The Constant of Proportionality in Lower Bound Constructions of Point-Line Incidences

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Abstract

Let $I(n,l)$ denote the maximum possible number of incidences between n points and l lines. It is well known that $I(n,l) = \Theta((nl)^{2/3} + n + l)$ [2,3,7]. Let C_{SzTr} denote the constant of proportionality of the $(nl)^{2/3}$ term. The known lower bound, due to Elekes [2], is $C_{\mathrm{SzTr}} \geq 2^{-2/3} = 0.63$. With a slight modification of Elekes' construction, we show that it can give a better lower bound of $C_{\mathrm{SzTr}} \geq 1$, i.e., $I(n,l) \geq (nl)^{2/3}$. Furthermore, we analyze a different construction given by Erdős [3], and show its constant of proportionality to be even better, $C_{\mathrm{SzTr}} \geq 3/(2^{1/3}\pi^{2/3}) = 1.11$.

1 Overview

The Szemerédi-Trotter bound [7] asserts that $I(n,l) = O((nl)^{2/3} + n + l)$, and this bound is tight, as shown in different lower bound constructions by Erdős [3] and Elekes [2] (See also [1,6] for simpler proofs of the upper bound). Let C_{SzTr} denote the constant of proportionality of the $(nl)^{2/3}$ term. The known upper bound on C_{SzTr} at present, due to Pach et al. [4], is $C_{\mathrm{SzTr}} \leq 2.5$. The known lower bound, due to Elekes [2], is $C_{\mathrm{SzTr}} \geq 2^{-2/3} = 0.63$. We modify Elekes' construction, and show that this modification gives a lower bound on the constant of proportionality of $C_{\mathrm{SzTr}} \geq 1$, i.e., $I(n,l) \geq (nl)^{2/3}$. Next, we analyze the construction of Erdős [3], and show its constant of proportionality to be even better, $C_{\mathrm{SzTr}} \geq 3/(2^{1/3}\pi^{2/3}) = 1.11$. This is an improvement upon a previous analysis of the Erdős construction [5], which gives the bound $C_{\mathrm{SzTr}} \geq (3/(4\pi^2))^{1/3} \approx 0.42$.

2 The Elekes construction

We present a slightly different construction than that of Elekes [2]. It is similar in principle, but more exhaustive. Let k and m be some positive integers. We denote by $\mathrm{Elekes}(k,m) = (P,L)$ the following set of points P, and family of lines L. P is defined as a $k \times km$ lattice section:[1]

$$P = \{0,\ldots,k-1\} \times \{0,\ldots,km-1\},$$

We put L to be all x-monotone lines that contain k points of P. These are lines of the form $y = ax + b$ with integer parameters[2] as follows. The b parameter is an integer in the range

$$0 \leq b \leq km - 1,$$

[1] Elekes uses a $k \times 2km$ lattice section, $\{1,\ldots,k\} \times \{1,\ldots,2km\}$.

[2] Elekes uses only some of the possible lines, $1 \leq a \leq m$, and $1 \leq b \leq km$. This gives $n = 2k^2m$ points, $l = km^2$ lines and $I = 2k^2m^2 = 2^{-2/3}(nl)^{2/3}$ incidences.

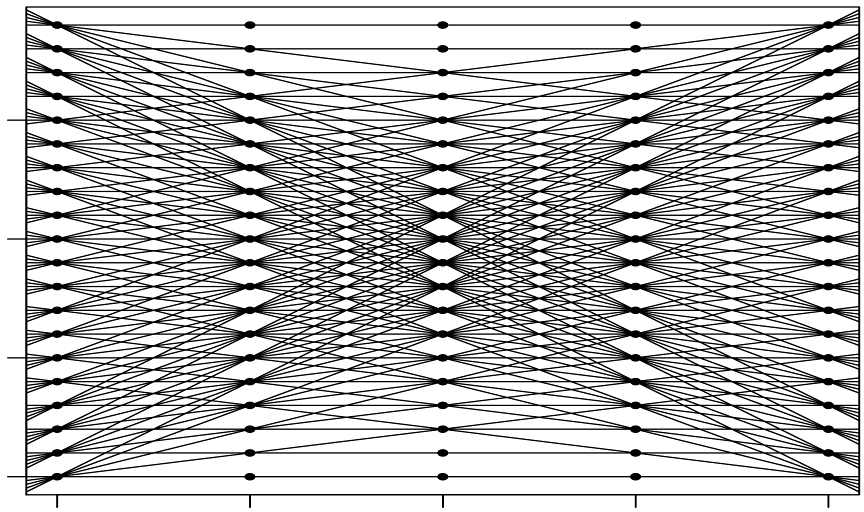

Figure 1: An $\text{Elekes}(5, 4)$ configuration. $n = 100$ points, $l = 100$ lines, and $I = 500$ incidences.

and the a parameter, given b, is restricted as follows. For $x = k-1$ we have $0 \leq a(k-1)+b \leq km-1$, or

$$-\frac{b}{k-1} \leq a \leq m + \frac{m-1}{k-1} - \frac{b}{k-1}.$$

The difference between the upper and lower bounds of a is $m+(m-1)/(k-1)$, and the number of integer values in this range is either $m + \lfloor (m-1)/(k-1) \rfloor$, or $m + 1 + \lfloor (m-1)/(k-1) \rfloor$. The latter case happens about $1 + ((m-1) \mod (k-1))$ out of $k-1$ times. The resulting number of lines is

$$\begin{aligned} l &\approx km\left(m + \left\lfloor \frac{m-1}{k-1} \right\rfloor + \frac{1 + ((m-1) \mod (k-1))}{k-1}\right) \\ &\approx km^2, \end{aligned}$$

and the number of incidences then comes out

$$I \approx k^2 m^2.$$

Together with the fact that the number of points is $n = k^2 m$, we get

$$I \approx (nl)^{2/3}.$$

An $\text{Elekes}(k, k-1)$ has an equal number of points and lines, $n = l = k^2(k-1)$, and $I = k^3(k-1) \approx n^{4/3}$ incidences.

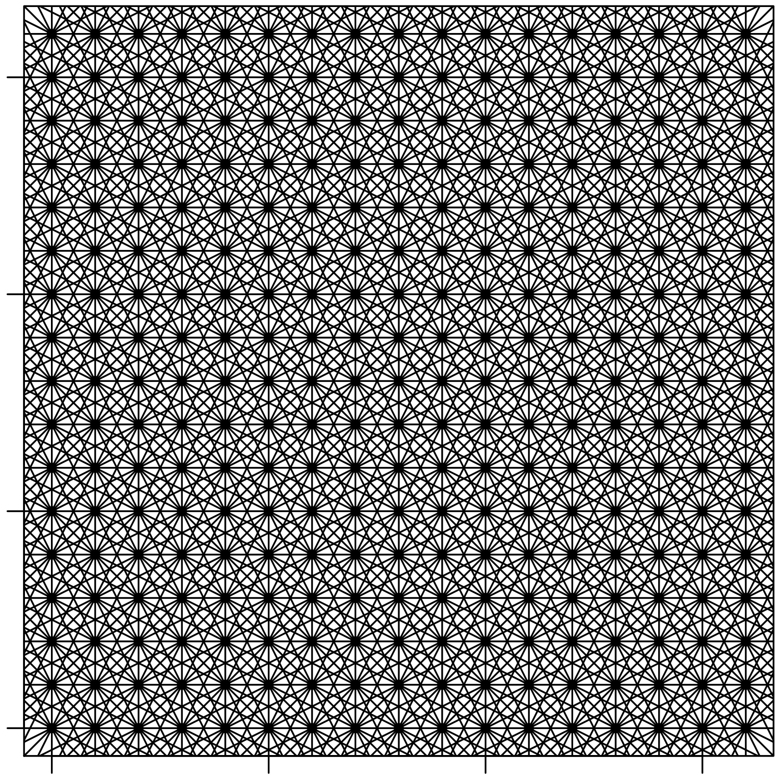

Figure 2: An $\mathrm{Erdos}(17, 3)$ configuration. $n = 289$ points, $l = 296$ lines, and $I = 2312$ incidences.

3 The Erdős construction

Erdős [3] considered n points on a $n^{1/2} \times n^{1/2}$ lattice section, together with the n lines that contain the most points. He noted that there are $\Theta(n^{4/3})$ incidences in this configuration, and conjectured that it is asymptotically optimal. His conjecture was settled in the affirmative as a corollary of the Szemerédi-Trotter bound [7]. Pach and Tóth [5] analyzed, in more generality, the square lattice section together with the lines with the most incidences, where the number of lines l is not necessarily equal to the number of points n. Their analysis yielded the bound $I \geq 0.42(nl)^{2/3}$. In this section we will analyze the same setting in a different way and get an improved bound of $I \geq 1.11(nl)^{2/3}$.

For two integers k and m, we denote by $\mathrm{Erdos}(k, m) = (P, L)$ the following set of points P, and family of lines L. We put P to be a $k \times k$ lattice section:

$$P = \{0, \dots, k-1\}^2 .$$

The number of points is $n = k^2$. Next, we put L to be all lines of the form $ax + by = c$ that pass through the square, where:

1. a, b, and c are integers.
2. a and b are coprime.
3. $a \geq 0$.
4. $|a| + |b| \leq m$.

The probability of a random pair (a, b) to be coprime is about $\frac{6}{\pi^2}$ [8]. There are $(m + 1)^2$ integer pairs in the range $\{(a, b) \mid |a| + |b| \leq m, a \geq 0\}$, so there are about $\frac{6m^2}{\pi^2}$ coprime pairs. Each pair (a, b) determines the direction of a pencil of parallel lines, $ax + by = c$, and each of the k^2 points is incident to a line in each of these directions. That is, each point is incident to about $\frac{6m^2}{\pi^2}$ lines, so in total

$$I \approx \frac{6k^2m^2}{\pi^2}.$$

It remains to estimate the number of lines. Consider a positive coprime pair (a, b). This pair generates lines $ax + by = c$, where:

1. The minimal value of c is 0, and the line $ax + by = 0$ passes through $(0, 0) \in P$.
2. The maximal value of c is $(a + b)(k - 1)$, and the line $ax + by = (a + b)(k - 1)$ passes through $(k - 1, k - 1) \in P$.

It follows that there are $(|a| + |b|)(k - 1) + 1$ values of c that generate lines that pass through the square. This number of lines is true also for negative b with a different range of c-values. The total number of lines $|L| = l$ is thus

$$\begin{aligned}
l &= \sum_{a,b}((|a| + |b|)(k - 1) + 1) && (3.1)\\
&\approx \sum_{j=1}^{m} \sum_{|a|+|b|=j} j(k - 1) + \frac{6m^2}{\pi^2} && (3.2)\\
&\approx \sum_{j=1}^{m} \frac{12j}{\pi^2} j(k - 1) + \frac{6m^2}{\pi^2} && (3.3)\\
&\approx \frac{12(k - 1)}{\pi^2} \sum_{j=1}^{m} j^2 + \frac{6m^2}{\pi^2} && (3.4)\\
&\approx \frac{4m^3(k - 1)}{\pi^2} + \frac{6m^2}{\pi^2}. && (3.5)
\end{aligned}$$

(3.1) is a sum over all coprime pairs (a, b) as above. (3.2) is the same sum in a different order of summation. In (3.3) we estimate the number of coprime pairs (a, b) such that $|a| + |b| = j$ as follows. There are $2j + 1$ integer pairs (a, b), such that $a \geq 0$ and $|a| + |b| = j$, and the probability of a pair from this subset to be coprime is, as already noted, $6/\pi^2$, so there should be an expected number of $(12j + 6)/\pi^2 \approx 12j/\pi^2$

coprime pairs. In (3.5) we use the approximation $\sum_{j=1}^{m} j^2 = m(m+1)(2m+1)/6 \approx m^3/3$. The dominant term in the final equation is

$$l \approx \frac{4m^3k}{\pi^2}.$$

From the values of n, l, and I in terms of k and m, we get that

$$I \approx \frac{3}{2^{1/3}\pi^{2/3}}(nl)^{2/3} \approx 1.11(nl)^{2/3}.$$

We note that L is not quite the family of lines with the most incidences, but rather, an approximation of it. Indeed, there are lines here, such as $x + y = 0$ with just one incidence. There are even lines with no incidences, like $2x + 3y = 1$. However, most lines do have many incidences, and a line $ax + by = c$, on average has about $k/(|a| + |b|)$ incidences.

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