Finite Automata

204213 Theory of Computation

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Outline

An automatic door

- Recall our automatic door example from last time?

An automatic door

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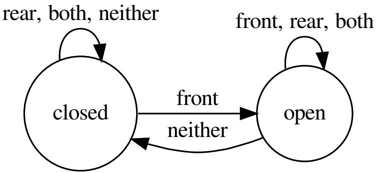
- Let's see a simulation.

What did you see?



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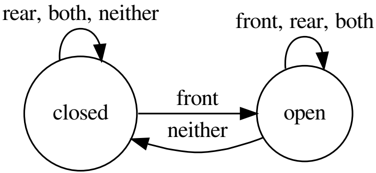

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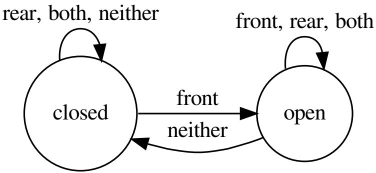

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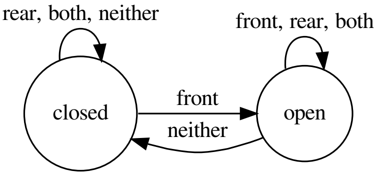

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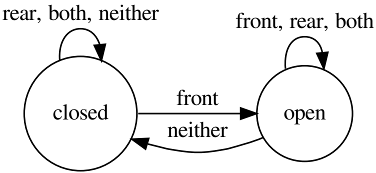
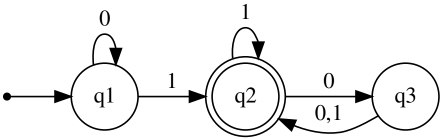


- There are two **states**: closed and open
- There are 4 possible inputs, and the state of the machine changes (or remains) after each input.
- See that in table form:

	neither	front	rear	both
closed	closed	open	closed	closed
open	closed	open	open	open

Another automaton: M_1



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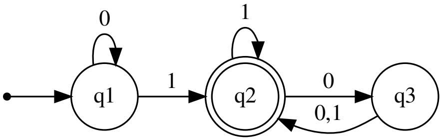

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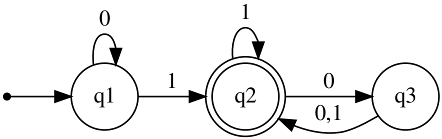

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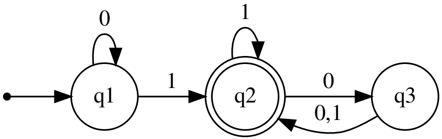

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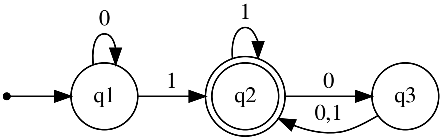

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- q_2 is the **accept state**. (see the double circle)
- Arrows are **transitions**.

Formal definition: why?

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Formal definition gives

- Precision

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Formal definition gives

- Precision
- Notation

Transition function: δ

- The rule for moving.

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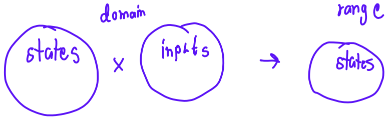
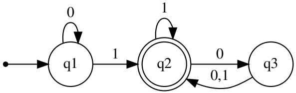


Definition [finite automaton]

A **finite automaton** is a 5-tuple $(Q, \Sigma, \delta, q_0, F)$ where

1. Q is a finite set called the *states*,
2. Σ is a finite set called the *alphabet*, set ของ input ทั้งหมด
3. $\delta : Q \times \Sigma \longrightarrow Q$ is the *transition function*,
4. $q_0 \in Q$ is the *start state*, and
5. $F \subseteq Q$ is the *set of accept states*.

Formal definition of M_1



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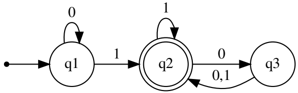


$M_1 = (Q, \Sigma, \delta, q_1, F)$, where

1. $Q = \{q_1, q_2, q_3\}$,
2. $\Sigma = \{0, 1\}$,
3. δ can be described as

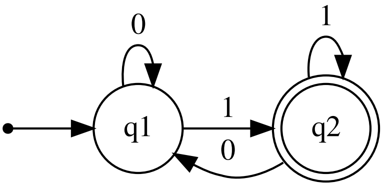
	0	1
q_1	q_1	q_2
q_2	q_3	q_2
q_3	q_2	q_2

4. q_1 is the start state, and
5. $F = \{q_2\}$.

Finite automaton M_2

$M_2 = (\{q_1, q_2\}, \{0,1\}, \delta, q_1, \{q_2\})$ โดย δ นิยามเป็น

δ	0	1
q_1	q_1	q_2
q_2	q_1	q_2



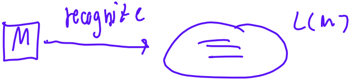
$L(M_2) = \{w \mid$ w เป็นชุด string ที่ลงท้ายด้วย 1 $\}$

Language of a machine

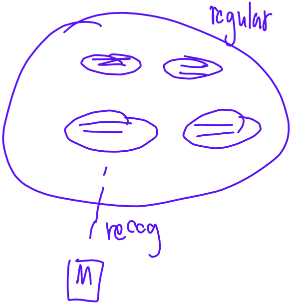
set of string

- A set A of strings is called the **language of machine** M if A is the set of all strings that M accepts.

Language of a machine



- A set A of strings is called the **language of machine** M if A is the set of all strings that M accepts.
- We write $L(M) = A$.
- We also say that M **recognizes** A.



ของ A

machine

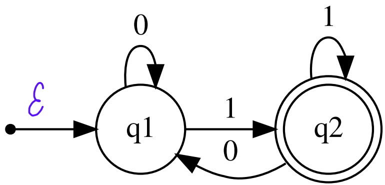
Language of machine M_1

- Let $A = \{w | w$ contains at least one 1 and an even number of 0's follow the last 1$\}$.
- $L(M_1) = A$

Language of machine M_1

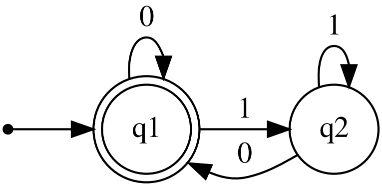
- Let $A = \{w | w$ contains at least one 1 and an even number of 0's follow the last 1$\}$.
- $L(M_1) = A$
- Or, we can say that M_1 recognizes A.

Finite automaton M_2



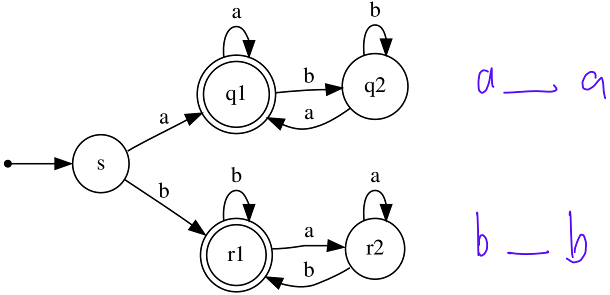
What is the language that M_2 recognizes?

Finite automaton M_3



What is the language that M_3 recognizes?

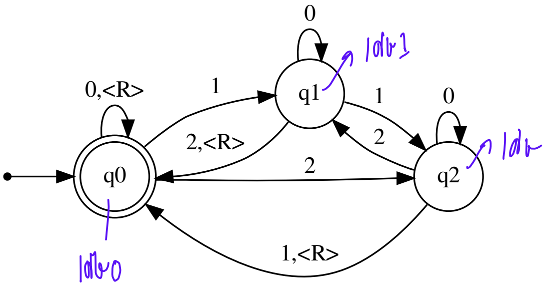
Finite automaton M_4



What is the language that M_4 recognizes?

Finite automaton M_5

$L(M_5) = \{ w \mid w$ เป็นสตริงตัวเลขที่รวมกันแล้วหาร 3 ลงตัว $\}$



What is the language that M_5 recognizes?

Formal definition of computation

Let $M = (Q, \Sigma, \delta, q_0, F)$ be a finite automaton and let $w = w_1 w_2 \cdots w_n$ be a string over alphabet Σ. M **accepts** w if

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1. $r_0 = q_0$,
2. $\delta(r_i, w_{i+1}) = r_{i+1}$ for $i = 0, \ldots, n-1$, and ← รับประกันว่าเกิด states ที่เกิดขึ้นจริง
3. $r_n \in F$.

Definition [regular language]

- **M recognizes language A** if $A = \{w | M \text{ accepts } w\}$.

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- M **recognizes language** A if $A = \{w | M \text{ accepts } w\}$.
- A language is called a **regular language** if some finite automaton recongnizes it.

set of language

{"01", "11"}

Designing finite automata

Tips:

- Pretending that you are the automaton.

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Tips:

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- You get one input at a time.
- Think about what you have to remember to make decision correctly. (That would be a set of states.)

Practice

Language consisting of all strings with an odd number of 1's.

Building more complex finite automata

- Let $\Sigma = \{0, 1, 2\}$.
- Can you build a finite automaton M_3 that accepts all strings whose sums are divisible by 3?

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Building more complex finite automata

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- Can you build a finite automaton M_5 that accepts all strings whose sums are divisible by 3 or 5?

$L_3 \cup L_5$ eg. regular operations

Construction from smaller building boxes

This is one of important ideas in computer science.

Regular operations

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A set and operations

- A collection of objects is **closed under some operation** if applying that operation to objects in that set only result in object in that set.

A set and operations

- A collection of objects is **closed under some operation** if applying that operation to objects in that set only result in object in that set.
- E.g., a set of natural number $\mathcal{N}$ is closed under multiplication.

Definition [regular operations]

For a language A and B, the regular operations **union**, **concatenation**, and **star** can be defined as follows.

- Union: $A \cup B = \{x | x \in A \ \text{ or } \ x \in B\}$
- Concatenation: $A \circ B = \{xy | x \in A \ \text{ and } \ y \in B\}$ เอา string มาต่อกัน
- Star: $A^* = \{x_1 x_2 \cdots x_k | k \geq 0 \ \text{ and each } \ x_i \in A\}$

↳ ทำซ้ำ, 0 ต่อก็ได้

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M_1 M_2

M_1 recognize A_1

M_2 —— A_2

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  - **Goal:** to show that there exists a finite automaton recognizing $A_1 \cup A_2$,
  - **Given that:** there are finite automata M_1 and M_2 such that M_1 recognizes A_1 and M_2 that recognizes A_2.

Union operation

"set"

Theorem

The class of regular languages is closed under the union operation

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Approach for proving it.

- Proof by construction.
- We know that there are finite automata M_1 that recognizes A_1 and M_2 that recognizes A_2.
- We shall construct M that recognizes $A_1 \cup A_2$.

How can we show that?

- Again, pretend that you are a finite automaton trying to recognizes $A_1 \cup A_2$.

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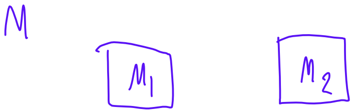

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  - Can we use them to recognize $A_1 \cup A_2$?

Given

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- Machine $M_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$ recognizing A_1
- Machine $M_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)$ recognizing A_2

Machine M recognizing $A_1 \cup A_2$

Machine $M = (Q, \Sigma, \delta, q_0, F)$, such that

- $Q = Q_1 \times Q_2$,

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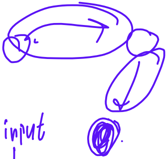
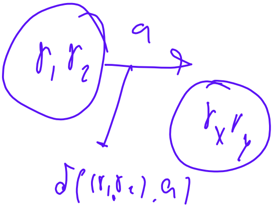


state input

$$\delta((r_1, r_2), a) = (\delta_1(r_1, a), \delta_2(r_2, a)),$$: state ถัดไป

- $q_0 = (q_1, q_2)$,
- $F = \{(r_1, r_2) | \; r_1 \in F_1 \text{ or } r_2 \in F_2\}$



Other regular operations

concatenate

- Can we use the same technique to prove that $A_1 \circ A_2$ is regular?

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    - To construct a finite automaton M for $A_1 \circ A_2$ from M_1 and M_2 that recognize A_1 and A_2 we need to simulate M_1 to the end of x and start simulating M_2 right after that. **And it is hard to "tell" where x ends.**

A machine that always guesses correctly

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- Can machine guess?

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  - Maybe?

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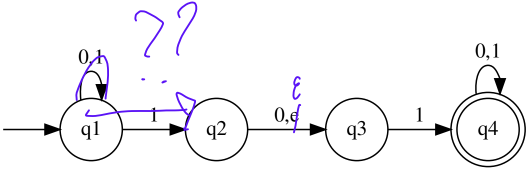
Example: Nondeterministic Finite Automaton N_1



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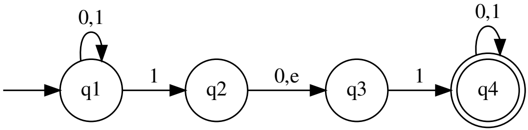
Note: e in the figure is ε.

Differences



- Duplicate symbols
- Missing symbols
- Empty string: ε

Deterministic and Nondeterministic Finite Automata

- Previously, we only consider finite automata whose next states are **determined** by their input alphabet and their current states.

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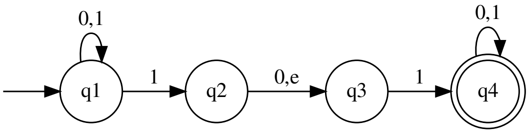
- On the other hand, in **nondeterministic** computation, many choices may exist.
- Therefore, we have deterministic finite automata (**DFA**) and nondeterministic finite automata (**NFA**).

How does N_1 compute?



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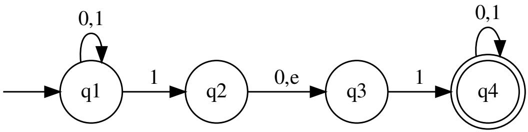

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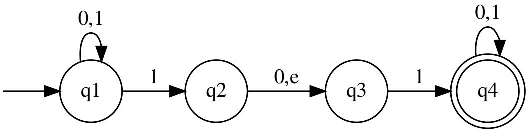

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See simulation.

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Rules for computation of nondeterministic finite automata

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- When to accept a string:
    - At the end of the input, if **any** of the copies is in an accept state, it **accept** the input.

N_1 on 010110

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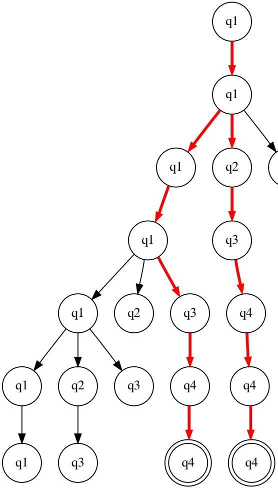
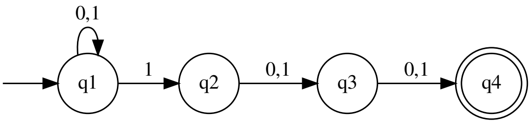


NFA N_2: what are the strings accepted by N_2?



NFA N_3: what are the strings accepted by N_3?

Let $\{0\}$ be the alphabet for N_3.