 0 & b & c & 0 \\ 0 & c & b & 0 \\ 1 & 0 & 0 & 0 \end{smallmatrix}\right]$, where $c^2 - b^2 \neq 0$ or 1.

We construct a series of gadgets f_s by a chain of s copies of f linked by double DISEQUALITY N. f_s has the signature matrix

$$M(f_s) = M(f)(NM(f))^{s-1} = N(NM(f))^s = N \begin{bmatrix} 1 & \mathbf{0} & 0 \\ \mathbf{0} & \begin{bmatrix} c & b \\ b & c \end{bmatrix}^s & \mathbf{0} \\ 0 & \mathbf{0} & 1 \end{bmatrix}.$$

We diagonalize $\left[\begin{smallmatrix} c & b \\ b & c \end{smallmatrix}\right]^s$ using $H = \frac{1}{\sqrt{2}}\left[\begin{smallmatrix} 1 & 1 \\ 1 & -1 \end{smallmatrix}\right]$ (note that $H^{-1} = H$), and get $M(f_s) = NP\Lambda_s P$, where

$$P = \begin{bmatrix} 1 & \mathbf{0} & 0 \\ \mathbf{0} & H & \mathbf{0} \\ 0 & \mathbf{0} & 1 \end{bmatrix}, \quad \text{and} \quad \Lambda_s = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & (c+b)^s & 0 & 0 \\ 0 & 0 & (c-b)^s & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The the signature matrix Λ_s has a good form for polynomial interpolation. Suppose we have a problem Pl-Holant($\neq_2|\ \hat{f}$), where $M(\hat{f}) = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & \hat{b} & \hat{c} & 0 \\ 0 & \hat{c} & \hat{b} & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$ to be reduced to Pl-Holant($\neq_2|\ f$). Suppose $\hat{f}$ appears m times in an instance $\hat{\Omega}$ of Pl-Holant($\neq_2|\ \hat{f}$). We replace each appearance of $\hat{f}$ by a copy of gadget f_s to get an instance Ω_s of Pl-Holant($\neq_2|\ f, f_s$), which is also an instance of Pl-Holant($\neq_2|\ f$). We can treat each of the m appearances of f_s as a new gadget composed of four functions in sequence N, P, Λ_s and P, and denote this new instance by Ω'_s. We divide Ω'_s into two parts. One part consists of m signatures $\Lambda_s^{\otimes m}$. Here, $\Lambda_s^{\otimes m}$ is expressed as a column vector. The other part is the rest of Ω'_s and its signature is represented by A which is a tensor expressed as a row vector. Then the Holant value of Ω'_s is the dot product $\langle A, \Lambda_s^{\otimes m}\rangle$, which is a summation over $4m$ bits. That is, the value of the $4m$ edges connecting the two parts. We can stratify all $0, 1$ assignments of these $4m$ bits having a nonzero evaluation of a term in Pl-Holant$_{\Omega'_s}$ into the following categories:

- There are i many copies of Λ_s receiving inputs 0000 or 1111;
- There are j many copies of Λ_s receiving inputs 0110;
- There are k many copies of Λ_s receiving inputs 1001;

where $i + j + k = m$.

For any assignment in the category with parameter (i, j, k), the evaluation of $\Lambda_s^{\otimes m}$ is clearly $(c+b)^{sj}(c-b)^{sk}$. Let a_{ijk} be the summation of values of the part A over all assignments in the category (i, j, k). Note that a_{ijk} is independent on the value of s. Since $i + j + k = m$, we can denote a_{ijk} by a_{jk}. Then we rewrite the dot product summation and get

$$\text{Pl-Holant}_{\Omega_s} = \text{Pl-Holant}_{\Omega'_s} = \langle A, \Lambda_s^{\otimes m}\rangle = \sum_{0 \leqslant j+k \leqslant m} a_{jk}(c+b)^{sj}(c-b)^{sk}.$$

Under this stratification, correspondingly we can define $\hat{\Omega}'$ and $\hat{\Lambda}$. Then we have

$$\text{Pl-Holant}_{\hat{\Omega}} = \text{Pl-Holant}_{\hat{\Omega}'} = \langle A, \hat{\Lambda}^{\otimes m}\rangle = \sum_{0 \leqslant j+k \leqslant m} a_{jk}(\hat{c}+\hat{b})^{j}(\hat{c}-\hat{b})^{k}.$$

Let $\phi = \hat{c} + \hat{b}$ and $\psi = \hat{c} - \hat{b}$. If we can obtain the value of $p(\phi, \psi) = \sum_{0 \leqslant j+k \leqslant m} a_{jk}\phi^j\psi^k$ in polynomial time, then we will have

$$\text{Pl-Holant}(\neq_2|\ \hat{f}) \leqslant_T \text{Pl-Holant}(\neq_2|\ f).$$

Let $\alpha = c + b$ and $\beta = c - b$. Since $c^2 - b^2 \neq 0$ or 1, we have $\alpha \neq 0$, $\beta \neq 0$ and $\alpha\beta \neq 1$. Also, by assumption $|c + b| \neq |c - b|$, we have $|\alpha| \neq |\beta|$. Define $L = \{(j, k) \in \mathbb{Z}^2 \mid \alpha^j\beta^k = 1\}$. This is a sublattice of $\mathbb{Z}^2$. Every lattice has a basis. There are 3 cases depending on the rank of L.

- $L = \{(0, 0)\}$. All $\alpha^j\beta^k$ are distinct. It is an interpolation reduction in full power. That is, we can interpolate $p(\phi, \psi)$ for any ϕ and ψ in polynomial time. Let $\phi = 4$ and $\psi = 0$, that is $\hat{b} = 2$ and $\hat{c} = 2$, and hence $M(\hat{f}) = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 2 & 2 & 0 \\ 0 & 2 & 2 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$. That is, $\hat{f}$ is non-singular redundant. By Theorem 2.21, Pl-Holant($\neq_2|\ \hat{f}$) is #P-hard, and hence Pl-Holant($\neq_2|\ f$) is #P-hard.
- L contains two independent vectors (j_1, k_1) and (j_2, k_2) over $\mathbb{Q}$. Then the nonzero vectors $j_2(j_1, k_1) - j_1(j_2, k_2) = (0, j_2k_1 - j_1k_2)$ and $k_2(j_1, k_1) - k_1(j_2, k_2) = (k_2j_1 - k_1j_2, 0)$ are in L. Hence, both α and β are roots of unity. That is $|\alpha| = |\beta| = 1$. Contradiction.

- $L = \{(ns, nt) \mid n \in \mathbb{Z}\}$, where $s, t \in \mathbb{Z}$ and $(s, t) \neq (0, 0)$. Without loss of generality, we may assume $t \geqslant 0$, and $s > 0$ when $t = 0$. Also, we have $s + t \neq 0$, otherwise $|\alpha| = |\beta|$. Contradiction. By Lemma 2.7, for any numbers ϕ and ψ satisfying $\phi^s\psi^t = 1$, we can obtain $p(\phi, \psi)$ in polynomial time. Since $\phi = \hat{c} + \hat{b}$ and $\psi = \hat{c} - \hat{b}$, we have $\hat{b} = \frac{\phi-\psi}{2}$ and $\hat{c} = \frac{\phi+\psi}{2}$. That is $M(\hat{f}) = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & \frac{\phi-\psi}{2} & \frac{\phi+\psi}{2} & 0 \\ 0 & \frac{\phi+\psi}{2} & \frac{\phi-\psi}{2} & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$. There are three cases depending on the values of s and t.
  - If $s \geqslant 0$ and $s + t \geqslant 2$. Consider the function $q(x) = (2-x)^s x^t - 1$. Since $s \geqslant 0$ and $t \geqslant 0$, it is a polynomial. Clearly, 1 is one of its roots and 0 is not its root. If $q(x)$ has no other roots, then for some constant $\lambda \neq 0$,
    $$q(x) = (2-x)^s x^t - 1 = \lambda(x-1)^{s+t} = (-1)^{s+t}\lambda((2-x)-1)^{s+t}.$$
    Notice that $x^t | q(x) + 1$, while $x^t \nmid \lambda(x-1)^{s+t} + 1$ for $t \geqslant 2$. Also, notice that $(2-x)^s | q(x) + 1$, while $(2-x)^s \nmid (-1)^{s+t}\lambda((2-x)-1)^{s+t}$ for $s \geqslant 2$. Hence, $t = s = 1$, which means $\alpha\beta = 1$. Contradiction.
    Therefore, $q(x)$ has a root x_0, where $x_0 \neq 1$ or 0. Let $\psi = x_0$ and $\phi = 2 - x_0$. Then $\phi^s\psi^t = 1$ and $M(\hat{f}) = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1-x_0 & 1 & 0 \\ 0 & 1 & 1-x_0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$. Note that $M_{x_2x_3,x_1x_4}(\hat{f}) = \begin{bmatrix} 0 & 0 & 0 & 1-x_0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1-x_0 & 0 & 0 & 0 \end{bmatrix}$. Since $1 - x_0 \neq 0$, $\hat{f}$ is non-singular redudant. By Theorem 2.21, Pl-Holant$(\neq_2| \hat{f})$ is #P-hard and hence Pl-Holant $(\neq_2| f)$ is #P-hard.
  - If $s < 0$ and $t > 0$. Consider the function $q(x) = x^t - (2-x)^{-s}$. Since $t > 0$ and $-s > 0$, it is a polynomial. Clearly, 1 is one of its roots and 0 is not its root. Since $t + s \neq 0$, the highest order term of $q(x)$ is either x^t or $-(-x)^{-s}$, which means the coefficient of the highest order term is ± 1. While the constant term of $q(x)$ is $-2^{-s} \neq \pm 1$. Hence, $q(x)$ can not be of the form $\lambda(x-1)^{\max(t,-s)}$ for some constant $\lambda \neq 0$. Moreover, since $t + s \neq 0$, $\max(t, -s) \geqslant 2$, which means $q(x)$ has a root x_0, where $x_0 \neq 1$ or 0. Similarly, let $\psi = x_0$ and $\phi = 2 - x_0$, and we have Pl-Holant $(\neq_2| f)$ is #P-hard.
  - If $s \geqslant 0$ and $s + t = 1$. In this case, we have $s = 0, t = 1$ or $s = 1, t = 0$ due to $t \geqslant 0$.
    * $s = 1, t = 0$. Let $\phi = 1$ and $\psi = \frac{1}{2}$. Then we have $\phi^1\psi^0 = 1$ and $M(\hat{f}) = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & \frac{1}{4} & \frac{3}{4} & 0 \\ 0 & \frac{3}{4} & \frac{1}{4} & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$. Let $M(f') = 4M_{x_2x_3,x_1x_4}(\hat{f}) = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 4 & 3 & 0 \\ 0 & 3 & 4 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$. Clearly, Pl-Holant$(\neq_2| f') \leqslant_T$ Pl-Holant $(\neq_2| f)$. For $M(f')$, correspondingly we define $\alpha' = 3 + 4 = 7$ and $\beta' = 3 - 4 = -1$. Obviously, $\alpha' \neq 0$, $\beta' \neq 0$, $\alpha'\beta' \neq 1$, and $|\alpha'| \neq |\beta'|$. Let $L' = \{(j, k) \in \mathbb{Z}^2 \mid \alpha'^j\beta'^k = 1\}$. Then we have $L' = \{(ns', nt') \mid n \in \mathbb{Z}\}$, where $s' = 0$ and $t' = 2$. Therefore, $s' \geqslant 0$ and $s' + t' \geqslant 2$. As we have showed above, we have Pl-Holant $(\neq_2| f')$ is #P-hard, and hence Pl-Holant $(\neq_2| f)$ is #P-hard.
    * $s = 0, t = 1$. Let $\phi = 3$ and $\psi = 1$. Then we have $\phi^0\psi^1 = 1$ and $M(\hat{f}) = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 2 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$.
      By Theorem 2.22, Pl-Holant$(\neq_2| \hat{f})$ is #P-hard, and hence Pl-Holant $(\neq_2| f)$ is #P-hard.

Case 2: If $c^2 - b^2 \neq 0$ and $|c + b| = |c - b|$, or $c^2 - a^2 \neq 0$ and $|c + a| = |c - a|$. By rotational symmetry, we may assume $c^2 - b^2 \neq 0$ and $|c + b| = |c - b|$. Normalizing f by assuming $c = 1$, we have $M(f) = \begin{bmatrix} 0 & 0 & 0 & a \\ 0 & b & 1 & 0 \\ 0 & 1 & b & 0 \\ a & 0 & 0 & 0 \end{bmatrix}$, where $1^2 - b^2 \neq 0$ and $1^2 - b^2 \neq a^2$ due to $f \notin \mathscr{M}$. Since $|1 + b| = |1 - b|$, b is a pure imaginary number.

Connect variables x_4, x_3 of a copy of signature f with variables x_1, x_2 of another copy of signature f both using $(\neq_2)$. We get a signature f_1 with the signature matrix

$$M(f_1) = M_{x_1x_2,x_4x_3}(f)NM_{x_1x_2,x_4x_3}(f) = \begin{bmatrix} 0 & 0 & 0 & a^2 \\ 0 & 2b & b^2+1 & 0 \\ 0 & b^2+1 & 2b & 0 \\ a^2 & 0 & 0 & 0 \end{bmatrix}.$$

a. If $c^2 - a^2 = 0$, that is $a^2 = 1$, and then $M(f_1) = \left[\begin{smallmatrix} 0 & 0 & 0 & 1 \\ 0 & 2b & b^2+1 & 0 \\ 0 & b^2+1 & 2b & 0 \\ 1 & 0 & 0 & 0 \end{smallmatrix}\right]$. Since $b^2 < 0$, we have $(b^2+1)^2 - (2b)^2 = (b^2-1)^2 > 1 = (a^2)^2$, which means $f_1 \notin \mathscr{M}$.

- If $b^2 = -1$, then $M(f_1) = \left[\begin{smallmatrix} 0 & 0 & 0 & 1 \\ 0 & \pm 2i & 0 & 0 \\ 0 & 0 & \pm 2i & 0 \\ 1 & 0 & 0 & 0 \end{smallmatrix}\right]$. By Theorem 4.6, Pl-Holant$(\neq_2|\, f_1)$ is #P-hard, and hence Pl-Holant$(\neq_2|\, f)$ is #P-hard.
- If $b^2 = -2$, then $M(f_1) = \left[\begin{smallmatrix} 0 & 0 & 0 & 1 \\ 0 & \pm 2\sqrt{2}i & -1 & 0 \\ 0 & -1 & \pm 2\sqrt{2}i & 0 \\ 1 & 0 & 0 & 0 \end{smallmatrix}\right]$. Connect two copies of f_1, and we have a signature f_2 with the signature matrix

$$M(f_2) = M_{x_1x_2,x_4x_3}(f_1)NM_{x_1x_2,x_4x_3}(f_1) = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & \mp 4\sqrt{2}i & -7 & 0 \\ 0 & -7 & \mp 4\sqrt{2}i & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

  It is easy to check $f_2 \notin \mathscr{M}$. Then, f_2 belongs to Case 1. Therefore, Pl-Holant$(\neq_2|\, f_2)$ is #P-hard, and hence Pl-Holant$(\neq_2|\, f)$ is #P-hard.
- If $b^2 \neq -1$ or -2, then $b^2+1 \neq -1$, and hence $1^2 - (b^2+1)^2 \neq 0$ due to $b \neq 0$. Also, since b^2+1 is a real number and $b^2+1 \neq 0$, we have $|(b^2+1)+1| \neq |(b^2+1)-1|$. Then, $f_1 \notin \mathscr{M}$ has the signature matrix of form $\left[\begin{smallmatrix} 0 & 0 & 0 & a_1 \\ 0 & b_1 & c_1 & 0 \\ 0 & c_1 & b_1 & 0 \\ a_1 & 0 & 0 & 0 \end{smallmatrix}\right]$, where $a_1b_1c_1 \neq 0$, $c_1^2 - a_1^2 \neq 0$ and $|c_1 + a_1| \neq |c_1 - a_1|$. That is, f_1 belongs to Case 1. Therefore, Pl-Holant$(\neq_2|\, f_1)$ is #P-hard, and hence Pl-Holant$(\neq_2|\, f)$ is #P-hard.

b. If $c^2 - a^2 \neq 0$ and $|c+a| = |c-a|$, then a is also a pure imaginary number. Connect variables x_1, x_4 of a copy of signature f with variables x_2, x_3 of another copy of signature f. We get a signature f_3 with the signature matrix

$$M(f_3) = M_{x_2x_3,x_1x_4}(f)NM_{x_2x_3,x_1x_4}(f) = \begin{bmatrix} 0 & 0 & 0 & b^2 \\ 0 & 2a & a^2+1 & 0 \\ 0 & a^2+1 & 2a & 0 \\ b^2 & 0 & 0 & 0 \end{bmatrix}.$$

Note that $f_3 \in \mathscr{M}$ implies $(a^2-1)^2 = (b^2)^2$. Since $f \notin \mathscr{M}$, $1 - a^2 \neq b^2$. Hence, $f_3 \in \mathscr{M}$ implies $a^2 - 1 = b^2$. Similarly, $f_1 \in \mathscr{M}$ implies $b^2 - 1 = a^2$. Clearly, f_1 and f_3 can not be both in $\mathscr{M}$. Without loss of generality, we may assume $f_3 \notin \mathscr{M}$.

- If $a^2 \neq -1$. There are two subcases.
  - $(a^2+1)^2 - (b^2)^2 = 0$. Since a is a pure imagianry number, $|a^2+1+2a| = |a+1|^2 = |a-1|^2 = |a^2+1-2a|$. Then f_3 has the signature matrix of form $\left[\begin{smallmatrix} 0 & 0 & 0 & a_3 \\ 0 & b_3 & c_3 & 0 \\ 0 & c_3 & b_3 & 0 \\ a_3 & 0 & 0 & 0 \end{smallmatrix}\right]$,

where $a_3b_3c_3 \neq 0$, $c_3^2-b_3^2 \neq 0$, $|c_3+b_3| = |c_3-b_3|$ and $c_3^2-a_3^2 = 0$. That is, f_3 belongs to Case 2.a. Therefore, Pl-Holant($\neq_2|\ f_3$) is #P-hard, and hence Pl-Holant($\neq_2|\ f$) is #P-hard.

  – $(a^2+1)^2-(b^2)^2 \neq 0$. Since a^2+1 and b^2 are both nonzero real numbers due to a and b are both pure imaginary numbers, we have $|a^2+1+b^2| \neq |a^2+1-b^2|$. Then f_3 has the signature matrix of form $\begin{bmatrix} 0 & 0 & 0 & a_3 \\ 0 & b_3 & c_3 & 0 \\ 0 & c_3 & b_3 & 0 \\ a_3 & 0 & 0 & 0 \end{bmatrix}$, where $a_3b_3c_3 \neq 0$, $c_3^2-a_3^2 \neq 0$ and $|c_3+a_3| \neq |c_3-a_3|$. That is, f_3 belongs to Case 1. Therefore, Pl-Holant($\neq_2|\ f_3$) is #P-hard, and hence Pl-Holant($\neq_2|\ f$) is #P-hard.

- If $a^2=-1$ and $b^2 \neq -2$, then $M(f_3) = \begin{bmatrix} 0 & 0 & 0 & b^2 \\ 0 & 2a & 0 & 0 \\ 0 & 0 & 2a & 0 \\ b^2 & 0 & 0 & 0 \end{bmatrix}$, where $|2a| = 2 \neq |b^2|$. By Theorem 4.6, Pl-Holant($\neq_2|\ f_3$) is #P-hard, and hence Pl-Holant($\neq_2|\ f$) is #P-hard.
- If $a^2=-1$ and $b^2=-2$, then $M(f_1) = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & \pm 2\sqrt{2}i & -1 & 0 \\ 0 & -1 & \pm 2\sqrt{2}i & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}$. Note that $M_{x_2x_3,x_1x_4}(f_1) = \begin{bmatrix} 0 & 0 & 0 & \pm 2\sqrt{2}i \\ 0 & -1 & -1 & 0 \\ 0 & -1 & -1 & 0 \\ \pm 2\sqrt{2}i & 0 & 0 & 0 \end{bmatrix}$. We have f_1 is non-singular redudant. Therefore, Pl-Holant($\neq_2|\ f_1$) is #P-hard, and hence Pl-Holant($\neq_2|\ f$) is #P-hard.

c. If $c^2-a^2 \neq 0$ and $|c+a| \neq |c-a|$. This is Case 1. Done.

Case 3: $c^2-b^2=0$ and $c^2-a^2=0$. If $c=b$ or $c=a$, then f is non-singular redudant, and hence Pl-Holant($\neq_2|\ f$) is #P-hard. Otherwise, $a=b=-c$. By normalization, we have $M(f) = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}$, and then $M(f_1) = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & -2 & 2 & 0 \\ 0 & 2 & -2 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$. Notice that $2^2-1^2 \neq 0$ and $|2+1| \neq |2-1|$. That is, f_1 belongs to Case 1. Therefore, Pl-Holant($\neq_2|\ f_1$) is #P-hard, and hence Pl-Holant($\neq_2|\ f$) is #P-hard. ☐

Lemma 6.2. *Let f be a 4-ary signature with the signature matrix*

$$M(f) = \begin{bmatrix} 0 & 0 & 0 & a \\ 0 & b & c & 0 \\ 0 & z & y & 0 \\ x & 0 & 0 & 0 \end{bmatrix}, \ abcxyz \neq 0.$$

If $by-cz=0$ or $ax-cz=0$, then Pl-Holant($\neq_2|\ f$) *is #P-hard.*

Proof. By rotational symmetry, we assume $by-cz=0$. By normalization, we assume $b=1$, and then $y=cz$. That is, $M_{x_1x_2,x_4x_3}(f) = \begin{bmatrix} 0 & 0 & 0 & a \\ 0 & 1 & c & 0 \\ 0 & z & cz & 0 \\ x & 0 & 0 & 0 \end{bmatrix}$.

- If $1+c \neq 0$. Connect the variables x_4 with x_3 of f using ($\neq_2$), and we get a binary signature g_1, where
$$g_1 = M_{x_1x_2,x_4x_3}(f)(0,1,1,0)^T = (0,1+c,(1+c)z,0)^T.$$
Note that $g_1(x_1,x_2)$ can be normalized as $(0,z^{-1},1,0)^T$. That is $g(x_2,x_1) = (0,1,z^{-1},0)^T$. Connect the variable x_1 of g_1 with the variable x_1 of f, and we get a signature f_1 with the signature matrix $\begin{bmatrix} 0 & 0 & 0 & a \\ 0 & 1 & c & 0 \\ 0 & 1 & c & 0 \\ xz^{-1} & 0 & 0 & 0 \end{bmatrix}$. Connect the variable x_1 with x_2 of f_1 using ($\neq_2$), and we get a binary signature g_2, where
$$g_2 = ((0,1,1,0)M_{x_1x_2,x_4x_3}(f))^T = (0,2,2c,0)^T.$$

Note that $g_2(x_1, x_2)$ can be normalized as $(0, c^{-1}, 1, 0)^T$, That is $g_2(x_2, x_1) = (0, 1, c^{-1}, 0)^T$. Connect the variable x_1 of g_2 with the variable x_3 of f_1, and we get a signature f_2 with the signature matrix $\left[\begin{smallmatrix} 0 & 0 & 0 & ac^{-1} \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ xz^{-1} & 0 & 0 & 0 \end{smallmatrix}\right]$. It is non-singular redudant. By Lemma 2.21, we have Pl-Holant($\neq_2 | f_2$) is #P-hard, and hence Pl-Holant($\neq_2 | f$) is #P-hard.

- If $1 + z \neq 0$, then connect the variable x_1 with x_2 of f using $(\neq_2)$, and we get a binary signature g'_1, where

$$g'_1 = ((0, 1, 1, 0)M_{x_1x_2,x_4x_3})^T = (0, 1 + z, (1 + z)c, 0)^T.$$

Note that $g'_1(x_1, x_2)$ can be normalized as $(0, c^{-1}, 1, 0)^T$. Same as the analysis of the case $1 + c \neq 0$, we still have Pl-Holant($\neq_2 | f$) is #P-hard.

- Otherwise, $1 + c = 1$ and $1 + z = 1$, that is $c = z = -1$. Then $M_{x_1x_2,x_4x_3}(f) = \left[\begin{smallmatrix} 0 & 0 & 0 & a \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ x & 0 & 0 & 0 \end{smallmatrix}\right]$, and $M_{x_3x_4,x_2x_1}(f) = \left[\begin{smallmatrix} 0 & 0 & 0 & x \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ a & 0 & 0 & 0 \end{smallmatrix}\right]$. Connect variables x_4, x_3 of a copy of signature f with variables x_3, x_4 of another copy of signature f, and we get a signature f_3 with the signature matrix

$$M(f_3) = M_{x_1x_2,x_4x_3}(f)NM_{x_3x_4,x_2x_1}(f) = \begin{bmatrix} 0 & 0 & 0 & ax \\ 0 & -2 & 2 & 0 \\ 0 & 2 & -2 & 0 \\ ax & 0 & 0 & 0 \end{bmatrix},$$

Clearly, $ax \neq 0$ and $f_3 \notin \mathscr{M}$. By Lemma 6.1, Pl-Holant $(\neq_2 | f_3)$ is #P-hard and hence Pl-Holant $(\neq_2 | f)$ is #P-hard. □

In the following Lemmas 6.3, 6.4, 6.6 and Corollaries 6.5, 6.7, let f be a 4-ary signature with the signature matrix

$$M(f) = \begin{bmatrix} 0 & 0 & 0 & a \\ 0 & b & c & 0 \\ 0 & z & y & 0 \\ x & 0 & 0 & 0 \end{bmatrix},$$

where $abxyz \neq 0$, $\det \left[\begin{smallmatrix} b & c \\ z & y \end{smallmatrix}\right] = by - cz \neq 0$ and $\det \left[\begin{smallmatrix} a & z \\ c & x \end{smallmatrix}\right] = ax - cz \neq 0$. Moreover $f \notin \mathscr{M}$, that is $cz - by \neq ax$.

Lemma 6.3. *Let $g = (0, 1, t, 0)^T$ be a binary signature, where $t \neq 0$ is not a root of unity. Then* Pl-Holant($\neq_2 | f, g$) *is #P-hard.*

Proof. Let $\mathcal{B} = \{g_1, g_2, g_3\}$ be a set of three binary signatures $g_i = (0, 1, t_i, 0)^T$. By Lemma 2.5, we have Pl-Holant $(\neq_2 | \{f\} \cup \mathcal{B}) \leqslant$ Pl-Holant $(\neq_2 | f, g)$. We will show Pl-Holant $(\neq_2 | \{f\} \cup \mathcal{B})$ is #P-hard and it follows Pl-Holant $(\neq_2 | f, g)$ is #P-hard.

Connect the variable x_2 of g_i $(i = 1, 2)$ with the variable x_1 of f using $(\neq_2)$ separately. We get two signatures f_{t_i} with the signature matrix $M(f_{t_i}) = \left[\begin{smallmatrix} 0 & 0 & 0 & a \\ 0 & b & c & 0 \\ 0 & t_iz & t_iy & 0 \\ t_ix & 0 & 0 & 0 \end{smallmatrix}\right]$. Note that $\det M_{\text{In}}(f_{t_i}) = t_i \det M_{\text{In}}(f)$ and $\det M_{\text{Out}}(f_{t_i}) = t_i \det M_{\text{Out}}(f)$. Connect variables x_4, x_3 of f with variables x_1,

x_2 of f_{t_1} both using $(\neq_2)$. We get a signature f_1 with the signature matrix

$$M(f_1) = \begin{bmatrix} 0 & 0 & 0 & a_1 \\ 0 & b_1 & c_1 & 0 \\ 0 & z_1 & y_1 & 0 \\ x_1 & 0 & 0 & 0 \end{bmatrix} = M(f)NM(f_{t_1}) = \begin{bmatrix} 0 & 0 & 0 & a^2 \\ 0 & t_1bz+bc & t_1by+c^2 & 0 \\ 0 & t_1z^2+yb & t_1yz+yc & 0 \\ t_1x^2 & 0 & 0 & 0 \end{bmatrix}.$$

We first show that there is a $t_1 \neq 0$ such that $b_1y_1c_1z_1 \neq 0$ and $(b_1z)(y_1c)-(c_1b)(z_1y) \neq 0$. Consider the quadratic function $p(t) = (tbz+bc)(tyz+yc)cz - (tby+c^2)(tz^2+yb)by$. Then, we have $p(t_1) = (b_1z)(y_1c)-(c_1b)(z_1y)$. Notice that the coefficient of the quadratic term in $p(t)$ is $byz^2(cz-by)$. It is not equal to zero since $byz^2 \neq 0$ and $cz-by \neq 0$. That is, $p(t)$ has degree 2, and hence it has at most two roots. Also, $b_1y_1 = 0$ implies $t_1 = -\frac{c}{z}$, $c_1 = 0$ implies $t_1 = -\frac{c^2}{by}$, and $z_1 = 0$ implies $t_1 = -\frac{yb}{z^2}$. Therefore we can choose such a t_1 that does not take these values $0, -\frac{c}{z}, -\frac{c^2}{by}$ and $-\frac{yb}{z^2}$, and t_1 is not a root of $p(t)$. That is, $t_1 \neq 0$, $b_1y_1c_1z_1 \neq 0$ and $(b_1z)(y_1c)-(c_1b)(z_1y) \neq 0$.

Connect variables x_4, x_3 of f_1 with variables x_1, x_2 of f_{t_2} both using $(\neq_2)$. We get a signature f_2 with the signature matrix

$$M(f_2) = \begin{bmatrix} 0 & 0 & 0 & a_2 \\ 0 & b_2 & c_2 & 0 \\ 0 & z_2 & y_2 & 0 \\ x_2 & 0 & 0 & 0 \end{bmatrix} = M(f_1)NM(f_{t_2}) = \begin{bmatrix} 0 & 0 & 0 & a_1a \\ 0 & t_2b_1z+c_1b & t_2b_1y+c_1c & 0 \\ 0 & t_2z_1z+y_1b & t_2z_1y+y_1c & 0 \\ t_2x_1x & 0 & 0 & 0 \end{bmatrix}$$

Since $b_1z \neq 0$ and $c_1b \neq 0$, we can let $t_2 = -\frac{c_1b}{b_1z}$ and $t_2 \neq 0$. Then $b_2 = t_2b_1z + c_1b = 0$. Since $(b_1z)(y_1c)-(c_1b)(z_1y) \neq 0$, we have $y_2 = t_2z_1y + y_1c \neq 0$. Notice that

$$\begin{aligned} \det M_{\text{In}}(f_2) &= \det M_{\text{In}}(f_1) \cdot (-1) \cdot \det M_{\text{In}}(f_{t_2}) \\ &= \det M_{\text{In}}(f) \cdot (-1) \cdot \det M_{\text{In}}(f_{t_1}) \cdot (-1) \cdot \det M_{\text{In}}(f_{t_2}) \\ &= t_1t_2 \det M_{\text{In}}(f)^3 \\ &\neq 0. \end{aligned}$$

We have $\det M_{\text{In}}(f_2) = b_2y_2 - c_2z_2 = -c_2z_2 \neq 0$. Similarly, we have $\det M_{\text{Out}}(f_2) = -a_2x_2 = t_1t_2 \det M_{\text{Out}}(f)^3 \neq 0$. Therefore, $M(f_2)$ is of the form $\begin{bmatrix} 0 & 0 & 0 & a_2 \\ 0 & 0 & c_2 & 0 \\ 0 & z_2 & y_2 & 0 \\ x_2 & 0 & 0 & 0 \end{bmatrix}$, where $a_2x_2y_2c_2z_2 \neq 0$. That is, f_2 is a signature in Case II. If $f_2 \notin \mathscr{M}$, then Pl-Holant $(\neq_2|\ f_2)$ is #P-hard by Theorem 5.2, and hence Pl-Holant $(\neq_2|\ \{f\} \cup \mathcal{B})$ is #P-hard.

Otherwise, $f_2 \in \mathscr{M}$, which means $\dfrac{\det M_{\text{In}}(f_2)}{\det M_{\text{Out}}(f_2)} = 1$. That is $\dfrac{\det M_{\text{In}}(f)^3}{\det M_{\text{Out}}(f)^3} = 1$. Since $f \notin \mathscr{M}$, $\dfrac{\det M_{\text{In}}(f)}{\det M_{\text{Out}}(f)} \neq 1$, and hence $\dfrac{\det M_{\text{In}}(f)^7}{\det M_{\text{Out}}(f)^7} \neq 1$. Similar to the construction of f_1, we construct f_3. First, connect the variable x_2 of g_3 with the variable x_1 of f_1 using $(\neq_2)$. We get a signature f_{1t_3} with the signature matrix $M(f_{1t_3}) = \begin{bmatrix} 0 & 0 & 0 & a_1 \\ 0 & b_1 & c_1 & 0 \\ 0 & t_3z_1 & t_3y_1 & 0 \\ t_3x_1 & 0 & 0 & 0 \end{bmatrix}$. Note that $\det M_{\text{In}}(f_{1t_3}) = t_3 \det M_{\text{In}}(f_1)$ and $\det M_{\text{Out}}(f_{1t_3}) = t_3 \det M_{\text{Out}}(f_1)$. Then connect variables x_4, x_3 of f_1 with variables x_1, x_2 of

f_{1t_3} both using ($\neq_2$). We get a signature f_3 with the signature matrix

$$M(f_3) = \begin{bmatrix} 0 & 0 & 0 & a_3 \\ 0 & b_3 & c_3 & 0 \\ 0 & z_3 & y_3 & 0 \\ x_3 & 0 & 0 & 0 \end{bmatrix} = M(f_1)NM(f_{1t_3}) = \begin{bmatrix} 0 & 0 & 0 & a^2 \\ 0 & t_3b_1z_1 + b_1c_1 & t_3b_1y_1 + c_1^2 & 0 \\ 0 & t_3z_1^2 + y_1b_1 & t_3y_1z_1 + y_1c_1 & 0 \\ t_3x_1^2 & 0 & 0 & 0 \end{bmatrix}.$$

Since $c_1 \neq 0$ and $z_1 \neq 0$, we can define $t_3 = -\frac{c_1}{z_1}$ and $t_3 \neq 0$. Then $b_3 = b_1(t_3z_1 + c_1) = 0$ and $y_3 = y_1(t_3z_1 + c_1) = 0$. Notice that

$$\begin{aligned}
\det M_{\text{In}}(f_3) &= \det M_{\text{In}}(f_1) \cdot (-1) \cdot \det M_{\text{In}}(f_{1t_3}) \\
&= -\det M_{\text{In}}(f_1) \cdot t_3 \det M_{\text{In}}(f_1) \\
&= -t_3(\det M_{\text{In}}(f) \cdot (-1) \cdot \det M_{\text{In}}(f_{t_1}))^2 \\
&= -t_3t_1^2 \det M_{\text{In}}(f)^4 \\
&\neq 0
\end{aligned}$$

We have $\det M_{\text{In}}(f_3) = -c_3z_3 \neq 0$ and similarly, $\det M_{\text{Out}}(f_3) = -a_3x_3 = -t_3t_1^2 \det M_{\text{Out}}(f)^4 \neq 0$. That is, $M(f_3)$ is of the form $\left[\begin{smallmatrix} 0 & 0 & 0 & a_3 \\ 0 & 0 & c_3 & 0 \\ 0 & z_3 & 0 & 0 \\ x_3 & 0 & 0 & 0 \end{smallmatrix}\right]$ where $a_3x_3c_3z_3 \neq 0$.

Connect variables x_4, x_3 of f_2 with variables x_1, x_2 of f_3 both using ($\neq_2$). We get a signature f_4 with the signature matrix

$$M(f_4) = \begin{bmatrix} 0 & 0 & 0 & a_4 \\ 0 & b_4 & c_4 & 0 \\ 0 & z_4 & y_4 & 0 \\ x_4 & 0 & 0 & 0 \end{bmatrix} = M(f_2)NM(f_3) = \begin{bmatrix} 0 & 0 & 0 & a_2a_3 \\ 0 & 0 & c_2c_3 & 0 \\ 0 & z_2z_3 & y_2c_3 & 0 \\ x_2x_3 & 0 & 0 & 0 \end{bmatrix}.$$

Clearly, f_4 is a signature in Case II. Also, notice that

$$\begin{aligned}
\det M_{\text{In}}(f_4) &= \det M_{\text{In}}(f_2) \cdot (-1) \cdot \det M_{\text{In}}(f_3) \\
&= t_1t_2 \det M_{\text{In}}(f)^3 \cdot t_3t_1^2 \det M_{\text{In}}(f)^4 \\
&= t_3t_2t_1^3 \det M_{\text{In}}(f)^7.
\end{aligned}$$

and

$$\det M_{\text{Out}}(f_4) = t_3t_2t_1^3 \det M_{\text{Out}}(f)^7.$$

We have

$$\frac{\det M_{\text{In}}(f_4)}{\det M_{\text{Out}}(f_4)} = \frac{\det M_{\text{In}}(f)^7}{\det M_{\text{Out}}(f)^7} \neq 1,$$

which means $f_4 \notin \mathscr{M}$. By Theorem 5.2, Pl-Holant ($\neq_2 |\ f_4$) is #P-hard, and hence Pl-Holant ($\neq_2 |\ \{f\} \cup \mathcal{B}$) is #P-hard. □

Lemma 6.4. *Let $g = (0, 1, t, 0)^T$ be a binary signature where t is an n-th primitive root of unity, and $n \geq 5$. Then* Pl-Holant($\neq_2 |\ f, g$) *is #P-hard.*

Proof. Note that $M_{x_1,x_2}(g) = \left[\begin{smallmatrix} 0 & 1 \\ t & 0 \end{smallmatrix}\right]$. Connect the variable x_2 of a copy of signature g with the variable x_1 of another copy of signature g using $(\neq_2)$. We get a signature g_2 with the signature matrix

$$M_{x_1,x_2}(g_2) = \begin{bmatrix} 0 & 1 \\ t & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ t & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ t^2 & 0 \end{bmatrix}.$$

That is, $g_2 = (0, 1, t^2, 0)^T$. Similarly, we can construct $g_i = (0, 1, t^i, 0)^T$ for $1 \leqslant i \leqslant 5$. Here, g_1 denotes g. Since the order $n \geqslant 5$, g_i are all distinct.

Connect variables x_4, x_3 of signature f with variables x_1, x_2 of g_i for $1 \leqslant i \leqslant 5$ respectively. We get binary signatures h_i, where

$$h_i = M_{x_1x_2,x_4x_3}(f)g_i = \begin{bmatrix} 0 & 0 & 0 & a \\ 0 & b & c & 0 \\ 0 & z & y & 0 \\ x & 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} 0 \\ 1 \\ t^i \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ b + ct^i \\ z + yt^i \\ 0 \end{pmatrix}.$$

Let $\varphi(\mathfrak{z}) = \dfrac{z + y\mathfrak{z}}{b + c\mathfrak{z}}$. Since $\det \left[\begin{smallmatrix} b & c \\ z & y \end{smallmatrix}\right] = by - cz \neq 0$, $\varphi(\mathfrak{z})$ is a Möbius transformation of the extended complex plane $\widehat{\mathbb{C}}$. We rewrite h_i in the form of $(b+ct^i)(0, 1, \varphi(t^i), 0)^T$, with the understanding that if $b + ct^i = 0$, then $\varphi(t^i) = \infty$, and we define $(b + ct^i)(0, 1, \varphi(t^i), 0)^T$ to be $(0, 1, z + yt^i, 0)^T$. If there is a t^i such that $\varphi(t^i)$ is not a root of unity, and $\varphi(t^i) \neq 0$ and $\varphi(t^i) \neq \infty$, by Lemma 6.3, we have Pl-Holant $(\neq_2|\, f, h_i)$ is #P-hard, and hence Pl-Holant $(\neq_2|\, f, g_1)$ is #P-hard. Otherwise, $\varphi(t^i)$ is 0, ∞ or a root of unity for $1 \leqslant i \leqslant 5$. Since $\varphi(\mathfrak{z})$ is a bijection of $\widehat{\mathbb{C}}$, there is at most one t^i such that $\varphi(t^i) = 0$ and at most one t^i such that $\varphi(t^i) = \infty$. That means, there are at least three t^i such that $|\varphi(t^i)| = 1$. Since a Möbius transformation is determined by any 3 distinct points, mapping 3 distinct points from S^1 to S^1 implies that this $\varphi(\mathfrak{z})$ maps S^1 homeomorphically onto S^1. Such a Möbius transformation has a special form: $\mathcal{M}(\alpha, e^{\mathrm{i}\theta}) = e^{\mathrm{i}\theta}\dfrac{(\mathfrak{z} + \alpha)}{1 + \bar{\alpha}\mathfrak{z}}$, where $|\alpha| \neq 1$.

By normalization in signature f, we may assume $b = 1$. Compare the coefficients, we have $c = \bar{\alpha}$, $y = e^{\mathrm{i}\theta}$ and $z = \alpha e^{\mathrm{i}\theta}$. Here $\alpha \neq 0$ due to $z \neq 0$. Also, since $M_{x_2x_3,x_1x_4}(f) = \left[\begin{smallmatrix} 0 & 0 & 0 & y \\ 0 & a & z & 0 \\ 0 & c & x & 0 \\ b & 0 & 0 & 0 \end{smallmatrix}\right]$ and $\det \left[\begin{smallmatrix} a & z \\ c & x \end{smallmatrix}\right] = ax - cz \neq 0$, we have another Möbius transformation $\psi(\mathfrak{z}) = \dfrac{c + x\mathfrak{z}}{a + z\mathfrak{z}}$. Plug in $c = \bar{\alpha}$ and $z = \alpha e^{\mathrm{i}\theta}$, we have

$$\psi(\mathfrak{z}) = \frac{\bar{\alpha} + x\mathfrak{z}}{a + \alpha e^{\mathrm{i}\theta}\mathfrak{z}} = \frac{\frac{\bar{\alpha}}{a} + \frac{x}{a}\mathfrak{z}}{1 + \frac{\alpha e^{\mathrm{i}\theta}}{a}\mathfrak{z}}.$$

By the same proof for $\varphi(\mathfrak{z})$, we get Pl-Holant $(\neq_2|\, f, g)$ is #P-hard, unless $\psi(\mathfrak{z})$ also maps S^1 to S^1. Hence, we can assume $\psi(\mathfrak{z})$ has the form $\mathcal{M}(\beta, e^{\mathrm{i}\theta'}) = e^{\mathrm{i}\theta'}\dfrac{(\mathfrak{z} + \beta)}{1 + \bar{\beta}\mathfrak{z}}$, where $|\beta| \neq 1$. Compare the coefficients, we have

$$\begin{cases} \dfrac{\alpha e^{\mathrm{i}\theta}}{a} = \bar{\beta} \\ \dfrac{\bar{\alpha}}{a} = e^{\mathrm{i}\theta'}\beta \\ \dfrac{x}{a} = e^{\mathrm{i}\theta'} \end{cases}.$$

Solving this equation, we get $a = \frac{\alpha}{\bar{\beta}}e^{\mathrm{i}\theta}$ and $x = \frac{\bar{\alpha}}{\beta}$. Let $\gamma = \frac{\alpha}{\bar{\beta}}$, and we have $a = \gamma e^{\mathrm{i}\theta}$ and $x = \bar{\gamma}$, where $|\gamma| \neq |\alpha|$ since $|\beta| \neq 1$ and $\gamma \neq 0$ since $x \neq 0$. Then, we have signature matrices

$M_{x_1x_2,x_4x_3}(f) = \left[\begin{smallmatrix} 0 & 0 & 0 & \gamma e^{\mathrm{i}\theta} \\ 0 & 1 & \bar{\alpha} & 0 \\ 0 & \alpha e^{\mathrm{i}\theta} & e^{\mathrm{i}\theta} & 0 \\ \bar{\gamma} & 0 & 0 & 0 \end{smallmatrix}\right]$, $M_{x_2x_3,x_1x_4}(f) = \left[\begin{smallmatrix} 0 & 0 & 0 & e^{\mathrm{i}\theta} \\ 0 & \gamma e^{\mathrm{i}\theta} & \alpha e^{\mathrm{i}\theta} & 0 \\ 20 & \bar{\alpha} & \bar{\gamma} & 0 \\ 1 & 0 & 0 & 0 \end{smallmatrix}\right]$, $M_{x_3x_4,x_2x_1}(f) = \left[\begin{smallmatrix} 0 & 0 & 0 & \bar{\gamma} \\ 0 & e^{\mathrm{i}\theta} & \bar{\alpha} & 0 \\ 0 & \alpha e^{\mathrm{i}\theta} & 1 & 0 \\ \gamma e^{\mathrm{i}\theta} & 0 & 0 & 0 \end{smallmatrix}\right]$ and $M_{x_4x_1,x_3x_2}(f) = \left[\begin{smallmatrix} 0 & 0 & 0 & 1 \\ 0 & \bar{\gamma} & \alpha e^{\mathrm{i}\theta} & 0 \\ 0 & \bar{\alpha} & \gamma e^{\mathrm{i}\theta} & 0 \\ e^{\mathrm{i}\theta} & 0 & 0 & 0 \end{smallmatrix}\right]$. Connect variables x_4, x_3 of a copy of signature f with variables x_3, x_4 of another copy of signature f using $(\neq_2)$. We get a signature f_1 with the signature matrix

$$M(f_1) = M_{x_1x_2,x_4x_3}(f) N M_{x_3x_4,x_2x_1}(f) = \begin{bmatrix} 0 & 0 & 0 & \gamma\bar{\gamma}e^{\mathrm{i}\theta} \\ 0 & (\alpha+\bar{\alpha})e^{\mathrm{i}\theta} & 1+\bar{\alpha}^2 & 0 \\ 0 & (1+\alpha^2)e^{\mathrm{i}2\theta} & (\alpha+\bar{\alpha})e^{\mathrm{i}\theta} & 0 \\ \gamma\bar{\gamma}e^{\mathrm{i}\theta} & 0 & 0 & 0 \end{bmatrix}.$$

- If $\alpha + \bar{\alpha} \neq 0$, normalizing $M_{x_1x_2,x_4x_3}(f_1)$ by dividing by $(\alpha+\bar{\alpha})e^{\mathrm{i}\theta}$, we have

$$M(f_1) = \begin{bmatrix} 0 & 0 & 0 & \dfrac{\gamma\bar{\gamma}}{(\alpha+\bar{\alpha})} \\ 0 & 1 & \dfrac{(1+\bar{\alpha}^2)e^{-\mathrm{i}\theta}}{(\alpha+\bar{\alpha})} & 0 \\ 0 & \dfrac{(1+\alpha^2)e^{\mathrm{i}\theta}}{(\alpha+\bar{\alpha})} & 1 & 0 \\ \dfrac{\gamma\bar{\gamma}}{(\alpha+\bar{\alpha})} & 0 & 0 & 0 \end{bmatrix}.$$

  Note that $\dfrac{(1+\alpha^2)e^{\mathrm{i}\theta}}{(\alpha+\bar{\alpha})}$ and $\dfrac{(1+\bar{\alpha}^2)e^{-\mathrm{i}\theta}}{(\alpha+\bar{\alpha})}$ are conjugates. Let $\delta = \dfrac{(1+\alpha^2)e^{\mathrm{i}\theta}}{(\alpha+\bar{\alpha})}$, and then $\bar{\delta} = \dfrac{(1+\bar{\alpha}^2)e^{-\mathrm{i}\theta}}{(\alpha+\bar{\alpha})}$. We have $|\delta|^2 = \delta\bar{\delta} = \dfrac{(1+\alpha^2)(1+\bar{\alpha}^2)}{(\alpha+\bar{\alpha})^2} \neq 1$ due to $\det M_{\mathrm{In}}(f_1) \neq 0$, and $\delta \neq 0$ due to $|\alpha|^2 \neq 1$. Consider the inner matrix of $M(f_1)$, we have $M_{\mathrm{In}}(f_1) = \left[\begin{smallmatrix} 1 & \bar{\delta} \\ \delta & 1 \end{smallmatrix}\right]$. Notice that the two eigenvalues of $M_{\mathrm{In}}(f_1)$ are $1+|\delta|$ and $1-|\delta|$, and obviously $\left|\frac{1-|\delta|}{1+|\delta|}\right| \neq 1$, which means there is no integer n and complex number C such that $M^n_{\mathrm{In}}(f_1) = CI$. Note that $\varphi_1(\mathfrak{z}) = \dfrac{\delta+\mathfrak{z}}{1+\bar{\delta}\mathfrak{z}}$ is a Möbius transformation of the form $\mathcal{M}(\delta, 1)$ mapping S^1 to S^1.
  Connect variables x_4, x_3 of signature f_1 with variables x_1, x_2 of signatures g_i. We get binary signatures $g_{(i,\varphi_1)}$, where

$$g_{(i,\varphi_1)} = M_{x_1x_2,x_4x_3}(f_1) g_i = \begin{bmatrix} 0 & 0 & 0 & * \\ 0 & 1 & \bar{\delta} & 0 \\ 0 & \delta & 1 & 0 \\ * & 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} 0 \\ 1 \\ t^i \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1+\bar{\delta}t^i \\ \delta + t^i \\ 0 \end{pmatrix} = (1+\bar{\delta}t^i) \begin{pmatrix} 0 \\ 1 \\ \varphi_1(t^i) \\ 0 \end{pmatrix}.$$

  Since φ_1 is a Möbius transformation mapping S^1 to S^1 and $|t^i| = 1$, we have $|\varphi_1(t^i)| = 1$, which means $1 + \bar{\delta}t^i \neq 0$. Hence, $g_{(i,\varphi_1)}$ can be normalized as $(0, 1, \varphi_1(t^i), 1)^T$. Successively construct binary signatures $g_{(i,\varphi_1^n)}$ by connecting f_1 with $g_{(i,\varphi_1^{n-1})}$. We have

$$g_{(i,\varphi_1^n)} = M(f_1) g_{(i,\varphi_1^{n-1})} = M^n(f_1) g_i = C_{(i,n)}(0, 1, \varphi_1^n(t^i), 1)^T,$$

  where $C_{(i,n)} = \prod\limits_{0 \leqslant k \leqslant n-1} \left(1 + \bar{\delta}\varphi_1^k(t^i)\right)$. We know $C_{(i,n)} \neq 0$, because for any k, $1+\bar{\delta}\varphi_1^{k-1}(t^i) \neq 0$ due to $|\varphi_1^k(t^i)| = \dfrac{|\delta + \varphi_1^{k-1}(t^i)|}{|1+\bar{\delta}\varphi_1^{k-1}(t^i)|} = 1$. Hence, $g_{(i,\varphi_1^n)}$ can be normalized as $(0, 1, \varphi_1^n(t^i), 1)^T$.

Notice that the nonzero entries $(1, \varphi_1^n(t^i))^T$ of $g_{(i,\varphi_1^n)}$ are completely decided by the inner matrix $M_{\text{In}}(f_1)$. That is

$$M_{\text{In}}^n(f_1)\begin{pmatrix}1\\ t^i\end{pmatrix} = C_{(i,n)}\begin{pmatrix}1\\ \varphi_1^n(t^i)\end{pmatrix}.$$

If for each $i \in \{1,2,3\}$, there is some $n_i \geqslant 1$ such that $(1, \varphi_1^{n_i}(t^i))^T = (1, t^i)^T$, then $\varphi_1^{n_0}(t^i) = t^i$, where $n_0 = n_1n_2n_3$ for $1 \leqslant i \leqslant 3$, i.e., the Möbius transformation $\varphi_1^{n_0}$ fixes three distinct complex numbers t, t^2, t^3. So the Möbius transformation is the identity map, i.e., $\varphi_1^{n_0}(\mathfrak{z}) = \mathfrak{z}$ for all $\mathfrak{z} \in \mathbb{C}$. This implies that $M_{\text{In}}^{n_0}(f_1) = C\left[\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right]$ for some constant C. This contradicts the fact that the ratio of the eigenvalues of M_{In} is not a root of unity. Therefore, there is an i such that $(1, \varphi_1^n(t^i))^T$ are all distinct for $n \in \mathbb{N}$. Then, we can realize polynomially many distinct binary signatures of the form $(0, 1, \varphi_1^n(t^i), 1)^T$. By Lemma 2.6, we have Pl-Holant($\neq_2|\ f, g$) is #P-hard.

- Otherwise $\alpha + \bar{\alpha} = 0$, which means α is a pure imaginary number. Suppose $\alpha = m\mathfrak{i}$, where $m \in \mathbb{R}$ and $|m| \neq 0$ or 1. Connect variables x_1, x_4 of a copy of signature f with variables x_4, x_1 of another copy of signature f, we get a signature f_2 with the signature matrix

$$\begin{aligned}
M(f_2) &= M_{x_2x_3,x_1x_4}(f) N M_{x_4x_1,x_3x_2}(f)\\
&= \begin{bmatrix}0 & 0 & 0 & e^{\mathfrak{i}\theta}\\ 0 & \gamma e^{\mathfrak{i}\theta} & m\mathfrak{i}e^{\mathfrak{i}\theta} & 0\\ 0 & -m\mathfrak{i} & \bar{\gamma} & 0\\ 1 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{bmatrix}\begin{bmatrix}0 & 0 & 0 & 1\\ 0 & \bar{\gamma} & m\mathfrak{i}e^{\mathfrak{i}\theta} & 0\\ 0 & -m\mathfrak{i} & \gamma e^{\mathfrak{i}\theta} & 0\\ e^{\mathfrak{i}\theta} & 0 & 0 & 0\end{bmatrix}\\
&= \begin{bmatrix}0 & 0 & 0 & e^{\mathfrak{i}\theta}\\ 0 & (-\gamma+\bar{\gamma})m\mathfrak{i}e^{\mathfrak{i}\theta} & (\gamma^2-m^2)e^{\mathfrak{i}2\theta} & 0\\ 0 & \bar{\gamma}^2 - m^2 & (-\gamma+\bar{\gamma})m\mathfrak{i}e^{\mathfrak{i}\theta} & 0\\ e^{\mathfrak{i}\theta} & 0 & 0 & 0\end{bmatrix}.
\end{aligned}$$

  - If $-\gamma + \bar{\gamma} \neq 0$, normalizing $M(f_2)$ by dividing by $(-\gamma+\bar{\gamma})m\mathfrak{i}e^{\mathfrak{i}\theta}$, we have

$$M_{\text{In}}(f_2) = \begin{bmatrix} 1 & \dfrac{(\gamma^2-m^2)e^{\mathfrak{i}\theta}}{(-\gamma+\bar{\gamma})m\mathfrak{i}}\\ \dfrac{(\bar{\gamma}^2-m^2)e^{-\mathfrak{i}\theta}}{(-\gamma+\bar{\gamma})m\mathfrak{i}} & 1\end{bmatrix}.$$

    Note that $\dfrac{(\gamma^2-m^2)e^{\mathfrak{i}\theta}}{(-\gamma+\bar{\gamma})m\mathfrak{i}}$ and $\dfrac{(\bar{\gamma}^2-m^2)e^{-\mathfrak{i}\theta}}{(-\gamma+\bar{\gamma})m\mathfrak{i}}$ are conjugates. Let $\zeta = \dfrac{(\bar{\gamma}^2-m^2)e^{-\mathfrak{i}\theta}}{(-\gamma+\bar{\gamma})m\mathfrak{i}}$, and then $|\zeta| \neq 1$ due to $\det M_{\text{In}}(f_2) \neq 0$, and $\zeta \neq 0$ due to $|\gamma| \neq |m|$. Same as the analysis of $M_{\text{In}}(f_1)$, the ratio of the two eigenvalues of $M_{\text{In}}(f_2)$ is also not equal to 1, which means there is no integer n and complex number C such that $M_{\text{In}}^n(f_2) = CI$. Notice that $\varphi_2(w) = \dfrac{\zeta + w}{1 + \bar{\zeta}w}$ is also a Möbius transformation of the form $\mathcal{M}(\zeta, 1)$ mapping S^1 to S^1. Similarly, we can realize polynomially many distinct binary signatures, and hence Pl-Holant($\neq_2|\ f, g$) is #P-hard.
  - Otherwise, $-\gamma + \bar{\gamma} = 0$, which means γ is a real number. Suppose $\gamma = n$, where $n \in \mathbb{R}$ and $|n| \neq 0$ or $|m|$. Connect variables x_4, x_3 of a copy of signature f with variables x_1,

x_2 of another copy of signature f, we get a signature f' with the signature matrix

$$
\begin{aligned}
M(f') &= M_{x_1x_2,x_4x_3}(f) N M_{x_1x_2,x_4x_3}(f) \\
&= \begin{bmatrix} 0 & 0 & 0 & ne^{\mathfrak{i}\theta} \\ 0 & 1 & -m\mathfrak{i} & 0 \\ 0 & m\mathfrak{i}e^{\mathfrak{i}\theta} & e^{\mathfrak{i}\theta} & 0 \\ n & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & ne^{\mathfrak{i}\theta} \\ 0 & 1 & -m\mathfrak{i} & 0 \\ 0 & m\mathfrak{i}e^{\mathfrak{i}\theta} & e^{\mathfrak{i}\theta} & 0 \\ n & 0 & 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & 0 & n^2e^{\mathfrak{i}2\theta} \\ 0 & (e^{\mathfrak{i}\theta}-1)m\mathfrak{i} & e^{\mathfrak{i}\theta}-m^2 & 0 \\ 0 & e^{\mathfrak{i}\theta}-e^{\mathfrak{i}2\theta}m^2 & (e^{\mathfrak{i}2\theta}-e^{\mathfrak{i}\theta})m\mathfrak{i} & 0 \\ n^2 & 0 & 0 & 0 \end{bmatrix}.
\end{aligned}
$$

* If $e^{\mathfrak{i}\theta}=1$, then $M(f) = \left[\begin{smallmatrix} 0 & 0 & 0 & \gamma \\ 0 & 1 & \bar{\alpha} & 0 \\ 0 & \alpha & 1 & 0 \\ \bar{\gamma} & 0 & 0 & 0 \end{smallmatrix}\right]$, and $M_{\mathrm{In}}(f) = \left[\begin{smallmatrix} 1 & \bar{\alpha} \\ \alpha & 1 \end{smallmatrix}\right]$. Since $|\alpha| \neq 1$, same as the analysis of $M_{\mathrm{In}}(f_1)$, we can realize polynomially many binary signatures, and hence Pl-Holant($\neq_2|\ f, g$) is #P-hard.
* Otherwise $e^{\mathfrak{i}\theta} \neq 1$, normalizing $M(f')$ by dividing by $(e^{\mathfrak{i}\theta}-1)m\mathfrak{i}$, we have

$$
M(f') = \begin{bmatrix} 0 & 0 & 0 & \frac{n^2e^{\mathfrak{i}\theta}}{(e^{\mathfrak{i}\theta}-1)m\mathfrak{i}} \cdot e^{\mathfrak{i}\theta} \\ 0 & 1 & \frac{e^{\mathfrak{i}\theta}-m^2}{(e^{\mathfrak{i}\theta}-1)m\mathfrak{i}} & 0 \\ 0 & \frac{1-e^{\mathfrak{i}\theta}m^2}{(e^{\mathfrak{i}\theta}-1)m\mathfrak{i}} \cdot e^{\mathfrak{i}\theta} & e^{\mathfrak{i}\theta} & 0 \\ \frac{n^2}{(e^{\mathfrak{i}\theta}-1)m\mathfrak{i}} & 0 & 0 & 0 \end{bmatrix}.
$$

Note that $\frac{1-e^{\mathfrak{i}\theta}m^2}{(e^{\mathfrak{i}\theta}-1)m\mathfrak{i}}$ and $\frac{e^{\mathfrak{i}\theta}-m^2}{(e^{\mathfrak{i}\theta}-1)m\mathfrak{i}}$ are conjugates, and $\frac{n^2e^{\mathfrak{i}\theta}}{(e^{\mathfrak{i}\theta}-1)m\mathfrak{i}}$ and $\frac{n^2}{(e^{\mathfrak{i}\theta}-1)m\mathfrak{i}}$ are conjugates. Let $\alpha' = \frac{1-e^{\mathfrak{i}\theta}m^2}{(e^{\mathfrak{i}\theta}-1)m\mathfrak{i}}$ and $\gamma' = \frac{n^2e^{\mathfrak{i}\theta}}{(e^{\mathfrak{i}\theta}-1)m\mathfrak{i}}$. Then $M(f') = \left[\begin{smallmatrix} 0 & 0 & 0 & \gamma' e^{\mathfrak{i}\theta} \\ 0 & 1 & \bar{\alpha'} & 0 \\ 0 & \alpha' e^{\mathfrak{i}\theta} & e^{\mathfrak{i}\theta} & 0 \\ \bar{\gamma'} & 0 & 0 & 0 \end{smallmatrix}\right]$. Notice that $M(f')$ and $M(f)$ have the same forms. Similar to the construction of f_2, we can construct a signature f_2' using f' instead of f. Since $-\gamma' + \bar{\gamma'} = -\frac{n^2e^{\mathfrak{i}\theta}}{(e^{\mathfrak{i}\theta}-1)m\mathfrak{i}} + \frac{n^2}{(e^{\mathfrak{i}\theta}-1)m\mathfrak{i}} = -\frac{n^2}{m\mathfrak{i}} \neq 0$, by the analysis of f_2, we can still realize polynomially many binary signatures and hence Pl-Holant($\neq_2|\ f, g$) is #P-hard. □

Remark: The order $n \geqslant 5$ promises that there are at least three points mapped to points on S^1, since at most one point can be mapped to 0 and at most one can be mapped to ∞. When the order n is 3 or 4, if no point is mapped to 0 or ∞, then there are still at least three points mapped to points on S^1. So, we have the following corollary.

Corollary 6.5. *Let $g = (0, 1, t, 0)^T$ be a binary signature where t is an n-th primitive root of unity, and $n = 3$ or 4. Let g_m denote $(0, 1, t^m, 0)^T$. For any (i, j, k, ℓ) which is a cyclic permutation* *of* $(1,2,3,4)$*, if there is no* g_m *such that* $M_{x_ix_j,x_\ell x_k}(f)g_m = d_1(0,1,0,0)^T$ *or* $d_2(0,0,1,0)^T$*, where* $d_1, d_2 \in \mathbb{C}$ *, then* Pl-Holant$(\neq_2|\ f, g)$ *is #P-hard.*

Lemma 6.6. *Let* $g = (0,1,0,0)^T$ *be a binary signature. Then* Pl-Holant$(\neq_2|\ f, g)$ *is #P-hard.*

Proof. Connect variables x_4, x_3 of the signature f with variables x_2 and x_1 of g both using $(\neq_2)$. We get a binary signature g_1, where

$$g_1 = M_{x_1x_2,x_4x_3}(f)(0,1,0,0)^T = (0,1,z,0)^T.$$

Note that $g_1(x_1,x_2)$ can be normalized as $(0,z^{-1},1,0)^T$ since $z \neq 0$. That is, $g_1(x_2,x_1) = (0,1,z^{-1},0)$. Then connect the variable x_1 of g_1 with the variable x_1 of f. We get a signature f_1 with the signature matrix $M(f_1) = \begin{bmatrix} 0 & 0 & 0 & a \\ 0 & 1 & c & 0 \\ 0 & 1 & yz^{-1} & 0 \\ xz^{-1} & 0 & 0 & 0 \end{bmatrix}$ denoted by $\begin{bmatrix} 0 & 0 & 0 & a \\ 0 & 1 & c & 0 \\ 0 & 1 & y_1 & 0 \\ x_1 & 0 & 0 & 0 \end{bmatrix}$, where $x_1y_1 \neq 0$.

- If $c = 0$, connect variables x_4, x_3 of f_1 with variables x_1, x_2 of g both using $(\neq_2)$. We get a binary signature h_1, where

  $$h_1 = M_{x_1x_2,x_4x_3}(f_1)(0,0,1,0)^T = (0,1,y_1,0)^T.$$

  Also, connect the variable x_4 with x_3 of f_1 using $(\neq_2)$. We get a binary signature h_2, where

  $$h_2 = M_{x_1x_2,x_4x_3}(f_1)(0,1,1,0)^T = (0,2,y_1,0)^T.$$

  Note that h_2 can be normalized as $(0,1,\frac{y_1}{2},0)^T$. Clearly, $|y_1| \neq |\frac{y_1}{2}|$, so they can not both be roots of unity. By Lemma 6.3, Pl-Holant $(\neq_2|\ f, h_1, h_2)$ is #P-hard, and hence Pl-Holant $(\neq_2|\ f, g)$ is #P-hard.
- Otherwise $c \neq 0$. Connect variables x_2, x_1 of g with variables x_1, x_2 of f both using $(\neq_2)$. We get a binary signature g_2, where

  $$g_2 = ((0,1,0,0)M_{x_1x_2,x_4x_3}(f_1))^T = (0,1,c,0)^T.$$

  Note that $g_2(x_1,x_2)$ can be normalized as $(0,c^{-1},1,0)^T$ since $c \neq 0$. That is, $g_2(x_2,x_1) = (0,1,c^{-1},0)^T$. Then connect the variable x_1 of g_2 with the variable x_3 of f. We get a signature f_2 with the signature matrix $M(f_2) = \begin{bmatrix} 0 & 0 & 0 & ac^{-1} \\ 0 & 1 & 1 & 0 \\ 0 & 1 & yz^{-1}c^{-1} & 0 \\ xz^{-1} & 0 & 0 & 0 \end{bmatrix}$ denoted by $\begin{bmatrix} 0 & 0 & 0 & a_2 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & y_2 & 0 \\ x_2 & 0 & 0 & 0 \end{bmatrix}$, where $a_2x_2y_2 \neq 0$. Notice that $M_{x_2x_3,x_1x_4}(f_2) = \begin{bmatrix} 0 & 0 & 0 & y_2 \\ 0 & a_2 & 1 & 0 \\ 0 & 1 & x_2 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$. Connect variables x_1, x_4 of signature f_2 with variables x_2, x_1 of g both using $(\neq_2)$. We get a binary signature h_3, where

  $$h_3 = M_{x_2x_3,x_1x_4}(f_2)(0,1,0,0)^T = (0,a_2,1,0)^T.$$

  h_3 can be normalized as $(0,1,\frac{1}{a_2},0)^T$. Also connect variables x_1, x_4 of signature f_2 with variables x_1, x_2 of g both using $(\neq_2)$. We get a binary signature h_4, where

  $$h_4 = M_{x_2x_3,x_1x_4}(f_2)(0,0,1,0)^T = (0,1,x_2,0)^T.$$

  If $|a_2| \neq 1$ or $|x_2| \neq 1$, then a_2 or x_2 is not a root of unity. By Lemma 6.3, Pl-Holant $(\neq_2|\ f, h_3, h_4)$ is #P-hard, and hence Pl-Holant $(\neq_2|\ f, g)$ is #P-hard. Otherwise, $|a_2| = |x_2| = 1$. Same as the construction of h_1 and h_2, construct binary signatures h'_1 and h'_2 using f_2 instead of f_1. We get

$$h'_1 = M_{x_1x_2,x_4x_3}(f_2)(0,0,1,0)^T = (0,1,y_2,0)^T,$$

and

$$h'_2 = M_{x_1x_2,x_4x_3}(f_2)(0,1,1,0)^T = (0,2,1+y_2,0)^T.$$

Note that h'_2 can be normalized as $(0,1,\frac{1+y_2}{2},0)^T$.

- If y_2 is not a root of unity, then by Lemma 6.3, Pl-Holant $(\neq_2|\ f, h'_1)$ is #P-hard, and hence Pl-Holant $(\neq_2|\ f,g)$ is #P-hard.
- If y_2 is an n-th primitive root of unity and $n \geqslant 5$, then by Lemma 6.4, Pl-Holant $(\neq_2|\ f, h'_1)$ is #P-hard, and hence Pl-Holant $(\neq_2|\ f,g)$ is #P-hard.
- If $y_2 = \omega(\omega = \frac{-1+\sqrt{3}\mathfrak{i}}{2}), \omega^2$ or $\pm\mathfrak{i}$, then $0 < |\frac{1+y_2}{2}| < 1$, which means it is not zero neither a root of unity. By Lemma 6.3, Pl-Holant $(\neq_2|\ f, h'_2)$ is #P-hard, and hence Pl-Holant $(\neq_2|\ f,g)$ is #P-hard.
- If $y_2 = 1$, then f_2 is non-singular redudant and hence Pl-Holant $(\neq_2|\ f,g)$ is #P-hard.
- If $y_2 = -1$. Connect two copies of f_2, we get a signature f_3 with the signature matrix

$$M(f_3) = M_{x_1x_2,x_4x_3}(f_2)NM_{x_1x_2,x_4x_3}(f_2) = \begin{bmatrix} 0 & 0 & 0 & a_2^2 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ x_2^2 & 0 & 0 & 0 \end{bmatrix}.$$

  Since $|a_2| = |x_2| = 1$, $|a_2^2x_2^2| = 1 \neq 4$. Therefore, $\{a_2^2, 2, x_2^2, 2\}$ does not belong to case (i) or case (ii) in Theorem 4.6. Hence, Pl-Holant $(\neq_2|\ f_3)$ is #P-hard, and hence Pl-Holant $(\neq_2|\ f,g)$ is #P-hard. □

Combine Lemma 6.4, Corollary 6.5 and Lemma 6.6. We have the following corollary.

Corollary 6.7. *Let $g = (0,1,t,0)^T$ be a binary signature where t is an n-th primitive root of unity, and $n \geqslant 3$. Then* Pl-Holant$(\neq_2|\ f,g)$ *is #P-hard.*

Now, we are able to prove the following theorem for Case IV.

Theorem 6.8. *Let f be a 4-ary signature with the signature matrix*

$$M(f) = \begin{bmatrix} 0 & 0 & 0 & a \\ 0 & b & c & 0 \\ 0 & z & y & 0 \\ x & 0 & 0 & 0 \end{bmatrix},$$

where $abxyz \neq 0$. Pl-Holant$(\neq_2|\ f)$ *is #P-hard unless $f \in \mathscr{M}$, in which case,* Pl-Holant$(\neq_2|\ f)$ *is tractable.*

Proof. Tractability follows by 2.15.

Now suppose $f \notin \mathscr{M}$. Connect the variable x_4 with x_3 of f using $(\neq_2)$, and we get a binary signature g_1, where

$$g_1 = M_{x_1x_2,x_4x_3}(0,1,1,0)^T = (0, b+c, z+y, 0)^T.$$

Connect the variable x_1 with x_2 of f using $(\neq_2)$, and we get a binary signature g_2, where

$$g_2 = ((0,1,1,0)M_{x_1x_2,x_4x_3})^T = (0, b+z, c+y, 0)^T.$$

- If one of g_1 and g_2 is of the form $(0,0,0,0)$, then $by = (-c)(-z) = cz$. That is $by - cz = 0$. Here $c \neq 0$ due to $by \neq 0$. By Lemma 6.2, Pl-Holant($\neq_2|\, f$) is #P-hard.
- If one of g_1 and g_2 can be normalized as $(0,1,0,0)$ or $(0,0,1,0)$. By Lemma 6.6, Pl-Holant($\neq_2|\, f$) is #P-hard.
- If one of g_1 and g_2 can be normalized as $(0,1,t,0)^T$, where $t \neq 0$ is not a root of unity, then by Lemma 6.3, Pl-Holant($\neq_2|\, f$) is #P-hard.
- If one of g_1 and g_2 can be normalized as $(0,1,t,0)^T$, where t is an n-th primitive root of unity and $n \geqslant 3$, then by Corollary 6.7, Pl-Holant($\neq_2|\, f$) is #P-hard.
- Otherwise, g_1 and g_2 do not belong to those cases above, which means both g_1 and g_2 both can be normalized as $(0,1,\epsilon_1,0)$ and $(0,1,\epsilon_2,0)$, where $\epsilon_1 = \pm 1$ and $\epsilon_2 = \pm 1$. That is, $b + c = \epsilon_1(z+y) \neq 0$ and $b + z = \epsilon_2(c+y) \neq 0$.
  - If $b+c = z+y$ and $b+z = c+y$, then $b = y$ and $c = z$. To be proved below.
  - If $b+c = -(z+y)$ and $b+z = c+y$, then $b+z = c+y = 0$. Contradiction.
  - If $b+c = z+y$ and $b+z = -(c+y)$, then $b+c = z+y = 0$. Contradiction.
  - If $b+c = -(z+y)$ and $b+z = -(c+y)$, we have $g_1 = (0,1,-1,0)^T$. Connect the variable x_2 of g_1 with the variable x_1 of f, and we get a signature f' with the signature matrix $M(f') = \begin{bmatrix} 0 & 0 & 0 & a \\ 0 & b & c & 0 \\ 0 & -z & -y & 0 \\ -x & 0 & 0 & 0 \end{bmatrix}$. Connect the variable x_1 with x_2 of f' using ($\neq_2$), and we get a binary signature $g' = (0, b-z, c-y, 0)^T$. Same as the analysis of g_1 and g_2 above, we have Pl-Holant($\neq_2|\, f'$) is #P-hard unless g' can be normalized as $(0,1,\epsilon_3,0)$, where $\epsilon_3 = \pm 1$. That is, $b - z = \epsilon_3(c-y) \neq 0$,
    - If $b - z = c - y$, combine with $b + c = -(z+y)$. We have $b = -y$ and $c = -z$. To be proved below.
    - If $b - z = -(c-y)$, combine with $b + c = -(z+y)$. We have $b + c = z + y = 0$. Contradiction.

    Therefore, Pl-Holant($\neq_2|\, f'$) is #P-hard and hence Pl-Holant($\neq_2|\, f$) is #P-hard

So far, we have Pl-Holant($\neq_2|\, f$) is #P-hard unless $b = \epsilon y$ and $c = \epsilon z$, where $\epsilon = \pm 1$. Similarly, connect the variable x_2 with x_3 of f using ($\neq_2$), and we get a binary signature $g_3 = (0, a+c, z+x, 0)^T$. Connect the variable x_1 with x_4 of f using ($\neq_2$), and we get a binary signature $g_4 = (0, a+z, c+x, 0)^T$. Same as the analysis of g_1 and g_2, we have Pl-Holant($\neq_2|\, f$) is #P-hard unless $a = \epsilon' x$ and $c = \epsilon' z$, where $\epsilon' = \pm 1$. Therefore, Pl-Holant($\neq_2|\, f$) is #P-hard unless $a = \epsilon x$, $b = \epsilon y$ and $c = \epsilon z$, where $\epsilon = \pm 1$. In this case, since $z \neq 0$, we have $abc \neq 0$. By Lemma 6.1, Pl-Holant($\neq_2|\, f$) is #P-hard. $\square$

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