A Note on Multiparty Communication Complexity and the Hales-Jewett Theorem

Adi Shraibman
The Academic College
of Tel-Aviv-Yaffo
Tel-Aviv, Israel
adish@mta.ac.il

Abstract

For integers n and k, the *density Hales-Jewett number* $c_{n,k}$ is defined as the maximal size of a subset of $[k]^n$ that contains no combinatorial line. We prove a lower bound on $c_{n,k}$, similar to the lower bound in [16], but with better dependency on k. The bound in [16] is roughly $c_{n,k}/k^n \geq exp(-O(\log n)^{1/\lceil \log_2 k \rceil})$ and we show $c_{n,k}/k^n \geq exp(-O(\log n)^{1/(k-1)})$. The proof of the bound uses the well-known construction of Behrend [3] and Rankin [17], as the one in [16], but does not require the recent refinements [7, 11, 14]. Instead our proof relies on an argument from communication complexity.

In addition, we show that for $k \geq 3$ the density Hales-Jewett number $c_{n,k}$ is equal to the maximal size of a cylinder intersection in the problem $Part_{n,k}$ of testing whether k subsets of $[n]$ form a partition. It follows that the communication complexity, in the Number On the Forehead (NOF) model, of $Part_{n,k}$, is equal to the minimal size of a partition of $[k]^n$ into subsets that do not contain a combinatorial line. Thus, Tesson's bound on the problem using the Hales-Jewett theorem [19] is in fact tight, and the density Hales-Jewett number can be thought of as a quantity in communication complexity.

1 Introduction

For any integers $n \geq 1$ and $k \geq 1$, consider the set $[k]^n$, of *words* of length n over the alphabet $[k]$. Define a *combinatorial line* in $[k]^n$ as a subset of k distinct words such that if we place these words in a $k \times n$ table, then all columns in this table belong to the set $\{(x, x, \ldots, x) : x \in [k]\} \cup \{(1, 2, \ldots, k)\}$. The *density Hales-Jewett number* $c_{n,k}$ is defined to be the maximal cardinality of a subset of $[k]^n$ which does not contain a combinatorial line.

Clearly, $c_{n,k} \leq k^n$, and a deep theorem of Furstenberg and Katznelson [9, 10] says that $c_{n,k}$ is asymptotically smaller than k^n:

Theorem 1 (Density Hales-Jewett theorem) *For every positive integer k and every real number $\delta > 0$ there exists a positive integer $DHJ(k, \delta)$ such that if $n \geq DHJ(k, \delta)$ then any subset of $[k]^n$ of cardinality at least δk^n contains a combinatorial line.*

The above theorem is a density version of the Hales-Jewett theorem:

Theorem 2 (Hales-Jewett theorem) *For every pair of positive integers k and r there exists a positive number $HJ(k, r)$ such that for every $n \geq HJ(k, r)$ and every r-coloring of the set $[k]^n$ there is a monochromatic combinatorial line.*

The density Hales-Jewett theorem is a fundamental result of Ramsey theory. It implies several well known results, such as van der Waerden's theorem [20], Szemerédi's theorem on arithmetic progressions of arbitrary length [18] and its multidimensional version [8].

The proof of Furstenberg and Katznelson used ergodic-theory and gave no explicit bound on $c_{n,k}$. Recently, additional proofs of this theorem were found [15, 1, 6]. The proof of [15] is the first combinatorial proof of the density Hales-Jewett theorem, and also provides effective bounds for $c_{n,k}$. In a second paper [16] in this project, several values of $c_{n,3}$ are computed for small values of n. Using ideas from recent work [7, 11, 14] on the construction of Behrend [3] and Rankin [17], they also prove the following asymptotic bound on $c_{n,k}$. Let $r_k(n)$ be the maximal size of a subset of $[n]$ without an arithmetic progression of length k, then:

Theorem 3 ([16]) *For each $k \geq 3$, there is an absolute constant $C > 0$ such that*

$$c_{n,k} \geq Ck^n \left(\frac{r_k(\sqrt{n})}{\sqrt{n}} \right)^{k-1} = k^n exp \left(-O(\log n)^{1/\lceil \log_2 k \rceil} \right).$$

We prove

Theorem 4 *For each $k \geq 3$, there is an absolute constant $C > 0$ such that*

$$c_{n,k} \geq Ck^n \frac{r_k(kn)}{kn \log kn} = k^n exp \left(-O(\log n)^{1/(k-1)} \right).$$

Our proof uses a communication complexity point of view, in the Number On the Forehead (NOF) model [5]. In this model k players compute together a boolean function $f : X_1 \times \cdots \times X_k \to \{0, 1\}$. The input, $(x_1, x_2, \ldots, x_k) \in X_1 \times \cdots \times X_k$, is presented to the players in such a way that the i-th player sees the entire input except x_i. A protocol is comprised of rounds, in each of which every player writes one bit (0 or 1) on a board that is visible to all players. The choice of the written bit may depend on the player's input and on all bits previously written by himself and others on the board. The protocol ends when all players know $f(x_1, x_2, \ldots, x_k)$. The cost of a protocol is the number of bits written on the board, for the worst input. The deterministic communication complexity of f, $D(f)$, is the cost of the best protocol for f.

Two key definitions in the number on the forehead model are a *cylinder* and a *cylinder intersection*. We say that $C \subseteq X_1 \times \cdots \times X_k$ is a cylinder in the i-th coordinate if membership in C does not depend on the i-th coordinate. Namely, for every y, y' and $x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_k$ there holds $(x_1, x_2, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_k) \in C$ iff $(x_1, x_2, \ldots, x_{i-1}, y', x_{i+1}, \ldots, x_k) \in C$. A cylinder intersection is a set C of the form $C = \cap_{i=1}^{k} C_i$ where C_i is a cylinder in the i-th coordinate.

Every c-bit communication protocol for a function f partitions the input space into at most 2^c cylinder intersections that are monochromatic with respect to f (see [12] for more details). Thus, one way to relax $D(f)$ is to view it as a coloring problem. Denote by $\alpha(f)$ is the largest size of a 1-monochromatic cylinder intersection with respect to f, and by $\chi(f)$ the least number of monochromatic cylinder intersections that form a partition of $f^{-1}(1)$. Obviously, $D(f) \geq \log \chi(f)$, and as we shell see, for special families of functions this bound is nearly tight, including the function $Part_{n,k}$ that we are interested in.

The function $Part_{n,k} : (2^{[n]})^k \to \{0, 1\}$ is defined as follows, $Part_{n,k}(S_1, \ldots, S_k) = 1$ if and only if $(S_1, \ldots, S_k)$ is a partition of $[n]$. Tesson [19] used the Hales-Jewett theorem to prove that $D(Part_{n,k}) \geq \omega(1)$. We observe that in fact the Hales-Jewett theorem is equivalent to this statement. This follows from the following strong relation $Part_{n,k}$ has with the Hales-Jewett number.

Theorem 5 *For every $k \geq 3$ and $n \geq 1$ there holds:*

1. *$c_{n,k} = \alpha(Part_{n,k})$, and*
2. *$\chi(Part_{n,k})$ is equal to the minimal numbers of colors required to color $[k]^n$ so that there is no monochromatic combinatorial line.*

Theorem 5 entails an alternative characterization of the Hales-Jewett theorem and its density version:

Theorem 6 (Hales-Jewett theorem) *For every fixed $k \geq 3$, one has $D(Part_{n,k}) = \omega(1)$.*

Theorem 7 (Density Hales-Jewett theorem) *For every $k \geq 3$ there holds*

$$\lim_{n\to\infty} \alpha(Part_{n,k})/k^n = 0.$$

Given the central role the Hales-Jewett theorem plays in Ramsey Theory, and the intricacy of its proof, it would be very nice to find a proof of the Hales-Jewett theorem in the framework of communication complexity.

The relation between $c_{n,k}$ and communication complexity also suggests a way to prove a lower bound on $c_{n,k}$, such a bound will follow from an efficient communication protocol for $Part_{n,k}$. We show indeed that $D(Part_{n,k}) \leq O\left(\log n\right)^{1/(k-1)}$, and thus the lower bound follows. We prove the relationship between $c_{n,k}$ and $\alpha(Part_{n,k})$ in Section 2, and Theorem 4 is proved in Section 3.

2 A communication complexity version of Hales-Jewett

We start with the definition of a *star*: A star is a subset of $X_1 \times \cdots \times X_k$ of the form

$$\{(x_1', x_2, \ldots, x_k), (x_1, x_2', \ldots, x_k), \ldots, (x_1, x_2, \ldots, x_k')\},$$

where $x_i \neq x_i'$ for each i. We refer to $(x_1, x_2, \ldots, x_k)$ as the star's *center*. Cylinder intersections can be easily characterized in terms of stars.

Lemma 8 ([12]) *A subset $C \subseteq X_1 \times \cdots \times X_k$ is a cylinder intersection if and only if for every star that is contained in C, its center also belongs to C.*

The function $Part_{n,k}$ has the property that for every $S_1, \ldots, S_{k-1} \in 2^{[n]}$ there is at most one set $S \subset [n]$ such that $Part_{n,k}(S_1, \ldots, S_{k-1}, S) = 1$. We call such a function a *weak graph function*, as opposed to a *graph function* [2] where there is always exactly one such S.

Graph functions have some particularly convenient properties, one of which is that cylinder intersections are characterized simply by the existence of stars, as proved in [13]. The same proof also works for weak graph functions and gives:

Lemma 9 ([13]) *Let $f : X_1 \times \cdots \times X_k \to \{0, 1\}$ be a weak graph function and $C \subseteq f^{-1}(1)$. The set C is a (1-monochromatic) cylinder intersection with respect to f if and only if it does not contain a star.*

Proof [of Theorem 5] As in [19], define a bijection ψ from $Part_{n,k}^{-1}(1)$ to $[k]^n$. A k-tuple $(S_1, \ldots, S_k)$ is mapped to $(j_1, \ldots, j_n) \in [k]^n$ where j_i is the index of the set S_{j_i} that contains i. Since $S_1, \ldots, S_k$ form a partition of $[n]$ this map is a bijection.

Now consider a 1-monochromatic star $(S'_1, \ldots, S_k), \ldots, (S_1, \ldots, S'_k)$ with respect to $Part_{n,k}$. Since this star is 1-monochromatic, it implies that in each of the families $(S'_1, \ldots, S_k), \ldots, (S_1, \ldots, S'_k)$, all subsets are pairwise disjoint. As a result, because $k \geq 3$, we get that the subsets $(S_1, \ldots, S_k)$ are also pairwise disjoint. This determines S'_j uniquely and implies that $S'_j = S_j \cup ([n] \setminus (\cup_{j=1}^k S_j))$, for every $j = 1, \ldots, k$. Therefore, if we consider $\psi(S'_1, \ldots, S_k), \ldots, \psi(S_1, \ldots, S'_k)$ and place them in a $k \times n$ table, then the columns of this table all belong to $\{(x, x, \ldots, x) : x \in [k]\} \cup \{(1, 2, \ldots, k)\}$. The i-th column of this table is in $\{(x, x, \ldots, x) : x \in [k]\}$ if $i \in S_j$ for some $j \in [k]$ and otherwise the i-th column is equal to $(1, 2, \ldots, k)$. Hence the stars in $Part_{n,k}^{-1}(1)$ are in one-to-one correspondence with combinatorial lines in $[k]^n$.

It follows that $c_{n,k} = \alpha(Part_{n,k})$, and that $\chi(Part_{n,k})$ is equal to the minimal number of colors required to color $[k]^n$ so that there is no monochromatic combinatorial line. ■

It is left to show the equivalence between Theorem 6 and the Hales-Jewett theorem, and Theorem 7 with its density version. The latter equivalence follows immediately from part 1 of Theorem 5. The equivalence of Theorem 6 to the Hales-Jewett theorem follows from part 2 of Theorem 5, and the following lemma [1]:

Theorem 10 ([13]) *For every weak graph function $f : X_1 \times \cdots \times X_k \to \{0, 1\}$, there holds*

$$\log \chi(f) \leq D(f) \leq \log \chi(f) + k.$$

3 A lower bound on $c_{n,k}$

We first give an efficient protocol for $Part_{n,k}$, and then explain how it implies Theorem 4.

Lemma 11 *For every fixed $k \geq 3$ it holds that*

$$D(Part_{n,k}) \leq O\left(\log \frac{kn \log kn}{r_k(kn)}\right) = O\left(\log n\right)^{1/(k-1)}.$$

Proof The protocol is a reduction to the Exactly-n function defined by *Exactly-n*$(x_1, \ldots, x_k) = 1$ if and only if $\sum_{i=1}^k x_i = n$, where $(x_1, \ldots, x_k)$ are non-negative integers. The reduction is simple, given an instance $(S_1, \ldots, S_k)$ to be computed, the players do the following:

1. The k-th player checks whether $S_1, \ldots, S_{k-1}$ are pairwise disjoint. If they are not pairwise disjoint the protocol ends with rejection.
2. The first player checks whether $S_2, \ldots, S_{k-1}$ are each disjoint from S_k. If this is not the case then the protocol ends with rejection.
3. The second player checks whether $S_1 \cap S_k = \emptyset$ and rejects if not.
4. The players use a protocol for Exactly-n to determine whether $\sum_{i=1}^k |S_i| = n$. The protocol accepts if and only if equality holds, and the sum is exactly n.

The first three steps of the above protocol require three bits of communication, and the last part uses a protocol for Exactly-n. Chandra, Furst and Lipton [5] gave a surprising protocol for Exactly-n with at most $O(\log \frac{kn \log kn}{r_k(kn)})$ bits of communication. It was later observed by Beigel, Gasarch and Glenn [4] that when plugging in the

[1] Lemma 10 was proved in [13] for graph functions, but the same proof works for weak graph functions. The only slight change needed is that the k-th player first checks whether there is a value y for which $f(x_1, \ldots, x_{k-1}, y) = 1$. If such a value does not exists then obviously $f(x_1, \ldots, x_{k-1}, x_k) = 0$.

bounds on $r_k(n)$ given by the construction of Rankin [17] one gets $O(\log \frac{kn \log kn}{r_k(kn)}) = O\left(\log n\right)^{1/(k-1)}$. ■

Proof [of Theorem 4] As mentioned before $\log \chi(f) \leq D(f)$ holds for every function f, combined with Lemma 11 this gives

$$\chi(Part_{n,k}) \leq exp\left(D(Part_{n,k})\right) \leq O\left(\frac{kn \log kn}{r_k(kn)}\right) = exp\left(O\left(\log n\right)^{1/(k-1)}\right).$$

Since $\alpha(f) \geq |f^{-1}(1)|/\chi(f)$ holds also for every f and $|Part^{-1}_{n,k}(1)| = k^n$, we get

$$\alpha(Part_{n,k}) \geq \frac{k^n}{\chi(Part_{n,k})} \geq O\left(k^n \frac{r_k(kn)}{kn \log kn}\right) = k^n exp\left(-O(\log n)^{1/(k-1)}\right).$$

The lower bound on $c_{n,k}$ now follows from part 1 of Theorem 5. ■

Theorem 4 actually proves a lower bound on the maximal size of a *Fujimura set* in $\Delta_{n,k}$, similarly to the proof in [16], where the following definitions are taken from: Let $\Delta_{n,k}$ denote the set of k-tuples $(a_1, \ldots, a_k) \in \mathbb{N}^k$ such that $\sum_{i=1}^k a_i = n$. Define a *simplex* to be a set of k points in $\Delta_{n,k}$ of the form $(a_1 + r, a_2, \ldots, a_k), (a_1, a_2 + r, \ldots, a_k), \ldots, (a_1, a_2, \ldots, a_k + r)$ for some $0 < r \leq n$. Define a Fujimura set to be a subset $B \subset \Delta_{n,k}$ that contains no simplices. Observe that

- $\Delta_{n,k} = (\textit{Exactly-n})^{-1}(1)$.
- A simplex in $\Delta_{n,k}$ is equivalent to a star.
- $\alpha(\textit{Exactly-n})$ is equal to the maximal size of a Fujimura set in $\Delta_{n,k}$, which is denoted by $c^{\mu}_{n,k}$ in [16].

The proof of Theorem 4 gives essentially a lower bound for $\alpha(\textit{Exactly-n})$ via an efficient protocol for Exactly-n, and the lower bound for $c_{n,k}$ is implied from the fact that $D(Part_{n,k}) \leq D(\textit{Exactly-n})+3$. It is an interesting question whether this bound is tight, or is it the case that $D(Part_{n,k})$ can be significantly smaller than $D(\textit{Exactly-n})$.

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