

# On The Communication Complexity of High-Dimensional Permutations

Nati Linial\*  
Hebrew University of Jerusalem  
Jerusalem, ISRAEL  
nati@cs.huji.ac.il

Adi Shraibman  
The Academic College  
of Tel-Aviv-Yaffo  
Tel-Aviv, ISRAEL  
adish@mta.ac.il

## Abstract

We study the multiparty communication complexity of high dimensional permutations, in the Number On the Forehead (NOF) model. This continues and extends the study of the Exact- $T$  functions [9], and in particular the Exactly- $n$  function from Chandra, Furst and Lipton's seminal paper [11] in which the NOF model was introduced.

We improve the known lower bounds on Exact- $T$  functions and extend these bounds to all high dimensional permutations. To this end we introduce several new tools into this area. In addition, we discover new and unexpected connections between the NOF communication complexity of high dimensional permutations and a variety of known and well studied problems in combinatorics.

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# 1 Introduction

In the Number On the Forehead (NOF) model [11],  $k$  players  $P_1, \dots, P_k$  compute together a boolean function  $f : X_1 \times \dots \times X_k \rightarrow \{0, 1\}$ . The instance of the problem to be solved,  $(x_1, x_2, \dots, x_k) \in X_1 \times \dots \times X_k$  is presented to the players in such way that each player  $P_i$  sees the entire input except  $x_i$ . A protocol is comprised of rounds, in each of which every player writes one bit (0 or 1) on a board that is visible to all players. The choice of the written bit may depend on the player's input and on all bits previously written by himself and others on the board (the communication transcript). The protocol ends when all players know  $f(x_1, x_2, \dots, x_k)$ . The cost of a protocol is the length of the transcript for the worst input. The deterministic communication complexity of  $f$ ,  $D_k(f)$ , is the cost of the best protocol for  $f$ .

As usual in computational models, and similarly to the 2-players communication complexity model, one can also define nondeterministic and randomized communication complexity. We denote the  $k$ -players NOF nondeterministic communication complexity of  $f$  by  $N_k(f)$ , and the randomized communication complexity with error bound  $1/3$  by  $R_k(f)$ .

This computational model is notoriously difficult to study. The generous share of information that the players initially receive in NOF makes this model very powerful and consequently it is hard to establish lower bounds here. In addition, in the absence of a good notion of rank and other useful matrix parameters in dimension three and above, new conceptual ideas are required. Indeed, even basic problems that are easily solved for  $k = 2$  are widely open for  $k \geq 3$ .

One such basic problem that is still wide open for  $k > 2$  is to separate deterministic from randomized communication complexity for an explicit function. For  $k = 2$  this is easy. A simple protocol [20] yields  $R_2(I_n) = O(1)$  where  $I_n$  is the equality (aka identity) function with  $X_1 = X_2 = [n] = \{1, \dots, n\}$ . On the other hand,  $D_2(I_n) \geq N_2(I_n) \geq \log n$  by the rank lower bound [23].

However, for  $k \geq 3$ , it is hard to separate deterministic from randomized communication complexity. It is not even clear how to choose a good candidate function for which such a lower bound would hold. The equality function no longer qualifies, since for  $k \geq 3$  there is a simple deterministic NOF protocol of constant cost for the identity function. One option is to seek an example among *graph functions*  $f : [n]^{k-1} \times [N] \rightarrow \{0, 1\}$ , as defined by Beame, David, Pitassi and Woelfel [7]. The property that defines such functions is that for every  $(x_1, \dots, x_{k-1}) \in [n]^{k-1}$  there is exactly one  $b \in [N]$  such that  $f(x_1, \dots, x_{k-1}, b) = 1$ .

Graph functions naturally suggest themselves toward separating deterministic and randomized communication complexity, since their randomized communication complexity is bounded. This follows by a simple reduction to the 2-players equality problem. On the other hand, a simple counting argument from [7] shows that most graph functions have high deterministic communication complexity<sup>1</sup>. Still, even for  $k = 3$  it remains open to find *explicit* graph functions with high deterministic communication complexity.

Proving a lower bound on  $D_3(f)$  for an explicit graph function  $f$ , turns out to be quite a challenge. The reason for this difficulty is that for  $k \geq 3$  almost all lower bound methods that serve us well when  $k = 2$ , break down. The only method that survives is the discrepancy method [6] and its variants. But the discrepancy method cannot be used to the end of separating deterministic from randomized communication complexity, since it applies to deterministic as well as to randomized communication complexity. We therefore remain only with the basic combinatorial tools, the partition bound and the cylinder intersection bound.

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<sup>1</sup>To be precise, the counting argument in [7] shows that most graph functions  $f : [n]^{k-1} \times [N] \rightarrow \{0, 1\}$  with  $N \cong \sqrt{\frac{n}{k}}$  have deterministic communication complexity  $\Omega(\log \frac{n}{k})$ .

Currently, the best lower bound on the deterministic communication complexity of a graph function  $f : [n]^{k-1} \times [N] \rightarrow \{0, 1\}$  for  $k \geq 3$  is  $\Omega(\log \log n)$  proved in [7] (using also results from [5]). But the technique of [7] apply only for  $n \gg N$ , whereas we are interested in the range  $N \geq n$ . As described in the next section, in this range only much weaker lower bounds are known.

Our focus here is on *high dimensional permutations* originally defined in [21]. A  $d$ -dimensional permutation is a map  $f : [n]^{d+1} \rightarrow \{0, 1\}$  with the property that for every index  $d+1 \geq i \geq 1$  and for every choice of  $x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_{d+1} \in [n]$  there is exactly one value of  $x_i \in [n]$  for which  $f(x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_{d+1}) = 1$ . Note that: (i) High-dimensional permutations are graph functions, (ii) A one-dimensional permutation is synonymous with a permutation matrix (iii) A two-dimensional permutation is equivalent to a *Latin square*.

Usually a Latin square is defined as an  $[n] \times [n]$  matrix  $A$  with entries from  $[n]$  such that every row and every column of  $A$  contains each value in  $[n]$  exactly once. The two definitions are seen to be equivalent as follows: Associated with  $A$  is the trivariate function  $f$  where  $f(x, y, b) = 1$  if and only if  $A(x, y) = b$ . More generally, a  $d$ -dimensional permutation  $f : [n]^{d+1} \rightarrow \{0, 1\}$  can also be represented as a map  $A : [n]^d \rightarrow [n]$  such that for every choice of  $x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_d \in [n]$ , as  $y$  varies over  $[n]$  the function  $A(x_1, \dots, x_{i-1}, y, x_{i+1}, \dots, x_d)$  takes each value in  $[n]$  exactly once. This dual perspective of high-dimensional permutations is rather useful for us. We maintain throughout both points of view and switch freely between the two.<sup>2</sup>

It is obvious that for every classical (i.e., one-dimensional) permutation the associated 2-players communication problem is the equality problem. The high-dimensional scene is substantially richer and more challenging, with plenty of different communication problems associated with the various high dimensional permutations. With two players, every permutation matrix can be turned into the identity matrix by renaming the rows (which clearly does not change the communication problem). The case for  $k > 2$  is entirely different. There are  $((1 + o(1)) \frac{n}{e^2})^{n^2}$  distinct 2-dimensional permutations (see [36]), and it is easy to see that this asymptotic expression stays unchanged even if we divide it by  $(n!)^3$  to account for possible renaming of rows, columns, and shafts.

A counting argument that we present in Section 2.3 implies that almost every  $k$ -dimensional permutation has communication complexity  $\Omega(\frac{\log n}{k})$ . This is, of course, up to the factor  $\frac{1}{k}$  as high as this quantity can get. The argument improves the analog argument regarding graph functions from [7] which does not work for permutations. Our proof relies on a recent lower bound of Potapov [27] on the number of high-dimensional permutations.

On the other hand, the best lower bounds for explicit permutations are very poor even for  $k = 3$ . Neither do we know the possible *range* of values for communication complexity of high dimensional permutations. It is easy to see that there exist graph functions with bounded communication complexity, but we do not know whether there exist high-dimensional permutations with very low communication complexity. As proved in [11], there are two-dimensional permutations with  $D_3(A) = O(\sqrt{\log n})$ , but we do not know how small  $D_3(A)$  can be. This question and the analogous  $k$ -players problem for  $k > 3$  motivate much of what we do here.

To recap, we study the NOF communication complexity of high-dimensional permutations with a tool set that is limited to the bare combinatorial parameters of such permutations. To facilitate reading, we momentarily concentrate on two-dimensional permutations, in which case many of the emerging difficulties already show up. Let  $A : [n] \times [n] \rightarrow [n]$  be a Latin square. We call a triple of distinct entries  $(x, y), (x', y), (x, y')$  an *A-star* if  $A(x', y) = A(x, y')$ . Much of our work concerns  $\alpha(A)$ , the largest size of a

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<sup>2</sup>Mind the gap! The communication problem associated with a  $d$ -dimensional permutation involves  $(d+1)$  players.

subset of  $[n]^2$  that contains no  $A$ -star and  $\chi(A)$ , the least number of parts in a partition of  $[n]^2$  into  $A$ -star free subsets. Obviously  $\chi(A) \geq \frac{n^2}{\alpha(A)}$ . We have purposely named these parameters  $\alpha$  and  $\chi$  so as to reflect their resemblance with independence numbers and chromatic numbers of graphs, respectively. Indeed, as we observe (Theorem 8), for every 2-dimensional permutation  $f$ , there holds  $\log \chi(f) \leq D_3(f) \leq \lceil \log \chi(f) \rceil + 2$ . The connection with communication complexity is that  $\alpha$  is the largest size of a 1-monochromatic cylinder intersection and  $\chi$  is the partition bound.

A function  $A : [n] \times [n] \rightarrow [N]$  is called a *linjection* if its restriction to any line is an injection. That is, each  $y \in [N]$  appears at most once in every row or column of the matrix associated with  $A$ . We denote  $\alpha(n, N)$  the largest value of  $\alpha(A)$  among all linjections  $A : [n] \times [n] \rightarrow [N]$ . We also define  $\chi(n, N)$  as the smallest possible  $\chi(A)$  for a linjection  $A$ . The inequality  $\chi(n, N) \geq n^2/\alpha(n, N)$  clearly holds. Note that if  $f$  is a linjection then  $N \geq n$ , and that a linjection is also a permutation if and only if  $N = n$ .

To deal with the general  $k \geq 3$ , we introduce in Section 2 the higher-dimensional notion of an  $A$ -star. We subsequently define similarly  $\alpha_k(A), \chi_k(A), \alpha_k(n, N), \chi_k(n, N)$ . These parameters are our main objects of study.

## 1.1 Previous work

The Number On the Forehead model was introduced by Chandra, Furst and Lipton in [11]. One of the functions they consider is *Exactly- $n$*  :  $[n]^3 \rightarrow \{0, 1\}$  where, given  $x, y, z \in [n]$ , we let *Exactly- $n$* ( $x, y, z$ ) = 1 if and only if  $x + y + z = n$ . Surprisingly, they proved that the communication complexity of this function is only  $O(\sqrt{\log n})$ , but their proof yields no explicit protocol. This function is not a permutation, not even a graph function. Nevertheless, as observed in [9], the proofs go through as well if we work modulo  $n$ , in which case we deal with a 2-dimensional permutation. This implies

$$\chi_3(n, n) \leq 2^{O(\sqrt{\log n})} \quad \text{and} \quad \alpha_3(n, n) \geq \frac{n^2}{2^{\Omega(\sqrt{\log n})}}.$$

As far as we know, these are presently the best upper bound on  $\chi_3(n, n)$  and lower bound on  $\alpha_3(n, n)$  that are known.

The upper bound in [11] is based on Behrend's famous construction [8] of a large subset of  $[n]$  with no three-term arithmetic progression. In addition, they prove an inexplicit lower bound of  $\omega_n(1)$  on the complexity of *Exactly- $n$* . This is based on Gallai's result [18, p. 38] that every finite coloring of a Euclidean space contains a monochromatic homothet of every finite set in that space.

Beigel, Gasarch and Glenn [9] have refined the study of *Exactly- $n$* , and considered the more general *Exact- $T$*  problem. The definition of the function  $f_{k,T}^G$  involves an abelian group  $G$ , an element  $T \in G$  and an integer  $k \geq 2$ . Here  $k$  players need to decide whether  $x_1 + x_2 + \dots + x_k = T$ , where the inputs  $x_1, \dots, x_k \in G$  are given to them in the NOF format. They showed that

$$\chi_3(f_{k,T}^G) \geq \Omega(\log \log n)$$

for every abelian group  $G$ . For the case  $G = \mathbb{Z}_n$ , this follows as well from [17] (which is a paper in additive combinatorics). Also in the context of additive combinatorics, a recent result of Shkerdov [32] yields

$$\alpha_3(f_{k,T}^G) \leq \frac{n^2}{(\log \log n)^c},$$

for every abelian  $G$  and  $c < 1/22$ .

For general  $k \geq 3$  and for an abelian group  $G$  that is the product of  $t$  cyclic groups, it is shown in [9] that

$$D_k(f_{k,T}^G) = \omega_t(1).$$

The proof is by reduction to a lower bound from [35], that is based on the Hales-Jewett Theorem (see [18]). This lower bound is again not explicit, and yields only that the complexity is not a constant.

Note that the Exact- $T$  problem can be defined as well in non-abelian groups  $G$ . Namely,  $f_{k,T}^G(x_1, \dots, x_k) = 1$  iff  $x_1 \cdot x_2 \cdot \dots \cdot x_k = T$ , where now the order of multiplication matters. Note also that the function  $f_{k,T}^G$  is a permutation for every group  $G$ , every  $T \in G$  and  $k \geq 2$ .

Consider the following three classes of functions to be computed in the NOF model: (i) Permutations that come from Abelian groups, (ii) Those that come from general groups, (iii) Latin squares. We consider each such function up to an arbitrary renaming of rows and columns. These three classes differ very substantially in their sizes. For a given order  $n$  the size of the relevant class is (i)  $\exp(O(\sqrt{\log n}))$ , (ii) At most  $\exp((\frac{2}{27} + o(1)) \log^3 n)$ , and (iii)  $((1 + o(1)) \frac{n}{e^2})^{n^2}$ .

## 1.2 New results

Our first result is a graph theoretic characterization of  $\alpha_k(n, N)$  and  $\chi_k(n, N)$ . Using this characterization we prove:

**Theorem 1** *For all integers  $k > 3$ ,  $n$ , and  $N$  there holds*

$$\alpha_k(n, N) \leq O\left(\frac{kn^{k-2}N}{\log^*(n)}\right).$$

*In addition, there is a constant  $c > 0$  so that*

$$\alpha_3(n, N) \leq \frac{n^2}{2^{c \log^*(n)}},$$

*where  $n$  and  $N$  are integers with  $n \leq N \leq 2^{c \log^*(n)} n$ .*

This upper bound on  $\alpha$  (and the implied lower bound on  $\chi$ ) improves our understanding of the situation in that: (i) It is the first explicit bound in this domain <sup>3</sup>, (ii) It applies to a non-constant number of players, (iii) Previous results were limited to the Exact- $T$  for abelian groups with many factors, whereas ours works for general permutations.

Our main tool in proving Theorem 1 is the graph and hypergraph removal lemmas. We recall that previous lower bounds were based on the Hales-Jewett theorem. <sup>4</sup>

Recall the multidimensional Szemerédi theorem (aka the corners theorem): For every integer  $d \geq 2$  and every  $\delta > 0$  there is an  $n_0$  such that if  $n > n_0$ , every  $A \subset [n]^d$  of cardinality  $|A| > \delta \cdot n^d$  contains a *corner*. Namely,  $A \supseteq \{x, x + t \cdot e_1, \dots, x + t \cdot e_d\}$  for some integer  $t \neq 0$  and some  $x \in [n]^d$  (Here  $e_i$  are the standard unit vectors in  $\mathbb{R}^d$ ). The multidimensional Szemerédi theorem is a special case of Theorem 1 which corresponds to the instance where the function under consideration is Exactly- $n$ . In particular, a corner is an  $A$ -star in Exactly- $n$ .

Several proofs of the multidimensional Szemerédi theorem are now known, and it has been generalized to any abelian group (and not just  $\mathbb{Z}_n$ ), i.e., Exact- $T$  functions. Our proof of Theorem 15 is an adaptation of Solymosi's [33] proof of the multidimensional Szemerédi theorem.

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<sup>3</sup>A recent proof of the density version of Hales-Jewett theorem [26] does give a quantitative bound for  $k = 3$ , but this bound is weaker than the bound in Theorem 1 for  $k = 3$ , thus we omit the details here.

<sup>4</sup>Gallai's Theorem that was used in [11] is a direct consequence of Hales-Jewett (see [18] for details).

For  $k = 3$  we can say more:

**Theorem 2** *For every natural number  $n$ ,*

$$\chi_3(n, n) \geq \Omega(\log \log n).$$

This extends the lower bound of [9] from the realm of abelian groups to all permutations. The proof of Theorem 2 uses only elementary counting arguments, and is closely related to the result of [17] on monochromatic corners on the integer grid.

Theorem 2 also implies a result of Meshulam that was derived toward the study of shared directional multi-channels. This result appears as Proposition 4.3 in [4], where further background can be found. This connection is also related to the relationship between  $\alpha_3(n, N)$  and Ruzsa-Szemerédi graphs, that we describe next. We say that  $G = (V, E)$  is an  $n$ -vertex  $(r, t)$ -Ruzsa-Szemerédi graph if  $E$  can be partitioned into  $t$  induced matchings, each of size  $r$ . This concept is interesting when both  $r$  and  $t$  grow with  $n$ .

Ruzsa-Szemerédi graphs appear in various contexts in Combinatorics, Computer Science and Information Theory, thus highlighting new connections between communication complexity and these various problems. For example, an efficient deterministic communication protocol for any permutation would yield an efficient wiring schemes for shared directional multi-channels. More relevant information can be found, e.g., in [10] and [4].

The connection to Ruzsa-Szemerédi graphs is seen as follows: Let  $A : [n] \times [n] \rightarrow [N]$  be a linjection. Corresponding to every  $A$ -star free  $S \subseteq [n]^2$  is a subset of  $|S|$  edges of  $K_{n,n}$  that is comprised of at most  $N$  induced matchings. When  $N \geq 2n - 1$  this correspondence can be reversed. The problem of constructing dense Ruzsa-Szemerédi graphs is notoriously difficult. Much work was dedicated to the search for such graphs and to the study of their possible densities. This provides extra evidence to the difficulty of proving lower bounds on the communication complexity of permutations even for  $k = 3$ , as such bounds are strongly related to the possible ranges of density of Ruzsa-Szemerédi graphs. This also highlights the difficulty of finding efficient protocols for any 2-dimensional permutation (or linjection) as such a protocol would yield a dense Ruzsa-Szemerédi graph.

Indeed, work on Ruzsa-Szemerédi graphs has a bearing on communication complexity in the NOF model. For example, results from [4] yield:

**Theorem 3** *For all  $\epsilon > 0$  and large enough  $n$  there holds*

$$2^{O(\frac{1}{\epsilon})} \geq \chi_3(n, n^{1+\epsilon}) \geq \Omega(\log \frac{1}{\epsilon}).$$

Finally, we prove two results concerning the Exact- $T$  problem for the Abelian group  $G = \mathbb{Z}_2^n$ . First, we show a lower bound on  $\alpha_3(f_{3,T}^G)$  and explain how it implies a non-trivial protocol. Then we provide an alternative characterization of  $\alpha_3(f_{k,T}^G)$ :

**Theorem 4**  $\alpha_3(A_3^{\mathbb{Z}_3^n})$  is equal to the largest cardinality of a subset  $W \subseteq \mathbb{Z}_4^n$  such that for every three distinct members  $\mathbf{x}, \mathbf{y}, \mathbf{z} \in W$  there is an index  $1 \leq i \leq n$  for which  $(x_i, y_i, z_i) \notin X$ , where

$$X = \{(0, 0, 0), (1, 1, 1), (2, 2, 2), (3, 3, 3), (0, 1, 2), (1, 0, 3), (2, 3, 0), (3, 2, 1)\}.$$

There are numerous important problems in combinatorics with a similar flavor, for different choices of  $X$ . The density Hales-Jewett theorem applies to the set  $X = \{(0, 0, 0), (1, 1, 1), (2, 2, 2), (3, 3, 3), (0, 1, 2)\}$ . In the cap-set problem for  $\mathbb{Z}_3^n$  the set  $X$  is comprised of all triplets  $(a, b, c) \in \mathbb{Z}_3^3$  with  $a + c = 2b$ . This problem was recently settled in breakthrough papers by Croot, Lev, and Pach, and by Ellenberg and Gijswijt [12, 13].

## 2 Basics

### 2.1 Communication complexity

We start with a key definition: We say that  $C \subseteq [n]^{k-1} \times [N]$  is a *cylinder in the  $i$ -th coordinate* if membership in  $C$  does not depend on the  $i$ -th coordinate. Namely, for every  $\beta, \gamma$  there holds  $(a_1, \dots, a_{i-1}, \beta, a_{i+1}, \dots, a_k) \in C$  iff  $(a_1, \dots, a_{i-1}, \gamma, a_{i+1}, \dots, a_k) \in C$ . A cylinder intersection is a set  $C$  of the form  $C = \cap_{i=1}^k C_i$  where  $C_i$  is a cylinder in the  $i$ -th coordinate.

The relevance to our problem is that what a single bit of communication from player  $P_i$  conveys is precisely membership in some cylinder in the  $i$ -th coordinate. Likewise, a cylinder intersection is a set of entries identifiable by one round of communication (i.e., one bit of communication from each player).

We also need the notion of a *star*.

**Definition 5** A star  $Star(\mathbf{x}, \mathbf{x}')$  is a subset of  $[n]^{k-1} \times [N]$  of the form

$$\{(x'_1, x_2, \dots, x_k), (x_1, x'_2, \dots, x_k), \dots, (x_1, x_2, \dots, x'_k)\},$$

where  $x_i \neq x'_i$  for each  $i$ . We refer to  $\mathbf{x} = (x_1, x_2, \dots, x_k)$  as the star's center, and note that the center does not belong to the star. Also  $\mathbf{x}' = (x'_1, x'_2, \dots, x'_k)$  is called the vector of shifts. When the vector of shifts  $\mathbf{x}'$  is clear from the context, we simply write  $Star(\mathbf{x})$  or  $Star(x_1, x_2, \dots, x_k)$ .

Cylinder intersections can be easily characterized in terms of stars.

**Lemma 6 ([20])** A subset  $C \subseteq [n]^{k-1} \times [N]$  is a cylinder intersection if and only if for every star  $Star(x_1, x_2, \dots, x_k)$  that is contained in  $C$  also  $(x_1, x_2, \dots, x_k) \in C$ . Namely, if a star is contained in  $C$  then its center also belongs to  $C$ .

Let  $f : [n]^{k-1} \times [N] \rightarrow \{0, 1\}$  be a function. Any  $c$ -bit communication protocol for  $f$  partitions the input space  $[n]^{k-1} \times [N]$  into at most  $2^c$  cylinder intersections that are monochromatic with respect to  $f$  (see e.g. [20]).

For a graph function, 1-monochromatic cylinder intersections are particularly simple.

**Lemma 7** Let  $f : [n]^{k-1} \times [N] \rightarrow \{0, 1\}$  be a graph function and  $C \subseteq f^{-1}(1)$ . The set  $C$  is a (1-monochromatic) cylinder intersection with respect to  $f$  if and only if it does not contain a star.

**Proof** If  $C$  does not contain a star, then  $C$  is a cylinder intersection by Lemma 6. On the other hand, if  $C$  contains a star  $Star(x_1, x_2, \dots, x_k)$  then by definition of a graph function,  $f(x_1, x_2, \dots, x_k) = 0$ . Thus,  $C$  does not contain the center of star, and therefore  $C$  is not a cylinder intersection (again using Lemma 6). ■

Given a subset  $S \subseteq [n]^{k-1} \times [N]$ , we define its *closure*  $\bar{S}$  as the minimal cylinder intersection containing  $S$ . Note that if  $S = Star(\mathbf{x})$  then  $\bar{S} = Star(\mathbf{x}) \cup \{\mathbf{x}\}$ . In general, by lemma 6, the closure of  $S$  includes all the centers of stars that are contained in  $S$ . In particular  $\bar{S} = S$  if and only if  $S$  is a cylinder intersection.

### 2.2 Graph functions and permutations

A *line*  $L$  in  $[n]^{k-1} \times [N]$  is defined by choosing an index  $i \in [k]$  and values  $a_j$  for every  $j \neq i$ . Namely,  $L := \{(x_1, x_2, \dots, x_k) : x_j = a_j \text{ for all } j \neq i\}$ .

Recall that  $f : [n]^{k-1} \times [N] \rightarrow \{0, 1\}$  is a *graph function* if for every  $(x_1, \dots, x_{k-1})$  there is a unique  $b \in [N]$  such that  $f(x_1, \dots, x_{k-1}, b) = 1$ . Namely, every line of the form  $\{(x_1, x_2, \dots, x_k) : x_1 = a_1, x_2 = a_2, \dots, x_{k-1} = a_{k-1}\}$  contains exactly one point  $x$  for which  $f(x) = 1$ . A graph function  $f : [n]^k \rightarrow \{0, 1\}$  is called a  $(k-1)$ -dimensional *permutation* (see [21]), if in addition, **every line** has exactly one point at which  $f = 1$ .

We need the notion of a *linjection*. This is a graph function  $f : [n]^{k-1} \times [N] \rightarrow \{0, 1\}$  with  $N \geq n$  such that every line contains **at most** one point at which  $f = 1$ .

Graph functions  $f : [n]^{k-1} \times [N] \rightarrow \{0, 1\}$  and maps  $A : [n]^{k-1} \rightarrow [N]$  are equivalent notions, which we denote by  $A = A(f)$  and  $f = f(A)$ . This equivalence is defined by the condition that  $f(x_1, \dots, x_{k-1}, A(x_1, \dots, x_{k-1})) = 1$  for every  $(x_1, \dots, x_{k-1}) \in [n]^{k-1}$ .

A function  $f$  is a permutation (linjection) if and only if the restriction of  $A(f)$  to any line is a bijection (injection). For example,  $f$  is a 1-dimensional permutation if and only if  $A(f)$  is a standard permutation, and  $f$  is a 2-dimensional permutation if and only if  $A(f)$  is a Latin square.

In the equivalence between a linjection  $f : [n]^{k-1} \times [N] \rightarrow \{0, 1\}$  and  $A = A(f)$ , a star  $S \subset f^{-1}(1)$  is mapped to a closed star  $\bar{C} \subseteq [n]^{k-1}$  such that  $A$  is constant on  $C$ . Note that  $A$  must attain a different value on the center of  $C$ . We call such a subset  $\bar{C}$  an *A-star*, and say that a subset of  $[n]^{k-1}$  is *A-star free* if it contains no *A-star*.<sup>5</sup>

For example, let  $A : [n] \times [n] \rightarrow [N]$  be a linjection. An *A-star* is a triple of distinct positions  $(x, y), (x', y), (x, y')$  such that  $A(x', y) = A(x, y')$ . This *A-star* corresponds to the star  $(x, y, z'), (x', y, z), (x, y', z)$  in  $[n] \times [n] \times [N]$ , where  $z' = A(x, y)$  and  $z = A(x', y) = A(x, y')$ .

We study here the communication complexity of high-dimensional permutations and, more generally, of linjections. As we observe in the next section, the communication complexity of such functions is completely characterized by the partition bound. This explains the significance of the following quantities.

For a linjection  $A : [n]^{k-1} \rightarrow [N]$ , denote by  $\alpha(A)$  the largest size of an *A-star free* subset of  $[n]^{k-1}$ . Also  $\chi(A)$  is the least number of parts in a partition of  $[n]^{k-1}$  into *A-star free* subsets. In other words, it is the least number of colors with which  $f^{-1}(1)$  can be colored so that every color class is star-free. Note that  $\alpha(A)$  is the largest size of a 1-monochromatic cylinder intersection with respect to  $f$ , and  $\chi(A)$  is the least number of monochromatic cylinder intersections whose union is  $f^{-1}(1)$ .

Denote  $\alpha_k(n, N) = \max_A \alpha(A)$ , and let  $\chi_k(n, N) = \min_A \chi(A)$ , both taken over all linjections  $A : [n]^{k-1} \rightarrow [N]$ . Obviously  $\chi_k(n, N) \geq n^{k-1} / \alpha_k(n, N)$ . We omit the subscript  $k$  when it is clear from context.

## 2.3 The partition bound and communication complexity, for graph functions

The next theorem is an adaptation of a proof from [11]. We simply observe that their proof holds for every graph function. See also [9, 7] for similar arguments.

**Theorem 8** *For every graph function  $f : [n]^{k-1} \times [N] \rightarrow \{0, 1\}$ , there holds*

$$\log \chi_k(f) \leq D_k(f) \leq \lceil \log \chi_k(f) \rceil + k - 1.$$

**Proof** The lower bound is standard and holds for any function. It follows from Lemma 7 and the fact that every  $c$ -bit communication protocol for a function  $f$  partitions the input space into at most  $2^c$  cylinder intersections that are monochromatic with respect to  $f$  (see [20] for more details).

To prove the upper bound  $D_k(f) \leq \lceil \log \chi_k(f) \rceil + k - 1$ , fix a  $\chi_k(f)$ -coloring of  $f^{-1}(1)$  where every color class is star-free. On input  $x_1, x_2, \dots, x_{k-1}, y$ , the last player computes  $y'$  such that  $f(x_1, x_2, \dots, x_{k-1}, y') = 1$  and publishes the color  $b$  of  $(x_1, x_2, \dots, x_{k-1}, y')$ . Since  $f$  is a graph function  $y'$  is unique, and so is  $b$ .

For each  $i = 1, \dots, k - 1$ , player  $P_i$  checks whether there is a value  $x'_i$  such that  $f(x_1, x_2, \dots, x_{i-1}, x'_i, x_{i+1}, \dots, y) = 1$  and  $(x_1, x_2, \dots, x_{i-1}, x'_i, x_{i+1}, \dots, y)$  is colored  $b$ . He writes 1 on the board if such an  $x'_i$  exists and writes 0 otherwise. The protocol's value is 1 if and only if all players wrote 1 on the board.

<sup>5</sup>The notion of an *A-star* generalizes the well studied notion of a *Corner* [2, 33] which corresponds to the special case when  $f$  is the Exactly- $n$ , or Exact- $T$  for some abelian group.



The total number of bits communicated in this protocol is  $\lceil \log \chi_k(f) \rceil + k - 1$ . We turn to prove that the protocol is correct. When  $f(x_1, x_2, \dots, x_{k-1}, y) = 1$ , the protocol clearly outputs 1. Now suppose that it outputs 1, even though  $f(x_1, x_2, \dots, x_{k-1}, y) = 0$ . This means that there is a choice of  $x'_1, x'_2, \dots, y'$  for which

$$f(x'_1, x_2, \dots, x_{k-1}, y) = f(x_1, x'_2, \dots, x_{k-1}, y) = \dots = f(x_1, x_2, \dots, x_{k-1}, y') = 1.$$

But then these elements of  $f^{-1}(1)$  cannot constitute an star-free set, a contradiction. ■

The protocol in the proof above is one-way (a protocol in which players write only one message on the board, of arbitrary length). Thus, in particular, it follows that:

**Corollary 9** *For every graph function  $f : [n]^{k-1} \times [N] \rightarrow \{0, 1\}$ , it holds that*

$$D_k(f) \leq D_k^1(f) \leq D_k(f) + k.$$

Where  $D_k^1(f)$  is the one-way communication complexity of  $f$ .

Note that in general one-way protocols can be much more powerful than standard protocols [25, 5]. But for graph functions, one-way protocols and regular protocols are equally powerful. In addition, nondeterminism also does not add much power for graph functions:

**Corollary 10** *For every graph function  $f : [n]^{k-1} \times [N] \rightarrow \{0, 1\}$ , it holds that*

$$\log \chi_k(f) \leq N_k^1(f) \leq D_k(f) \leq N_k^1(f) + k.$$

**Proof** The lower bound  $N_k^1(f) \leq D_k(f)$  is obvious, since nondeterministic protocols are at least as powerful as deterministic ones. We turn to the rest of the bounds.

It is known that a  $c$ -bit nondeterministic protocol for  $f$  induces a cover of  $f^{-1}(1)$  by at most  $2^c$  sets each of which is a cylinder intersection. By Lemma 7, since  $f$  is a graph function, a subset of  $f^{-1}(1)$  is a cylinder intersection if and only if it is star-free.

Thus, a  $c$ -bit nondeterministic protocol for  $f$  induces a covering of  $f^{-1}(1)$  by at most  $2^c$  star-free sets. But for graph functions, a subset of a star-free set is also star-free, so we obtain a coloring of  $f^{-1}(1)$  by at most  $2^c$  colors where every color class is star-free. Consequently  $\log \chi_k(f) \leq N^1(f)$ , which, combined with Theorem 8 yields that  $D_k(f) \leq \log \chi_k(f) + k \leq N^1(f) + k$ . ■

Finally we prove a nearly tight lower bound on the communication complexity of random high-dimensional permutations.

**Theorem 11** *For every integer  $k \geq 2$ , and for most  $(k-1)$ -dimensional permutations  $f : [n]^k \rightarrow \{0, 1\}$ ,*

$$\log \chi_k(f) \geq \Omega\left(\frac{\log n}{k}\right).$$

**Proof** The lower bound on the number of high-dimensional permutations was recently improved by Potapov [27]. He showed that there are at least  $2^{\Omega(n^d \log n)}$   $d$ -dimensional permutations. If we view a permutation as a map  $[n]^k \rightarrow \{0, 1\}$ , this means at least  $2^{\Omega(n^{k-1} \log n)}$  permutations. In the spirit of the proof for Lemma 3.5 in [7], we now estimate the number of such permutation for which  $\chi_k(f)$  is bounded. Note that we cannot simply use the estimate from [7] since it only works for functions  $f : [n]^{k-1} \times [N] \rightarrow \{0, 1\}$  with  $N$  that is much smaller than  $n$ , roughly  $N \leq \sqrt{\frac{n}{k}}$ .

Let  $f : [n]^k \rightarrow \{0, 1\}$  be a  $(k-1)$ -dimensional permutation, and let  $\{C_1, \dots, C_\chi\}$  be a partition of  $f^{-1}(1)$  into  $\chi = \chi_k(f)$  cylinder intersections. For  $i \in [k]$  define a function  $A_i : [n]^{k-1} \rightarrow [\chi]$  as follows: For  $(a_1, \dots, a_{k-1}) \in [n]^{k-1}$ , let  $L$  be the line given by the equations

$$x_1 = a_1, \dots, x_{i-1} = a_{i-1}, x_{i+1} = a_i, \dots, x_k = a_{k-1}.$$

There is a unique 1 entry in  $L$  and this entry is in exactly one of the cylinder intersections  $\{C_1, \dots, C_\chi\}$ , say  $C_j$ . In this case we define  $A_i(a_1, \dots, a_{k-1}) = j$ .

As seen in the proof of Theorem 8, it is possible to recover  $f$  from knowledge of the functions  $A_1, \dots, A_k$ . Namely,  $f(x_1, \dots, x_k) = 1$  if and only if all the values  $A_i(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_{k-1})$  for  $i = 1, \dots, k$  are equal. But for every  $i \in [k]$  there are  $\chi^{n^{k-1}}$  possible functions  $A_i : [n]^{k-1} \rightarrow [\chi]$ . Thus, the number of  $(k-1)$ -dimensional permutations  $f : [n]^k \rightarrow \{0, 1\}$  with  $\chi_k(f) \leq \chi$  is at most  $(\chi^{n^{k-1}})^k = 2^{kn^{k-1} \cdot \log \chi}$ . Combining this with Potapov's lower bound we get that

$$\log \chi \geq \Omega\left(\frac{\log n}{k}\right).$$

for most  $(k-1)$ -dimensional permutations. ■

A simple corollary of Theorem 11, Theorem 8 and Corollary 10 is:

**Corollary 12** *For every integer  $k \geq 2$ , almost all  $(k-1)$ -dimensional permutations  $f : [n]^k \rightarrow \{0, 1\}$  satisfy*

$$D_k(f) \geq N_k^1(f) \geq \Omega\left(\frac{\log n}{k}\right).$$

It also follows from Theorem 11 that  $\chi_k(n, n) \geq 2^{\Omega(\frac{\log n}{k})}$ . It is interesting to find out how this extends for  $\chi_k(n, N)$  with  $n < N$ . It is also interesting to see whether the dependency on  $k$  can be removed.

Finally we turn to the case  $k = 2$ . The number of 2-dimensional permutations (aka Latin squares) is known to be  $((1 + o(1)) \frac{n}{e^2})^{n^2}$  (see [36]). It follows that for most 2-dimensional permutations  $f$  there holds  $\log \chi_2(f) \geq \frac{1}{3} \log n - \Theta(1)$ .

### 3 A graph theoretic characterization of $\alpha_k(n, N)$

#### 3.1 The case $k = 3$

As mentioned, we also view every linjection  $A : [n] \times [n] \rightarrow [N]$  as an  $n \times n$  matrix. We now associate with it a tripartite graph  $G(A)$  with parts  $R, C$  and  $W$ . Here  $R$  (resp.  $C$ ) is the set of rows (columns) in  $A$  and  $W \subseteq [N]$  is the set of all entries that appear in  $A$ , i.e., it is the range of  $A$ . As for the edge set: There is a complete bipartite graph with parts  $R$  and  $C$ . Also, vertices  $i \in R$  and  $w \in W$  are adjacent iff there is a  $w$  entry in row  $i$  of  $A$ . Likewise for  $C$  to  $W$  edges. Note that  $G(A)$  is a subgraph of  $K_{n,n,N}$ , and if  $N = n$  then  $G(A) = K_{n,n,n}$ .

We now focus on the triangles in  $G(A)$ . We denote the triangle on  $x \in R, y \in C, b \in W$  by  $\langle x, y, b \rangle$ . This triangle can be *trivial*, reflecting the fact that  $A(x, y) = b$ . Nontrivial triangles in  $G(A)$  correspond to stars. Namely there are  $x', y' \in [n]$  with  $x \neq x', y \neq y'$  such that  $A(x, y') = A(x', y) = b$ . Since  $A$  is a linjection it follows that  $A(x, y) \neq b$  and this triangle corresponds to the star  $\{(x', y), (x, y')\}$ . Note that whether a triangle is trivial is not a property of  $G(A)$ , but rather depends on the underlying linjection  $A$ .

A  $G$ -star in a subgraph  $G$  of  $K_{n,n,N}$  is a triple of triangles of the form

$$\langle x, y, b' \rangle, \langle x', y, b \rangle, \langle x, y', b \rangle.$$

The point is that while these triangles are edge-disjoint, their union contains the additional triangle  $\langle x, y, b \rangle$ . Define  $\overline{\alpha}(G)$  to be the largest cardinality of a family of edge-disjoint triangles that contains no  $G$ -star. In other words, a family of edge-disjoint triangles the union of which contains no additional triangle.

Let  $\overline{\alpha}(n, N) = \max_G \overline{\alpha}(G)$  where the maximum is over subgraphs of  $K_{n,n,N}$ . Then:

**Theorem 13** *For every two integers  $n \leq N$  there holds  $\alpha_3(n, N) \leq \overline{\alpha}(n, N)$ . If  $N \geq 2n - 1$ , then  $\alpha_3(n, N) = \overline{\alpha}(n, N)$ .*

**Proof** We show first that  $\alpha_3(n, N) \leq \overline{\alpha}(n, N)$ . Let  $A : [n] \times [n] \rightarrow [N]$  be a linjection and let  $S \subseteq [n] \times [n]$  be an  $A$ -star free subset of entries. We prove the claim by constructing a  $G$ -star free family  $T$  of  $|S|$  edge-disjoint triangles in  $G = G(A)$ . Let

$$T = \{ \langle x, y, A(x, y) \rangle \mid (x, y) \in S \}.$$

The claim follows, since it is easily verified that  $\{(x, y), (x', y), (x, y')\}$  is an  $A$ -star in  $S$  iff  $\{ \langle x, y, A(x, y) \rangle, \langle x', y, A(x', y) \rangle, \langle x, y', A(x, y') \rangle \}$  is a  $G$ -star in  $T$ .

Next we prove the reverse inequality  $\alpha_3(n, N) \geq \overline{\alpha}(n, N)$  when  $N \geq 2n - 1$ .

Given a  $G$ -star free family  $T$  of edge-disjoint triangles in a subgraph  $G$  of  $K_{n,n,N}$ , we find a linjection  $A : [n] \times [n] \rightarrow [N]$  that contains an  $A$ -star free subset  $S \subset [n]^2$  of size  $|T|$ . In the proof we actually first construct  $S$  and only then proceed to define  $A$  in full.

We define  $S$  to be the projection of  $T$  to its first two coordinates. Namely,

$$S = \{ (x, y) \mid \langle x, y, b \rangle \in T \text{ for some } b \}.$$

To define  $A$ , we first let  $A(x, y) = b$  for every  $\langle x, y, b \rangle \in T$ .

Since  $T$  is  $G$ -star free, it follows that  $S$  is  $A$ -star free. What is missing is that  $A$  is only partially defined. We show that when  $N \geq 2n - 1$  this partial definition can be extended to a linjection. Since the triangles in  $T$  are edge-disjoint it follows that in the partially defined  $A$ , no value appears more than once in any row or column. It remains to define  $A$  on all the entries outside of  $S$  and maintain this property. Indeed this can be done entry by entry. At worst there are  $2n - 2$  values that are forbidden for the entry of  $A$  that we attempt to define next, and therefore there is always an acceptable choice. ■

### 3.2 General $k$

The construction for general  $k$  is a natural extension of the case  $k = 3$ . We associate with every linjection  $A : [n]^{k-1} \rightarrow [N]$  a  $k$ -partite  $(k - 1)$ -uniform hypergraph  $H(A)$ . The parts of the vertex set are denoted  $Q_1, \dots, Q_{k-1}$  and  $W$ . Each  $Q_i$  is a copy of  $[n]$  and, as above,  $W$  is the range of  $A$ . There is a complete  $(k - 1)$ -partite hypergraph on the  $k - 1$  parts  $Q_1, \dots, Q_{k-1}$ . Given  $x_1 \in Q_1, \dots, x_{i-1} \in Q_{i-1}, x_{i+1} \in Q_{i+1}, \dots, x_{k-1} \in Q_{k-1}$  and  $w \in W$ , we put the hyperedge  $x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_{k-1}, w$  in  $H(A)$  iff there is a (necessarily unique)  $x_i^* \in [n]$  for which  $A(x_1, \dots, x_{i-1}, x_i^*, x_{i+1}, \dots, x_{k-1}) = w$ .

We proceed to investigate *cliques* in  $H(A)$ , i.e., sets of  $k$  vertices, every  $k - 1$  of which form an edge. For  $k = 3$ , we distinguished between those triangles in  $G(A)$  that correspond to an entry in  $[n]^2$  and those that form a star, and we make a similar distinction for general  $k$ .

It is easy to see that if  $A(x_1, \dots, x_{k-1}) = w$ , then  $x_1, \dots, x_{k-1}, w$  form a clique. Such a clique is considered *trivial*.

In contrast,  $x_1, \dots, x_{k-1}, w$  is a nontrivial clique iff for every  $i$  there exists an  $x'_i \neq x_i$  such that  $A(x_1, \dots, x_{i-1}, x'_i, x_{i+1}, \dots, x_{k-1}) = w$ .

As above, we define for  $H = H(A)$  the parameter  $\overline{\alpha}_k(H)$ . It is the largest size of a family  $K$  of nontrivial cliques in  $H$  such that: (i) No two share a hyperedge, and (ii) The hypergraph comprised of all cliques in  $K$  contains no additional nontrivial cliques. Let  $\overline{\alpha}_k(n, N) = \max_H \overline{\alpha}_k(H)$  over all  $k$ -partite  $(k - 1)$ -uniform hypergraphs  $H$ . Then

**Theorem 14** *For every two integers  $n \leq N$  there holds  $\alpha_k(n, N) \leq \overline{\alpha}_k(n, N)$ , and if  $N > (k - 1)(n - 1)$  then  $\alpha_k(n, N) = \overline{\alpha}_k(n, N)$ .*

**Proof** It is not hard to check that a family of hyperedge-disjoint nontrivial cliques induces an additional nontrivial clique if and only if it contains  $k$  cliques of the form:

$$\langle x_1, \dots, x_{k-1}, b' \rangle, \langle x'_1, \dots, x_{k-1}, b \rangle, \dots, \langle x_1, \dots, x'_{k-1}, b \rangle.$$

We call such a set of cliques an  $H$ -star.

The proof is similar to the proof of Theorem 13. To further simplify matters, we recall that  $\alpha_k(A)$  is the largest size of a star-free subset of  $[n]^{k-1} \times [N]$  that is 1-monochromatic with respect to  $f = f(A)$ .

Let  $f : [n]^{k-1} \times [N] \rightarrow \{0, 1\}$  be a linjection, and let  $S$  be a star-free subset of  $[n]^{k-1} \times [N]$ . Define the following family of (trivial) cliques in  $H = H(A)$ :

$$K = \{\langle x_1, \dots, x_{k-1}, b \rangle \mid (x_1, \dots, x_{k-1}, b) \in S\}.$$

Since the cliques in  $K$  are trivial, they are hyperedge-disjoint. Also, since  $S$  is star-free, it follows that  $K$  contains no  $H$ -stars, as  $H$ -stars directly correspond to stars in  $[n]^{k-1} \times [N]$ .

For the reverse inequality, given a family  $K$  of edge-disjoint cliques with no  $H$ -stars in the complete  $k$ -partite  $(k-1)$ -uniform hypergraph, define similarly

$$S = \{(x_1, \dots, x_{k-1}, b) \mid \langle x_1, \dots, x_{k-1}, b \rangle \in K\}.$$

and let  $f(x_1, \dots, x_{k-1}, b) = 1$  for every  $(x_1, \dots, x_{k-1}, b) \in S$ .

Since the cliques in  $K$  are pairwise edge-disjoint,  $S$  contains at most one entry in every line, and  $S$  is star-free since  $K$  does not contain a  $H$ -star.

It remains to show that  $f$  can be extended to a linjection when  $N > (k-1)(n-1)$ . We omit this argument which is similar to the proof of Theorem 13 and only note that it is better to formulate it in terms of  $A = A(f)$ .  $\blacksquare$

The proofs of Theorem 14 and 13 make it interesting to better understand the relationship between  $\bar{\alpha}_k(n, N)$  and  $\alpha_k(n, N)$ . As the proofs show,  $\bar{\alpha}_k(n, N)$  is the largest cardinality of a star-free subset of  $[n]^{k-1} \times [N]$  that meets every line in  $[n]^{k-1} \times [N]$  at most once. To qualify for  $\alpha_k(n, N)$  this subset must, in addition, be extendable to a linjection, so clearly  $\bar{\alpha}_k(n, N) \geq \alpha_k(n, N)$ . We wonder whether this additional requirement creates a substantial difference between the two parameters. Specifically, how are  $\bar{\alpha}_k(n, N)$  and  $\alpha_k(n, N)$  related in the range  $n \leq N \leq (k-1)(n-1)$ ? These two parameters need not be equal in this range, since  $\alpha_3(4, 4) = 8$  and  $\bar{\alpha}_3(4, 4) = 9$ , as we show in Section 7.1.

## 4 An upper bound on $\alpha_k(n, N)$

We prove an upper bound on  $\alpha_k(n, N)$ , using its graph theoretic interpretation from Section 3. Again we start with the case  $k = 3$ , and then proceed to  $k > 3$ .

### 4.1 The case $k = 3$

**Theorem 15** *Let  $A : [n] \times [n] \rightarrow [N]$  be a linjection, where  $N \leq n \cdot 2^{c \log^*(n)}$ . Then,*

$$\alpha_3(A) \leq O\left(\frac{n^2}{2^{c \log^*(n)}}\right).$$

Here  $c > 0$  is an absolute constant.

The proof of Theorem 15 is an adaptation of Solymosi's [33], simplification of a proof of Ajtai and Szemerédi's [2] Corners Theorem. We use along the way the triangle removal lemma [31] in its improved version due to Fox [14].

**Lemma 16 (Triangle removal lemma)** *For every  $\epsilon > 0$  there is a  $\delta > 0$  such that every  $n$ -vertex graph with at most  $\delta n^3$  triangles can be made triangle-free by removing  $\epsilon n^2$  edges. Specifically  $\delta^{-1}$  can be taken as a tower of twos of height  $405 \log \epsilon^{-1}$ .*

**Proof** [of Theorem 15] Let  $G = G(A)$ ,  $V = V(G)$ . Notice that  $|V| = 2n + N$ . Let  $S \subset [n]^2$  be an  $A$ -star free subset of size  $\alpha_3(A)$ . As in the proof of Theorem 13 we let  $T = \{ \langle x, y, A(x, y) \rangle \mid (x, y) \in S \}$  be the family of triangles in  $G$  that corresponds to  $S$ . Let  $F$  be that subgraph of  $G$  whose edge set is the union of all triangles in  $T$ . This graph contains the  $|S|$  edge-disjoint triangles in  $T$ , and no additional triangles.

Thus, if we denote  $\delta = |S|/|V|^3$  and  $\epsilon = |S|/|V|^2$ , then  $F$  contains exactly  $\delta|V|^3$  triangles and it cannot be made triangle free by removing fewer than  $\epsilon|V|^2$  edges. Lemma 16 yields  $\log^*(\delta^{-1}) \leq 405 \log(\epsilon^{-1})$ , and since  $\delta < \frac{n^2}{(2n+N)^3} < \frac{1}{N}$  we conclude that

$$\epsilon \leq 2^{\frac{-1}{405} \log^*(N)}.$$

But  $|S| = \epsilon|V|^2 \leq 9\epsilon N^2$ , so that for  $N \leq 2^{c \log^*(n)} n$ , with  $c = (3 \cdot 405)^{-1}$ , there holds

$$|S| \leq O\left(\frac{n^2}{2^{c \log^*(n)}}\right).$$

■

## 4.2 The case of general $k$

**Theorem 17** *For every natural numbers  $k \geq 3$ ,  $n$  and  $N$  it holds that*

$$\alpha_k(n, N) \leq O\left(\frac{kn^{k-2}N}{\log^*(n)}\right).$$

To this end we need the hypergraph removal lemma.

**Theorem 18 ([16, 24, 28, 29, 34])** *Let  $k$  be a positive integer. For every  $\epsilon > 0$  there exists  $\delta > 0$  with the following property. Let  $H$  be a  $k$ -partite  $(k-1)$ -uniform hypergraph with parts  $X_1, \dots, X_k$  and at most  $\delta \prod_{i=1}^k |X_i|$  cliques. There exists for each  $i$ , a subset  $R_i \subseteq \prod_{j \neq i} X_j$  of at most  $\epsilon \prod_{j \neq i} |X_j|$  hyperedges of  $H$  so that the hypergraph  $H \setminus \cup R_i$  is clique-free. Specifically one can take  $\delta^{-1}$  to be a tower of twos of height  $O(\epsilon^{-1})$ .*

**Proof** [of Theorem 17] By Theorem 14  $\alpha_k(n, N) \leq \bar{\alpha}_k(n, N)$ , so it suffices to prove that

$$\bar{\alpha}_k(n, N) \leq O\left(\frac{kn^{k-2}N}{\log^*(n)}\right).$$

By definition of  $\bar{\alpha}_k$ , there is a  $k$ -partite  $(k-1)$ -uniform hypergraph  $H$  with vertex sets  $X_1 = \dots = X_{k-1} = [n]$  and  $X_k = [N]$ , containing exactly  $\bar{\alpha}_k(n, N)$  disjoint  $k$ -cliques and no additional cliques. Consequently, at least  $\bar{\alpha}_k(n, N)$  hyperedges must be removed to make  $H$  clique-free, whence

$$\bar{\alpha}_k(n, N) \leq \epsilon kn^{k-2}N.$$

But  $\delta^{-1} \geq n$ , so that  $\epsilon = O(\frac{1}{\log^* \delta^{-1}}) = O(\frac{1}{\log^* n})$ . The claim follows. ■

## 5 Disjoint union of induced matchings

A graph is called an  $(r, t)$ -Ruzsa-Szemerédi graph if its edge set can be partitioned into  $t$  edge-disjoint induced matchings, each of size  $r$ . These graphs were introduced in 1978

and have been extensively studied since then. Of particular interest are dense Ruzsa-Szemerédi graphs, with  $r$  and  $t$  large, in terms of  $n$ , the number of vertices. Such graphs have applications in Combinatorics, Complexity theory and Information theory. Also, there are several known interesting constructions, relying on different techniques.

Let  $G$  be a tripartite graph with parts  $R, C, W$  of cardinalities  $n, n, N$  respectively. Let  $T$  be a  $G$ -star free family of edge disjoint triangles in  $G$ . Let  $F$  be the bipartite graph with parts  $R$  and  $C$  where there is an edge between  $r \in R$  and  $c \in C$  iff there is some  $b \in W$  such that  $(r, c, b) \in T$ . Then  $F$  is the union of at most  $N$  edge-disjoint induced matchings, since all the edges that correspond to a given  $b \in W$  form an induced matching.

This construction can easily be reversed: Let  $F$  be a subgraph of  $K_{n,n}$  that is the union of  $N$  edge disjoint induced matchings, with a total of  $\bar{\alpha}$  edges. We can construct a tripartite  $G$  (a subgraph of  $K_{n,n,N}$ ) that contains a family of  $\bar{\alpha}$  pairwise disjoint triangles, and has no  $G$ -stars. We conclude that

**Observation 19** *Let  $n \leq N$  be positive integers, then  $\bar{\alpha}_3(n, N)$  is the largest number of edges in a union of  $N$  edge-disjoint induced matchings in  $K_{n,n}$ .*

This observation exhibits a strong connection between (i) The problem of constructing dense  $(r, t)$ -Ruzsa-Szemerédi graphs, and (ii) The construction of a large star-free subset  $S \subseteq [n] \times [n] \times [t]$  that meets every line at most once. The two problems differ only slightly. In one, the underlying graph is bipartite and in the other all induced matching must have the same cardinality. But these differences can be bridged quite easily, as observed in the following lemma.

**Lemma 20** *1. If there exists an  $(r, t)$ -Ruzsa-Szemerédi graph on  $n$  vertices, then  $\bar{\alpha}_3(\frac{n}{2}, t) \geq \frac{rt}{2}$ .  
2. If  $\bar{\alpha}_3(n, t) \geq rt$  then there exists a  $(\frac{r}{2}, t)$ -Ruzsa-Szemerédi graph on  $n$  vertices.*

**Proof** For the first claim, let  $G = (V, E)$  be a  $(r, t)$ -Ruzsa-Szemerédi graph on  $n$  vertices, and let  $E_1, E_2, \dots, E_t$  be the partition of  $E$  into induced matchings. We can find (e.g., by a random choice) a subset  $A \subset V$  of  $\lfloor \frac{n}{2} \rfloor$  vertices, so that at least  $|E|/2$  edges are in the cut  $C = (A, \bar{A})$ . Also,  $C \cap E_1, C \cap E_2, \dots, C \cap E_t$  is a partition of the edges of the bipartite graph  $(A, \bar{A}, C)$  into  $t$  disjoint induced matchings. Therefore,  $\bar{\alpha}_3(\frac{n}{2}, t) \geq \frac{rt}{2}$ .

For the second part, suppose that  $\bar{\alpha}_3(n, t) \geq rt$ . Namely, there is a collection of disjoint induced matchings  $M_1, \dots, M_t \subseteq E(K_{n,n})$  with  $\sum_1^t |M_i| \geq rt$ . We split each  $M_i$  into  $\lfloor \frac{2|M_i|}{r} \rfloor$  sets of  $\geq r/2$  edges each. Note that  $\sum_1^t a_i \geq rt$  implies that  $\sum_1^t \lfloor \frac{2a_i}{r} \rfloor \geq t$  and a subset of an induced matching is an induced matching, so we finally have a family of at least  $t$  disjoint induced matchings each of size  $\frac{r}{2}$ . ■

Ruzsa-Szemerédi graphs have various applications in several fields [33, 4, 31, 3, 19, 10]. In [10] they are applied to Information Theory, and the study of *shared directional multi-channels*, a subject that is strongly related to communication complexity. Such a channel is comprised of a set of inputs and a set of outputs to which are connected transmitters and receivers respectively. Associated with each input is a set of outputs, that receive any signal placed at that input. A message is received successfully at an output of the channel if and only if it is addressed to the receiver connected to that output and no other signals concurrently reach that output. Therefore, when communicating over a shared channel, we want the edges (corresponding to messages sent in one round) to form an induced matching. The challenge is to partition  $K_{n,n}$  into families of pairwise disjoint induced matchings. The number of parts correspond to the number of receivers allowed at each output, and the number of matchings in each partition corresponds to the number of rounds.

The relation to communication complexity is as follows: A  $c$ -bit communication protocol for any linjection  $A : [n] \times [n] \rightarrow [N]$  induces a partition of  $K_{n,n}$  into  $c$  such families of disjoint induced matchings. Thus, such a communication protocol, gives an  $N$  round protocol for the shared directional multi-channel, with  $c$  receivers per station, and vise-versa.

In constructing a shared directional multi-channel, we seek to minimize the number of rounds required for a given number of transmitters. Alon, Moitra, and Sudakov [4] showed that for any  $\epsilon > 0$  there is partition of  $K_{n,n}$  into at most  $2^{O(\frac{1}{\epsilon})}$  graphs each of which is a family of at most  $O(n^{1+\epsilon})$  induced matchings. This gives an  $O(n^{1+\epsilon})$  round protocol for shared directional multi-channel with  $2^{O(\frac{1}{\epsilon})}$  receivers.

Translated to the language of NOF protocols and combining with Corollary 24, we conclude:

**Theorem 21** *For all  $\epsilon > 0$  and all large enough  $n$ , there holds:*

$$2^{O(\frac{1}{\epsilon})} \geq \chi_3(n, n^{1+\epsilon}) \geq \Omega(\log \frac{1}{\epsilon}).$$

## 6 A lower bound on $\chi_3(n, N)$

**Theorem 22**  $\chi_3(n, n) \geq \log \log n - O(\log \log \log n)$ .

This is clearly the case  $N = n$  of the following lemma.

**Lemma 23** *Let  $L = \chi_3(n, N)$  for some integers  $N \geq n$ , then*

$$\log n < (2^{L+1} - 1) \cdot \log(4NL/n).$$

**Proof** [of Lemma 23] Let  $A : [n] \times [n] \rightarrow [N]$  be a linjection with  $\chi_3(A) = L$ . This means that  $A$ 's entries can be  $L$ -colored so that every color class is  $A$ -star free. We pick  $v_1 \in [N]$ , the most frequent value that appears in  $A$ , and then  $c_1 \in [L]$ , the most abundant color among  $A$ 's  $v_1$ -entries. Clearly,  $|S_1| \geq n^2/(NL)$ , where  $S_1$  is the set of  $c_1$ -colored  $v_1$ -entries in  $A$ . As usual we denote the closure of  $S_1$  by  $\bar{S}_1$ , and note that since  $S_1$  meets every row and column in  $A$  at most once, there is a combinatorial rectangle  $R_1 \subseteq \bar{S}_1 \setminus S_1$  of sides  $\frac{|S_1|}{2} \times \frac{|S_1|}{2}$ . Clearly the color  $c_1$  is missing from  $R_1$ .

Now we recurse: Let  $v_i \in [N]$  be the most frequent value that appears in  $R_{i-1}$ , and  $c_i$  the most abundant color among these entries. Let  $S_i$  be the set of  $c_i$ -colored  $v_i$ -entries in  $R_{i-1}$ . Finally,  $R_i \subseteq \bar{S}_i \setminus S_i$  is a combinatorial rectangle of sides  $\frac{|S_i|}{2} \times \frac{|S_i|}{2}$  that misses colors  $c_1, \dots, c_{i-1}, c_i$ . It follows that for all  $i \geq 1$  there holds

$$|S_{i+1}| \geq \frac{|S_i|^2}{4NL},$$

which yields by induction that

$$|S_i| \geq \frac{n^{2^i}}{4^{2^i-1}-1(NL)^{2^i-1}}.$$

But since we eliminate one letter each time, this reduction process can last at most  $L$  steps, namely  $|S_{L+1}| \leq 1$ , whence

$$1 > \frac{n^{2^{L+1}}}{(4NL)^{2^{L+1}-1}} = n \cdot \left( \frac{n}{4NL} \right)^{2^{L+1}-1}$$

as claimed. ■

Another simple corollary of Lemma 23 is due to Meshulam and is reproduced in [4].

**Corollary 24** *If  $\chi_3(n, N) \leq L$  for some integers  $N \geq n$ , then  $N \geq \frac{1}{4L} \cdot n^{1+1/(2^L-1)}$ .*

## 6.1 A note on the case $k > 3$

As we have just seen  $\chi_3(A) \geq \Omega(\log \log n)$  for every 2-dimensional permutation  $A$ . It is conceivable that a similar bound holds for higher dimensions as well. This was previously conjectured in [9] for the Exact- $T$  problem. If we try to adapt the proof of Lemma 23 to higher  $k$ , exactly one difficulty arises which we formulate as a question.

**Question 25** *Let  $S \subseteq [n]^k$  be a set of cardinality  $m$  that meets every line at most once. Determine, or estimate  $\phi_k(n, m)$ , the least possible cardinality  $|\bar{S}|$  of its closure. We use the shorthand  $\phi_k(m)$  when appropriate.*

For  $k = 2$  the answer is easy:  $\phi_2(m) = m^2$ , since  $|\bar{S}| = |S|^2$ . But for  $k > 2$  the problem becomes very hard and no lower bound is known. In fact, for  $k \geq 3$ , and for large enough  $m$  there holds  $\phi_k(m) = m$ . In other words, unlike the case  $k = 2$  it may happen that  $\bar{S} = S$  for large  $S$ . For example, as shown in [11],  $\phi_3(m) = m$  when  $m = n^2/2^{\Omega(\sqrt{\log n})}$ , whereas it is shown in [32] that  $\phi_3(m) > m$  when  $m \geq n^2/(\log \log n)^{\frac{1}{22}}$ . For  $k > 3$  the situation is even worse, and all we have are the very weak lower bounds from Section 4.2. Namely, it follows from Theorem 17 that  $\phi_k(m)$  must be larger than  $m$  when  $m \geq \Omega\left(\frac{kn^{k-1}}{\log^*(n)}\right)$ , and that is all we know.

It should be clear that proving any non-trivial bounds on  $\phi_k(m)$  is a very interesting challenge. We raise the following conjecture in an attempt of improving the lower bounds on  $\chi_3(n, n)$ .

**Conjecture 26** *There are constants  $c_1, c_2 > 0$  such that if  $S \subseteq [n]^3$  meets every line at most once, and if  $|S| \geq n^2/(\log \log n)^{c_1}$ , then  $|\bar{S}| \geq n^3/(\log \log n)^{c_2}$ .*

## 7 $\alpha_3(f_{3,T}^{\mathbb{Z}_2^n})$ and $\chi_3(f_{3,T}^{\mathbb{Z}_2^n})$

In this section we focus on the exact- $T$  problem for the abelian group  $\mathbb{Z}_2^n$ . In other words, we study the permutation  $f_{k,T}^{\mathbb{Z}_2^n}$ . In Section 7.1 we prove a lower bound on  $\alpha_3(f_{3,T}^{\mathbb{Z}_2^n})$ , and in Section 7.2 we show that this lower bound implies the existence of an efficient protocol for  $f_{3,T}^{\mathbb{Z}_2^n}$ . In the last section we give an alternative characterization of  $\alpha_3(f_{3,T}^{\mathbb{Z}_2^n})$  which brings forth the relation between this problem and several known combinatorial objects. The complexity of  $f_{k,T}^{\mathbb{Z}_2^n}$  is independent of  $T$ , so we can and will omit the subscript  $T$  in this section. Without loss of generality one can take  $T = 0$ . Also, throughout this section we let  $A_k^G = A(f_k^G)$ .

### 7.1 A lower bound for $\alpha_3(f_3^{\mathbb{Z}_2^n})$

First we prove that  $A_t^{\mathbb{Z}_2^n}$ -star freeness is preserved under tensor product. Let  $S \subset (\mathbb{Z}_2^n)^{k-1}$ , denote by  $S \otimes S$  the subset of  $(\mathbb{Z}_2^{2n})^{k-1}$  comprised of all vectors  $(x_1, y_1, \dots, x_{k-1}, y_{k-1})$  such that  $x_i, y_i \in S$  for  $i = 1, \dots, k-1$ .

**Lemma 27** *If  $S$  is  $A_k^{\mathbb{Z}_2^n}$ -star free then  $S \otimes S$  is  $A_k^{\mathbb{Z}_2^{2n}}$ -star free.*

**Proof** Let  $A = A_k^{\mathbb{Z}_2^{2n}}$  and let

$$(z_1, \dots, z_{k-1}), (z_1 + d, \dots, z_{k-1}), \dots, (z_1, \dots, z_{k-1} + d)$$

be an  $A$ -star in  $S \times S$ , where for each  $1 \leq i \leq k-1$ ,  $z_i = (x_i, y_i)$  with  $x_i, y_i \in S$ . Denote also  $d = (d^1, d^2)$  where  $d^1, d^2 \in \mathbb{Z}_2^n$ . Then either

$$(x_1, \dots, x_{k-1}), (x_1 + d^1, \dots, x_{k-1}), \dots, (x_1, \dots, x_{k-1} + d^1)$$



is an  $A_k^{\mathbb{Z}_2^n}$ -star in  $S$ , or

$$(y_1, \dots, y_{k-1}), (y_1 + d^2, \dots, y_{k-1}), \dots, (y_1, \dots, y_{k-1} + d^2)$$

is an  $A_k^{\mathbb{Z}_2^n}$ -star in  $S$ , since either  $d^1 \neq 0$  or  $d^2 \neq 0$ .  $\blacksquare$

It follows that if, for some fixed  $m$ , we can find a large  $A_k^{\mathbb{Z}_2^m}$ -star free subset  $S$ , then tensor powers of  $S$  are large  $A_k^{\mathbb{Z}_2^n}$ -star free sets. We show:

**Lemma 28**  $\alpha_3(A_3^{\mathbb{Z}_2^2}) = \alpha_3(n, n) = 8$ .

Together with Lemma 27 this yields:

**Corollary 29** *For every integer  $n \geq 2$ , there holds  $\alpha_3(A_3^{\mathbb{Z}_2^n}) \geq 2^{3n/2}$ .*

**Proof** Let  $S$  be a star-free subset in  $A_3^{\mathbb{Z}_2^2}$  of cardinality  $8 = 4^{3/2}$  as in Lemma 28. The claim follows by taking the tensor powers of  $S$  as in Lemma 27.  $\blacksquare$

**Proof** [of Lemma 28] We denote the elements of  $\mathbb{Z}_2^2$  as follows  $(0, 0) = 0$ ,  $(0, 1) = 1$ ,  $(1, 0) = 2$  and  $(1, 1) = 3$ . The matrix associated with  $A_3^{\mathbb{Z}_2^2}$  is:

$$\begin{array}{cccc} \mathbf{0} & \mathbf{1} & \mathbf{2} & \mathbf{3} \\ 1 & 0 & \mathbf{3} & \mathbf{2} \\ \mathbf{2} & \mathbf{3} & 0 & 1 \\ 3 & 2 & \mathbf{1} & \mathbf{0} \end{array}$$

The entries in bold are a star-free subset of cardinality 8, so that  $\alpha_3(A_3^{\mathbb{Z}_2^2}) \geq 8$ , and consequently  $\alpha_3(4, 4) \geq 8$ . One can verify that in fact  $\alpha_3(4, 4) = \alpha_3(A_3^{\mathbb{Z}_2^2}) = 8$ . To see this first notice that if there is a star-free subset of cardinality 9 then one of the values must appear three times which already determines 10 out of the 16 entries. One can now rule out the existence of a size 9 star-free subset by exhaustive search.  $\blacksquare$

It is interesting to determine  $\alpha_k(n, n)$  for some small values of  $n$ . For example:

- Determine  $\alpha_3(8, 8)$ , in particular compute  $\alpha_3(A_3^{\mathbb{Z}_2^3})$ .
- Determine  $\alpha_k(4, 4)$ , in particular compute  $\alpha_k(A_k^{\mathbb{Z}_2^2})$ , for  $k > 3$ .

It is interesting to note that, while as shown,  $\alpha_3(4, 4) = 8$ , there holds  $\overline{\alpha}_3(4, 4) = 9$ . The fact that  $\overline{\alpha}_3(4, 4) \leq 9$  is easy to verify, and the following example shows the equality:

$$\begin{array}{cccc} \mathbf{1} & * & * & \mathbf{3} \\ * & \mathbf{1} & * & \mathbf{4} \\ * & * & \mathbf{1} & \mathbf{2} \\ \mathbf{2} & \mathbf{3} & \mathbf{4} & * \end{array}$$

Thus, continuing the discussion at the end of Section 3,  $\overline{\alpha}_3(n, N)$  and  $\alpha_3(n, N)$  need not be equal when  $N < 2n - 1$ .

## 7.2 $\alpha(f_k^G)$ vs. $\chi(f_k^G)$

The following theorem is a simple generalization of Theorem 4.3 in [11].

**Theorem 30** *If  $G$  is a group of order  $n$ , then*

$$\chi_k(f_k^G) \leq O\left(\frac{kn^{k-1} \log n}{\alpha_k(f_k^G)}\right).$$

**Proof** The proof is in two steps:

**Step I:**  $A$ -star freeness is preserved under translation, where  $A = A_k^G$ . Indeed, let  $S \subset G^{k-1}$  and let  $\mathbf{a} = (a_1, \dots, a_{k-1}) \in G^{k-1}$ . If

$$(x_1, \dots, x_{k-1}), (x_1 + d, \dots, x_{k-1}), \dots, (x_1, \dots, x_{k-1} + d)$$

is an  $A$ -star in  $S + \mathbf{a}$ , then

$$(x_1, \dots, x_{k-1}) - \mathbf{a}, (x_1 + d, \dots, x_{k-1}) - \mathbf{a}, \dots, (x_1, \dots, x_{k-1} + d) - \mathbf{a}$$

is an  $A$ -star in  $S$ .

**Step II:** Every  $S \subset G^{k-1}$  has  $O(\frac{kn^{k-1} \log n}{|S|})$  translates whose union covers all of  $G^{k-1}$ . This follows from the integrality gap for covering [22], but for completeness here is a proof. Pick at random  $t$  translates  $\mathbf{a}_1, \dots, \mathbf{a}_t \in [n]^{k-1}$  of  $S$ . The probability that a given element  $\mathbf{x} \in [n]^{k-1}$  is covered by a random translate of  $S$  is exactly  $\frac{|S|}{n^{k-1}}$ . Therefore, and since the translates are picked independently uniformly at random, the expected number of uncovered elements of  $G^{k-1}$  is

$$n^{k-1} \cdot \left(1 - \frac{|S|}{n^{k-1}}\right)^t.$$

Taking  $t = O(\frac{kn^{k-1} \log n}{|S|})$  makes the expectation less than 1, which proves the lemma. ■

**Corollary 31** *There holds*

$$\chi_3(f_3^{\mathbb{Z}_2^m}) \leq O\left(m \cdot 2^{m/2}\right).$$

**Proof** Follows from Theorem 30 and Corollary 29. ■

The bound in Corollary 31 is similar to the bound of Ada, Chattopadhyay, Fawzi and Nguyen [1] for the case  $k = 3$ , with slight improvement in the log factors. Ada et al. proved  $\chi_3(f_3^{\mathbb{Z}_2^m}) \leq O(m^{k+1} 2^{m/2^{k-2}})$ , by observing that this function is a composed function of the form  $NOR \circ XOR$  and giving non trivial protocols for such cases.

Note that the proof of Theorem 30 yields a cover of  $[n]^{k-1}$  by  $A$ -star free sets, but this is easily turned into a partition, since a subset of an  $A$ -star free set is also  $A$ -star free. Therefore, any lower bound on  $\alpha_k(f_k^G)$  can be translated into an upper bound on  $\chi_k(f_k^G)$  which in turn implies an efficient (non-explicit) protocol for  $f_k^G$  (By Theorem 8). Another interesting consequence of Theorem 30 is that any lower bound on  $\chi_k(f_k^G)$  significantly larger than  $\log n$  improves the known bounds for the size of a corner-free subset of  $G$ . This clearly boosts our interest in the multiparty communication complexity of  $f_k^G$ .

We wonder whether there are analogs of Theorem 30 for every permutation.

**Question 32** *How large can  $\chi_k(A) \cdot \alpha_k(A)/n^{k-1}$  be for an arbitrary permutation  $A$ ?*

### 7.3 An equivalent definition

Let  $X \subset \mathbb{Z}_4^3$ . We call a subset  $T$  of  $\mathbb{Z}_4^n$   $X$ -free if it does not contain an ordered triple of distinct elements  $\mathbf{x}, \mathbf{y}, \mathbf{z}$  such that  $(x_i, y_i, z_i) \in X$  for every  $1 \leq i \leq n$ . Then:

**Theorem 33** *Let*

$$X = \{(0, 0, 0), (1, 1, 1), (2, 2, 2), (3, 3, 3), (0, 1, 2), (1, 0, 3), (2, 3, 0), (3, 2, 1)\} \subset \mathbb{Z}_4^3,$$

*then  $\alpha_3(A_3^{\mathbb{Z}_4^3})$  is the largest cardinality of an  $X$ -free subset of  $\mathbb{Z}_4^n$ .*

**Proof** Recall that  $\alpha_3(A_3^{\mathbb{Z}_2^n})$  is the largest cardinality of an  $A_n$ -star free subset of  $(\mathbb{Z}_2^n)^2$ , where  $A_n = A_3^{\mathbb{Z}_2^n}$ . So it suffices to find a bijection  $\psi$  from  $(\mathbb{Z}_2^n)^2$  to  $\mathbb{Z}_4^n$  such that  $S \subseteq (\mathbb{Z}_2^n)^2$  is mapped to an  $X$ -free set if and only if  $S$  is  $A_n$ -star free.

We define  $\psi$  for  $n = 1$  and extend it entry-wise to a mapping from  $(\mathbb{Z}_2^n)^2$  to  $\mathbb{Z}_4^n$ . The definition for  $n = 1$  is as follows:  $\psi(0, 0) = 0$ ,  $\psi(0, 1) = 1$ ,  $\psi(1, 0) = 2$  and  $\psi(1, 1) = 3$ .

We need to show that if  $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in (\mathbb{Z}_2^n)^2$  is a  $A_n$ -star, then every coordinate in  $(\psi(x_1, y_1), \psi(x_2, y_2), \psi(x_3, y_3))$  belongs to  $X$ , and vice versa. Since the map  $\psi$  is defined coordinate-wise it suffices to check this for  $n = 1$ , and also for the trivial case where  $(x_1, y_1) = (x_2, y_2) = (x_3, y_3)$ . A triple  $(x_1, y_1), (x_1, y_1 + d), (x_1 + d', y_1)$  is a (trivial or non-trivial) star in  $A_1$  iff  $x_1 + (y_1 + d) = (x_1 + d') + y_1$ , i.e.,  $d = d'$ , and thus an  $A_1$ -star is a triple of the form  $(x_1, y_1), (x_1, y_1 + d), (x_1 + d, y_1)$ . If  $d = 0$  then obviously  $(\psi(x_1, y_1), \psi(x_1, y_1 + d), \psi(x_1 + d, y_1)) \in \{(0, 0, 0), (1, 1, 1), (2, 2, 2), (3, 3, 3)\} \subset X$ . When  $d = 1$  there are four cases to check:

1.  $x_1 = 0$  and  $y_1 = 0$  then  $(\psi(x_1, y_1), \psi(x_1, y_1 + d), \psi(x_1 + d, y_1)) = (0, 1, 2) \in X$ .
2.  $x_1 = 0$  and  $y_1 = 1$  then  $(\psi(x_1, y_1), \psi(x_1, y_1 + d), \psi(x_1 + d, y_1)) = (1, 0, 3) \in X$ .
3.  $x_1 = 1$  and  $y_1 = 0$  then  $(\psi(x_1, y_1), \psi(x_1, y_1 + d), \psi(x_1 + d, y_1)) = (2, 3, 0) \in X$ .
4.  $x_1 = 1$  and  $y_1 = 1$  then  $(\psi(x_1, y_1), \psi(x_1, y_1 + d), \psi(x_1 + d, y_1)) = (3, 2, 1) \in X$ .

On the other hand it is not hard to check that for each  $(a, b, c) \in X$  the triplet  $\psi^{-1}(a), \psi^{-1}(b), \psi^{-1}(c)$  is a star in  $A_1$  or  $a = b = c$ . This proves the claim.  $\blacksquare$

Fix an integer  $s \geq 2$  and let  $HJ(n, s)$  denote the largest size of a  $Y_s$ -free subset of  $[s]^n$ , where  $Y_s$  is the following set of  $s$ -tuples:  $\{(1, \dots, s)\} \cup \{(i, i, \dots, i) | i = 1, 2, \dots, s\}$ . The density Hales-Jewett theorem states that  $HJ(n, s) = o(s^n)$  for every fixed  $s$  [15, 26].

Theorem 33, and the observation that the first three coordinates of the 4-tuples in  $Y_4$  all belong to  $X$ , imply that  $\alpha_3(A_3^{\mathbb{Z}_2^n}) \leq HJ(n, 4)$ .

The cap-set problem for  $\mathbb{Z}_4^n$  also belongs to the same circle of problems. It concerns the largest size of an arithmetic-triple-free set in  $\mathbb{Z}_4^n$ . We mention in passing the recent breakthrough [12, 13] in this area which showed that this size is at most  $4^{(\gamma + o(1)) \cdot n}$  with  $\gamma \approx 0.926$ . Let  $Z \subset \mathbb{Z}_4^3$  be the set of all ordered triplets  $(a, b, c) \in \mathbb{Z}_4^3$  satisfying  $a + c = 2b$ . The cap set problems concerns exactly the largest possible cardinality of a  $Z$ -free subset of  $\mathbb{Z}_4^n$ . Since  $X \subset Z$  it follows that this size is bounded by  $\alpha_3(A_3^{\mathbb{Z}_2^n})$ .

The proof of Theorem 33 extends verbatim to general  $k \geq 3$ . It yields a subset  $X \subset \mathbb{Z}_{2^{k-1}}^k$  such that  $\alpha_k(A_k^{\mathbb{Z}_2^n})$  is the largest cardinality of an  $X$ -free subset of  $\mathbb{Z}_{2^{k-1}}^n$ .

By taking  $X$  that includes all vectors  $(a, a, \dots, a) \in \mathbb{Z}_{2^{k-1}}^k$  for  $a \in \mathbb{Z}_{2^{k-1}}$  and the vector  $(0, 1, 2, 4, \dots, 2^{k-2})$  we can maintain the relation between  $\alpha_k(A_k^{\mathbb{Z}_2^n})$  and the density Hales-Jewett theorem for every  $k$ .

## 8 Conclusion and open problems

This paper raises numerous open problems. Below we collect some of the major ones and explain some implications that would follow from progress on these questions.

**Question 34** *Improve the lower bound  $\chi_3(n, n) \geq \Omega(\log \log n)$ .*

Implications:

- Any lower bound  $\chi_3(n, n) \geq \omega(\log \log n)$  yields an improvement to the best known bound on the number of colors required to color the  $n \times n$  grid with no monochromatic equilateral right triangles. This subject goes back to Ajtai, Komlos and Szemerédi's corners theorem [2] and its implications in additive combinatorics due to Solymosi [33].

- A lower bound  $\chi_3(n, n) \geq \omega(\log n)$  would improve the best known gap between randomized and deterministic communication complexity in the 3-players NOF model.
- A lower bound  $\chi_3(n, n) \geq \Omega(\log n \cdot \log \log n)$  will improve the best known upper bound on the size of corner-free subsets of  $G^2$  for any abelian group  $G$ .
- A lower bound  $\chi_3(n, n) \geq \Omega(\log^2 n)$  will improve the best bounds on the size of a subset of  $\mathbb{Z}_n$  with no three-term arithmetic progression. This is a classic problem that goes back at least to the 1950's [30].

**Question 35** *Improve the upper bound  $\chi_3(n, n) \leq 2^{O(\sqrt{\log n})}$ .*

Implications:

- The construction of denser Ruzsa-Szemerédi graphs than currently known. Namely,  $n$ -vertex graphs which are the disjoint union of  $n$  induced matchings, all of the same size  $r$ . This, in turn, reflects on the many applications of these.
- That would improve our understanding regarding the limits of the triangle removal lemma. Note that the current gaps between the bound in this lemma are huge.

**Question 36** *Improve the bounds on  $\chi_k(n, n)$  for  $k > 3$ .*

**Question 37** *Improve the bounds on  $\alpha_k(n, n)$  for  $k > 3$ .*

That would improve our state of knowledge regarding the bounds for the hypergraph removal lemma.

It is also interesting to determine  $\alpha_k(n, n)$  for some small values of  $n$ . For example:

- Determine  $\alpha_3(8, 8)$ , in particular compute  $\alpha_3(A_3^{\mathbb{Z}_2^3})$ .
- Determine  $\alpha_k(4, 4)$ , in particular compute  $\alpha_k(A_k^{\mathbb{Z}_2^2})$ , for  $k > 3$ .

**Question 38** *What is the relationship between  $\overline{\alpha}_k(n, N)$  and  $\alpha_k(n, N)$  in the whole range  $n \leq N \leq (k-1)(n-1)$ ?*

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