NFA, DFA, and regular expressions

204213 Theory of Computation

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Outline

Review Homework 2

8. (Sipser 1.41) Let $B_n = \{a^k \mid k \text{ is multiple of } n\}$. Show that the language B_n is regular.

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Regular operations

Last time, we defined 3 regular operations:

- Union: $A \cup B = \{x | x \in A \text{ or } x \in B\}$,
- Concatenation: $A \circ B = \{xy | x \in A \text{ and } y \in B\}$,
- Star: $A^* = \{x_1 x_2 \cdots x_k | k \geq 0 \text{ and each } x_i \in A\}$,

and proved the following theorem.

Theorem 1

The class of regular languages is closed under the union operation.

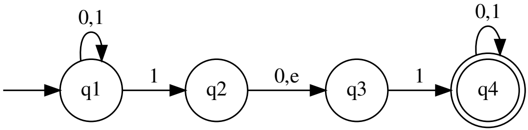
What about other operations?

- We prove Theorem 1 by simulating two finite automata with one finite automaton.
- This approach cannot be used directly to prove that the set of regular languages is closed under concatenation. Why?
  - For string $w \in A_1 \circ A_2$, there exists a pair x and y such that $w = xy$ and $x \in A_1$ and $y \in A_2$.
  - To construct a finite automaton M for $A_1 \circ A_2$ from M_1 and M_2 that recognize A_1 and A_2 we need to simulate M_1 to the end of x and start simulating M_2 right after that. **And it is hard to "tell" where x ends.**

A machine that always guesses correctly

- Suppose that our machine can guess where x ends.
- It can
  - simulate M_1 on the input string up to the end of x,
  - jump to the start state in M_2 right after x ends, and
  - accept string $w = xy$ when the machine stops at some accept state in M_2.
- Can machine guess?
  - Maybe?
  - But guess correctly?
  - Ummm.. it definitely can, **in theory.**

Differences

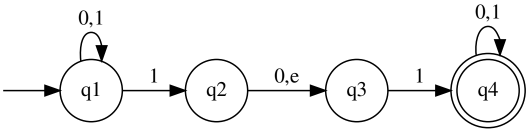


- Duplicate symbols
- Missing symbols
- Empty string: ε

Deterministic and Nondeterministic Finite Automata

- Previously, we only consider finite automata whose next states are **determined** by their input alphabet and their current states.
- Computation where each next step is fully determined is called **deterministic** computation.
- On the other hand, in **nondeterministic** computation, many choices may exist.
- Therefore, we have deterministic finite automata (**DFA**) and nondeterministic finite automata (**NFA**).

How does N_1 compute?



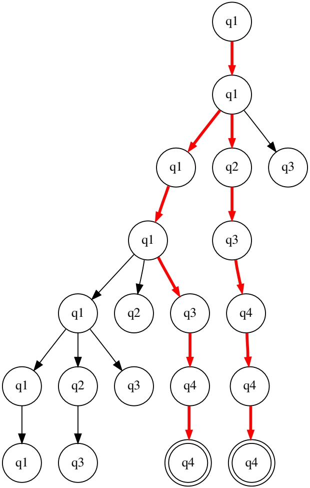
$\delta(q_1, 1) = \{q_1, q_2\}$

At any point where there are many choices for the next step, the machine **splits** itself into many copies and follow all possible steps in parallel.

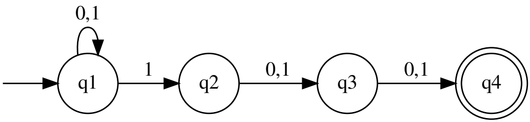
Rules for computation of nondeterministic finite automata

- If there are many choices, split.
- Copies die if they can't move according to the input.
- When to accept a string:
  - At the end of the input, if **any** of the copies is in an accept state, it **accept** the input.

N_1 on 010110

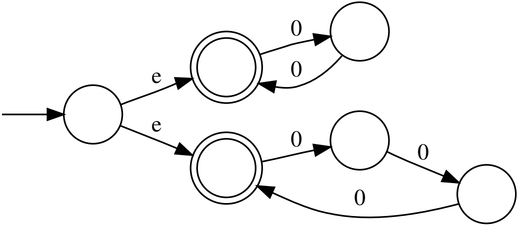


NFA N_2: what are the strings accepted by N_2?

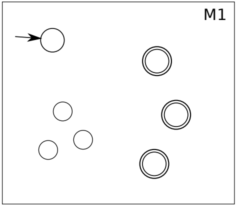


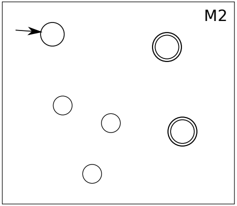
NFA N_3: what are the strings accepted by N_3?

Let $\{0\}$ be the alphabet for N_3.



Union (if NFA is allowed)

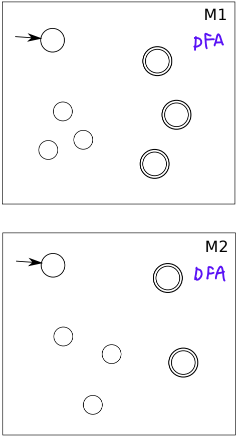


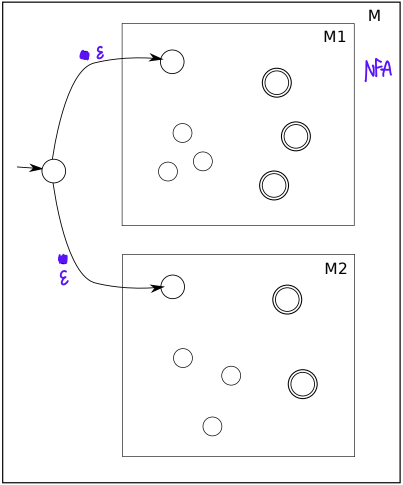


Union (if NFA is allowed)

ยังไม่สามารถพิสูจน์คุณสมบัติปิดทาง regular ได้

เพราะยังสามารถสร้างได้แต่ NFA





Formal definition of NFAs: δ

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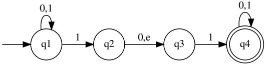
- We have that the state transition δ for an NFA is a function from $Q \times \Sigma_\varepsilon$ to $\mathcal{P}(Q)$.

Definition [NFA]

A **nondeterministic finite automaton** is a 5-tuple $(Q, \Sigma, \delta, q_0, F)$, where

1. Q is a finite set of states,
2. Σ is a finite alphabet, ความแตกต่างของ
3. * $\delta : Q \times \Sigma_\varepsilon \longrightarrow \mathcal{P}(Q)$ is the transition function,
4. $q_0 \in Q$ is the start state, and
5. $F \subseteq Q$ is the set of accept states.

Example: N_1



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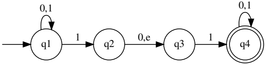

N_1 is $(Q, \Sigma, \delta, q_1, F)$ where

1. $Q = \{q_1, q_2, q_3, q_4\}$,
2. $\Sigma = \{0, 1\}$,
3. δ is defined as

	0	1	$\emptyset$
q_1	$\{q_1\}$	$\{q_1, q_2\}$	$\emptyset$
q_2	$\{q_3\}$	$\emptyset$	$\{q_3\}$
q_3	$\emptyset$	$\{q_4\}$	$\emptyset$
q_4	$\{q_4\}$	$\{q_4\}$	$\emptyset$

4. q_1 is the start state, and
5. $F = \{q_4\}$.

Formal definition of computation of NFAs

Let $N = (Q, \Sigma, \delta, q_0, F)$ be an NFA and let w be a string over alphabet Σ.

We say that ***N* accepts *w*** if we can write $w = w_1 w_2 \cdots w_n$ where each w_i is a member of Σ_ε and

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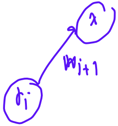
1. $r_0 = q_0$,
2. $r_{i+1} \in \delta(r_i, w_{i+1})$ for $i = 0, \ldots, n-1$, and
3. $r_n \in F$.



Are NFAs more powerful than DFAs?

- With the power of nondeterminism, NFAs seem to be more powerful.

NFA มีพลังกว่า DFA หรือป่าว

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- With the power of nondeterminism, NFAs seem to be more powerful.
- In fact, DFAs and NFAs recognize the **same** class of languages! Spoils → ไม่ได้เก่งขึ้น

Are NFAs more powerful than DFAs?

- With the power of nondeterminism, NFAs seem to be more powerful.
- In fact, DFAs and NFAs recognize the **same** class of languages!
- We say that two machines are *equivalent* if they recognize the same language.

Proving equivalence

Two directions:

כולם

- Given a DFA, construct an NFA recognizing the same language.

שון

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ทำไงล่ะทีนี้

Proving equivalence

Two directions:

- Given a DFA, construct an NFA recognizing the same language.
  - Easy! DFA is also an NFA.
- Given an NFA, construct a DFA recognizing the same language.
  - No that easy.

Main theorem

Theorem 2

Every nondeterministic finite automaton has an equivalent deterministic finite automaton.

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How can we prove that?

- Recall "reader as automaton"?
- Given an NFA N, think of a DFA M as a manager who operates N.
- What does M have to **remember** in order to simulate N correctly?

ดูลออยู่ที่ไหน "บ้าง"

จำ state อย่างเดียวไม่ได้

→ ต้องจำ set of state

Proof of Theorem 2

Let $N = (Q, \Sigma, \delta, q_0, F)$ be an NFA.

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Let $N = (Q, \Sigma, \delta, q_0, F)$ be an NFA. We shall construct a DFA $M = (Q', \Sigma, \delta', q_0', F')$ recognizing the same language.

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Easy case: N has no ε arrows.

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2. If N is at state $q \in Q$ and receives input symbol $a \in \Sigma$, N may moves to any states in $\delta(q, a)$.
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3. $q'_0 = \{q_0\}$, and
4. $F' = \{R \in Q' | R \text{ contains an accept state of } N\}$.

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- If N is at q, it can move freely to any states in $D(q)$.

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- Fix start states $q_0' = E(\{q_0\})$.

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- At any point on the computation of M, the state of M is the set of all possible states that N can be in at that point.

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- M correctly simulates N.
- Thus, our proof is complete.

Note on the correctness proof

- Our previous proof of Theorem 2 is quite short and does not give out all the details.
- This is okay for now, since our construction is simple enough so that it is quite obvious that it is correct.
- For more complicated constructions, we need to be more formal.

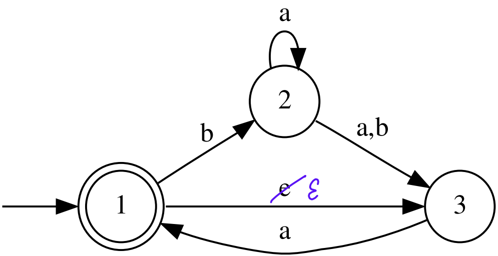
A more general definition of regular languages

Corollary 3

A language is regular iff some nondeterministic finite automaton recognizes it.

เนื่องจาก DFA eq กับ NFA

Example



แปลงเป็น DFA

Closure under the regular operations

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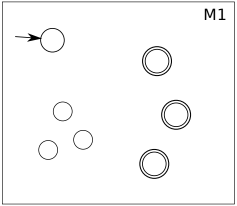
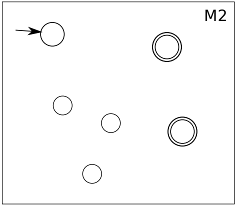
- We'll look at each operation.

Union





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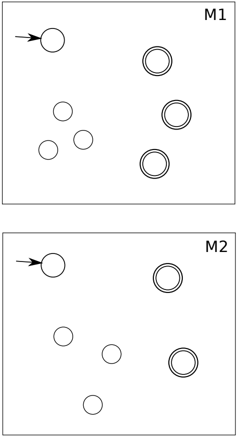
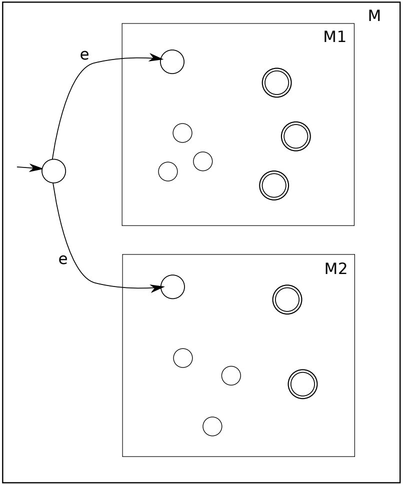
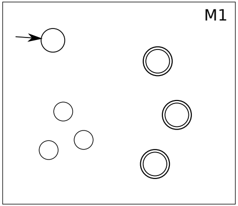
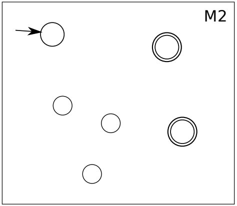




Concatenation





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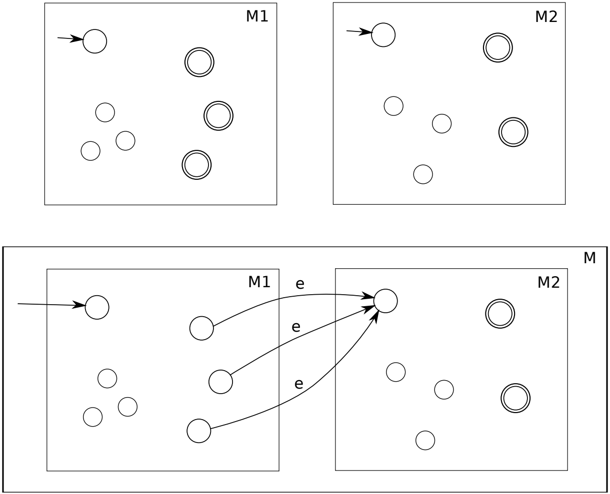
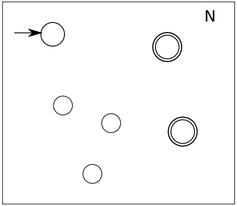


Star



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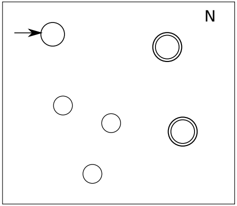
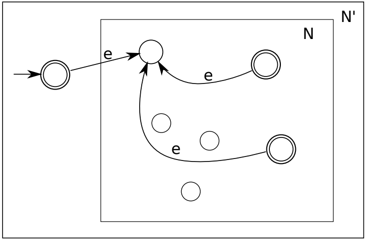
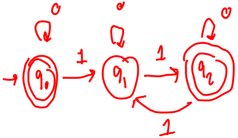




Practice

- Let alphabet $\Sigma = \{0, 1\}$.
- $A_1 = \{w \mid w$ contains even number of 1's $\}$.



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  Find an FA M_2 that recognizes A_2.
- Construct an NFA N_1 recognizing $A_1 \cup A_2$.
- Construct an NFA N_2 recognizing $A_1 \circ A_2$.
- Construct an NFA N_3 recognizing $(A_1^*) \circ A_2$.

Building up patterns

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- For example,

$$(\{0\} \cup \{1\}) \circ \{0\}^*,$$

0
1 0 0 0 0
00 . . .

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$$(\{0\} \cup \{1\}) \circ \{0\}^*,$$

or in a shorter form $(0 \cup 1)0^*$

Building up patterns

- Regular operations can be used to build expressions describing languages.
- For example,
$$(\{0\} \cup \{1\}) \circ \{0\}^*,$$
or in a shorter form $(0 \cup 1)0^*$
- This is called a **regular expression**.
- Note: 0 denotes $\{0\}$, 1 denotes $\{1\}$, and $\circ$ is omitted.

Another examples

$(0 \cup 1)^*$

ทุก binary string รวม ε

$= 111111$

$= 00000$

$= \varepsilon$

$= 0101$

Another examples

$$(0 \cup 1)^*$$

- All possible strings (including ε).

Another examples

$$(0 \cup 1)^*$$

- All possible strings (including ε).
- If $\Sigma = \{0, 1\}$, we can write Σ for $(0 \cup 1)$, and write Σ^* for any strings from alphabet Σ.

ε מכיל

Definition [regular expression]

นิพจน์เรกูลาร์

R is a **regular expression** if R is

1. a for some $a \in \Sigma$,
2. ε,
3. $\emptyset$,

Base case

4. $(R_1 \cup R_2)$ where R_1 and R_2 are regular expressions,
5. $(R_1 \circ R_2)$ where R_1 and R_2 are regular expressions, and
6. (R_1^*) where R_1 is a regular expression.

สมบัติปิด

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6. (R_1^*) where R_1 is a regular expression.

This is an **inductive definition**.

Precedence

ลำดับการทำงาน

Operations are performed in this order:

- $*$
- $\circ$
- $\cup$

Shorthands

- RR^*

Shorthands

ต้องมีอย่างน้อย 1

- RR^* can be written as R^+
- $RRRR$ = R^4

Shorthands

- RR^* can be written as R^+
- $RRRR$ can be written as R^4, in general R^k is the concatenation of R to itself for k times.

Examples of regular expressions

1. 0^*10^*.

$= 0\,1\,0$

$= 1$

Examples of regular expressions

1. 0^*10^*.
2. $\Sigma^*1\Sigma^*$.

Examples of regular expressions

1. 0^*10^*.
2. $\Sigma^*1\Sigma^*$.
3. $\Sigma^*001\Sigma^*$.

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Examples of regular expressions

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2. $\Sigma^*1\Sigma^*$.
3. $\Sigma^*001\Sigma^*$.
4. $(01^+)^*$
5. $(\Sigma\Sigma)^*$
6. $01 \cup 10$
7. $0\Sigma^*0 \cup 1\Sigma^*1 \cup 0 \cup 1$

Examples of regular expressions

1. 0^*10^*.
2. $\Sigma^*1\Sigma^*$.
3. $\Sigma^*001\Sigma^*$.
4. $(01^+)^*$
5. $(\Sigma\Sigma)^*$
6. $01 \cup 10$
7. $0\Sigma^*0 \cup 1\Sigma^*1 \cup 0 \cup 1$
8. $1^*\emptyset$

Examples of regular expressions

1. 0^*10^*. → มี 1 แล้ว 0 อยู่ ซ้าย/ขวา
2. $\Sigma^*1\Sigma^*$.
3. $\Sigma^*001\Sigma^*$.
4. $(01^+)^*$
5. $(\Sigma\Sigma)^*$
6. $01 \cup 10$
7. $0\Sigma^*0 \cup 1\Sigma^*1 \cup 0 \cup 1$
8. $1^*\emptyset = \emptyset$ $a \circ \phi = \phi$ ไม่มี $x \in a$ $y \notin \phi$ ที่ $xy = w$
9. $\emptyset^*$ $= \{\varepsilon\}$

Examples of regular expressions

1. 0^*10^*. → มี 1 จำนวน 1 ตัว และมี 0 ที่เหลือ
2. $\Sigma^*1\Sigma^*$. → มี 1 อย่างน้อย 1 ตัว
3. $\Sigma^*001\Sigma^*$. → มี 001 อยู่ข้างใน
4. $(01^+)^*$ → ทุกๆ 0 จะตามด้วย 1 อย่างน้อย 1 ตัว
5. $(\Sigma\Sigma)^*$ → ความยาวเป็นคู่
6. $01 \cup 10$ → $\{01, 10\}$
7. $0\Sigma^*0 \cup 1\Sigma^*1 \cup 0 \cup 1$ → ขึ้นต้นและลงท้ายเหมือนกัน
8. $1^*\emptyset = \emptyset$
9. $\emptyset^* = \{\varepsilon\}$

Properties

- $R \cup \emptyset$ = R

Properties

- $R \cup \emptyset = R$

Properties

- $R \cup \emptyset = R$
- $R \circ \varepsilon$

Properties

- $R \cup \emptyset = R$
- $R \circ \varepsilon = R$

Properties

- $R \cup \emptyset = R$
- $R \circ \varepsilon = R$
- $R \cup \varepsilon$

Properties

- $R \cup \emptyset = R$
- $R \circ \varepsilon = R$
- $R \cup \varepsilon$ might not equal R.

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- $R \circ \varepsilon = R$
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- $R \circ \varepsilon = R$
- $R \cup \varepsilon$ might not equal R.
- $R \circ \emptyset$ might not equal R.

regular languages

[illegible] FA

[illegible] describes [illegible]

DFA, NFA

(Machine)

language

regular expression

(grammar)

Equivalence

Theorem 4

A language is regular iff some regular expression describes it.

Machine Recognize language

RegEx describe language

Equivalence

พิสูจน์ 2 ทาง

Theorem 4

A language is regular iff some regular expression describes it.

There are two directions to prove the theorem:

- If a language is described by a regular expression, then it is regular.
- If a language is regular, then it can be described by a regular expression.

Equivalence

Theorem 4

A language is regular iff some regular expression describes it.

There are two directions to prove the theorem:

- If a language is described by a regular expression, then it is regular. $\Leftarrow$
- If a language is regular, $\Rightarrow$ then it can be described by a regular expression.

Today we'll prove only the first direction.

[illegible] FA 可

recognize 的

Practice

- Let alphabet $\Sigma = \{0, 1\}$.

Practice

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- Find an FA M_1 that recognizes 01^+

Practice

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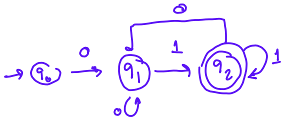
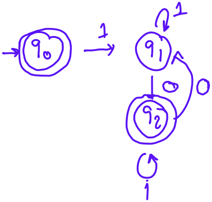
Practice

- Let alphabet $\Sigma = \{0, 1\}$.
- I Find an FA M_1 that recognizes 01^+
- II Find an FA M_2 that recognizes $(10)^*$
- III Find an FA M_3 that recognizes $(01^+) \cup (10)^*$

I



II



III = I ∪ II

Practice

- Let alphabet $\Sigma = \{0, 1\}$.
- Find an FA M_1 that recognizes 01^+
- Find an FA M_2 that recognizes $(10)^*$
- Find an FA M_3 that recognizes $(01^+) \cup (10)^*$
- Find an FA M_4 that recognizes $1^+ \circ ((01^+) \cup (10)^*)$

A regular expression describes a regular language

Lemma 5

If a language is described by a regular expression, then it is regular.

A regular expression describes a regular language

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If a language is described by a regular expression, then it is regular.

To prove this, we'll look at how we a regular expression is constructed.

Rule 1

R is a **regular expression** if R is a for some $a \in \Sigma$.

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What is a DFA that recognizes R?

Rule 2

R is a **regular expression** if R is ε.

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Rule 6

R is a **regular expression** if R is (R_1^*) where R_1 is a regular expression.

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Given an NFA N_1 that recognizes R_1, what is a NFA that recognizes R?

Proof by Structural Induction

- In proving Lemma 5, we show how to construct an NFA of a regular expression given NFAs of its subexpressions.

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- The proof is again an inductive proof.
- Sometimes, this kind of inductive proofs is called structural induction.
  - Inductive Hypothesis (when considering a regular expression R): Assume that for all smaller regular expressions R', the language described by R' can be recognized by some NFA N'.

Practice

1. Find an NFA recognizing $(01 \cup 0)^*$.

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   Try to build an NFA using the construction discussed in class.

Practice

1. Find an NFA recognizing $(01 \cup 0)^*$.
   Try to build an NFA using the construction discussed in class.
2. Find an NFA recognizing $(0 \cup 1)^* 010$.