OPTIMAL ENVELOPE APPROXIMATION IN FOURIER BASIS WITH APPLICATIONS IN TV WHITE SPACE

Animesh Kumar

Department of Electrical Engineering
Indian Institute of Technology, Bombay
Mumbai, India – 400076
animesh@ee.iitb.ac.in

ABSTRACT

Lowpass *envelope approximation* of smooth continuous-variable signals are introduced in this work. Envelope approximations are necessary when a given signal has to be approximated always to a larger value (such as in TV white space protection regions). In this work, a near-optimal approximate algorithm for finding a signal's envelope, while minimizing a mean-squared cost function, is detailed. The sparse (lowpass) signal approximation is obtained in the linear Fourier series basis. This approximate algorithm works by discretizing the envelope property from an infinite number of points to a large (but finite) number of points. It is shown that this approximate algorithm is near-optimal and can be solved by using efficient convex optimization programs available in the literature. Simulation results are provided towards the end to gain more insights into the analytical results presented.

***Index Terms*—** Approximation methods, signal analysis, signal approximation, TV white space

1. INTRODUCTION

This work introduces a *fundamental topic* in some Electrical Engineering applications—envelope approximations. First, this problem is motivated. TV white space devices are required to consult a TV white space database [1], which in turn computes the *protection region* of the TV transmitters. The TV white space database service providers are licensed by a regulatory body such as the FCC in United States. The protection regions for the TV transmitters is smooth and can be non-circular in shape; for example, see Fig. 1, which illustrates protection regions obtained from the iconectiv website [2] for Channel 22 in the New York region. iconectiv is one of the database service providers licensed by the FCC. Observe that protection regions such as **2** and **3** are non-circular.

The protection region signifies a closed region where only the licensed TV transmitter can use the TV channel frequencies For example, in Fig. 1, in the region labeled **2** only licensed user can operate in Channel 22 of the TV band. If the TV white space database wishes to communicate the protection region by using a lowpass (sparse or rate-efficient) approximation, it needs to calculate a lowpass representation of shapes such as **2** and **3**. While performing the approximation, there are two possible errors: (i) a point in TV protection region is declared as unprotected; and (ii) a point in unprotected region is declared as TV protection region. To protect the licensed operation of TV transmitters type (i) errors are *not allowed* . So, any lowpass representation of TV protection region must have an "enveloping" structure. To address such problems, *envelope approximations* are studied in this work.

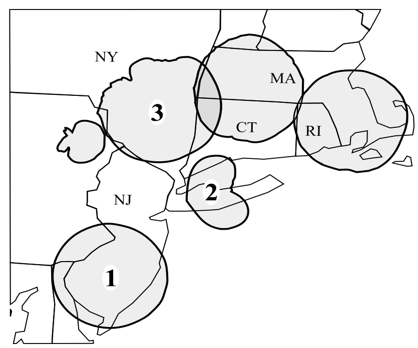

Fig. 1. The TV protection regions for Channel 22 near New York, from the website of a TV white space service provider *iconectiv* in the United States [2], are shown. Protection regions labeled **2** and **3** are non-circular in shape.

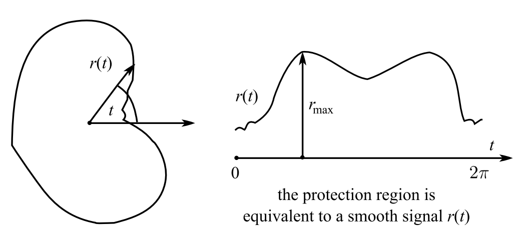

Fig. 2. A protection region can be viewed as a one-dimensional signal $r(t)$ with respect to the angle t as shown.

Consider a smooth TV protection region as depicted in Fig. 2. Let its centroid be the origin. Then, the protection region can be *parametrized* by a periodic signal $r(t)$ as shown in Fig. 2. With the knowledge of origin (the center), the signal $r(t)$ is equivalent to the protection region. This periodic and smooth signal can be approximated by orthogonal basis in a linear space; for example, Fourier series can be used [3]. In this work, $r(t)$ will be approximated to a bandlimited Fourier series $r_{\text{app}}(t)$, where $r_{\text{app}}(t)$ has only $(L+1)$ harmonics in its Fourier series. The envelope constraint requires $r_{\text{app}}(t) \geq r(t)$ for all $t \in [0, 2\pi]$, while minimizing a desirable *cost function*. For TV protection region approximation, the area enclosed by $r_{\text{app}}(t)$ should be minimized, which means

$$\begin{aligned} &\text{minimize } C(r_{\text{app}}, r) := \frac{1}{2}\int_0^{2\pi} r_{\text{app}}^2(t)\mathrm{d}t \\ &\text{subject to } r_{\text{app}}(t) \geq r(t). \end{aligned} \tag{1}$$

This is the core problem addressed in this work.

Main result: An *approximate algorithm*, which banks upon convex optimization program, is developed to address the optimization problem in (1). The approximate algorithm has two features: (i) it is provably near-optimal to the best solution of optimization in (1); and (ii) the nearness to optimality can be controlled by choosing the complexity of solving the approximate algorithm.

Related work: As far as we know, the topic of envelope approximation, subject to a cost function, has not been addressed in the literature. This is a fundamentally new topic. The topic of approximation or greedy approximation in linear basis, on the other hand, is classically well known [4, 5].

Organization: Section 2 discusses the signal and approximation model, and introduces the cost function. Section 3 presents the approximate algorithm for finding a near-optimal envelope of a signal. Section 4 presents simulation results while conclusions are in Section 5.

2. MODELING ASSUMPTIONS

A finite support real-valued field $f(t)$ will be considered, where $t \in [0, 1]$, without loss of generality. It will be assumed that a periodic repetition of $f(t)$, that is $\sum_{k\in\mathbb{Z}} f(t-k)$, is differentiable in $[0, 1]$ so that Fourier basis is sparse for the signal [3, 4]. It will be assumed that

$$|f'(t)| \leq c \tag{2}$$

for some constant $c > 0$. With Fourier basis, the pointwise representation for differentiable signals $f(t)$ is given by [3]

$$f(t) = \sum_{k\in\mathbb{Z}} a[k]\exp\left(j2\pi kt\right) \tag{3}$$

where the Fourier series coefficients are given by $a[k] = \int_0^1 f(t)\exp\left(-j2\pi kt\right)\mathrm{d}t$. Since $f(t)$ is real-valued, conjugate symmetry implies $a[k] = \bar{a}[-k]$. In general, to specify $f(t)$ completely, infinite number of coefficients $\{a[0], a[1], a[2], \ldots\}$ have to be specified.

In this work, an *optimal envelope approximation* of $f(t)$ will be designed. Let $f_{\text{app}}(t)$ be any $(L+1)$-complex coefficient based envelope approximation. Any $(L+1)$-coefficient envelope approximation will have the following form:

$$f_{\text{app}}(t) = \sum_{k=-L}^{L} b[k]\exp\left(j2\pi kt\right), \tag{4}$$

where the envelope approximation will satisfy:

$$f_{\text{app}}(t) \geq f(t) \quad \text{ for all } t \in [0, 1]. \tag{5}$$

Since $f_{\text{app}}(t)$ is real-valued, the coefficients $b[k]$ and $b[-k]$ are related by conjugate symmetry, that is $b[k] = \bar{b}[-k]$. The approximation $f_{\text{app}}(t)$ is specified by $L+1$ coefficients $\{b[0], \ldots, b[L]\}$. For compact notation let

$$\vec{b} := (b[-L], b[-L+1], \ldots, b[L])^T \tag{6}$$

where $\vec{b}$ is a column vector.

For the protection region approximation, a mean-squared cost will be minimized. The cost is defined by

$$\mathcal{C}(f_{\text{app}}, f) = \int_0^1 (f_{\text{app}}^2(t) - f^2(t))\mathrm{d}t \tag{7}$$

where $f_{\text{app}}(t)$ is the envelope approximation of $f(t)$. This cost represents the white space area lost as protection region.

3. OPTIMAL ENVELOPE APPROXIMATION

This section presents a framework to obtain a near-optimal envelope approximation for a smooth (differentiable) signal $f(t)$. The cost function is assumed to be

$$\begin{aligned} \mathcal{C}(f_{\text{app}}, f) &= \int_0^1 (f_{\text{app}}^2(t) - f^2(t))\mathrm{d}t &(8)\\ &= \sum_{|k|\leq L} |b[k]|^2 - \sum_{i\in\mathbb{Z}} |a[i]|^2 &(9) \end{aligned}$$

where $f_{\text{app}}(t) \geq f(t)$. Given a signal $f(t)$, its energy is fixed. The envelope approximation problem is equivalent to finding

$$\begin{aligned} &\vec{b}_{\text{opt}} \in \arg\min_{\vec{b}} \sum_{|k|\leq L} |b[k]|^2 \\ &\text{subject to } f_{\text{app}}(t) \geq f(t) \quad \forall t \in [0, 1]. \end{aligned} \tag{10}$$

The signal $f(t)$ is fixed in the above optimization. For any fixed $t = t_0$ the constraint is linear since

$$f_{\text{app}}(t_0) \geq f(t_0) \Leftrightarrow \vec{b}^T\vec{\phi}(t_0) \geq f(t_0) \tag{11}$$

where $\vec{\phi}(t) := (\exp(-j2\pi Lt), \exp(-j2\pi(L-1)t), \ldots, \exp(j2\pi Lt))^T$ is a vector of phasors. If the constraint in (10) was restricted to a finite number of points in $[0,1]$, then the optimization in (10) can be solved as a quadratic program with linear constraints.[1] The quadratic program is solvable using classical methods [6]. So, the *difficulty* in solving the optimization in (10) is an infinite number of constraints.

The signal $f(t)$ has been assumed to be differentiable. The approximation $f_{\text{app}}(t)$ will be infinitely differentiable due to bandlimitedness. This smoothness of $f_{\text{app}}(t) - f(t)$ suggests that if $f_{\text{app}}(t_0) - f(t_0) \geq 0$, it will be positive or near-zero in a small interval around t_0. This intuition motivates an n-point approximation to the constraint of optimization problem in (10). Consider the following optimization, which is an n-point approximation to the optimization in (10)

$$\vec{b}_{\text{appopt},n} = \arg\min_{\vec{b}} \sum_{|k|\leq L} |b[k]|^2$$
$$\text{subject to } f_{\text{app}}(t) \geq f(t) \quad \forall t \in \left\{0, \frac{1}{n}, \ldots, \frac{n-1}{n}\right\}. \quad (12)$$

The above optimization program has a quadratic cost with n linear constraints and it is solvable by a convex program solver [6]. Let $\vec{b}_{\text{opt}}$ and $\vec{b}_{\text{appopt}}$ be the unique arguments for which (10) and (12) are minimized. Then $\vec{b}_{\text{appopt},n}$ can be solved with a convex program. It is expected, though unproved so far, that $\vec{b}_{\text{opt}}$ and $\vec{b}_{\text{appopt},n}$ will be "close" as n becomes large. Their closeness is established next.

First note that there are more constraints in (10) than in (12). Therefore,

$$\sum_{|k|\leq L} |b_{\text{opt}}[k]|^2 \geq \sum_{|k|\leq L} |b_{\text{appopt},n}[k]|^2. \quad (13)$$

A sub-optimal approximation $f_{\text{subopt},n}(t)$ with Fourier series $\vec{b}_{\text{subopt},n}$ will be constructed using $\vec{b}_{\text{appopt},n}$ such that $f_{\text{subopt},n}(t) \geq f(t)$ for all $t \in [0,1]$. Assume that $|f'_{\text{appopt},n}(t)| \leq c'$, where c' is a finite constant, which is proved later in this section. Since $f_{\text{appopt},n}(t)$ is obtained by solving optimization in (12), therefore

$$f_{\text{appopt},n}\left(\frac{i}{n}\right) \geq f\left(\frac{i}{n}\right), \forall i \in \{0,1,\ldots,n-1\} \quad (14)$$

because of constraint equation. For any point $t \in [\frac{i}{n}, \frac{i+1}{n}]$

$$\begin{aligned} & f_{\text{appopt},n}(t) - f(t) \\ & \stackrel{(a)}{\geq} f_{\text{appopt},n}\left(\frac{i}{n}\right) - c'\left(t - \frac{i}{n}\right) - \left[f\left(\frac{i}{n}\right) + c\left(t - \frac{i}{n}\right)\right] \\ & \stackrel{(b)}{\geq} f_{\text{appopt},n}\left(\frac{i}{n}\right) - f\left(\frac{i}{n}\right) - \frac{c+c'}{n} \quad (15) \\ & \stackrel{(c)}{\geq} -\frac{c+c'}{n} \quad (16) \end{aligned}$$

where (a) follows by $f_{\text{appopt},n}(t) \geq f_{\text{appopt},n}(t_0) - c'(t-t_0)$ and $f(t) \leq f(t_0) + c(t-t_0)$ for $t \geq t_0$, (b) follows by $t - i/n \leq 1/n$ for $t \in [\frac{i}{n}, \frac{i+1}{n}]$, and (c) follows by (14). The above inequality holds for every i (uniformly), so

$$f_{\text{appopt},n}(t) - f(t) \geq -\frac{c+c'}{n} \quad \forall t \in [0,1]. \quad (17)$$

Define

$$f_{\text{subopt},n}(t) := f_{\text{appopt},n}(t) + \frac{c+c'}{n} \quad (18)$$

which means

$$\begin{aligned} b_{\text{subopt},n}[k] &= b_{\text{appopt},n}[k] \quad \text{for } k \neq 0 \\ &= b_{\text{appopt},n}[0] + \frac{c+c'}{n} \quad \text{for } k = 0 \quad (19) \end{aligned}$$

From (17) and (18), it follows that

$$f_{\text{subopt},n}(t) \geq f(t) \quad (20)$$

or $f_{\text{subopt}}(t)$ satisfies the constraint in (10), which means

$$\sum_{|k|\leq L} |b_{\text{opt}}[k]|^2 \leq \sum_{|k|\leq L} |b_{\text{subopt},n}[k]|^2. \quad (21)$$

From (13) and (21), we get the following key inequality

$$\sum_{|k|\leq L} |b_{\text{appopt},n}[k]|^2 \leq \sum_{|k|\leq L} |b_{\text{opt}}[k]|^2 \leq \sum_{|k|\leq L} |b_{\text{subopt},n}[k]|^2.$$

By this inequality,

$$\begin{aligned} & \sum_{|k|\leq L} |b_{\text{opt}}[k]|^2 - |b_{\text{appopt},n}[k]|^2 \quad (22) \\ & \leq \sum_{|k|\leq L} |b_{\text{subopt},n}[k]|^2 - |b_{\text{appopt},n}[k]|^2 \quad (23) \\ & = \left|b_{\text{appopt},n}[0] + \frac{c+c'}{n}\right|^2 - |b_{\text{appopt},n}[0]|^2 \quad (24) \\ & = 2b_{\text{appopt},n}[0]\frac{c+c'}{n} + \left(\frac{c+c'}{n}\right)^2 \quad (25) \\ & = 2b_{\text{appopt},n}[0]\frac{c+c'}{n} + o(1/n). \quad (26) \end{aligned}$$

Similarly,

$$\begin{aligned} & \sum_{|k|\leq L} |b_{\text{subopt}}[k]|^2 - |b_{\text{opt}}[k]|^2 \quad (27) \\ & \leq \sum_{|k|\leq L} |b_{\text{subopt},n}[k]|^2 - |b_{\text{appopt},n}[k]|^2 \quad (28) \\ & = 2b_{\text{appopt},n}[0]\frac{c+c'}{n} + o(1/n). \quad (29) \end{aligned}$$

The above results guarantee that the cost obtained by $f_{\text{subopt},n}(t)$, a suboptimal solution to the envelope approximation problem

[1] The reader would notice that $\vec{\phi}(t)$ is complex-valued, while quadratic program works with real valued linear constraints. If $b[k] = b_R[k] + jb_I[k]$, then conjugate symmetry implies that $b[-k] = b_R[k] - jb_I[k]$. These complex valued linear constraints can be re-cast into real valued linear constraints in terms of $b[0], b_R[1], b_I[1], \ldots, b_R[L], b_I[L]$. The details are omitted for simplicity of the exposition and due to space constraints.

obtained through $f_{\text{appopt},n}(t)$, is *at-most* $O(1/n)$ away from the true optimum. If n is large-enough, the above discussion guarantees that *a near-optimal solution to the envelope approximation problem can be obtained* in an efficient way.

It remains to show that $|f'_{\text{appopt},n}(t)| \leq c'$. First note that

$$|f'_{\text{appopt},n}(t)| = \left|\sum_{|k|\leq L} j2\pi k b_{\text{appopt},n}[k]\exp(j2\pi kt)\right| \tag{30}$$

$$\leq \sum_{|k|\leq L} 2\pi |k b_{\text{appopt},n}[k]| \tag{31}$$

$$\leq \sum_{|k|\leq L} 2\pi L |b_{\text{appopt},n}[k]| \tag{32}$$

$$\leq 2\pi L \left[\sum_{|k|\leq L} |b_{\text{appopt},n}[k]|^2\right]^{1/2} \tag{33}$$

Next, note that $f_{\text{wc}}(t) \equiv \|f\|_\infty$ is always a part of the constraint set in (10) and (12). Therefore,

$$\sum_{|k|\leq L} |b_{\text{appopt},n}[k]|^2 \leq \int_0^1 |f_{\text{wc}}(t)|^2 \mathrm{d}t = \|f\|_\infty^2. \tag{34}$$

Substitution of (34) in (33) results in

$$c' := |f'_{\text{appopt},n}(t)| \leq 2\pi L \|f\|_\infty \tag{35}$$

This concludes the proof of $c' < \infty$. Simulations are presented next.

4. SIMULATIONS ON TV PROTECTION REGION

To test the optimal envelope approximation method of the previous section, TV protection regions in Channel 2 of United States were examined using the TV white spaces US Interactive Map of Spectrum Bridge. Across United States, there are 57 protected service contours. One of these contours was hand-picked and its protection region was segmented (using image processing techniques) to obtain $r(t), t \in [-\pi, \pi]$. This protection region was picked since it has points with sudden change in derivative, and would be difficult to approximate. Then, the Fourier basis based envelope approximation technique was applied (see (19)). The results are shown in Fig. 3 and Fig. 4. In Fig. 3, $L = 1$ and n is increased from 3 onwards to obtain $r_{\text{subopt},n}(t)$. It is observed that $r_{\text{subopt},n}(t)$ nearly converges for $n \geq 10$. In Fig. 4, $L = 1$ is increased to $L = 5$. It is observed that with 11 Fourier series coefficients, the envelope is proximal to the original signal $r(t)$, except near derivative discontinuity.

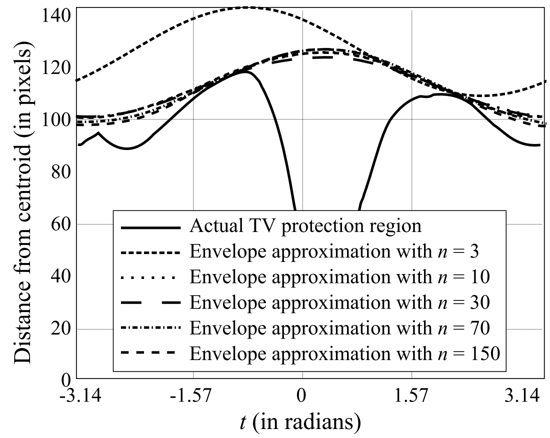

Fig. 3. The sub-optimal envelope approximation $r_{\text{subopt},n}(t)$ is obtained as a function of n. Here $L = 1$. It is observed that $r_{\text{subopt},n}(t)$ is proximal to optimal for $n \geq 10$. By design, the approximations are larger than $r(t)$ for each value of t.

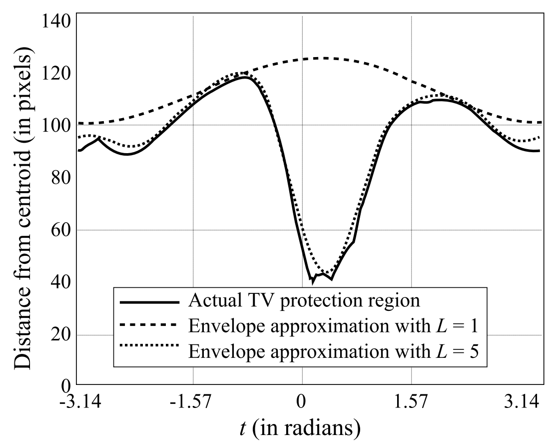

Fig. 4. The optimal envelope approximation, a solution to (10), is illustrated for $L = 1$ and $L = 5$. By design, the approximation is larger than $r(t), t \in [-\pi, \pi]$. This property also ensures that the approximate protection region is a superset of the actual protection region.

5. CONCLUSIONS

An approximate algorithm for finding a signal's envelope, while minimizing a mean-squared cost function, was detailed. A near-optimal envelope approximation was found in Fourier basis using linear space properties and efficient solvability of quadratic optimization subject to linear constraints. The approximate algorithm when subjected to n-constraints resulted in a near-optimal envelope signal with a gap of $O(1/n)$ in the cost function from the optimum. The results were verified with simulations on TV white space protection region.

6. REFERENCES

[1] D. Gurney, G. Buchwald, L. Ecklund, S.L. Kuffner, and J. Grosspietsch, "Geo-location database techniques for incumbent protection in the TV white space," in

Proc. of IEEE Symposium on Dynamic Spectrum Access Networks. Oct. 2008, pp. 1–9, IEEE, New York.

[2] ," https://spectrum.iconectiv.com/main/home/contour_vis.shtml.

[3] Stéphane Mallat, *A Wavelet Tour of Signal Processing: The Sparse Way*, Academic Press, Burlington, MA, USA, 2009.

[4] Ronald A. DeVore and George G. Lorentz, *Constructive Approximation*, Springer-Verlag, 1993.

[5] Vladimir Temlyakov, *Greedy Approximation*, Cambridge, New York, USA, 2011.

[6] Stephen Boyd and Lieven Vandenberghe, *Convex optimization*, Cambridge University Press, 2004.