Simulation & Modeling

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Credit Hours: 3hrs

Unit 3: Simulation of Continuous System

Class Load : 5 hrs

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Differential and Partial differential equations:

Differential Equations

- A differential equation is an equation which contains one or more terms and the derivatives (instantaneous rate of change) of one variable (i.e., **dependent variable**) with respect to the other variable (i.e., **independent variable**).

  dy/dx = f(x)

  Here "x" is an independent variable and "y" is a dependent variable

  For example, dy/dx = 5x

- An example of a linear differential equation with constant coefficients to describe the **wheel suspension system of an automobile** can be given as

  $$M\ddot{x} + D\dot{x} + Kx = KF(t)$$

- The dependent variable x appears with its first and second order derivative $\dot{x}$ & $\ddot{x}$ and the term involving these quantities are multiplied by constant coefficient and added. The quantity $F(t)$ is an input to the system depending upon the independent variable t.

Differential and Partial differential equations:

Partial Differential Equations

- A Partial Differential Equation commonly denoted as PDE is a differential equation containing **partial derivatives of the dependent variable (one or more)** with **more than one** **independent variable**.
- The simple PDE is given by;

$\partial u/\partial x\,(x,y) = 0$.

- It involves the derivative of the same dependent variable with respect to each of the independent variable.

1. $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$
2. $u_{xx} + u_{yy} = 0$

Differential and Partial differential equations:

Linear Equations

- A Linear equation can be defined as the equation having the **maximum only one degree**.
- A linear equation forms a **straight line** on the graph.
- Example: $x + 7 = 12$, $2x - 9 = 1$, and $x + 5 = 2x - 3$ are equation in one variable x.

Differential and Partial differential equations:

Non-Linear Equations

- A Nonlinear equation can be defined as the equation having the maximum degree **2 or more than 2**.
- A nonlinear equation forms a **curve** on the graph.
- **Example:**
  - $3x^2 + 2x + 1 = 0$, $3x + 4y = 5$, this are the example of nonlinear equations, because equation 1 have highest degree of 2 and second equation have variable x and y.

Significance of differential equations:

1. Differential equations describe various **exponential growth** and **decay**.
2. They are also used to describe the **change** in return on investment overtime.
3. They are used in field of medical science for modeling cancer **growth** or the **spread** of disease in the body.
4. **Movement** of electricity can be also described with the help of it.
5. They help economics in finding **optimum investment strategies**.
6. The **motion** of waves or a pendulum can also be described using these equations.

Continuous System Simulation:

Continuous System

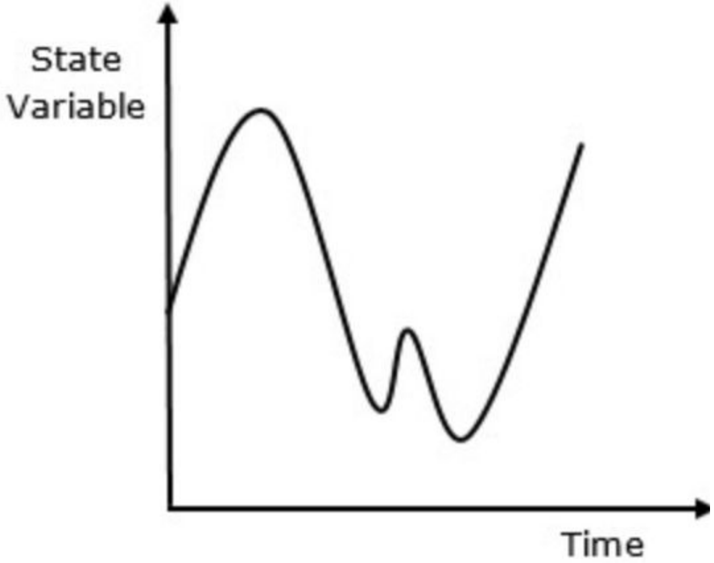
- A continuous system is one in which the **state variable(s) change continuously over time**.
- **Analog or continuous signals** are used, allowing for smooth transitions and continuous changes.
- Key characteristics of continuous systems include:
  - Variables can take on any value within a certain range.
  - Behavior can be represented by continuous representations.
  - Analog representation (e.g., electrical circuits, fluid dynamics).
  - Continuous values (e.g., real numbers).

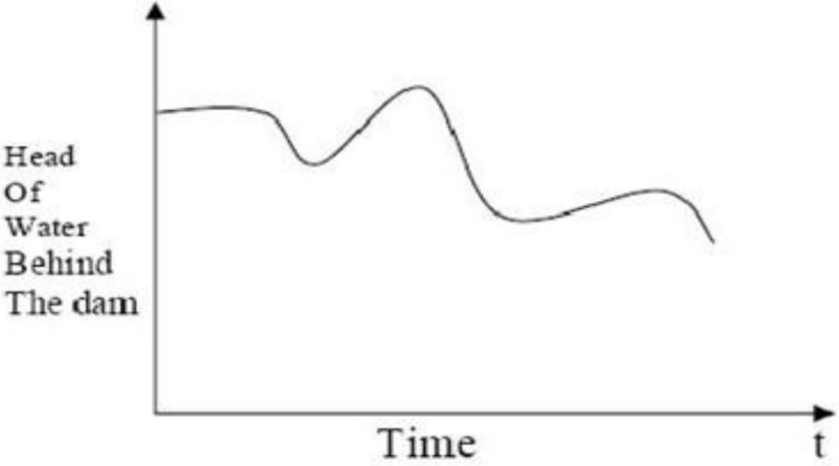
Continuous System Simulation:

- **Continuous System**
  - **Example: Head of Water behind the Dam**
    - During winter seasons level of which water decreases gradually and during rainy season level of water increase gradually. The change in water level is continuous. The figure below shows the change of water level over time.

Continuous System Simulation:

- **Continuous System**





Manual Simulation:

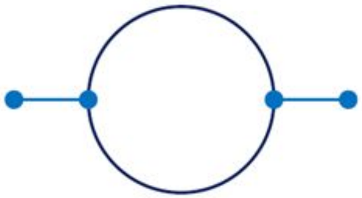
- Manual Simulation is a way of simulating a system **without** using a **digital computer** instead, the simulation is carried **out by hand** using basic tools like paper, pencil, calculator, dice, random number tables, or physical objects.

Analog Computer:

- **An analog computer is a type of computing device that operates on continuous signals and physical quantities to perform calculations and simulations**
- Before general availability of digital computers, there existed devices whose **behaviour is equivalent to a mathematical operation**, such as **addition or integration.**
- Putting together combination of such devices in a manner specified by a mathematical model of a system, allowed the system to be simulated.
- Analog computers can **simulate continuous systems**, such as the behavior of electrical circuits, fluid dynamics, mechanical systems, and other physical phenomena.
- By using electrical signals that continuously vary, analog computers can **represent and predict the behavior** of these systems over time.

Analog Methods:

- The general method to apply analog computers for the simulation of continuous system models involves following components:



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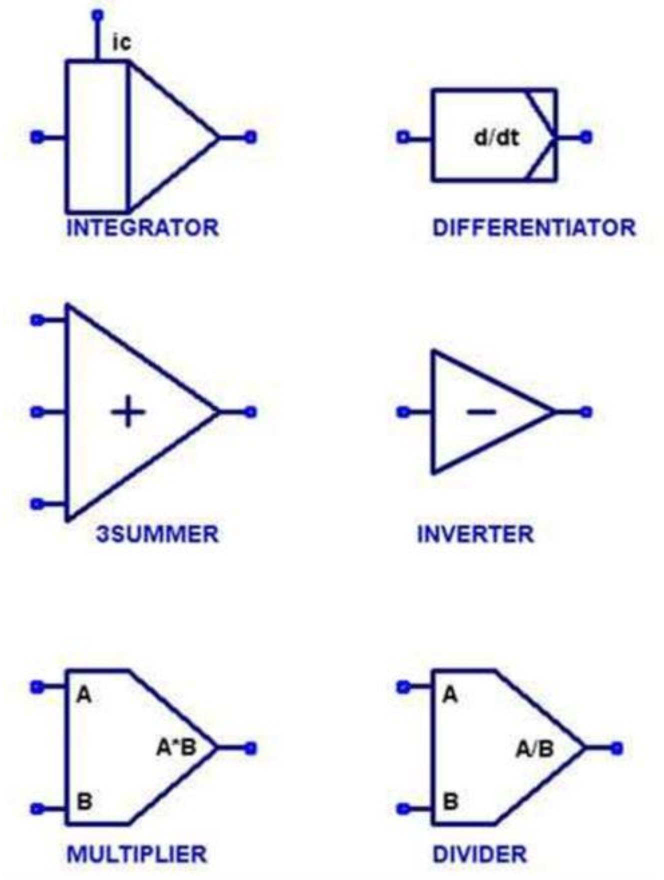


Analog Methods:

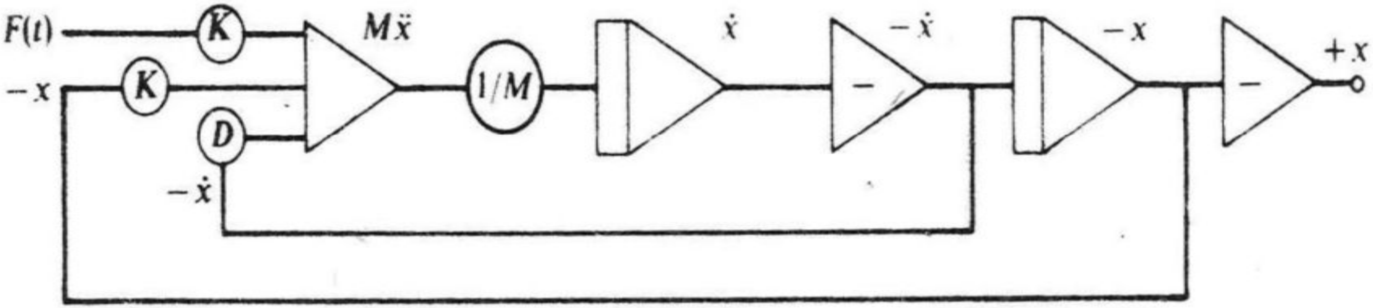
- The general method by which analog computers are applied can be demonstrated using **second order differential equation.** (**Example :** Automobile Suspension Problem)

$$M\ddot{x} + D\dot{x} + kx = kF(t)$$

- Solving the equation for the **highest order derivative** gives,

$$M\ddot{x} = kF(t) - D\dot{x} - kx$$

- The diagram to solve this problem in this manner is given in figure below as:



Analog Methods:

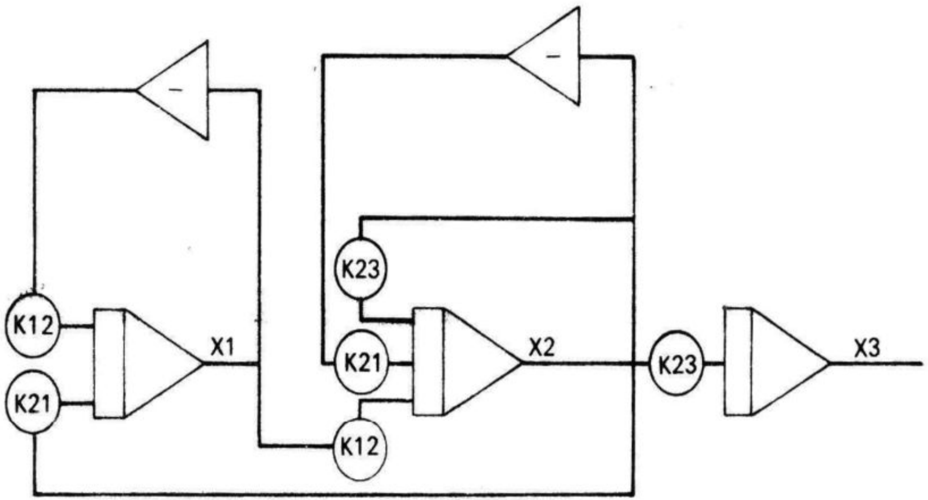
- Suppose a variable representing the **input** $F(t)$ is supplied, assume there exists variables representing $-x$, $-\dot{x}$ and $\ddot{x}$.
  - These three variables can be **scaled** and added to **produce** $M\ddot{x}$.
  - **Integrating** it with a scale factor $1/M$ produces $\dot{x}$.
  - Changing sign produces $-\dot{x}$, further integrating produces $-x$, a further sign inverter is included to produce $+x$ as **output**.

Analog Methods:

- Analog computers are those computers that are **unified** with devices like **adder (digital circuit) and integral** so as to simulate the **continuous mathematical model of the system**, which generates continuous outputs.
- Analog method of system simulation is for use of **analog computer** and **other analog devices** in the simulation of continuous system.
- The analog computation is sometimes called **differential analyser.** Electronics analog computers for simulation are based on the use of high gain DC amplifier called **operational amplifier (op amps)**.

Analog Methods:

- Q.> **Design analog computer of**
  - $x'_1 = -k_{12}x_1 + k_{21}x_2$
  - $x'_2 = k_{12}x_1 - (k_{21} - k_{23})x_2$
  - $x'_3 = k_{23}x_2$
- (Human Liver System)



Analog computer model of the liver

Analog Methods:

- There are **three integrators.** Reading from left to right, they solve the equations for x_1, x_2 & x_3. **Interconnections** between the three integrators with sign changers where necessary provides inputs that define the differential coefficients of the three variables.
- First integrator, for example is solving the equation,

$$dx_1/dy = -k_{12}x_1 + k_{21}x_2$$

- The second integrator is solving the equation

$$dx_2/dy = k_{12}x_1 - (k_{21} - k_{23})x_2$$

- In this case, the variable x_2 is being used twice as an input to the integrator, so that the two coefficients k_{21} and k_{23} can be changed independently. The last integrator solves the equation

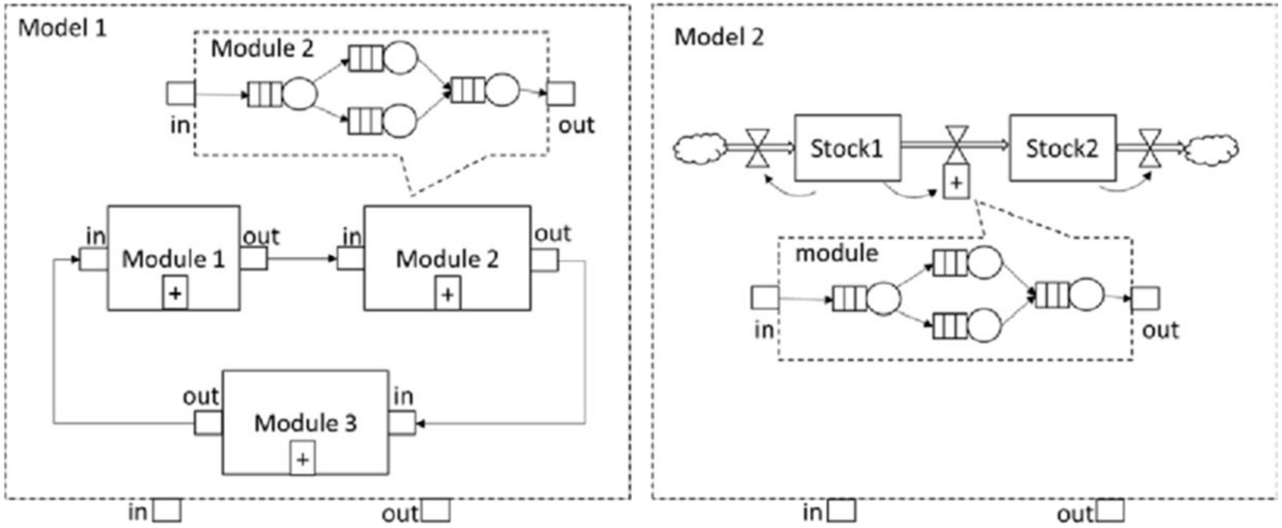
$$dx_3/dy = k_{23}x_2$$

Hybrid Simulation:

- For most studies, the model is clearly either of a **continuous or discrete** nature and that is the determining factor in deciding whether to use an **analog or digital computer for system simulation.** However, there are times when an analog and digital computers are **combined** to provide simulation.
- Currently the main application areas are healthcare, supply chain management and manufacturing, and the majority of published models combine **discrete-event simulation and system dynamics.**
- Here **one computer may be simulating the system** being studied while **other is providing a simulation of the environment** in which the system is to operate.
- It is also possible that the system being simulated is an **interconnection** of continuous and discrete subsystems, which can be modelled by an analog and digital computer **being linked together.**

Hybrid Simulation:

❖ Hybrid Simulation are the models that are simultaneously implemented on **both analog and digital computers**, or models that contain **both discrete and continuous variables**, or models that **combine simulation with an analytical method such as optimization.**



Hybrid Simulation:

- They can be used to model systems **that are too large or too complex to be modeled with a single model.**
- They can be used to **combine the strengths** of different modeling approaches. For example, a **physical model can be used to capture the dynamics of a system**, while a **mathematical model can be used to capture the uncertainty in the system.**
- They can be used to validate numerical models. By **comparing the output** of a numerical model to the output of a physical model, it is possible to verify that the numerical model is accurate.
- **Example:**
  - A **physical model of a car** could be used to capture the **dynamics of the car's handling,** while a **mathematical model** could be used to capture the **uncertainty in the car's fuel consumption**.
  - A **physical model of a patient's heart** could be used to capture the **dynamics of the heart's beating,** while a **mathematical model** could be used to capture the **uncertainty in the patient's response to medication**.

Analog vs Digital Computers:

Analog Computers

- A computer that uses a **continuous signal** to process is called an analog computer.
- The output of an analog computer in the form of **graphical or voltage signals**.
- Analog computers are **slower and less reliable**.
- Analog computers are used in specific fields such as mechanical engineering and medicine as they are **difficult to use**.
- Analog computers provide **accurate computation** results.
- **Example:** The best example of an analog device is **hands wrist watch** and **oscilloscope** (an instrument used to display and analyze the waveform of electronic signals) used by engineers for measurements.

Analog vs Digital Computers:

Digital Computers:

- ❖ A computer that uses a **discrete signal** for its operation is called a digital computer.
- ❖ The output of digital devices is in the form of **binary numbers.**
- ❖ Digital computers are **high-speed** devices that you can rely on.
- ❖ Digital computers **lose precision** due to the nature of discrete signals.
- ❖ These computers are designed to use for **general purposes**, thus can be used in all aspects of life.
- ❖ **Example:** The best example of digital devices are **PC, smartphones, calculators, a pedometer**, (which counts the exact number of steps while walking.)

Digital – Analog Simulators

- **To avoid the disadvantages of analog computers, many digital computer programming languages have been written to produce digital analog simulators.**
- They allow a continuous model to be programmed on a **digital computer** in essentially the same way as **it is solved on an analog** computer.
- A **program** is written to **link** the **macro instructions** (addition, integration etc), in essentially same manner as operational amplifier are connected in the analog computer.
- Digital Analog Simulation is a **technique which makes a digital computer operate much like an analog** computer.

A Pure Pursuit Problem:

- The **pure pursuit problem** refers to a tracking algorithm in which a moving vehicle **continuously adjusts** its path by steering toward a point ahead of it on a **given trajectory**. The goal is to **determine the curvature** that shifts the vehicle from its **current position** to a specified **target point**.
- **Concept:**
  - The algorithm **selects a goal position** located some distance forward along the path **from** the vehicle's current position.
  - The vehicle then "pursues" this goal point by **steering** toward it, **updating** the target as it moves forward.
  - This method is named pure pursuit because it resembles the idea of **continuously** "chasing" a moving point.

A Pure Pursuit Problem:

Applications in Aerial Combat

- In aerial combat, pursuit curves are strategic paths for **intercepting** or following a target aircraft. There are three main pursuit methods:
  - **Pure Pursuit**
    - The pursuing aircraft **points its nose directly** at the target aircraft.
    - Used mainly for simple following or visual tracking.
  - **Lead Pursuit**
    - The pursuing aircraft aims **ahead of the target** aircraft's current position.
    - Commonly used for gun attacks to intercept the target's path.
  - **Lag Pursuit**
    - The pursuing aircraft aims **behind the target**'s position.
    - Used for rear-aspect missile attacks or maintaining distance.

A Pure Pursuit Problem > Algorithm :

Pursuit Algorithm:

- The execution of the pure pursuit algorithm itself is quite simple. The pure pursuit algorithm can be summarized as below:
  1. Establish the **current location** of the vehicle.
  2. Locate the path point **closest** to the vehicle.
  3. Locate the **goal** point
  4. Convert the goal pole to **vehicle coordinates.**
  5. Compute the **curvature** and appeal the vehicle to set the **routing** to that curvature.
  6. **Update** the vehicle's location

A Pure Pursuit Problem > Example :

Pure-Pursuit Example: Fighter Intercepting a Bomber

- Suppose a fighter aircraft sights an enemy bomber and flies directly toward it, in order to catch up with the bomber and destroy it. Here the target is the bomber, it continues flying along a specified curve so the fighter, which is the pursuer, has to change its direction to keep pointed toward the target. Now, we are concerned in determining the attack course of the fighter and in knowing how long it would take for it to catch up with the bomber.
- If the bomber flies along a **straight line**, the problem can be solved directly with analytic techniques. On the other hand, if the path of the bomber is **curved**, then the problem is much more complicated and generally cannot be solved directly.

A Pure Pursuit Problem > Example :

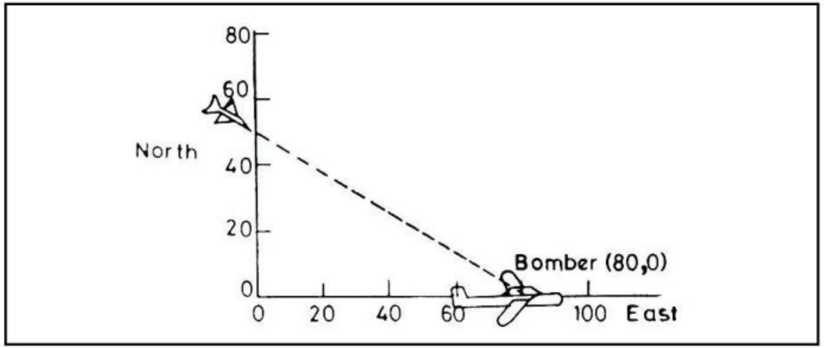
- To solve this **curved path problem** we will use simulation under the following simplifying conditions **(Assumptions)**:

  1. The bomber and the fighter are flying in the **same horizontal plane** when the fighter **first sights** the bomber, and both stay in that plane. This makes the pursuit model **two dimensional**.
  2. The fighter's speed VF is constant (20 kms/minute).
  3. The bomber's path is **known in advance** as discrete positions each minute.
  4. The fighter continuously **re-orients** to the bomber's current position.

A Pure Pursuit Problem > Example :

Coordinate System

- Origin at the point due south of the fighter and due west of the bomber at t=0.
  - **Initial positions (km):**
    - Fighter:
      - F0 =(0,50)
    - Bomber:
      - B0 =(80,0)



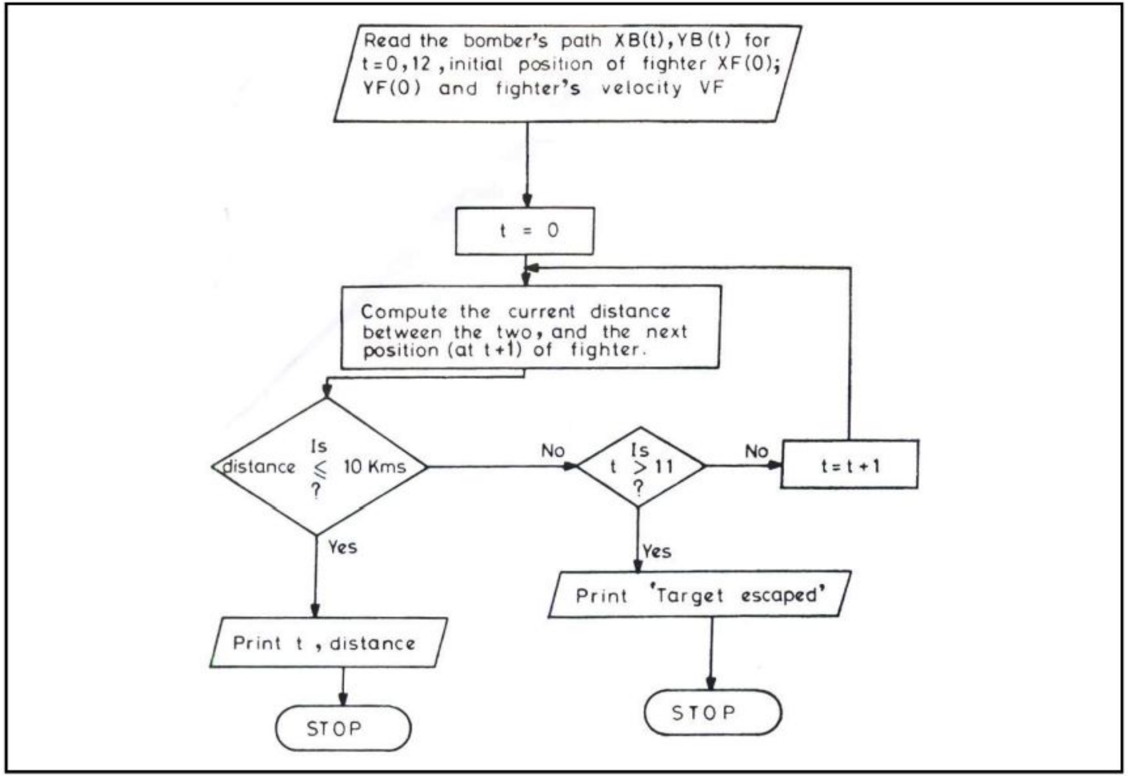
Now we will represent the path of the bomber which is known to us in advance by two arrays, the east coordinates and the north coordinates at specified moments or we can say each minute. We call these coordinates XB(t) and YB (t), respectively. They are presented in the form of a table (in kirometers) below.

Time, t	0	1	2	3	4	5	6	7	8	9	10	11	12
$XB(t)$	80	90	99	108	116	125	133	141	151	160	169	179	180
$YB(t)$	0	−2	−5	−9	−15	−18	−23	−29	−28	−25	−21	−20	−17

Likewise, we will represent the path of the fighter plane by two arrays XF(t) and YF(t). In this example, initially we are given

YF(O) = 50 kms, XF(O) = 0 kms.

A Pure Pursuit Problem > Simulation Flowchart :



A Pure Pursuit Problem > Simulation Flowchart Pseudocode :

1. Input Data
   - Bomber's path: XB (t), YB (t) for t=0,1,...,12 minutes.
   - Initial fighter position: XF(0), Y (0).
   - Fighter's velocity: VF (constant speed =20 kmps).
2. Initialize
   - Set time t=0.
3. Main Loop
   - Step A: Compute
     - The current distance between the fighter and the bomber:
       - $DIST\ (t) = \sqrt{(YB(t) - YF(t))^2 + (XB)(t) - XF(t))^2}$
       - The next position of the fighter at t+1 based on its heading toward the bomber.
   - Step B: Check interception condition:
     - If $d(t) \leq 10$ km → Target caught.
     - Output: interception time t and distance.
     - Stop simulation.
   - Step C: If not intercepted, check time limit:
     - If t >11 minutes → Target escaped.
     - Output: "Target escaped".
     - Stop simulation.
   - Step D: If neither condition is met:
     - Increment time t←t+1 and repeat from Step A.

A Pure Pursuit Problem:

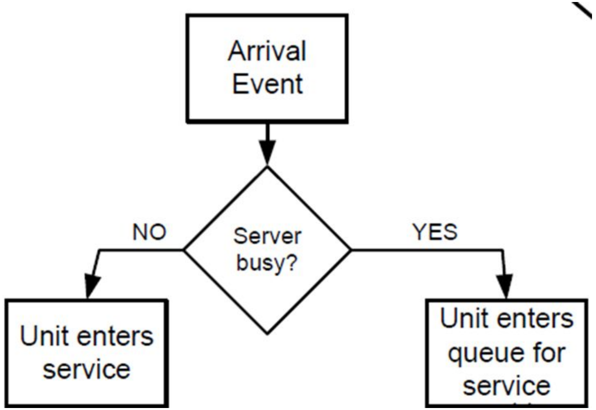
- **Key Observations in Pure Pursuit Simulation**
  - If the target moves in a **straight line** and is **slower** than the pursuer, interception is **guaranteed**.
  - In **curved or faster** target motion, interception may **fail**.
  - Smaller time steps yield **more accurate** trajectories.
- **Applications**
  - Military interception systems.
  - Autonomous vehicle and robotics navigation.
  - Sports motion analysis (e.g., chasing in football or hockey).
  - Wildlife predator–prey movement modeling.

Queuing System:

- The line where the **entities or customers wait is generally known as queue.**
- The combination of all entities in system **being served and waiting for services** will be called a **queuing system**.
- A queueing system consists of **one or more servers** that provide service of some kind to arriving customers.
- **Queuing system are the waiting lines in which the system attribute are waiting for a service.**
- The queue may be of the **customer waiting for the server** **or** **server waiting** for customer.

Queuing System:

- If the servers are busy, the **customers join one or more queues** in front of the servers
- The general diagram of queuing system can be shown as a queuing system **involves customers arriving at a constant or variable time rate for service at a service station.**
- **Example:** Customers can be **students waiting for registration in college, airplane queuing for landing at airfield, or jobs waiting in machines shop.**



Queuing System:

- They remain in queue **till they are provided the service**. Sometimes queue being too long, they will leave the queue and go, it **results a loss of customer**.

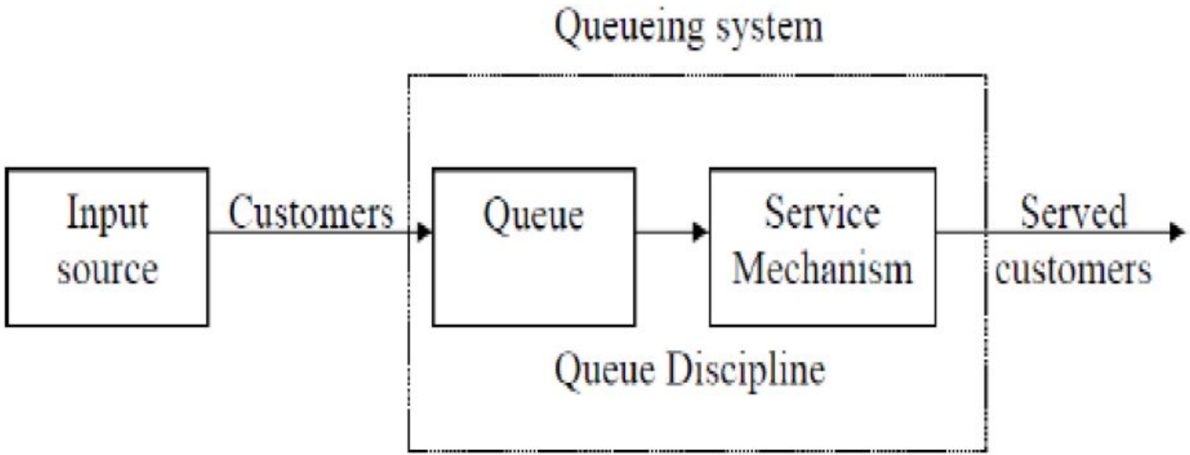
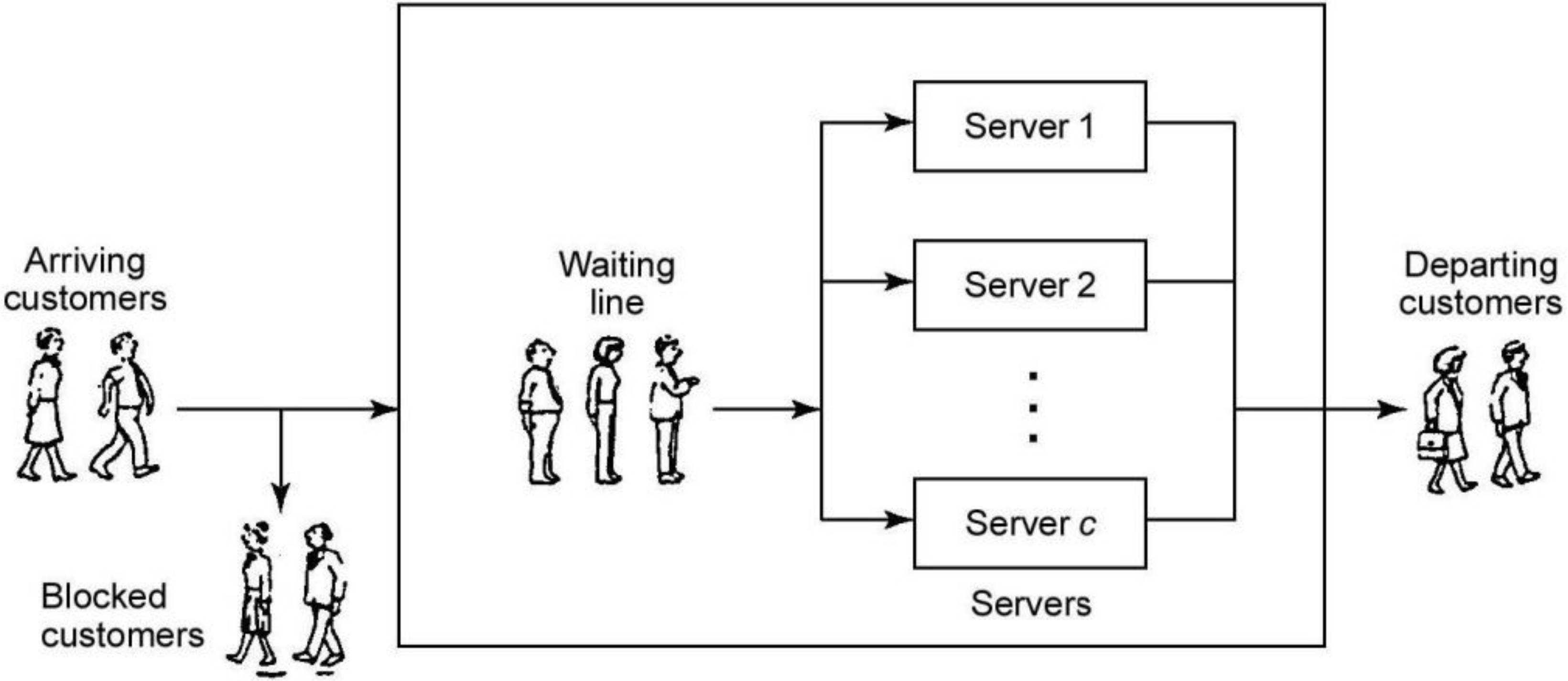


Figure 1. Basic queueing model structure (Balsam and Marin, 2007; Adan and Resing, 2015).

Queuing System:



Queuing System:

- The basic concept of queuing theory is the **optimization of wait time, queue length**, and the **service available** to those standing in a queue.
- **Cost** is one of the important factors in the queuing problem.
  - **Waiting in queues incur cost** (to lose money), whether human are waiting for services or machines waiting in a machine shop. On the other hand if service counter is waiting for customers **that also involves cost.**
  - **In order to reduce queue length**, **extra service centres** are to be provided but for extra service centers, **cost of service becomes higher.**

Queuing System:

- **A congestion system** is system in which there is **high demand for resources for a system**, and when the **resources become insufficient.**
- In queuing theory, traffic congestion is said to occur **if the arrival rate into a system exceeds the service rate of the system at a point in time**
- Congestion occurs **when number of server is insufficient and arrival rate exceeds capacity.**

Examples of Queuing System:

System	Customers	Server(s)
Reception desk	People	Receptionist
Repair facility	Machines	Repair person
Garage	Trucks	Mechanic
Airport security	Passengers	Baggage x-ray
Hospital	Patients	Nurses
Warehouse	Pallets	Fork-lift Truck
Airport	Airplanes	Runway
Production line	Cases	Case-packer
Warehouse	Orders	Order-picker
Road network	Cars	Traffic light
Grocery	Shoppers	Checkout station
Laundry	Dirty linen	Washing machines/dryers
Job shop	Jobs	Machines/workers
Lumberyard	Trucks	Overhead crane
Sawmill	Logs	Saws
Computer	Email	CPU, disk
Telephone	Calls	Exchange
Ticket office	Football fans	Clerk

Main Element of Queuing Systems :

- The key elements, of a queuing system are the **customers** and **servers**.
- The term "**customer**" can refer to **people, machines, trucks, mechanics, patients**—**anything that arrives** at a facility and **requires service.**
- The term "**server**" might refer to **receptionists, repair persons, CPUs in a computer, or washing machines**....any resource (person, machine, etc. which **provides the requested service**.

Elements/Characteristics of Queuing Systems

1. Calling Population
2. System Capacity
3. Arrival Process
4. Queue Discipline
5. Queue Maximum Size
6. Service Capacity
7. Number of Servers
8. Output

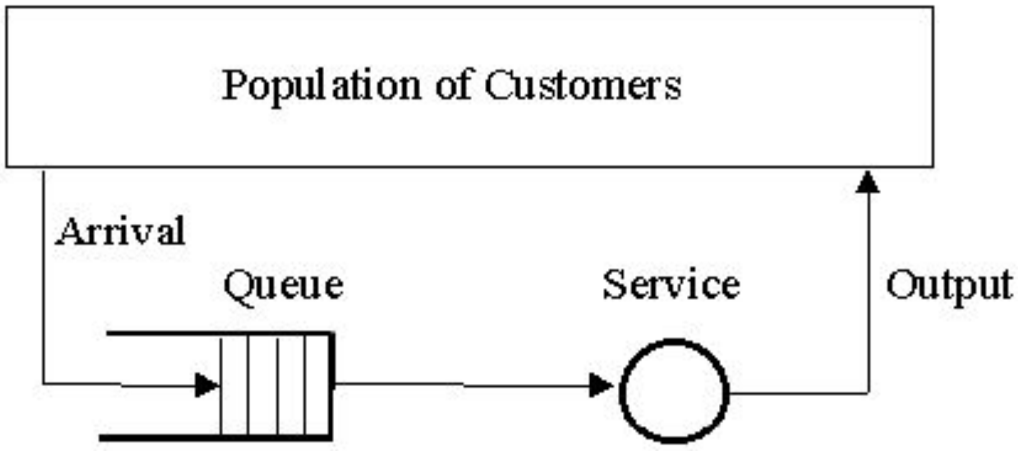
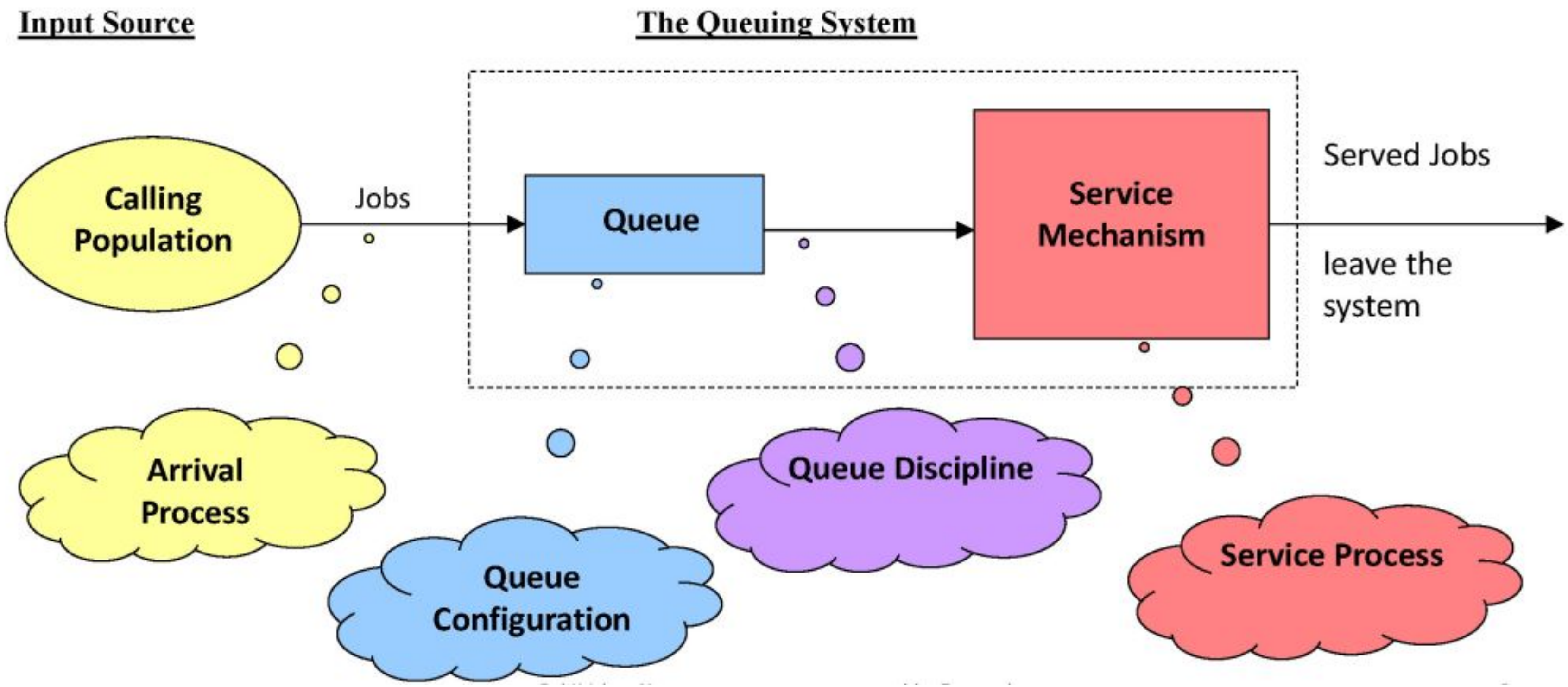


Figure: Queuing System / Model

Elements/Characteristics of Queuing Systems



Elements/Characteristics of Queuing Systems

- **Calling Population**
  - The population of **potential customers** those **requires service** from the system.
  - It may be finite or infinite.
  - In systems with a large population of potential customers, the calling population is usually assumed to be
    - **FInite Population Model (Closed System): Tyre Curing Machine, Doctor's Facility**
    - **Infinite Population Model (Open System): Restaurant Facility, Bank Facility**
- **System capacity**
  - A limit on the number of customers that may be waiting on line or system, may be limited or unlimited.
  - For eg. As finite paring system may be considered as having limited capacity, might have only 10 slot for parking ticket. There are no limits for the people waiting on line for tickets.

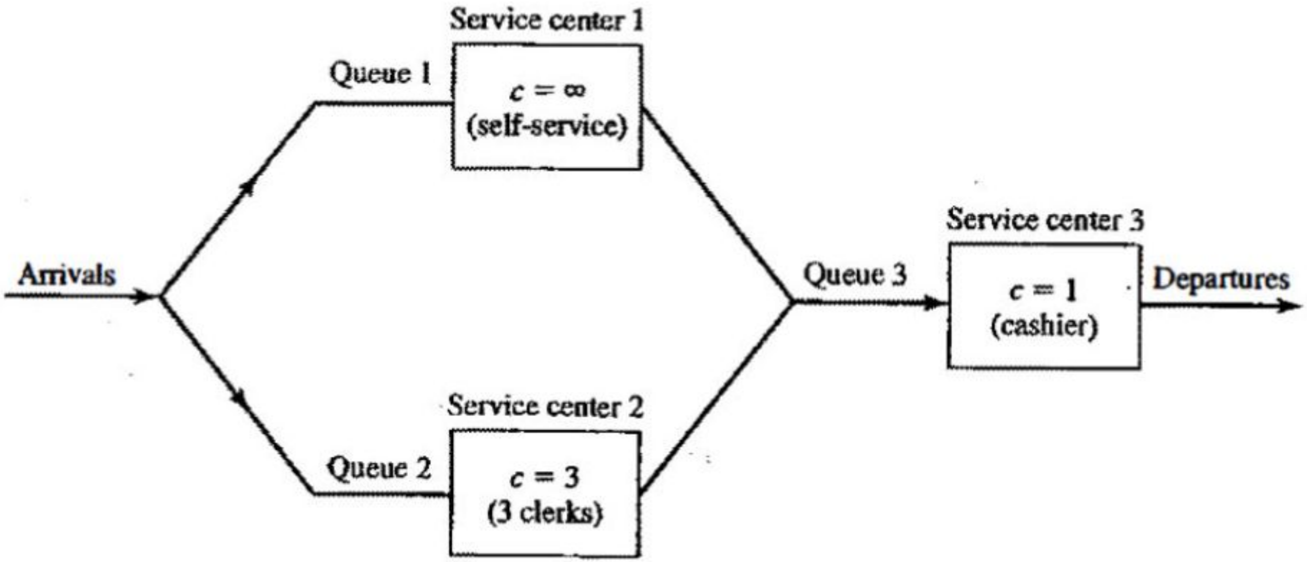
Elements/Characteristics of Queuing Systems

- **The Arrival Process**
  - Arrival defines the way customers enter the system.
  - Arrival may occur at **scheduled** times or at **random** times
  - Usually the arrivals are random with random intervals between two adjacent arrivals.
  - Typically the arrival is described by a random distribution of intervals also called arrival pattern.
- **Queue Discipline**
  - Queue discipline refers to the logical ordering of customers in a queue and determines which customer will be chosen for service when a server becomes free.
  - Examples or Queuing discipline are:
    - FIFO - First In First out
    - LIFO - Last In First out
    - SIRO - Service in Random Order
    - SPT - Shortest processing time First
    - PR - Service according to priority

Elements/Characteristics of Queuing Systems

- **Queue Maximum Size**
  - Maximum size of customers that can be accommodated in the queue.
- **Service Time**
  - The service times of successive arrivals may be constant or of random duration. Successive arrivals are usually characterized as a sequence of independent variable e.g. exponential, log normal and normal distribution.
  - Services **may** be identically distributed for all customers or **may not** be.
  - **Single server (c = 1)**, **multiple server (1 < c < ∞)**, or **unlimited servers (c = ∞)**. A self-service facility is usually characterized as having an unlimited number of servers.

Service Times and Service Mechanism:



Elements of queuing system :

- **No of servers**
  - There may be presence of signal or multiple servers in a queuing system. A system with multiple servers is able to provide parallel services to the customer.
- **Output:**
  - Output represents the way customers leave the system.
  - Output is mostly ignored by theoretical models but **sometimes the customers leaving the server enter the queue again**.

Queuing Theory > Terms :

Remember:

- **Balking:** Refers to customers deciding not to join a queue upon seeing it is too long.
- **Reneging:** Occurs when customers leave a queue after waiting for some time without being served.
- **Jockeying:** Happens when customers switch between queues in hopes of faster service.
- **Pooling:** Combines multiple service stations or queues into a single shared queue to improve efficiency.
- **Priority:** Allocates service based on predefined importance levels, serving higher-priority customers first.

Kendall's Queuing Notation

- Kendall purposed the most widely accepted queuing notation.
- Kendall's Notation is a system of notation according to which the **various characteristics of a queuing model are identified.**
- Kendall's notation is used for parallel server systems.

  - **Basic Format:** **A / B / c / N / K / D**

Queuing Notation

- **Basic Format:** **A / B / c / N / K / D**
  - A - Inter-arrival Pattern
  - B – Service Pattern
  - c – No. of Parallel Servers
  - N – System Capacity (**Types:** Finite or Infinite, **Default:** Infinite)
  - K – Size of Calling Population (**Types:** Finite or Infinite, **Default:** Infinite)
  - Q – Queue Discipline (Default: FIFO)

Queuing Notation

- ❖ **A & B** each may be following:
  - ➢ M = Exponential or Markov
  - ➢ M^x= Bulk Markov
  - ➢ G = General
  - ➢ E_k = Erlang of order k
  - ➢ D = Deterministic or Constant
  - ➢ N = Normal Distribution
  - ➢ PH = Phase Type Distribution
  - ➢ H = Hyper Exponential
  - ➢ GI = General Independent

❖ **Basic Format:** A / B / c / N / K / D

Note: If final 3 are not specified then:

- ❖ N = infinity
- ❖ K = infinity
- ❖ D = FIFO

Queuing Notation

- **Example:** M/G/3/7/10/FIFO
  - Interarrival Pattern – Exponential
  - Service Pattern – General
  - No. of Severs – 3
  - Total Capacity – 7
  - Calling Population - 10
  - Queue Discipline – FIFO

Queuing Notation

- Example:

  1. $D / M / 1$

- Example:

  2. $M / D / 2 / LIFO$

Queuing Notation

- Example:

  1. *D / M* / 1

     - Interarrival *pattern – Deterministic*
     - *Service pattern – Exponential Distribution*
     - *No. of Server –* 1
     - *System capacity – Infinite*
     - *Population size – Infinite*
     - *Queuing discipline – FIFO*

- Example:

  2. *M / D* / 2 / *LIFO*

     - Interarrival *pattern – Exponential Distribution*
     - *Service pattern – Deterministic*
     - *No. of Server –* 2
     - *System capacity – Infinite*
     - *Population size – Infinite*
     - *Queuing discipline – LIFO*

Queuing Notation

- Example:

3. ***G / Em* / 1 / 20**

   - Interarrival *pattern – General Distribution*
   - *Service pattern – Erlang Distribution*
   - *No. of Server –* 1
   - *System capacity –* 20
   - *Population size – Infinite*
   - *Queuing discipline – FIFO*

4. D/M/1/10/50/LIFO
5. M/G/3/20

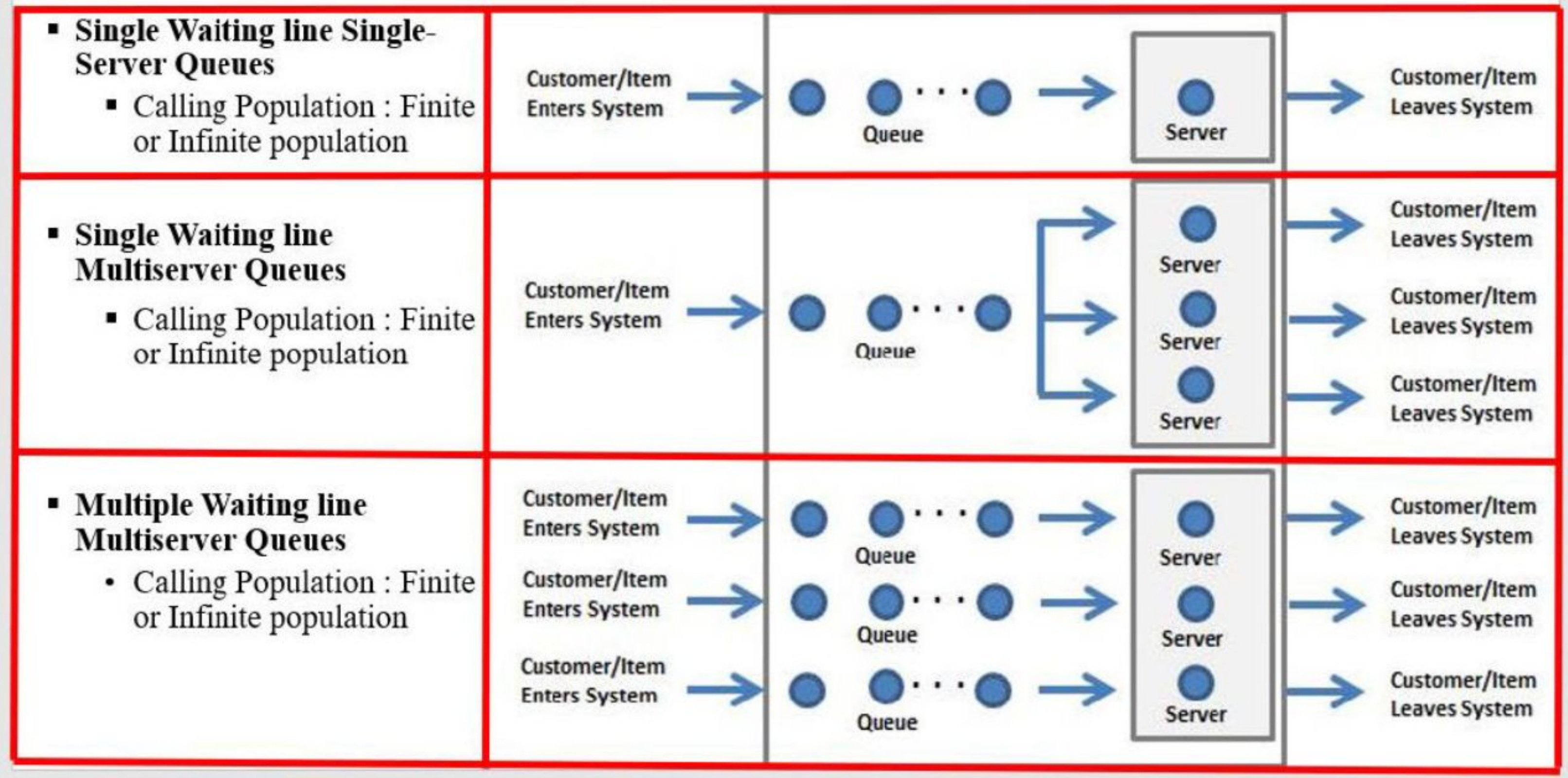
Kendalls Queuing Notation > Questions:

- A hospital emergency room has 6 doctors treating patients. Patient arrivals follow a general distribution, and service times also follow a general distribution. There is no system capacity limit, and patients are served in priority order. Express this system in Kendall notation.
- A computer network has a server cluster with 10 parallel processors. Jobs arrive according to a Poisson process, and service times follow an exponential distribution. The system has a total capacity of 100 jobs, and they are processed based on a shortest-job-first discipline. What is the correct Kendall notation?
- A barber shop has 3 barbers, each serving one customer at a time. The arrivals follow a Poisson process, and service times follow an exponential distribution. However, if a customer sees more than 5 people waiting, they leave without getting a haircut (balking). Express this system in Kendall notation.
- A manufacturing assembly line has a conveyor system with 4 robotic arms handling parts. Parts arrive based on a deterministic schedule, and each robotic arm takes a normally distributed time to process a part. The system can hold up to 50 parts at a time, and processing follows a FIFO (First-In-First-Out) discipline. What is the Kendall notation?

Models of Queuing System:

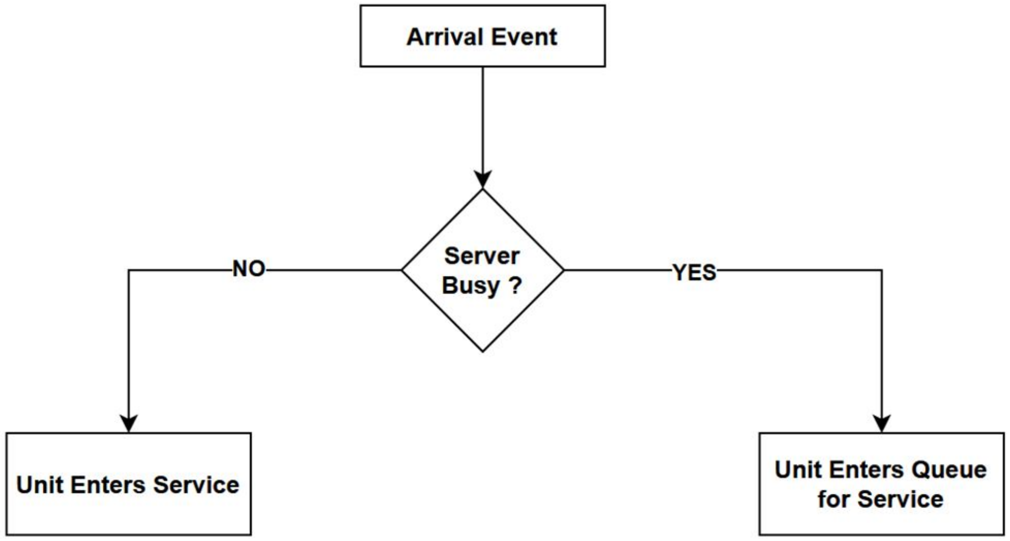
- A queuing system is described by its **calling population, the nature of arrivals, the service mechanism, system capacity and the queuing discipline.**
- Types:
  - **Single Server Queuing System**
    - Single Waiting Line Single Server Queue
    - Multiple Waiting Line Single Server Queue
  - **Multiple Server Queuing System**
    - Single Waiting Line Multiple Server Queue
    - Multiple Waiting Line Multiple Server Queue

Models of Queuing System:



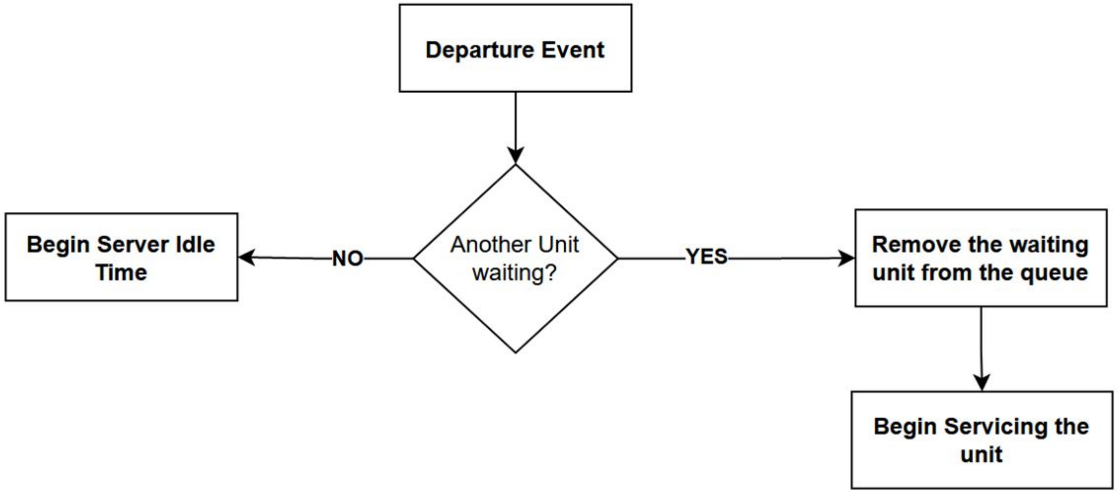
Queuing Model:

- **Arrival Event:**
  - If server is idle, unit gets service, otherwise unit enters queue.



Queuing Model:

- **Departure event:**
  - ➢ If queue is not empty begin servicing next unit, otherwise service will be idle.



Queuing Model:

Server utilization:

- Server utilization is the percentage of the time that all servers are busy. **It is the ratio between average arrival rate to the average service rate.**
- Traffic Intensity

Single Server (M/M/1):

$$\rho = \lambda / \mu$$

Multiple Server (M/M/c):

$$\rho = \lambda / (c * \mu)$$

where **λ = average arrival rate**

μ = the average service rate

c = number of servers

Performance of M/M/1 Model> Formulas :

- **Given:**

  λ: Arrival rate of jobs (packets on input link)

  μ: Service rate of the server (output link)

$$P_0 = 1 - (\lambda/\mu)$$
$$P_n = [1 - (\lambda/\mu)](\lambda/\mu)^n$$
$$L = \lambda/(\mu - \lambda)$$
$$L_q = \lambda^2/[\mu(\mu - \lambda)]$$
$$W = 1/(\mu - \lambda)$$
$$W_q = \lambda/[\mu(\mu - \lambda)]$$
$$P_w = \lambda/\mu$$
$$\rho = \lambda/\mu$$

Steady State Performance Measure
P_0 = Probability that there are no customers in the system (Server idle ratio).
P_n = Probability that there are "n" customers in the system.
L = Average number of customers in the system.
L_q = Average number of customers in the queue.
W = Average time a customer spends in the system.
W_q = Average time a customer spends in the queue.
P_w = Probability that an arriving customer must wait for service.
ρ = Server utilization ratio

Queuing Model > Example :

- Customers arrive at Mary's Shoes every 12 minutes on the average, according to a Poisson process. Service time is exponentially distributed with an average of 8 minutes per customer. Management is interested in determining the performance measures for this service system.
- Solution:

  Input:

  $\lambda = 1/12$ customers per minute $= 60/12 = 5$ per hour.

  $\mu = 1/8$ customers per minute $= 60/8 = 7.5$ per hour.

$$P_0 = 1 - (\lambda/\mu) = 1 - (5/7.5) = 0.3333$$

$$L = \lambda/(\mu - \lambda) = 2$$

$$L_q = \lambda^2/[\mu(\mu - \lambda)] = 1.3333$$

$$W = 1/(\mu - \lambda) = 0.4 \text{ hours} = 24 \text{ minutes}$$

$$W_q = \lambda/[\mu(\mu - \lambda)] = 0.26667 \text{ hours} = 16 \text{ minutes}$$

Queuing Model > Example :

- Customers arrive in a bank according to a Poisson's process with mean inter arrival time of 10 minutes. Customers spend an average of 5 minutes on the single available counter, and leave. Discuss
  I. What is the probability that a customer will not have to wait at the counter?
  II. What is the expected number of customers in the bank?
  III. How much time can a customer expect to spend in the bank?

  - **Solution: We will take an hour as the unit of time. Thus**

    $\lambda = 6$ customers/hour, and $\mu = 12$ customers/hour.

    ✓ The customer will not have to wait if there are no customers in the bank.

    $$P_0 = 1 - \lambda/\mu = 1 - 6/12 = 0.5$$

    ✓ Expected numbers of customers in the bank are given by

    $$L = \lambda/(\mu - \lambda) = 6/6 = 1$$

    ✓ Expected time to be spent in the bank is given by

    $$W = 1/(\mu - \lambda) = 1/(12-6) = 1/6 \text{ hour} = 10 \text{ minutes.}$$

Example 3: If arrivals are occurring at rate $\lambda = 10$ per hour and management has the choice of two servers, one who works at rate $\mu_1 = 20$ customers per hour and the second at rate $\mu_2 = 12$ customers per hour, then

Answers:

$S_1 = 0.909$, $S_2 = 0.833$

$L_1 = 10$, $L_2 = 5$

$Lq_1 = 9.09$ $Lq_2 = 4.1$

$W_1 = 1$, $W_2 = 0.5$

Measures of System Performance :

The performance of a queuing system can be evaluated in terms of a number of response parameters, however the following four are generally employed.

- Average number of customer in the queue or in the system
- Average waiting time of the customer in the queue or in the system
- System utilization (Server utilization)
- The cost of waiting time and idle time

Note: Too long waiting line may discourage the prospectus customers, while no queue may suggest that service offered is not good quality to attract customers.

Application of queuing system:

- It is used in designing and operating **transportation systems such as airports, freeways, ports, and subways.**
- It is used in evaluating designs for service organizations such as **call centers, fast-food restaurants, hospitals, and post offices.**
- An important application of a **mathematical queueing system** is determining the **minimum number of servers needed at a workstation or service center.**
- It is used in the analysis of **service facilities provided by organizations.**
- It is used in the analysis of **production and material handling systems.**
- It is used in **telephone and communications systems**.
- It is used in the analysis of **telecommunications, computer networks, predicting computer performance, traffic, etc.**

Examples of real world queuing system:

- **Commercial Queuing Systems:** Commercial organizations serving external customers.
  - Examples: Dentist, bank, ATM, gas stations, plumber, garage, etc.
- **Transportation Service Systems:** Vehicles are customers or servers.
  - Examples: Vehicles waiting at toll stations and traffic lights, trucks or ships waiting to be loaded, taxi cabs, fire engines, elevators, buses, etc.
- **Business-internal Service Systems:** Customers receiving service are internal to the organization providing the service.
  - Examples: Inspection stations, conveyor belts, computer support, etc.
- **Social Service Systems:**
  - Examples: Judicial process, at a hospital - waiting lists for organ transplants or student dorm rooms.

MARKOV CHAIN:

- **Markov Process:** If the future outcome depends only in the present outcome but not the past outcomes, then the process is called Markov Process.
- A **Markov chain** is a stochastic model describing a sequence of possible events in which the **probability of each event depends only on the state attained in the previous event**.
- A process with **finite number** of states (or outcomes or steps) in which being in a particular state at **step n+1, depends only** on the state occupied at **step n**.

Prof. Andrei A. Markov (1856-1922) , published his result in 1906.

Russian mathematician Andrey Markov

MARKOV CHAIN:

- Markov dependence is a concept attributed to the **Russian mathematician Andrei Andreyevich Markov** that at the start of the 20th century investigated the alternance of vowels and consonants in the poem Onegin by Poeshkin.
- **He developed a probabilistic model where successive results depended on all their predecessors only through the immediate predecessor.** The model allowed him to obtain good estimates of the relative frequency of vowels in the poem.

$$x_1 \rightarrow x_2 \rightarrow x_3 \rightarrow x_4 \rightarrow x_5$$

$$P(x_5 | x_4, x_3, x_2, x_1) = P(x_5 | x_4)$$

- **Note:** So the probability of a certain state being reached **(i.e transition)**, depends only on the previous state of the chain.
- The future behaviour of the system depends only on the current state ***i*** and not on any of the previous states.

Properties of Markov Chains:

- A sequence of trail of an experiment is a **Markov chain** if:
  - The set of all possible states of stochastic (probabilistic) system is **finite.**
  - The variables move from one state to another and the probability of transition from a given state is **dependent** only on the **present state** of the system, **not** in which it was reached.
  - The **probabilities** of reaching to various states from any given state are **measurable** and **remain constant** over time.

MARKOV CHAIN:

- A Markov chain is a process that consists of a **finite number of states** with the Markovian property and some **transition probabilities p_{ij}**, where p_{ij} is the probability of the process moving from state i to state j.

A Markov chain consists of state and transition probabilities.

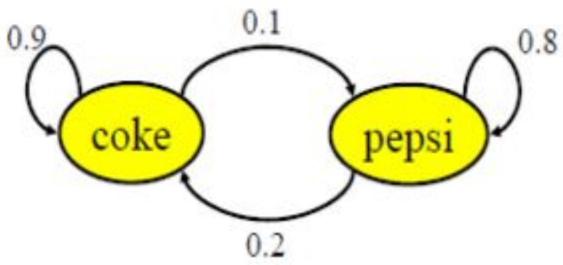
- **Each transition probabilities is the probability of moving from one state to another in one step.**
- The transition probabilities are **independent** of the past and depend only on the two states involved.
- **The matrix of transition probabilities are called transition matrix.**

MARKOV CHAIN > Transition Matrix :

- A Transition matrix has following features:
  1) It is square, since all possible states must be used both as rows and as columns.
  2) All entries are between 0 and 1.
  3) The sum of entries of any row must be 1, since the number in the row give the probability of changing from one state at the left to one of state across the top.

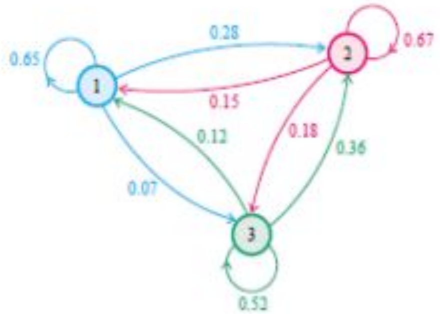
transition matrix:

$$P = \begin{bmatrix} 0.9 & 0.1 \\ 0.2 & 0.8 \end{bmatrix}$$



MARKOV CHAIN > Transition Matrix :

- If P^2 represent the matrix product P.P, then P^2 gives the probabilities of a transition from one state to another in two repetitions of an experiment.
- $P^3 = P.\ P^2$ gives the probabilities of change after three generations.
- In general, P^K gives the probabilities of a transition from one state to another in K repetitions of an experiment.



$$\begin{array}{c c} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 0.65 & 0.28 & 0.07 \\ 0.15 & 0.67 & 0.18 \\ 0.12 & 0.36 & 0.52 \end{bmatrix} = P. \end{array}$$

MARKOV CHAIN Diagram:

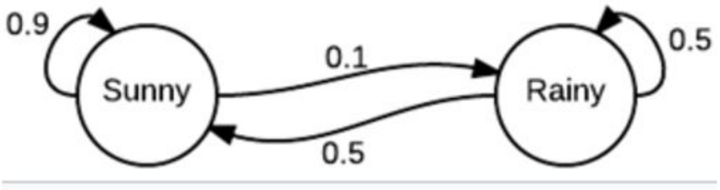
Example:A simple weather model

- The **probabilities of weather** can be represented by a transition matrix:
  - If it is sunny day today then there is 90% chance tomorrow will also be sunny day and 10% chance it will be rainy.
  - If it is rainy day today then there is 50% chance tomorrow will also be rainy and 50% chance it will be sunny.

MARKOV CHAIN Diagram:

A simple weather model

- The **probabilities of weather** can be represented by a transition matrix:
  - If it is sunny day today then there is 90% chance tomorrow will also be sunny day and 10% chance it will be rainy.
  - If it is rainy day today then there is 50% chance tomorrow will also be rainy and 50% chance it will be sunny.



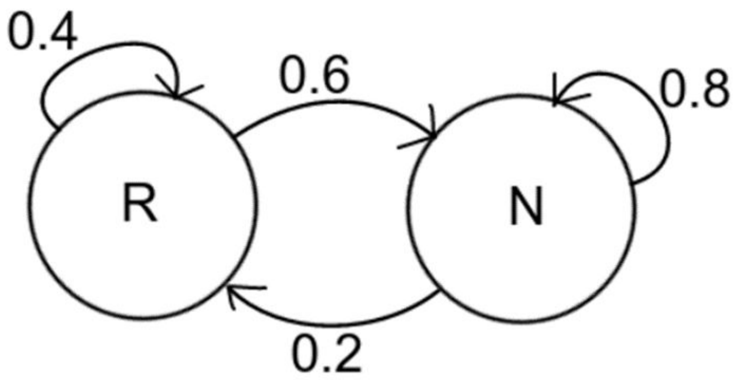
State diagram

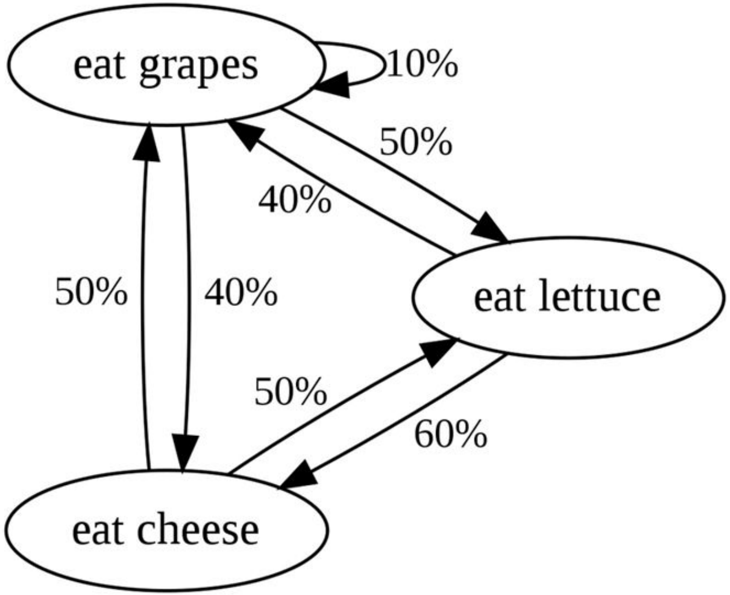
$$P = \begin{bmatrix} 0.9 & 0.1 \\ 0.5 & 0.5 \end{bmatrix}$$

Transition probability matrix

Markov Chain Diagram:

Observe the following state diagram and write transition matrix for them:



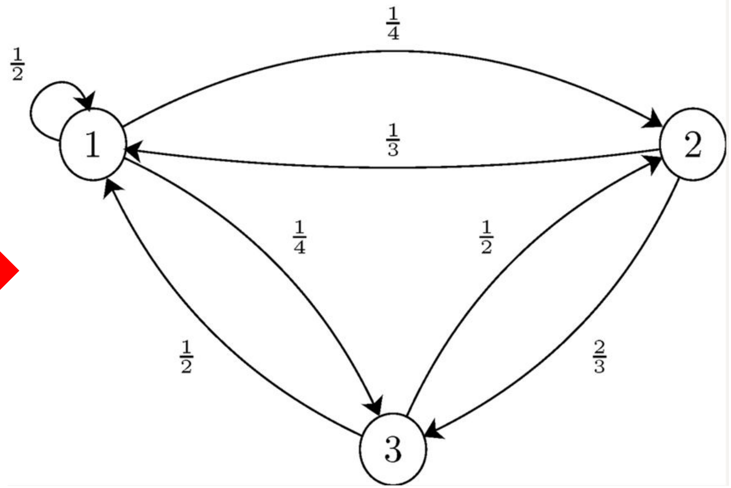


Markov Chain Diagram:

Consider the Markov chain with three states, S={1,2,3}, that has the following transition matrix:

$$P = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{3} & 0 & \frac{2}{3} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}.$$

Draw the state transition diagram for this chain.



Process Example:

❖ General rules of time n are:

$$x_{(n)} = x_{(n-1)} P \quad \textbf{or}$$

$$x_{(n)} = x_{(0)} P^{(n)}$$

$$x_{(1)} = x_0 P^1$$

❖ **Example:** The probabilities of weather conditions (modeled as either rainy or sunny), given the weather on the preceding day, can represented by a transition matrix:

$$P = \begin{pmatrix} 0.9 & 0.1 \\ 0.5 & 0.5 \end{pmatrix}$$

Process Example:

Predicting the weather:

The weather on day 0 (today) is known to be sunny. This is represented by an initial state vector in which the "sunny" entry is 100%, and the "rainy" entry is 0%:

$$\mathbf{x}^{(0)} = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

The weather on day 1 (tomorrow) can be predicted by multiplying the state vector from day 0 by the transition matrix:

$$\mathbf{x}^{(1)} = \mathbf{x}^{(0)} P = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 0.9 & 0.1 \\ 0.5 & 0.5 \end{bmatrix} = \begin{bmatrix} 0.9 & 0.1 \end{bmatrix}$$

Thus, there is a 90% chance that day 1 will also be sunny.

The weather on day 2 (the day after tomorrow) can be predicted in the same way, from the state vector we computed for day 1:

$$\mathbf{x}^{(2)} = \mathbf{x}^{(1)} P = \mathbf{x}^{(0)} P^2 = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 0.9 & 0.1 \\ 0.5 & 0.5 \end{bmatrix}^2 = \begin{bmatrix} 0.86 & 0.14 \end{bmatrix}$$

or

$$\mathbf{x}^{(2)} = \mathbf{x}^{(1)} P = \begin{bmatrix} 0.9 & 0.1 \end{bmatrix} \begin{bmatrix} 0.9 & 0.1 \\ 0.5 & 0.5 \end{bmatrix} = \begin{bmatrix} 0.86 & 0.14 \end{bmatrix}$$

General rules for day *n* are:

$$\mathbf{x}^{(n)} = \mathbf{x}^{(n-1)} P$$

$$\mathbf{x}^{(n)} = \mathbf{x}^{(0)} P^n$$

Example-1:

Given Data:

Not Rainy Today

- 40% Rainy tomorrow
- 60% not Rainy tomorrow

Rainy Today

- 20% Rainy tomorrow
- 80% not Rainy tomorrow

Q. What will be probability if today is not raining then rain after 3 days?

Example-1:

*What will be probability if **todays is not raining** then **rain after three days**?*

$$X3 = X_o * (P)^3$$

$$X3 = (0 \ \ 1)\begin{pmatrix}0.4 & 0.6\\0.2 & 0.8\end{pmatrix} * \begin{pmatrix}0.4 & 0.6\\0.2 & 0.8\end{pmatrix} * \begin{pmatrix}0.4 & 0.6\\0.2 & 0.8\end{pmatrix}$$

$$X3 = (0 \ \ 1)\begin{pmatrix}0.28 & 0.72\\0.24 & 0.76\end{pmatrix} * \begin{pmatrix}0.4 & 0.6\\0.2 & 0.8\end{pmatrix}$$

$$X3 = (0 \ \ 1)\begin{pmatrix}0.26 & 0.74\\0.25 & 0.75\end{pmatrix}$$

$$X3 = (\mathbf{0.25} \ \ 0.75)$$

Thus, the probability of rain after three days is 0.25

Example-2:

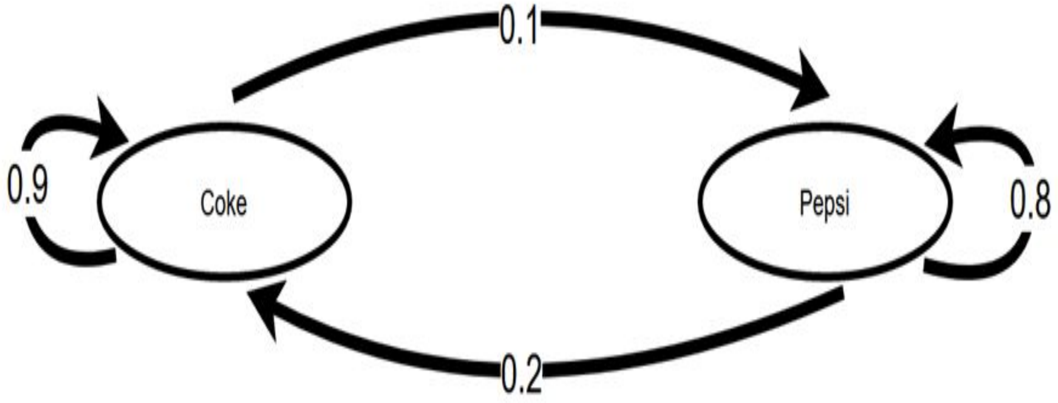
Given a person is currently a Pepsi purchaser. What is the probability of purchase of coke after **two purchases** from now?

- Coke => 90% Coke
- Pepsi => 20% Coke

P = $\begin{pmatrix} 0.9 & 0.1 \\ 0.2 & 0.8 \end{pmatrix}$;

Xo =??

X2 = ??



Example-2:

$$X_2 = X_o * (P)^2$$

$$X_2 = (0\ 1)\begin{pmatrix}0.9 & 0.1\\ 0.2 & 0.8\end{pmatrix} * \begin{pmatrix}0.9 & 0.1\\ 0.2 & 0.8\end{pmatrix}$$

$$X_2 = (0\ 1)\begin{pmatrix}0.81 + 0.2 & 0.9 + 0.8\\ 0.18 + 0.16 & 0.2 + 0.64\end{pmatrix}$$

$$X_2 = (0\ 1)\begin{pmatrix}0.83 & 0.17\\ 0.34 & 0.66\end{pmatrix}$$

$$X_2 = (\mathbf{0.34}\quad 0.66)$$

Thus, the probability purchase of coke after two purchases from now is 0.34.

Question-1: Given that a chance of Ford car user to buy a Ford card in next purchase is 70% and that his next purchase will be a Scorpio purchase is 30%, chance of a Scorpio car use to buy a Scorpio car at the next purchase is 80% and chance that his next purchase will be Ford car is 20%. What is the probability to buy a Scorpio car after three purchase of a current Ford car user? If 70% user uses Ford car today, what percentage of user will use Scorpio after 3 purchase?

Answers (For Reference):

1. [0.475 0.525]

2. [0.4375 0.525]

Question-2: Given that chance of a Sony user to buy Sony at next purchase is 80% and that his next purchase will be Samsung is 20% and chance of a Samsung user to buy Samsung at next purchase is 85% and chance that his next purchase will be Sony is 15%. What is the probability to buy Sony after three purchase of a current Samsung user? If 60% users uses Samsung after 3 purchase? (2074 Bhadra)

Answers (For Reference):

1. [0.310875 0.689125]

2. [0.310875 0.689125]

Question-3: Given that chance of a Sony user to buy Sony at next purchase is 75% and that his next purchase will be Samsung is 25% and chance of a Samsung user to buy Samsung at next purchase is 85% and chance that his next purchase will be Sony is 15%. What is the probability to buy Sony after three purchase of a current Samsung user? (2075 Magh)

Answers (For Reference):

1. [0.294 0.706]

- **Given that chance of a Honda Bike user to buy Honda bike at next purchase is 70%, and his next purchase will be Yamaha Bike is 30% and chance of a Yamaha bike user to buy Yamaha bike at next purchase is 80% and chance that his next purchase will be Honda Bike is 20%. What is the probability to buy Yamaha bike after three purchase of current Honda bike user?**

Answers (For Reference):

1. [0.475 0.525]

Applications:

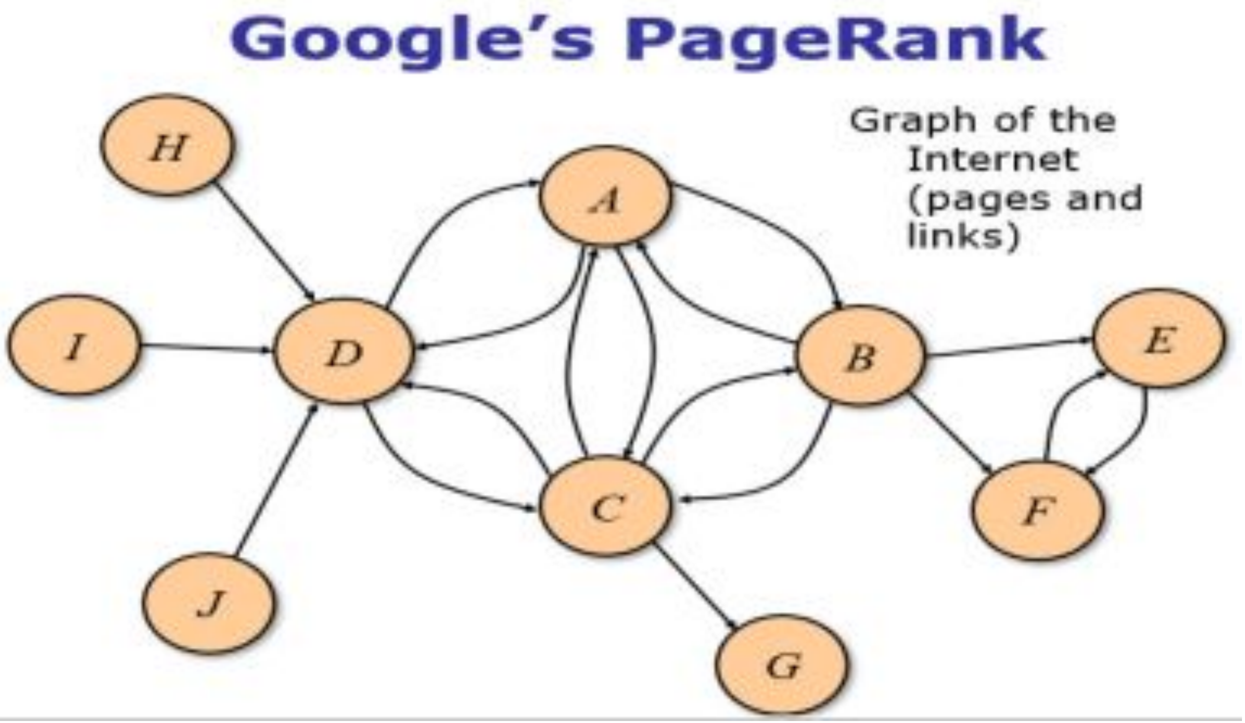
- Since Markov chains can be **designed to model many real-world processes**, they are used in a wide variety of situations.
- These fields range from the **mapping of animal life populations** to **search engine algorithms**, music composition and **speech recognition**.
- In **economics and finance**, they are often used to **predict macroeconomic situations** like **market crashes** and **cycles between recession and expansion**. Other areas of application include **predicting asset** and **option prices** and **calculating credit risks**.

Applications:

- Google's PageRank algorithm **treats the web like a Markov model**. You can say that all the **web p**ages are states, and the **links between them are transitions possessing specific probabilities.** In other words, we can say that no matter what you're searching on Google, there's a finite probability of you ending up on a particular web page.

- If you use **Gmail**, you must've noticed their **Auto-fill feature**. This feature automatically predicts your sentences to help you write emails quickly. Markov chains help in this sector considerably as they can provide **predictions** of this sort effectively.

Applications:



Applications:

- **Statistics:** Markov chain methods have also become very important for **generating sequences of random numbers** to accurately reflect very complicated desired probability distributions, via a process called Markov chain Monte Carlo (MCMC) And many more.
- **Queuing theory :** Markov chains are the **basis for the analytical treatment of queues** (queuing theory). This makes them critical for optimizing the performance of telecommunications networks, where **messages must often compete for limited resources** (such as bandwidth).
- **Physics :** Markovian systems appear extensively in thermodynamics and statistical mechanics, whenever **probabilities are used to represent unknown or unmodelled details of the system**, if it can be assumed that the dynamics are time-invariant, and that no relevant history need be considered which is not already included in the state description.

PYQs:

1. What do you mean by queuing system? Explain the characteristics of queuing system with example. Discuss briefly about balking, reneging and jockeying in queuing theory. **2025 PU (12)**
2. What is a Markov Chain? What is a transition matrix in a markov chain and how is it used to determine the behaviour of the system over the time? **2024 PU (8)**
3. What is a queuing system and what are the key components involved? **2024 PU (8)**
4. Explain about Kendall's Notation with an example **2024 PU (6)**
5. Define and describe Markov Chain with example **2024 PU (8)**
6. Short Note:
    a. Use of Partial differential equation in simulation model **2024 PU (4)**

References:

1. Jerry Banks, John S. Carson II, Barry L. Nelson, David M. Nicol – *Discrete-Event System Simulation* – Pearson (2013)
2. Averill M. Law – *Simulation Modeling and Analysis* – McGraw-Hill Education (2014)
3. Geoffrey Gordon – *System Simulation*

THANKS!

Do you have any questions?

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