Network Capacity Bound for Personalized PageRank in Multimodal Networks

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Abstract

In a former paper [13] the concept of Bipartite PageRank was introduced and a theorem on the limit of authority flowing between nodes for personalized PageRank has been generalized. In this paper we want to extend those results to multimodal networks. In particular we introduce a hypergraph type that may be used for describing multimodal network where a hyperlink connects nodes from each of the modalities. We introduce a generalisation of PageRank for such graphs and define the respective random walk model that can be used for computations. we finally state and prove theorems on the limit of outflow of authority for cases where individual modalities have identical and distinct damping factors.

Keywords: multimodal networks, social networks, PageRank

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1 Introduction

The notion of the PageRank as a measure of importance of a web page was introduced in [19]. The basic model was soon extended in diverse directions (personalized PageRank, topical PageRank, Ranking with Back-step, Query-Dependent PageRank, Lazy Walk Pagerank etc.) to support numerous applications (Web page ranking, client and seller ranking, clustering, classification of web pages, word sense disambiguation, spam detection, detection of dead pages etc.) [15].

In this paper our attention is leaned towards a certain aspect of personalized PageRank, related to its usage as a way to cluster nodes of an undirected graph¹. A cluster is frequently deemed to be a group of pages such that it is unlikely to be left by a random walker. This idea is directly related to an important theorem on limitation of authority flow, presented in e.g. [8].

In a number of application domains where the relationships between objects may be conveniently represented in the form of a graph, where we think of bimodal sets of objects (that is where an object may belong to one of two modalities and relationships only between the modalities are allowed). The success story of PageRank prompted many researchers to apply it also to those graphs. Let us just mention studies concerning mutual evaluations of students and lecturers [17], reviewers and movies in a movie recommender systems, or authors and papers in scientific literature or queries and URLs in query logs [10], or performing image tagging [1] or for tagging in social networks [14] or sentimental analysis [6].

For a number of reasons direct transfer of PageRank to domains closely related to social networks has various deficiencies. For example the PageRank was explicitly designed to remove periodicity from the graph structure, whereas already the bipartite graphs have explicitly this kind of structure. It ranks all the graph nodes in a single ordering, while one would prefer having separate ranks for each of the modalities. It assigns same default weights to each node so that one modality can be assigned preferential total weight, while we may be interested in separating them. Last not least in some cases social networks can have even more modalities than two so that traditional graph representation may not be adequate any more. For example one may have three modalities like: products, clients and labels assigned by clients

¹An unoriented graph may serve the representation of relationships spanned by a network of friends, telecommunication infrastructure or street network of a city

to products, or internet users, webpages and tags assigned to webpages by users etc.

Therefore a suitable generalization of PageRank to such structures is needed in order to retain both advantages of the multimodal graph representation and those of PageRank.

In a former paper [13] a concept of Bipartite PageRank was developed and a theorem on limits of authority flow was proposed for it.

In the current paper we are interested in a further generalization to multimodal networks where we have more modalities,

The fundamental issues here seems to be the fact that unlike networks with one or two modalities, networks with more modalities are in fact hypergraphs, and not ordinary graphs. Hence a new concept of a random walk needs to be developed.

This paper is structured as follows:

In section 2 we will recall the basic definitions applying to unimodal and bimodal networks.

In section 3 we will recall the concept of hypergraph and then in section 4 we will review the literature on various ways of generalizing PageRank to hypergraphs. Afterwards in section 5 we will present our own generalisation of PageRank to multimodal networks that we will call *MuMoRank* and we will present our major result on authority flow in such networks. In section 6 we will illustrate the introduced concept of MuMoRank with a numerical example. Section 7 contains some final remarks.

Our contribution is as follows:

- We propose a new ranking method for nodes in a multimodal hypergraph,
- we investigate the flow limits for personalized version of this ranking and prove respective theorems for
 - for dumping factors identical for each modality and
 - for dumping factors different for each modality

2 Basic concepts in Unimodal and Bimodal networks

Let consider a unimodal network first (e.g. Web) as a graph G = (V, E), where V is the set of nodes (e.g. Web pages) and $E \subset V \times V$ is the set of links. Let us assume that each node $n \in V$ has at least one outgoing edge.

One of the many interpretations of the traditional PageRank is the probability that a knowledgeable but mindless random walker will encounter a given Web page. Knowledgeable because he knows the addresses of all the web pages. Mindless because he chooses a next page to visit without paying attention to any hints on its contents. So upon entering a particular web page, if it has no outgoing links, the walker jumps to any Web page with uniform probability. If there are outgoing links, he chooses with uniform probability one of the outgoing links and goes to the selected web page, unless he gets bored. If he gets bored (which may happen with a fixed probability ζ on any page), he jumps to any Web page with uniform probability. A more detailed introduction random walk concept may be found in [16].

Personalized PageRank, on the other hand, prescribes that upon being bored, the random walker jumps to only a subset of the Web pages, related to his interests. We can consider a mindless page-u-fan random walker who is doing exactly the same, but in case of a jump out of boredom he does not jump to any page, but to the page u. we will speak about a uniform-set-U-fan if he jumps to any of the pages from the set U with uniform probability. we will talk about a hub-page-preferring-set-U-fan if he jumps to members of the set U of pages with probability proportional to their out-degrees.

There are many other variations of the traditional PageRank concept. we shall point here only at the so-called Lazy random walk. It was described e.g. by [7]. It differs from the traditional PageRank in that the random walker before choosing the next page to visit he fist tosses a coin and upon heads he visits the next page and upon tails he stays in the very same node of the network.

So let us introduce some notation. With \mathbf{r} we will denote a (column) vector of ranks: $r_j^{(t)}$ will mean the PageRank of page j. All elements of $\mathbf{r}^{(t)}$ are non-negative and their sum equals 1.

Let $\mathbf{P} = [p_{ij}]$ be a matrix such that if there is a link from page j to page

i, then $p_{i,j} = \frac{1}{outdeg(j)}$, where outdeg(j) is the out-degree of node j^2 . In other words, \mathbf{P} is column-stochastic matrix satisfying $\sum_i p_{ij} = 1$ for each column j. If a node had an out-degree equal 0, then prior to construction of \mathbf{P} the node is replaced by one with edges outgoing to all other nodes of the network.

Under these circumstances we have

$$\mathbf{r}^{(t)} = (1 - \zeta) \cdot \mathbf{P} \cdot \mathbf{r}^{(t)} + \zeta \cdot \mathbf{s} \tag{1}$$

where s is the so-called "initial" probability distribution (i.e. a column vector with non-negative elements summing up to 1) that is also interpreted as a vector of Web page preferences.

For a knowledgeable walker for each node j of the network $s_j = \frac{1}{|N|}$, where |N| is the cardinality of the set of nodes N constituting the network. For a page-u-fan we have $s_u = 1$, and $s_j = 0$ for any other page $j \neq u$. For a uniform-set-U-fan we get

$$s_j = \begin{cases} \frac{1}{|U|} & \text{if } j \in U \\ 0 & \text{otherwise} \end{cases}, \ j = 1, \dots |N|$$

and for a hub-page-preferring-set-U-fan we obtain

$$s_{j} = \begin{cases} \frac{outdeg(j)}{\sum_{k \in U} outdeg(k)} & \text{if } j \in U \\ 0 & \text{otherwise} \end{cases}, \ j = 1, \dots |N|$$
 (2)

The Lazy-Walk-PageRank differs from the above in the following way:

$$\mathbf{r}^{(l)} = (1 - \zeta) \cdot (0.5\mathbf{I} + 0.5\mathbf{P}) \cdot \mathbf{r}^{(l)} + \zeta \cdot \mathbf{s}$$
(3)

where \mathbf{I} is the identity matrix.

One can easily derive relation to the traditional PageRank.

$$\mathbf{r}^{(l)} = \frac{1-\zeta}{1+\zeta} \cdot (\mathbf{P}) \cdot \mathbf{r}^{(l)} + \frac{2\zeta}{1+\zeta} \cdot \mathbf{s}$$

This means that $\mathbf{r}^{(l)}$ for ζ is the same as $\mathbf{r}^{(t)}$ for $\frac{2\zeta}{1+\zeta}$ In a former paper [13] is was proven that

² For some versions of PageRank, like TrustRank $p_{i,j}$ would differ from $\frac{1}{outdeg(j)}$ giving preferences to some outgoing links over the other. We are not interested in such considerations here.

Theorem 1 For the preferential personalized PageRank we have

$$p_o\zeta \le (1-\zeta)\frac{|\partial(U)|}{Vol(U)}$$

where $\partial(U)$ is the set of edges leading from U to the nodes outside of U (the so-called "edge boundary of U"), hence $|\partial(U)|$ is the cardinality of the boundary, and Vol(U), called volume or capacity of U is the sum of outdegrees of all nodes from U.

It is easy to demonstrate that for Lazy walk:

Theorem 2 For the preferential lazy personalized PageRank we have

$$p_o\zeta \le \frac{1-\zeta}{2} \frac{|\partial(U)|}{Vol(U)}$$

where $\partial(U)$ is the set of edges leading from U to the nodes outside of U (the so-called "edge boundary of U"), hence $|\partial(U)|$ is the cardinality of the boundary, and Vol(U), called volume or capacity of U is the sum of outdegrees of all nodes from U.

Let us consider now bipartite graphs. Bipartite (as well as the multimodal ones that we will talk about later) graphs a non-directional, hence their out-degree and in-degrees are identical so that we will from now on consider only the node degrees.

Some non-directed graphs occurring e.g. in social networks are in a natural way bipartite graphs. That is there exist nodes of two modalities and meaningful links may occur only between nodes of distinct modalities (e.g. clients and items purchased by them).

Some literature exists already for such networks attempting to adapt PageRank to the specific nature of bipartite graphs, e.g. [10]. Whatever investigations were run, apparently no generalization of theorem 1 was approached.

One seemingly obvious choice would be to use the traditional PageRank, like it was done in papers [17, 1]. But this would be conceptually wrong because the nature of the super-node would cause authority flowing between nodes of the same modality which is prohibited by the definition of these networks.

Therefore in a former paper [13] a way was proposed to close this conceptual gap and a novel approach to Bipartite PageRank was introduced and the Theorem 1 was extended to this case.

Let us briefly recall the basic concepts. Consider the flow of authority in a bipartite network with two distinct super-nodes: one collecting the authority from items and passing them to clients, and the other the authority from clients and passing them to items.

$$\mathbf{r}^p = (1 - \zeta^{kp}) \cdot \mathbf{P}^{kp} \cdot \mathbf{r}^k + \zeta^{kp} \cdot \mathbf{s}^p \tag{4}$$

$$\mathbf{r}^k = (1 - \zeta^{pk}) \cdot \mathbf{P}^{pk} \cdot \mathbf{r}^p + \zeta^{pk} \cdot \mathbf{s}^k \tag{5}$$

The following notation is used in these formulas

- \mathbf{r}^p , \mathbf{r}^k , \mathbf{s}^p , and \mathbf{s}^k are stochastic vectors, i.e. the non-negative elements of these vectors sum to 1;
- the elements of matrix \mathbf{P}^{kp} are: if there is a link from page j in the set of *Clients* to a page i in the set of *Items*, then $p_{ij}^{kp} = \frac{1}{deg(j)}$, otherwise $p_{ij}^{kp} = 0$;
- the elements of matrix \mathbf{P}^{pk} are: if there is a link from page j in the set of Items to page i in the set of Clients, then $p_{ij}^{pk} = \frac{1}{deg(j)}$, and otherwise $p_{ij}^{pk} = 0$;
- $\zeta^{kp} \in [0,1]$ is the boring factor when jumping from *Clients* to *Items*;
- $\zeta^{pk} \in [0,1]$ is the boring factor when jumping from Items to Clients.

Definition 1 The solutions \mathbf{r}^p and \mathbf{r}^k of the equation system (4) and (5) will be called item-oriented and client-oriented bipartite PageRanks, resp.

For such settings we formulated a theorem for bipartite PageRank analogous to the classical theorem 1.

Theorem 3 For the preferential personalized bipartite PageRank we have

$$p_{p,o}\zeta^{pk} \le (1 - \zeta^{kp}) \frac{|\partial(\frac{U^k}{U^p})|}{\min(Vol(U^k), Vol(U^p))}$$

and

$$p_{k,o}\zeta^{kp} \le (1 - \zeta^{pk}) \frac{|\delta(\frac{U^p}{U^k})|}{\min(Vol(U^k), Vol(U^p))}$$

where

- $p_{k,o}$ is the sum of authorities from the set Clients\\U^k,
- $p_{p,o}$ is the sum of authorities from the set $Items \setminus U^p$,
- $\partial(\frac{U^k}{U^p})$ is the set of edges outgoing from U_k into nodes from Items\\U_p (that is "fan's border" of U^k),
- $\partial(\frac{U^p}{U^k})$ is the set of edges outgoing from U^p into nodes from Clients\ U^k (that is "fan's border" of U^p),
- $Vol(U^k)$ is the sum of out-degrees of all nodes from U^k (capacity of U^k)
- $Vol(U^p)$ is the sum of out-degrees of all nodes from U^p (capacity of U^p)

Note here that the above concept cannot be directly generalized to lazy-random-walks in bipartite graphs because a boring factor that is different for each of the modalities has been introduced above. Under such circumstances the amount of authority within one modality would grow at the expense of the other modality. So the introduction of a lazy random walk in bipartite graphs would require that the boring factor is equal in both modalities. This path will be followed in our generalization to multiple modalities.

3 Hypergraphs

To extend these results to multimodal networks we have first to introduce the concept of random walk through such a network.

Let us recall what the multiple modalities in a network may mean. One such situation when multiple modalities are used is when a consumer buys a product and attaches a label "good" or "bad" to it. Another case is when a community user evaluates a web page and attaches one or more tags to it. A teacher may evaluate a student. There are many other possibilities.

Note that in general case of a multimodal social network, a link ties together more than two objects so that it is not an edge but rather a hyperedge, so we have to handle hypergraphs.

Now let us turn to a more formal description of the issue. A hypergraph is generally defined as follows:

$$HG = (N, HE)$$

where N is the set of nodes, $HE \subseteq 2^N - \{\}$ is the set of hyperedges, each hyperedge being a non-empty subset of N

However, we are interested in a special subtype of the hypergraphs. Let M be the number of distinct modalities and

$$N = N_1 \cup N_2 \cup \ldots \cup N_M$$

where N_i is a set of nodes of one modality, and for $i \neq j$ the following holds: $N_j \cap N_i = \{\}$, that is intersection of modalities is empty. Then a multimodal hypergraph be defined as

$$HGM = (N, HEM)$$

where for any $h \in HEM \ card(h \cap N_j) = 1 \ \text{for} \ j = 1, ..., M$. So each multimodal hyperedge is of the same cardinality (M).

4 Various ways of generalizing PageRank to hypergraphs

Over the recent years various generalization approaches for PageRank occurred, including ones that are either named "HyperRank" or "HyperPageRank" or are related to ranking in hypergraphs.

As these generalisations often invoke the random walker interpretation of PageRank, let us recall it. PageRank is therein the probability that a random walker finds himself on a given node under stationary conditions. A walker in a graph performs the following operation. Being at a node, with some probability $(1-\zeta)$ he walks to any of its neighbours via outgoing links with uniform probability³. With the rest probability ζ he gets bored⁴ and hence

 $^{^3\}mathrm{If}$ this probability is not uniform then we talk about weighted PageRanks, like TrustRank

 $^{^{4}\}zeta$ is called boring or damping factor or (authority) emission rate.

jumps to any node of the network with probability described by some "initial vector" s. Probability distribution s differentiates the PageRank types. If no element of s is equal zero, we speak about traditional Pagerank, if only a subset of values is non-zero, then we speak about personalised PageRank. If the distribution over non-zero values is uniform, then we speak about uniform PageRank, if these probabilities are proportional to the number of (in-going or out-going) edges, then we speak about (authority-preferring or hub-preferring) preferential PageRank. If, prior to any of the mentioned activities, then walker may decide with some probability, that he will stay at the node instead of walking or jumping, then we talk about lazy PageRank.

One stream of research is not related to multimodal generalizations, but rather is concentrated on stabilizing or accelerating PageRank computations. This idea is represented e.g. by [20, 3]. Essentially instead of single pages, disjoint groups of them (like domains or hosts) are considered and with each link from a page A to page B a hyperedge is associated containing the node B and all the nodes from the group to which A belongs. Such an approach is claimed to contribute to fighting spam. In the similar stream, [4] uses the concept of hypergraph partitioning for assignment of tasks for efficient PageRank computation by available processors.

Authors of [11, 12] deal with generalisation of PageRank to folksonomies, that is hypergraphs with three modalities. They convert the hypergraph to an ordinary undirected graph, with the set of nodes being the union of all three modalities and edges occurring between nodes that belong to some common hyperedge. Then they apply the lazy version of the traditional PageRank. [2] defines, following the mentioned and other approaches, a random walk over the hypergraph in such a way as if there were two kinds of nodes, ordinary nodes and hyperedges which are connected by ordinary (undirected) edges whenever a hyperedge is incidental with a node. In such a graph the traditional PageRank is applied. A generalisation to weighted and personalised graphs is done in a straight-forward way. [9] considers random walks over hypergraphs in the same spirit concentrating on regular forms of the hypergraph (with a fixed size hypergraphs or nodes incident with a fixed number of hyperedges). the goal is to compute the cover time. [18] seeks communities in hypergraphs via random walks.

⁵In undirected networks, we speak only about preferential PageRank. But we can also treat the undirected links as pairs of links pointing in both directions. In this paper, when talking about generalised graphs, we will always think about undirected, that is buildirectional links.

Our approach to generalisation of PageRank to hypergraphs differs from the ones mentioned above in a number of ways. Unlike [9], we do not constrain ourselves to regular graphs. Unlike [11, 12] we do not restrict our approach to only three modalities. Note also that in case of three modalities only, [2] approach to PageRank computation is essentially equivalent to that of [11, 12]. The essential difference to our approach is the following: We consider all the modalities separately. We think that it is not a usable information whether or not a user has a higher rank than a web page, or a client than a product. We would rather compare members of modalities. As we handle the modalities separately, each can have a separate decay factor which is not the case in the abovementioned approaches. Hence the values of ranks will be different. The differences will become more visible, if the number of nodes being "personalised" differs between the modalities. Last not least we introduce the concept of separate supernodes for each of the modalities so that the authority is not lost by one modality and passed to another because we consider the modalities as separate, incomparable concepts. This separation is an important difference between our approach and the existent ones. And we consider such an assumption as a quite natural one.

It is worth noting that there exist notions of "HyperRank" not related to PageRank. [21] uses hypergraphs to associate images, their tags and geolocations. It creates its own "HyperRank" not being a generalisation of PageRank, but rather a ranking of objects in response to a user query, with an explicit ranking formula. [5] applies the idea of hypergraph in music recommendation. As previously, the ranking induced has not been derived from PageRank, but is rather a closed-form approach.

Apparently in none of these approaches the problem that we want to deal with here was considered:

- sealed off modalities from which authority does not flow out
- varying damping / boring factor in various modalities
- the problem of outflow of authority from a selected set of nodes

5 Ranks and Random Walks in Multimodal Networks

5.1 Basic definitions and concepts

Multimodality with more than 2 modalities means that we encounter a problem when we want to use the notion of random walker as a vehicle to generalise PageRank to social networks. A random walker cannot jump through a hyper edge to another object because there may be more than one "at the other end" to jump to.

So let us look at the multimodal network in a different way: let us define "generalized nodes" to be either normal nodes (objects) or hyperedges (links). Let us further introduce the "generalized edges", each linking a hyper edge with a node that it is adjacent to.

Now the "generalized graph" consisting of "generalized nodes" and "generalized edges" is an ordinary graph through which a random walker can go just like for the traditional Pagerank. Note anyway that in this case we have a bipartite graph, with two kinds of nodes. There are however two issues to be still resolved:

- how to interpret the jumps from the hyperedge back to the node that one was at before
- how the flow of authority is to be organized for the jumps out of boredom.

The first issue is closely related to the concept of so-called "lazy walk" that is that at a given moment one does or does not move or jump further (this may be thought of as simulation of a walker staying for a shorter or a longer time at a node). This would require to modify the theorem stated above for lazy random walk.

The second issue is of course that one can get bored at different rates at different modalities.

Now define the generalized graph as follows:

$$GG(HGM) = (GN, GE)$$

where $GE \subseteq N \times HEM$ where e = (n, hem) iff $n \in hem$ and $GN = N \cup HEM$.

Obviously, the GG is an ordinary bipartite graph with two node subsets N and HEM, but we cannot apply here directly the approach from [13] to bipartite graphs. This is because we do not want the hyper-edges HEM neither to emit nor to receive any authority to and from supernode.

Instead let us assume that each modality node subset N_i has its own supernode to which each node (of whatever modality) emits authority and it redistributes the authority among its own nodes only. For the sake of simplicity let us have a common emission rate ζ for all of them.

Let us further assume that each modality has a sum of authority equal to one.

Let us now consider a random walker through the graph GG. When in N_i , the random walker may decide either to perform a "boring" jump with probability ζ to any other node of the same modality while choosing the target node proportionally to its in/out degree. With probability $1-\zeta$ he chooses uniformly to go out through one of the edges leading to HEM set. When in HEM, he may go out through any of the outgoing edges with uniform probability (landing in N). For each modality for each node we define MuMoRank as the probability that the abovementioned random walker finds himself in this node given he is in this modality.

So let us introduce some notation. With $\mathbf{r_N}$ we will denote a (column) vector of MuMoRanks: $r_{N,j}$ will mean the MuMoRank of node j. All elements of $\mathbf{r_N}$ are non-negative and their sum equals 1 for each subset N_i of N (that is modality i). With $\mathbf{r_{HEM}}$ we will denote a (column) vector of supplementary ranks of hyperedges (Hyperedge-MuMoRanks).

Let

$$\mathbf{P}_{\mathbf{N}->\mathbf{HEM}} = [p_{N->HEM,ij}]$$

be a matrix such that if there is a link from node j to hyperedge i, then $p_{N->HEM,i,j} = \frac{1}{deg(j)}$, where deg(j) is the degree of node j. In other words, $\mathbf{P_{N->HEM}}$ is column-stochastic matrix satisfying

$$\sum_{i} p_{N->HEM,ij} = 1$$

for each column j. If a node had a degree equal 0, then prior to construction of $\mathbf{P}_{\mathbf{N}->\mathbf{HEM}}$ the node is removed from the network.

Let

$$\mathbf{P_{HEM->N}} = [p_{HEM->N,ij}]$$

be a matrix such that if there is a link from hyperedge j to node i, then $p_{HEM->N,i,j} = \frac{1}{deg(j)}$, where deg(j) is the degree of hyperedge j. In other words, $\mathbf{P_{HEM->N}}$ is column-stochastic matrix satisfying

$$\sum_{i} p_{HEM->N,ij} = 1$$

for each column j. There exists no hyperedge with degree equal 0. In fact each has the degree equal to number of modalities.

Under these circumstances we have

$$\mathbf{r}_{HEM} = (1 - \zeta) \cdot \mathbf{P}_{\mathbf{N} - > \mathbf{HEM}} \cdot \mathbf{r}_{\mathbf{N}}$$
 (6)

$$\mathbf{r}_N = \mathbf{P}_{\mathbf{HEM} - > \mathbf{N}} \cdot \mathbf{r}_{\mathbf{HEM}} + \zeta \cdot \mathbf{s} \tag{7}$$

where s is the so-called "initial" probability distribution (i.e. a column vector with non-negative elements summing up to 1 for each modality separately) that is also interpreted as a vector of node preferences.

The limit theorem for modalities is similar to lazy walk preferential theorem 2.

Let us now consider a generalization of personalised PageRank to hypergraphs, in the spirit just mentioned in case of bipartite networks. Let $U^{(i)}$ be the set of nodes of modality i to which one jumps preferentially upon being bored, and let such a set exist for each modality. This means that $s_j = \frac{1}{|U^{(i)}|}$ if node j belongs to modality i and lies in the set $U^{(i)}$, and is zero otherwise.

Let us now think about a fan of the group of nodes $U^{(1)}, \ldots, U^{(M)}$ who prefers the hubs, and assume first that

$$U^{(1)} = N_1, \dots, U^{(M)} = N_M$$

Assume further that at the moment t we have the following state of authority distribution: node j of modality i contains

$$r_j(t) = \frac{deg(j)}{\sum_{k \in N_i} deg(k)}$$

5.2 Dumping factors equal zero

Let us fix ζ at zero.

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Let us consider now the moment t+1. Node j of the modality i passes into each outgoing link the authority (to each incident hyperedge) $\frac{1}{\sum_{k \in N_i} deg(k)}$. This quantity is obviously identical for each node of the modality i. But as each hyperedge is connected to exactly one node of each modality, hence $\sum_{k \in N_i} deg(k)$ is also identical for each modality, and in effect each node gives each hyperedge the same quantity of authority. As the hyperedge distributes the authority evenly, the same amount returns to each node again. So we have equilibrium - the authority distribution does not change.

Let us now turn to the general case of $U^{(i)}$ not necessary identical with N_i , we will elaborate a couple of theorems limiting the flow of authority in the hypergraphs.

5.3 Dumping factors identical for all modalities

Let us make first the simplification that all the ζ s are the same - just equal ζ . Let us consider the way how we can limit the flow of authority in a single channel. The amount of authority passed to a node consists of two parts: a variable one being a share of the authority at the feeding end of the channel and a fixed one coming from a super-node. So, by increasing the variable part say in a vicinity of a node we come to the point that the receiving end gets less authority than was there on the other end of the channel because of the "redistribution" role of the supernode(s).

Let us seek the amount of authority d such that multiplied by the number of links of a sending node will be not lower than the authority of this node and that after the time step its receiving node would have also amount of authority equal or lower than d multiplied by the number of its in-links. That is we want to have that:

$$d \cdot (1 - \zeta) + \frac{\zeta}{\sum_{v \in U^{(i)}} deg(v)} \le d$$

The above relationship corresponds to the situation that on the one hand if a node of any modality has at most d amount of authority per link, then it sends to a hyperedge at most $d \cdot (1-\zeta)$ authority via the link . The receiving hyperedge redistributes to each of the links at most $d \cdot (1-\zeta)$. Any node belonging to an $U^{(i)}$ gets additionally from the supernode exactly $\frac{\zeta}{\sum_{v \in U^{(i)}} deg(v)}$ authority per its link. We seek a d such that these two components do not exceed d together.

This implies immediately, that

$$d \ge \frac{1}{\sum_{v \in U^{(i)}} deg(v)}$$

for each modality i.

So we obtain a satisfactory d when

$$d_{sat} = max_{i=1,...,M} \frac{1}{\sum_{v \in U^{(i)}} deg(v)}$$

Now we are ready to formulate a theorem for *MuMoRanks* (multimodal PageRank) limiting the outflow of authority. analogous to the classical theorem 2.

Theorem 4 For the preferential personalized MuMoRank we have

$$\sum_{i} p_{i,o}\zeta \le (1-\zeta) \frac{|\partial U|}{\min_{i=1,\dots,M}(HVol(U^{(i)}))}$$

where

- $p_{i,o}$ is the sum of authorities from the set $N_i \setminus U^{(i)}$,
- ∂U is the sum over all hyperedges of $\frac{l_n*l_o}{M}$, where l_n is the number of links from U intersecting with this hyperedge, l_o is the number of links from not U intersecting with this hyperedge, (note that $l_n + l_o = M$.)
- $HVol(U^{(i)})$ is the sum of degrees of all hyperedges intersecting with $U^{(i)}$ (capacity of $U^{(i)}$)

The proof is analogous as in case of classical PageRank presented in [13], using now the quantity d_{sat} we have just introduced. The idea is that the authority outflowing through outlinks from U to the remaining nodes must enter again the U via the respective supernodes. So the quantity $\sum_i p_{i,o} \zeta$ is the authority re-entering the U via the respective supernodes. On the other hand $(1-\zeta)d_{sat}$ is what at most leaves the U via a link leading outside of U. There are $|\partial U|$ such outlinks. So the outflow is

$$(1-\zeta)d_{sat}|\partial U| = (1-\zeta)max_{i=1,\dots,M} \frac{1}{\sum_{v \in U^{(i)}} deg(v)} |\partial U| =$$

$$(1-\zeta)\frac{|\partial U|}{\min_{i=1,\dots,M}\sum_{v\in U^{(i)}}deg(v)}$$

This completes the proof.

We can refine this reasoning assuming that we are not looking for a set of separate d's for each modality.

Consider the situation that on the one hand if a node of ith modality has at most $d^{(i)}$ amount of authority per link, then it sends to a hyperedge at most $d^{(i)} \cdot (1-\zeta)$ authority via the link . The receiving hyperedge redistributes to each of the links at most

$$\frac{1}{M} \sum_{i} d^{(i)} \cdot (1 - \zeta)$$

. Any node belonging to an $U^{(i)}$ gets additionally from the supernode exactly

$$\frac{\zeta}{\sum_{v \in U^{(i)}} deg(v)}$$

authority per its link. We seek $d^{(i)}$ such that these two components do not exceed $d^{(i)}$ together.

So

$$\frac{1}{M} \sum_{i} d^{(i)} \cdot (1 - \zeta) + \frac{\zeta}{\sum_{v \in U^{(i)}} deg(v)} \le d^{(i)}$$

Based on this formulation we can start to seek for such ds that

$$d^{(i)} = d_0 + \frac{\zeta}{\sum_{v \in U^{(i)}} deg(v)}$$

where d_0 is some base authority. This leads immediately to

$$\begin{split} d_0*(1-\zeta) + \frac{1}{M} \sum_i \frac{\zeta}{\sum_{v \in U^{(i)}} deg(v)} \cdot (1-\zeta) + \frac{\zeta}{\sum_{v \in U^{(i)}} deg(v)} \leq d_0 + \frac{\zeta}{\sum_{v \in U^{(i)}} deg(v)} \\ \frac{1}{M} \sum_i \frac{\zeta}{\sum_{v \in U^{(i)}} deg(v)} \cdot (1-\zeta) \leq d_0 \zeta \end{split}$$

So that the satisfactory d_0 would be

$$d_{0,sat} = \frac{1}{M} \sum_{i} \frac{1}{\sum_{v \in U^{(i)}} deg(v)} \cdot (1 - \zeta)$$

This implies in a similar way the theorem

Theorem 5 For the preferential personalized MuMoRank we have

$$\sum_{i} p_{i,o} \zeta \leq (1 - \zeta) ((\sum_{h \in H} \frac{l_{o,h}}{M} \sum_{i \in N_h} \frac{\zeta}{HVol(U^{(i)})}) + (\frac{1}{M} \sum_{i} \frac{\cdot (1 - \zeta)}{HVol(U^{(i)})} \sum_{h \in H} \frac{l_{o,h} * l_{n,h}}{M}))$$

where

- $p_{i,o}$ is the sum of authorities from the set $N_i \setminus U^{(i)}$,
- l_o is the number of links from not U intersecting with this hyperedge,
- $HVol(U^{(i)})$ is the sum of degrees of all hyperedges intersecting with $U^{(i)}$ (capacity of $U^{(i)}$)

5.4 Dumping factors different between modalities

We can repeat now these considerations assuming that the ζ s can differ between the modalities.

Let now consider nonzero $\zeta^{(i)}$ a for modalities i=1,...,M.

Under the same moment t conditions let us consider now the moment t+1. From the modality i node j to the modality l super-node the authority

$$\zeta^{(i)} \frac{deg(j)}{M * \sum_{k \in N_i} deg(k)}$$

flows, and into each outgoing link

$$(1 - \zeta^{(i)}) \frac{1}{\sum_{k \in N_i} deg(k)}$$

is passed to the hyperedge. This quantity is obviously identical for each node of the modality i. But as each hyperedge is connected to exactly one node of each modality, hence the hyperedge obtains

$$\sum_{i=1}^{M} (1 - \zeta^{(i)}) \frac{1}{\sum_{k \in N_i} deg(k)}$$

- same for each hyperedge. Each supernode gets on the other hand the same amount of authority:

$$\sum_{i=1}^{M} \zeta^{(i)}/M$$

. So upon redistribution each node gets the same amount of authority it had before. So again we have equilibrium - the authority distribution does not change.

So with the mentioned initial authority distribution we get a steady state. If we seek a single d for all modalities, then we would proceed as follows.

Let us seek the amount of authority d such that multiplied by the number of links of a sending node will be not lower than the authority of this node and that after the time step its receiving node would have also amount of authority equal or lower than d multiplied by the number of its in-links.

That is we want to have that:

$$d \cdot (1 - \zeta^{(i)}) + \frac{\frac{1}{M} \sum_{i} \zeta^{(i)}}{\sum_{v \in U^{(i)}} deg(v)} \le d$$

The above relationship corresponds to the situation that on the one hand if a node of modality i has at most d amount of authority per link, then it sends to a hyperedge at most

$$d \cdot (1 - \zeta^{(i)})$$

authority via the link . The receiving hyperedge redistributes to each of the links at most

$$d \cdot (1 - \frac{1}{M} \sum_{i} \zeta^{(i)})$$

. Any node belonging to an $U^{(i)}$ gets additionally from the supernode exactly

$$\frac{\frac{1}{M}\sum_{i}\zeta^{(i)}}{\sum_{v\in U^{(i)}}deg(v)}$$

authority per its link. We seek a d such that these two components do not exceed d together.

This implies immediately, that

$$\frac{\frac{1}{M}\sum_{i}\zeta^{(i)}}{\sum_{v\in U^{(i)}}deg(v)} \le d\zeta^{(i)}$$

$$d \ge \frac{\frac{1}{M} \sum_{i} \zeta^{(i)}}{\zeta^{(i)} \sum_{v \in U(i)} deg(v)}$$

for each modality i.

Hence we get a satisfactory d when

$$d_{sat} = max_{i=1,...,M} \frac{\frac{1}{M} \sum_{i} \zeta^{(i)}}{\zeta^{(i)} \sum_{v \in U^{(i)}} deg(v)}$$

So let us formulate the implications for authority outflow.

Theorem 6 For the preferential personalized MuMoRank we have

$$\sum_{i} p_{i,o}\zeta \le \frac{|\partial U^{\zeta}|}{\min_{i=1,\dots,M}(HVol(U^{(i)}))}$$

where

- $p_{i,o}$ is the sum of authorities from the set $N_i \setminus U^{(i)}$,
- $|\partial U^{\zeta}|$ is the sum over all hyperedges of $\frac{l_o*\sum_{i\in N}(1-\zeta^{(i)})}{M}$, where N is the set modalities from U intersecting with this hyperedge, l_o is the number of links from not U intersecting with this hyperedge, (note that card(N) + lo = M.)
- $HVol(U^{(i)})$ is the sum of degrees of all hyperedges intersecting with $U^{(i)}$ (capacity of $U^{(i)}$)

Alternatively think of separate ds for all the modalities. Consider the situation that on the one hand if a node of ith modality has at most $d^{(i)}$ amount of authority per link, then it sends to a hyperedge at most

$$d^{(i)} \cdot (1 - \zeta^{(i)})$$

authority via the link . The receiving hyperedge redistributes to each of the links at most

$$\frac{1}{M}\sum_{i}d^{(i)}\cdot(1-\zeta^{(i)})$$

. Any node belonging to an $U^{(i)}$ gets additionally from the supernode exactly

$$\frac{\overline{\zeta}}{\sum_{v \in U^{(i)}} deg(v)}$$

with $\overline{\zeta} = \frac{1}{M} \sum_{i} \zeta^{(i)}$ authority per its link. We seek $d^{(i)}$ such that these two components do not exceed $d^{(i)}$ together.

So

$$\frac{1}{M} \sum_{i} d^{(i)} \cdot (1 - \zeta^{(i)}) + \frac{\overline{\zeta}}{\sum_{v \in U^{(i)}} deg(v)} \le d^{(i)}$$

Based on this formulation we can start to seek for such ds that $d^{(i)} = d_0 + \frac{\overline{\zeta}}{\sum_{v \in U^{(i)}} deg(v)}$ where d_0 is some base authority. This leads immediately to

$$\begin{split} d_0*\frac{1}{M}\sum_{i}(1-\zeta^{(i)}) + \frac{1}{M}\sum_{i}\frac{\overline{\zeta}}{\sum_{v\in U^{(i)}}deg(v)}\cdot(1-\zeta^{(i)}) + \frac{\overline{\zeta}}{\sum_{v\in U^{(i)}}deg(v)} \leq d_0 + \frac{\overline{\zeta}}{\sum_{v\in U^{(i)}}deg(v)} \\ d_0*(1-\overline{\zeta}) + \frac{1}{M}\sum_{i}\frac{\overline{\zeta}}{\sum_{v\in U^{(i)}}deg(v)}\cdot(1-\zeta^{(i)}) \leq d_0 \\ \frac{1}{M}\sum_{i}\frac{\overline{\zeta}}{\sum_{v\in U^{(i)}}deg(v)}\cdot(1-\zeta^{(i)}) \leq d_0\overline{\zeta} \\ \frac{1}{M}\sum_{i}\frac{1-\zeta^{(i)}}{\sum_{v\in U^{(i)}}deg(v)} \leq d_0 \end{split}$$

From which the satisfactory d_0 can be derived like

$$d_{0,sat} = \frac{1}{M} \sum_{i} \frac{1 - \zeta^{(i)}}{\sum_{v \in U^{(i)}} deg(v)}$$

This implies in a similar way the theorem

Theorem 7 For the preferential personalized MuMoRank we have

$$\sum_{i} p_{i,o} \zeta^{(i)} \leq \left(\sum_{h \in H} \frac{l_{o,h}}{M} \sum_{i \in N_{i}} (1 - \zeta^{(i)}) \left(\frac{\overline{\zeta}}{HVol(U^{(i)})} + \frac{1}{M} \sum_{i} \frac{1 - \zeta^{(i)}}{HVol(U^{(i)})} \right) \right)$$

where

- $p_{i,o}$ is the sum of authorities from the set $N_i \setminus U^{(i)}$,
- $l_{o,h}$ is the number of links from not U intersecting with this hyperedge, (note that $l_n + lo = M$.)
- $HVol(U^{(i)})$ is the sum of degrees of all hyperedges intersecting with $U^{(i)}$ (capacity of $U^{(i)}$)

6 Numerical Example

Let us illustrate the concepts just introduced with a small example.

Assume a fictitious product evaluation database with three modalities:

- users (Eva, Mary, Bob, John, Jane, Ann, Henry, Max),
- products (TVset, VideoPlayer, Laptop, DVDPlayer, Smartphone, Netbook), and
- tags (handsome, welldesigned, beautiful, pretty, annoying, awful, worthless).

Assume that the users have tagged the products as in Table 1

We can construct a hypergraph corresponding to this product evaluation and compute the MuMoRanks for each member of each modality.

Assume that the boring factors ζ are equal 0.3 for users, 0.2 for products and 0.1 for tags. Assume also that our set of preferred nodes consists of: {Eva, Mary, Henry, beautiful, awful, Laptop, Netbook }. Hub-preferring walk is assumed.

In Table 2 we have the resulting MuMoRanks.

Within each modality the sum of MuMoRanks sums up to 1.

The observed outflow of authority from our preferred nodes to the other amounts to 0.2072. Note that

$$HVol(U^{users}) = 12$$

 $HVol(U^{products}) = 9$
 $HVol(U^{tags}) = 11$

Table 1: Product tagging example

User name	Product name	Tag
Eva	TVset	handsome
Eva	VideoPlayer	welldesigned
Eva	Laptop	awful
Eva	Netbook	awful
Mary	TVset	handsome
Mary	Smartphone	handsome
Mary	Laptop	beautiful
Mary	Netbook	beautiful
Bob	Laptop	beautiful
Bob	VideoPlayer	welldesigned
John	VideoPlayer	welldesigned
John	DVDPlayer	welldesigned
Jane	TVset	awful
Jane	VideoPlayer	beautiful
Jane	DVDPlayer	worthless
Jane	Smartphone	worthless
Ann	VideoPlayer	annoying
Ann	DVDPlayer	beautiful
Henry	Netbook	handsome
Henry	Laptop	awful
Henry	DVDPlayer	awful
Henry	Smartphone	awful
Max	Netbook	handsome
Max	Laptop	welldesigned

We get then

$$d_{sat} = 0.1818$$
$$|\partial U^{\zeta}| = 6.8666$$

therefore according to theorem 6 the upper authority outflow limit amounts to 0.7629 (which is much higher than the actual one, this is a typical issue with small networks).

Alternatively if we use theorem 7 to find the bounds on authority, we get the following estimates.

$$d_{0,sat} = 0.0763$$

$$d_{sat}^{users} = 0.0930$$

$$d_{sat}^{products} = 0.0985$$

$$d_{sat}^{tags} = 0.0945$$

and then we get a lower bound of 0.6516 on authority outflow, which is slightly better.

One topic was not touched above, namely that of convergence. But the convergence can be looked for in an analogous way as done for the HITS (consult e.g. [16, Ch. 11]).

7 Concluding Remarks

In this paper we proposed a hypergraph type usable for describing multimodal network where a hyperlink connects nodes from each of the modalities, like users, tags, products etc.. We have introduced a novel approach to the concept of multimodal PageRank for such a network, ranking each modality separately, because we consider the modalities as incomparable in their rankings. We defined the respective random walk model that can be used for computations.

We have proposed (upper) limits for the flow of authority in a multimodal hypergraph and proved theorems on the limit of outflow of authority for cases where individual modalities have identical and distinct damping factors. Finally we illustrated the concepts with an example.

These limits can be used in a number of ways, including verification of validity of clusters in hyper graphs. It is quite a common assumption that the better the cluster the less authority flows out of it. The theorems proven in this paper state that the outgoing authority has a natural upper limit.

Table 2: MuMoRanks of nodes in the hypergraph derived from Table 1

Node name	MuMoRank
Eva	0.2227237898750969
Mary	0.22777717270236
Bob	0.061828005075369515
John	0.033909153659620814
Jane	0.10046820687444284
Ann	0.0451464448214134
Henry	0.23951027791757953
Max	0.06863694887041327
TVset	0.0977834762379729
VideoPlayer	0.1053579150501943
Laptop	0.33408509623747196
DVDPlayer	0.10552136952069643
Smartphone	0.092695605367122
Netbook	0.2645565373828387
handsome	0.17491834988889507
welldesigned	0.11119309198650744
beautiful	0.288215407332984
pretty	0.0
annoying	0.015551677185920565
awful	0.37155624749822336
worthless	0.03856522590376586

This upper limit does not depend on the inner structure of a cluster but rather on its boundaries. So it may be an interesting further research direction to see to what extend this inner structure may influence getting closer or more far away from the theoretical limits.

As a further research direction it is also obvious that finding tighter limits is needed, or a proof that the found limits are the lowest ones possible. This would improve the evaluation of e.g. cluster quality.

In this paper we restricted ourselves to the case of preferential distribution of supernode authority among the nodes. A further research would be needed to cover the case of uniform distribution.

References

- [1] Christian Bauckhage. Image tagging using pagerank over bipartite graphs. In *Proc. of the 30th DAGM Symposium on Pattern Recognition*, pages 426–435, Berlin, Heidelberg, 2008. Springer-Verlag. doi:10.1007/978-3-540-69321-5_43.
- [2] Abdelghani Bellaachia and Mohammed Al-Dhelaan. Random walks in hypergraph. In *Proceedings of the 2013 International Conference on Applied Mathematics and Computational Methods*, 2013.
- [3] Klessius Berlt, Edleno Silva de Moura, André Carvalho, Marco Cristo, Nivio Ziviani, and Thierson Couto. Modeling the web as a hypergraph to compute page reputation. *Inf. Syst.*, 35(5):530–543, July 2010.
- [4] Jeremy T. Bradley, Douglas V. de Jager, William J. Knottenbelt, and Aleksandar Trifunović. Hypergraph partitioning for faster parallel pager-ank computation. In *LECTURE NOTES IN COMPUTER SCIENCE* 3670, pages 155–171. Springer, 2005.
- [5] Jiajun Bu, Shulong Tan, Chun Chen, Can Wang, Hao Wu, Lijun Zhang, and Xiaofei He. Music recommendation by unified hypergraph: Combining social media information and music content. In *Proceedings of the International Conference on Multimedia*, MM '10, pages 391–400, New York, NY, USA, 2010. ACM.
- [6] Fuhai Chen, Yue Gao, Donglin Cao, and Rongrong Ji. Multimodal hypergraph learning for microblog sentiment prediction. In 2015 IEEE

- International Conference on Multimedia and Expo, ICME 2015, Turin, Italy, June 29 July 3, 2015, pages 1–6. IEEE, 2015.
- [7] Fan Chung. Pagerank as a discrete Green's function. In *International Conference to Celebrate the Birthday of Shing-Tung Yau, August 27-September 1, 2008; also Geometry and Analysis I ALM 17, 285-302 (2010)*, 2008.
- [8] Fan Chung. Pagerank as a discrete Green's function. In Lizhen Ji, editor, Geometry and Analysis, I, volume 17 of Advanced Lectures in Mathematics (ALM), pages 285–302. International Press of Boston, 15 July 2011.
- [9] Colin Cooper, Alan M. Frieze, and Tomasz Radzik. The cover times of random walks on random uniform hypergraphs. *Theor. Comput. Sci.*, 509:51–69, 2013.
- [10] Hongbo Deng, Michael R. Lyu, and Irvin King. A generalized co-hits algorithm and its application to bipartite graphs. In Proc. of the 15th ACM SIGKDD International Conf. on Knowledge Discovery and Data Mining, KDD'09, pages 239–248, Paris, June 28-July 1 2009. ACM New York, NY, USA. doi: 10.1145/1557019.1557051.
- [11] Andreas Hotho, Robert Jäschke, Christoph Schmitz, and Gerd Stumme. Information retrieval in folksonomies: Search and ranking. In *Proceedings of the 3rd European Conference on The Semantic Web: Research and Applications*, ESWC'06, pages 411–426, Berlin, Heidelberg, 2006. Springer-Verlag.
- [12] Andreas Hotho, Robert Jäschke, Christoph Schmitz, and Gerd Stumme. Folkrank: A ranking algorithm for folksonomies. In *Proc. FGIR 2006*, 2006.
- [13] M. A. Kłopotek, S. T. Wierzchoń, R. A. Kłopotek, and E. A. Kłopotek. Personalized bipartite pagerank and the network capacity. In *Proceedings of Artificial Intelligence Studies*, volume 2012. University of Exact and Human Sciences, Siedlee, Poland, 2012.
- [14] Christian Körner, Roman Kern, Hans-Peter Grahsl, and Markus Strohmaier. Of categorizers and describers: An evaluation of quantitative measures for tagging motivation. In *Proceedings of the 21st ACM*

- Conference on Hypertext and Hypermedia, HT '10, pages 157–166, New York, NY, USA, 2010. ACM.
- [15] Amy N. Langville. An annotated bibliography of papers about Markov chains and information retrieval, 2005. URL: http://www.cofc.edu/~langvillea/bibtexpractice.pdf.
- [16] Amy N. Langville and Carl D. Meyer. Google's PageRank and beyond: the science of search engine rankings. Princeton University Press, 2006.
- [17] Stephan Link. Eigenvalue-based bipartite ranking. Bachelorarbeit/bachelor thesis, 2011. URL: http://www.pms.ifi.lmu.de/publikationen/#PA_Stephan.Link.
- [18] N. Neubauer and K. Obermayer. Towards community detection in k -partite k -uniform hypergraphs, 2009.
- [19] Lawrence Page, Sergey Brin, Rajeev Motwani, and Terry Winograd. The PageRank citation ranking: bringing order to the web. Technical Report 1999-66, Stanford InfoLab, November 1999. URL: http://ilpubs.stanford.edu:8090/422/.
- [20] Dan Petrovic. Introducing: Hyperpagerank. http://dejanseo.com.au/introducing-hyperpagerank/.
- [21] Jiejun Xu, Vishwakarma Singh, Ziyu Guan, and B. S. Manjunath. Unified hypergraph for image ranking in a multimodal context. In *ICASSP*, pages 2333–2336. IEEE, 2012.