A Game of Nontransitive Dice

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Abstract

We consider a two player simultaneous-move game where the two players each select any permissible n-sided die for a fixed integer n. A player wins if the outcome of his roll is greater than that of his opponent. Remarkably, for $n > 3$, there is a unique Nash Equilibrium in pure strategies. The unique Nash Equilibrium is for each player to throw the Standard n-sided die, where each side has a different number. Our proof of uniqueness is constructive. We introduce an algorithm with which, for any nonstandard die, we may generate another die that beats it.

Keywords: Efron's Dice, Nontransitive Dice, Probability Paradoxes, Game Theory

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1 Introduction

If Hercules and Lychas play at dice
Which is the better man, the greater throw
May turn by fortune from the weaker hand.

William Shakespeare,
The Merchant of Venice

Nontransitive dice are a fascinating topic in applied probability. They first came into the limelight as a result of a paper by Martin Gardner (Gardner (1970)) [6]. They are one of a larger class of nontransitivity "paradoxes" (see [11], [2]), which also include the well-known Condorcet Voting Paradox, as described in [5].

Recent papers published concerning nontransitive dice include [1], [3], and [10]. Another recent interpretation of the situation is by *Hetyei (2016)* [8] who instead characterizes the scenario as one modeled by throws of unfair *coins*. Finally, the issue of nontransitive dice is the subject of a polymath project proposal (at the time of this draft's writing) by Timothy Gowers [7].

One problem that may be considered is how nontransitive dice should be played in a strategic interaction of two or more players. *Rump (2001)* [9] was the first to explore this: he looks at the two player game where each player could choose one of four specific[1] nontransitive symmetric dice and finds the set of equilibria. He then extends the situation to one where the players each choose two dice.

Our paper examines a broader problem: we consider a two-player, one-shot, simultaneous move game in which each player selects a general n-sided die and rolls it. The player with the highest face showing wins a reward, which we may normalize to 1. The solution concept that we use for the game is that of a Nash Equilibrium. We show that for $n > 3$ there is a single, unique Nash Equilibrium in which both players play the standard die. Moreover, our proof of uniqueness is constructive and we present an algorithm that, for any non-standard die, generates a die that beats it. One notable implication of this is that for any non-standard die, there is at least one die that is the

[1] [9] formulates the problem using the four six-sided "Efron Dice".

result of a one-step, the most elementary step of our algorithm, applied to the standard die, that beats it.

Finkelstein and Thorp [4] consider the same problem, where for some fixed integer n, two players each choose a die and roll against each other. They also show the standard die ties any other die in expectation, and that every nonstandard die loses to some other die. They look at the optimal strategy in the game where the standard die has been removed.

Our paper differs from [4] in the following key ways. We provide different proofs of the existence and uniqueness of the Nash Equilibrium in the game[2], and we are able to do so using exclusively elementary mathematics. Additionally, our proof is constructive and we formulate a simple algorithm that allows us, for any non-standard die, to generate a die that beats it. Finally, the corollary that for any non standard die, there is a die that beats it that is merely a one-step away from the standard die is also novel.

This paper uses terminology from the potential Polymath project weblog post by Gowers ([7]) and from *Conrey et. all (2016)* [3].

The structure of this paper is follows: in Section 2 we introduce relevant terms, formulate the game, and present the main result of the paper, Theorem 1. In Sections 3 and 4 we prove Propositions 1 and 2, that (S_n, S_n) is a Nash Equilibrium and that it is unique (for $n \geq 4$), respectively.

2 The Basic Game

Define a general n-sided die (henceforth just "die") as an integer-valued random variable D_n that takes values in the finite set $\{1, 2, ..., n\}$, where the distribution must satisfy the following conditions:

1. For each possible value of D_n, m, the probability that it occurs, p_m, is a multiple of $1/n$.

[2][4] do not explicitly show this, but it clearly follows from their Lemma 1 and Theorem 2.

2.

$$\mathbb{E}_p(D_n) = \sum_m m p_m = \frac{n+1}{2} \tag{1}$$

For a given n, denote the set of all n-sided dice by $\mathcal{D}_n$.

Then, a standard n -sided die, S_n, is simply a die where each value occurs with equal probability, $1/n$. Naturally $S_n \in \mathcal{D}_n$.

We can represent any n-sided die, $D_n \in \mathcal{D}_n$, as a (discrete) uniformly distributed random-variable that takes values in the multiset of size n, with elements in $\{1, 2, ..., n\}$ and sum equal to $\frac{n(n+1)}{2}$:

$$D_n = \left\{ (d_1, d_2, \ldots, d_n) : \sum_{i=1}^{n} d_i = \frac{n(n+1)}{2}, \;\; p_{d_i} = 1/n \right\} \tag{2}$$

Example 1. *Three examples of four sided dice, S_4, X_4, Y_4 are:*

$$\begin{aligned} S_4 &= (1, 2, 3, 4) \\ X_4 &= (1, 1, 4, 4) \\ Y_4 &= (2, 2, 2, 4) \end{aligned} \tag{3}$$

2.1 The Game

Consider two players, Amy (A) and Bob (B). They play the following one shot game. Fix n, Amy and Bob each independently select any n-sided die, $A_n, B_n \in \mathcal{D}_n$ and then roll them against each other. Their expected payoffs are the probability that the realization of their roll is higher than the realization of their opponents roll[3].

Then, a **Strategy** for Amy, (and analogously for Bob), is simply a choice of die $A_n \in \mathcal{D}_n$. For any pair of strategies, (A_n, B_n), Amy's expected payoff, $U_A(A_n, B_n)$, is

$$U_A(A_n, B_n) = \Pr(A_n > B_n) + \frac{1}{2}\Pr(A_n = B_n) \tag{4}$$

[3]If the realizations of the rolls are the same, the winner is decided by a coin flip.

Example 2. *Suppose $n = 4$ and let Amy and Bob choose dice X_4 and Y_4 from Example 1, respectively.*

Then,

$$\begin{aligned} U_A(X_4, Y_4) &= 7/16 \\ U_B(X_4, Y_4) &= 9/16 \end{aligned} \tag{5}$$

We now present the following theorem:

Theorem 1. *For any n, the unique Nash Equilibrium of the two player game is where both players play the standard die S_n. That is, the unique Nash Equilibrium is the strategy pair (S_n, S_n).*

We shall prove this theorem by proving two propositions: Proposition 1, that (S_n, S_n) is a Nash Equilibrium; and Proposition 2, that (S_n, S_n) is the unique equilibrium (for $n \geq 4$).

3 (S_n, S_n) is a Nash Equilibrium

We prove the following proposition:

Proposition 1. *(S_n, S_n) is a Nash Equilibrium.*

Define a **One-step**, $\Phi(\cdot)$, as an operation on a die $D_n = (d_1, d_2, \dots, d_n)$ where the integer 1 is added to an element d_j and subtracted from an element d_i, $d_i \neq 1$, $d_j \neq n$. That is,

$$\Phi(D_n) = (d_1, \dots, d_i - 1, \dots, d_j + 1, \dots, d_n) \tag{6}$$

Evidently, $\Phi(D_n)$ satisfies the conditions in Equation 2 and so $\Phi(D_n)$ is a die. We may make the following remark:

Remark 1. *Any die can be reached in a sequence of one-steps from any other die. This of course implies that ny die can be reached in a sequence of one-steps from the standard die, S_n.*

Before, proving Proposition 1, we prove the following lemma:

Lemma 1. $\forall i \in \{A, B\}$ *and* $\forall D_n \in \mathcal{D}_n$,

$$U_i(S_n, D_n) = U_i(D_n, S_n) = \frac{1}{2} \tag{7}$$

Proof. We prove this lemma by induction. Evidently, $\forall i \in \{A, B\}$,

$$U_i(S_n, S_n) = \frac{1}{2} \tag{8}$$

First we show that $\forall i \in \{A, B\}$,

$$U_i(S_n, \Phi(S_n)) = U_i(\Phi(S_n), S_n) = \frac{1}{2} \tag{9}$$

We know that

$$\begin{aligned} U_i(S_n, S_n) = &\left(\frac{1}{n}\right)\left(\frac{n-1}{n}\right) + \left(\frac{1}{n}\right)\left(\frac{n-2}{n}\right) + \cdots + \left(\frac{1}{n}\right)\left(\frac{2}{n}\right) + \left(\frac{1}{n}\right)\left(\frac{1}{n}\right) \\ &+ n\left(\frac{1}{n}\right)\left(\frac{1}{n}\right)\left(\frac{1}{2}\right) = \frac{1}{2} \end{aligned} \tag{10}$$

WLOG we may suppose d_j is the $(n-1)$th term and d_i is the second term. Then,

$$\begin{aligned} U_i(\Phi(S_n), S_n) = 2&\left(\frac{1}{n}\right)\left(\frac{n-1}{n}\right) + \left(\frac{1}{n}\right)\left(\frac{n-3}{n}\right) + \cdots + \left(\frac{1}{n}\right)\left(\frac{2}{n}\right) \\ &+ n\left(\frac{1}{n}\right)\left(\frac{1}{n}\right)\left(\frac{1}{2}\right) \end{aligned} \tag{11}$$

We subtract equation 10 from equation 11, yielding:

$$\begin{aligned} U_i(\Phi(S_n), S_n) - U_i(S_n, S_n) &= \left(\frac{1}{n}\right)\left(\frac{n-1}{n}\right) - \left(\frac{1}{n}\right)\left(\frac{n-2}{n}\right) - \left(\frac{1}{n}\right)\left(\frac{1}{n}\right) \\ &= \left(\frac{1}{n}\right)\left(\frac{1}{n}\right) - \left(\frac{1}{n}\right)\left(\frac{1}{n}\right) = 0 \end{aligned} \tag{12}$$

By Remark 1, any die can be reached in a sequence of one-steps from the standard die, S_n. Denote by G_n the die reached as the image of a sequence of one-steps of arbitrary length r from the standard die S_n. That is, let

$$G_n = \Phi^1 \circ \Phi^2 \circ \cdots \circ \Phi^r \big(S_n\big) \tag{13}$$

for an arbitrary sequence of one-steps $(\Phi^1, \Phi^2, ..., \Phi^r)$. Naturally, (with some abuse of notation), we can write G_n as the multiset,

$$G_n = (g_1, g_2, \ldots, g_n) \tag{14}$$

We shall find it useful to define α_k as

$$\alpha_k = g_k - 1, \qquad \text{for } k \in \big\{1, 2, ..., n\big\} \tag{15}$$

Suppose that

$$U_i(G_n, S_n) = \frac{1}{2} \tag{16}$$

That is,

$$\begin{aligned} U_i(G_n, S_n) = &\left(\frac{1}{n}\right)\left(\frac{\alpha_n}{n}\right) + \left(\frac{1}{n}\right)\left(\frac{\alpha_{n-1}}{n}\right) + \cdots + \left(\frac{1}{n}\right)\left(\frac{\alpha_2}{n}\right) + \left(\frac{1}{n}\right)\left(\frac{\alpha_1}{n}\right) \\ &+ n\left(\frac{1}{n}\right)\left(\frac{1}{n}\right)\left(\frac{1}{2}\right) = \frac{1}{2} \end{aligned} \tag{17}$$

We apply the arbitrary one-step Φ to G_n:

$$\begin{aligned} U_i(\Phi(G_n), S_n) = &\left(\frac{1}{n}\right)\left(\frac{\alpha_n}{n}\right) + \left(\frac{1}{n}\right)\left(\frac{\alpha_{n-1}}{n}\right) + \cdots + \left(\frac{1}{n}\right)\left(\frac{\alpha_j + 1}{n}\right) + \cdots \\ &+ \left(\frac{1}{n}\right)\left(\frac{\alpha_i - 1}{n}\right) + \cdots + \left(\frac{1}{n}\right)\left(\frac{\alpha_2}{n}\right) + \left(\frac{1}{n}\right)\left(\frac{\alpha_1}{n}\right) + n\left(\frac{1}{n}\right)\left(\frac{1}{n}\right)\left(\frac{1}{2}\right) \end{aligned} \tag{18}$$

We subtract equation 17 from equation 18, yielding:

$$U_i(\Phi(G_n), S_n) - U_i(G_n, S_n = \left(\frac{1}{n}\right)\left(\frac{1}{n}\right) - \left(\frac{1}{n}\right)\left(\frac{1}{n}\right) = 0 \tag{19}$$

Thus, we conclude that $\forall i \in \{A, B\}$ and $\forall D_n \in \mathcal{D}_n$,

$$U_i(S_n, D_n) = U_i(D_n, S_n) = \frac{1}{2} \tag{20}$$

$\square$

Trivially, (and by Lemma 1),

$$U_i(S_n, S_n) = \frac{1}{2} \tag{21}$$

Suppose for the sake of contradiction that $U_i(S_n, S_n)$ is not a Nash Equilibrium. Then, at least one player has a strictly profitable deviation. WLOG we shall suppose that it is Amy. Then, Amy has a strategy $\hat{A}_n$ such that

$$U_A(\hat{A}_n, S_n) > U_A(S_n, S_n) = \frac{1}{2} \tag{22}$$

This implies that

$$U_B(\hat{A}_n, S_n) < \frac{1}{2} \tag{23}$$

However, this violates Lemma 1. We have established a contradiction and may conclude that (S_n, S_n) is a Nash Equilibrium. $\square$

4 Uniqueness of the Nash Equilibrium

We prove the following proposition:

Proposition 2. *The Nash Equilibrium, (S_n, S_n), is unique for $n \geq 4$.*

Suppose that there is another Nash equilibrium, given by (A_n, B_n). Then, by the definition of a Nash Equilibrium,

$$\begin{gathered} U_A(A_n, B_n) \geq U_A(D_n, B_n) \\ U_B(A_n, B_n) \geq U_B(A_n, D_n) \\ \forall D_n \in \mathcal{D}_n \end{gathered} \tag{24}$$

Consequently, to prove our result, it is sufficient to show that $\exists \hat{D}_n \in \mathcal{D}_n$ such that,

$$U_A(A_n, B_n) < U_A(\hat{D}_n, B_n) \tag{25}$$

Suppose that $(A_n, B_n) \neq (S_n, S_n)$ is a Nash Equilibrium. WLOG $B_n \neq S_n$. Clearly, $\forall i \in \{A, B\}$,

$$U_i(A_n, B_n) = \frac{1}{2} \tag{26}$$

Likewise, $\forall i \in \{A, B\}$,

$$U_i(B_n, B_n) = \frac{1}{2} \tag{27}$$

Pausing to interject with a brief aside,

Aside 1. *Consider a strategy pair (A_n, B_n). We say that player i has a profitable deviation if and only if there is a die that beats her opponents die.*

Writing the two n-sided dice $A_n, B_n \in \mathcal{D}_n$ as multisets,

$$\begin{aligned} A_n &= (a_1, a_2, \dots, a_n) \\ B_n &= (b_1, b_2, \dots, b_n) \end{aligned} \tag{28}$$

*we say A_n **Beats** B_n if the number of pairs (a_i, b_j) with $a_i > b_j$ exceeds the number of pairs with $a_i < b_j$.*

Thus, it is sufficient to show that there is a die $\hat{D}_n$ that beats B_n. Our proof of this is constructive. In the following proposition (Proposition 3), we present an algorithm that allows us, for any die $B_n \neq S_n$ to find a die, $\hat{D}_n$, that beats it.

Proposition 3. ***(Algorithm)*** *Given any die $B_n \neq S_n$, we can construct a die, $\hat{D}_n$, that beats it in the following manner:*

Let,

$$B_n = (b_1, b_2, \dots, b_n) \tag{29}$$

Recall,

$$S_n = (s_1, s_2, \ldots, s_n) = (1, 2, \ldots, n) \tag{30}$$

Define γ_k, $k \in \{1, 2, ..., n\}$ as the number of elements of B_n equal to $s_k = k$.

$$\gamma_k = \#\{b_i \in B_n | b_i = s_k = k\} \tag{31}$$

Define ξ_k as

$$\xi_k = \gamma_k + \gamma_{k+1} \tag{32}$$

Then, we write the following table:

b_1	b_2	$\ldots$	b_n	s_1	s_2	$\ldots$	s_n
			$+$	ξ_1	$\ldots$	ξ_{n-1}	X
			$-$	X	ξ_1	$\ldots$	ξ_{n-1}

This table is an accounting aid: The middle row (corresponding to the "+") should be interpreted as showing that if we were to add 1 to the element s_k of the die S_n, the resulting die, G_n, would have an increase of γ_k in the number of pairs (g_i, b_j) with $g_i > b_j$ as compared to die S_n versus B_n; and a decrease of γ_{k+1} in the number of pairs (g_i, b_j) with $g_i < b_j$ as compared to die S_n versus B_n. Thus the "net gain" of adding one to element s_k would be ξ_k. The X is just a place holder (since we may not add one to n)

Likewise, the bottom row (corresponding to the "−") should be interpreted as showing that if we were to subtract 1 from the element s_k of the die S_n, the resulting die, G_n, would have an decrease of γ_{k-1} in the number of pairs (g_i, b_j) with $g_i > b_j$ as compared to die S_n versus B_n; and an increase of γ_k in the number of pairs (g_i, b_j) with $g_i < b_j$ as compared to die S_n versus B_n. Thus the "net loss" of subtracting one from element s_k would be ξ_{k-1}. The X is just a placeholder (since we may not subtract one from 1).

Thus, to construct a die $\hat{D}_n$ that beats B_n, we need simply find a pair (ξ_i, ξ_j) with $i \neq j+1$ and $\xi_i > \xi_j$. Then, simply add 1 to s_i and take 1 away from s_j and the resulting die will beat B_n.

Evidently, if $\xi_a \neq \xi_b$ for some a, b, then our algorithm works. Thus, it is sufficient to prove the following lemma:

Lemma 2. *If $n \geq 4$, then $\exists$ a pair $a, b \in \{1, 2, ..., n\}$, for which $\xi_a \neq \xi_b$.*

Before proving this lemma, we present an example of the algorithm. Proof of the lemma is placed after Example 3, in Section 4.1.

Example 3. *Consider the 6 sided die, $B_n = (6, 4, 4, 4, 2, 1)$. Write the table:*

1 2 4 4 4 6	1	2	3	4	5	6
$+$	***2***	1	3	3	1	X
$-$	X	2	***1***	3	3	1

To see how this chart was filled in, consider for example the column under 1. Obviously, we may not subtract 1 from 1, hence the X in that space. If we were to add 1 to 1, the resulting entry 2, would now beat the 1 entry for B_n, and would no longer lose to the 2 entry on B_n, and so we fill in 2 into that space as the net gain.

For another example, consider the column under 4. If we were to subtract 1 from 4, the resulting entry, 3, would now lose to three more of the elements from B_n (and would still beat two elements) and so the net loss would be $3 + 0 = 3$. Then, if we were to add 1 to 4, the resulting entry, 5, would now beat three more of the elements of B_n (and would still lose to B_n's 6). Thus, the net gain would be $3 + 0$.

Having filled in the chart, our next step is to construct the die $\hat{D}_n$. One such die is obtained by adding and subtracting the, respective, bold entries. It is $(2, 2, 2, 4, 5, 6)$. Indeed we see that the following dice generated by our algorithm all beat B_n:

$$
\begin{array}{ll}
(2,2,2,4,5,6) & (2,2,3,4,5,5) \\
(1,1,4,4,5,6) & (1,2,4,4,5,5) \\
(1,1,3,5,5,6) & (1,2,2,5,5,6) \\
(1,2,3,5,5,5) &
\end{array}
\tag{33}
$$

4.1 Proof of Lemma 2

Evidently $\xi_a = \xi_b \quad \forall a, b \in \{1, 2, \ldots, n\}$ if and only if

$$\gamma_1 + \gamma_2 = \gamma_2 + \gamma_3 = \gamma_3 + \gamma_4 = \cdots = \gamma_{n-1} + \gamma_n \tag{34}$$

which holds if and only if

$$\begin{aligned} &\gamma_1 = \gamma_3 = \cdots = \gamma_k && \forall \text{ odd integers } k \in \{1, 2, \ldots, n\} \\ &\gamma_2 = \gamma_4 = \cdots = \gamma_j && \forall \text{ even integers } j \in \{1, 2, \ldots, n\} \end{aligned} \tag{35}$$

We also have the following two relationships:

$$\begin{aligned} \sum_{k \text{ odd}}^{n} \gamma_k + \sum_{j \text{ even}}^{n} \gamma_j &= n \\ \sum_{k \text{ odd}}^{n} k\gamma_k + \sum_{j \text{ even}}^{n} j\gamma_j &= \frac{n(n+1)}{2} \end{aligned} \tag{36}$$

We can combine Equations 35 and 36 to obtain:

$$\begin{aligned} \frac{n+1}{2}\gamma_1 + \frac{n-1}{2}\gamma_2 = n \\ \text{for odd n} \end{aligned} \tag{37}$$

and

$$\begin{aligned} \gamma_1 + \gamma_2 = 2 \\ \frac{n}{2}\gamma_1 + \frac{n+2}{2}\gamma_2 = n + 1 \\ \text{for even n} \end{aligned} \tag{38}$$

Now observe, we cannot have both $\gamma_1 \geq 1$ and $\gamma_2 \geq 1$ since if one were equal to 1 and the other were greater than 1, this would violate 36; and if they were both equal to 1, then $B_n = S_n$, a contradiction. Thus, either γ_1 or γ_2 must be equal to 0.

Suppose n is odd and that $\gamma_1 = 0$. Then from equation 37 we have

$$(n-1)\gamma_2 = 2n \tag{39}$$

which does not have a solution in integers n, γ_2 for $n > 3$. Now suppose that n is odd and that $\gamma_2 = 0$. Then from equation 37 we have

$$(n+1)\gamma_1 = 2n \tag{40}$$

which does not have a solution in integers n, γ_1 for $n > 1$. Thus, we conclude that n cannot be odd.

Suppose n is even and that $\gamma_1 = 0$. From equation 38 we must have $\gamma_2 = 2$ and

$$n + 2 = n + 1 \tag{41}$$

which is obviously a contradiction. Finally, suppose n is even and that $\gamma_2 = 0$. From equation 38 we must have $\gamma_1 = 2$ and

$$n = n + 1 \tag{42}$$

which is also a contradiction. Thus we have proved Lemma 2. $\square$

Propositions 3 and 1 immediately follow. We may also write the following Corollary, which we have proved "along the way".

Corollary 1. *Let $n \geq 4$. Then, for any die $B_n \neq S_n$, $\exists$ a die $\tilde{D}_n$ that is the image of a one-step Φ on the standard die S_n, i.e. $\Phi(S_n) = \tilde{D}_n$, that beats B_n.*

Note that given some die $B_n \neq S_n$, our algorithm yields every winning die (i.e. a die that beats B_n) that is a one-step away from S_n. Moreover, it is easy to see how using our algorithm, we could also obtain the set of losing dice that are a one-step away from S_n. Because our algorithm also allows us to find the net effect of any one-step applied to S_n, for any $B_n \neq S_n$, we are also able to find the "best" (and "worst") die/dice a one-step away from S_n to play versus B_n. The die or dice with the greatest net gain from the one-step would have the highest likelihood of beating S_n (and thus the greatest expected utility). In the same manner, the die or dice with the greatest (in absolute value) net loss from the one-step would have the lowest likelihood of beating S_n (and thus the lowest expected utility).

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