Simulation & Modeling

Course Code: BIT534CO
Year/ Semester: III/VI

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Credit Hours: 3hrs

Unit 5: Analysis of Simulation Output

Class Load : 10 hrs

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Output Analysis:

- **Output analysis is the examination of data generated by a simulation.**
- Its purpose is either to **predict** the performance of a system or to **compare** the performance of two or more alternative system designs.
- The greatest disadvantage of simulation is that we **don't get exact answers**, results are only estimates.
- So, careful **design and analysis** is needed to make these estimates as valid and precise as possible, and interpret their meaning properly.
- **Statistical methods** are used to analyze the results of simulation experiments.

Analysis of Simulation Output > WHY? :

- ❖ Average and worst time in system
- ❖ Average and worst time in queue
- ❖ Average hourly production
- ❖ Standard deviation of hourly production
- ❖ Proportion of time a machine is up, idle or down
- ❖ Maximum queue length
- ❖ Average number of parts in system

Nature of the Problem:

- ❖ One a **stochastic variable** has been introduced into simulation model all the system variables describing the system behaviour also become **stochastic**.
- ❖ Hence, it needs some **statistical method** to analyze the simulation output.
- ❖ A large body of statistical methods has been developed over the years to analyze results in science, engineering and other fields.
- ❖ It seems natural to attempt applying these methods to **analyze the simulation output** but most of them **pre-suppose** that the results are mutually independent (IID) and the simulation process always never produce raw output than IID **(Independently and Identically Distributed).**

Independently and Identically Distributed (IID):

- Usually a random variable is **drawn from an infinite population** that has probability distribution **with finite mean and finite variance.**
- This mean that the **population distribution is not affected** by the number of **sample already made** or **does it change with time**.
- If, further the **value of sample is not affected in anyway** by value of another sample, the **random variables are mutually independent**.
- Random variables that **meet all these conditions** are said to be **independently and identically distributed.**

Type of simulation on the basis of output:

a) Termination simulation or finite simulation:

- **The termination of a finite simulation takes place at a specified time or is caused by some specific events.**
- **For Example:** Banks opens at 8:30 am (time 0) with no customers present and 8 of the 11 teller working (initial conditions), and closes at 4:30 pm (Time T_E = 480 min)
- The terminating simulation **runs for some specified duration** of time T_E where E is **a specified event** that stops the simulation.
- It starts at time 0 under **well specified initial condition** and ends at the **stopping time, T_E** . For the banking system, the simulation analyst chooses to consider a terminating system because **object of interest** is one day's operations on the bank.

Type of simulation on the basis of output:

b) Non terminating Simulation (Steady state simulation):

- It runs continuously or at least over a very long period of time.
- The main purpose of steady study simulation is the study of **long run behavior of system.** Performance measure is called a steady state parameter if it is a characteristic of the equilibrium distribution of an output stochastic process.
- **Examples: Continuously operating communication system** where the objective of computation is **mean delay of packet** in the long run .

- *(**Note:** whether a simulation is considered to be terminating or non-terminating depends on both the **objective of study** and **nature of the system**).*

Type of simulation on the basis of output:

Non terminating Simulation:

1. Runs continuously or at least over a very long period of time.
2. Initial conditions defined by analyst.
3. Runs for some analyst specified period of time T_E
4. Study the steady state (long run) properties of the system, properties that are not influenced by the initial condition of model.

Terminating Simulation:

1. Runs for some duration of time T_e, where e is a specified event that stops the simulation
2. Starts at time 0 under well-specified initial conditions.
3. Ends at the stopping time T_e
4. The simulation analyst chooses to consider it a terminating system because the object of interest is one days operation

Example (Bank) : opens at 8:30 am (time 0) with no customers present and 8 of the 11 teller working (initial conditions), and closes at 4:30 pm (Time Te= 480 min)

Central Limit Theorem (CLT) :

- ❖ The theorem states that the sum of n **IID** variable drawn from a population that has **mean μ** and a **variance of σ^2** is approximately distributed as a normal variable with **mean $n^*\mu$** and **variance $n^* \sigma^2$** .
- ❖ The Central Limit Theorem (CLT) states that the **distribution of sample means approximates a normal distribution as the sample size increases**.
- ❖ **Sample sizes of 30** or more are usually considered sufficient for the CLT to apply.
- ❖ A key aspect of the CLT is that the **average of the sample means and standard deviations will equal** the **population mean and standard deviation**.
- ❖ With a **large enough sample size**, we can predict the population's characteristics with accuracy.

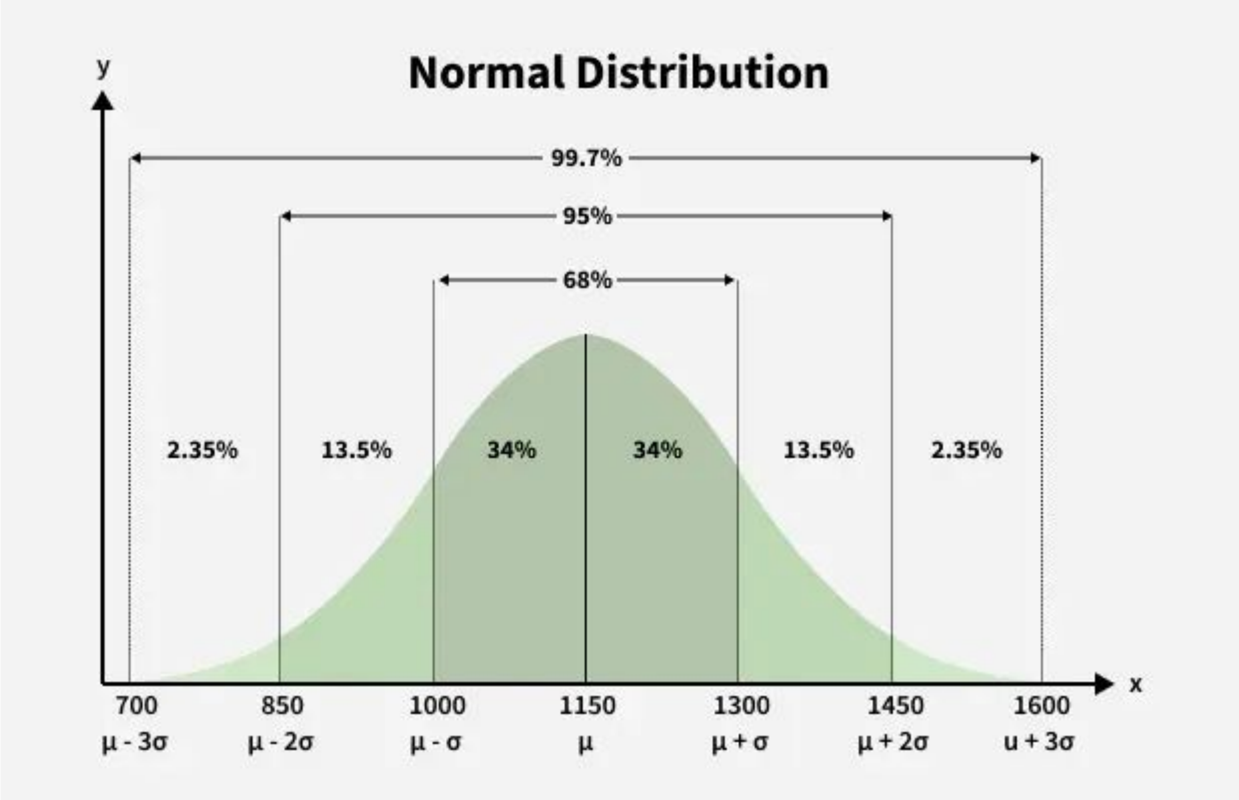
Estimation Methods:

- Estimation methods in simulation refer to techniques **used to estimate various properties or characteristics of a system, process, or phenomenon through the use of computer-based simulation models.**
- **The estimate method gives the desired range of the sample random variable from infinite population so that the desired output can be achieved.**
- These methods are employed to derive numerical or statistical estimates for specific parameters, performance measures, or outcomes within the simulated environment.
- **To estimate means to find something close to the correct answer.**
- Infinite population has a **stationary probability distribution** with a finite mean μ and finite variance σ^2.

Estimation Methods:

- A random variable is drawn from an infinite population that has a stationary probability distribution **with a finite mean μ and finite variance σ^2.**
- Random numbers **that meet all these conditions** are said to be **IID (Independently and Identically Distributed) variable** for which the **central limit theorem** can be applied.
- The theorem states that the sum of n **IID** variable drawn from a population that has **mean μ** and a **variance of σ^2** is approximately distributed as a normal variable with **mean $n^*\mu$** and **variance $n^*\sigma^2$**.
- Central limit theorem must be invoked to rely upon normal distribution of infinite population
- **Only then we can apply estimation method to that variable taken from infinite population. variable taken from infinite population.**

Normal Distribution or Gaussian Distribution:



Estimation Methods:

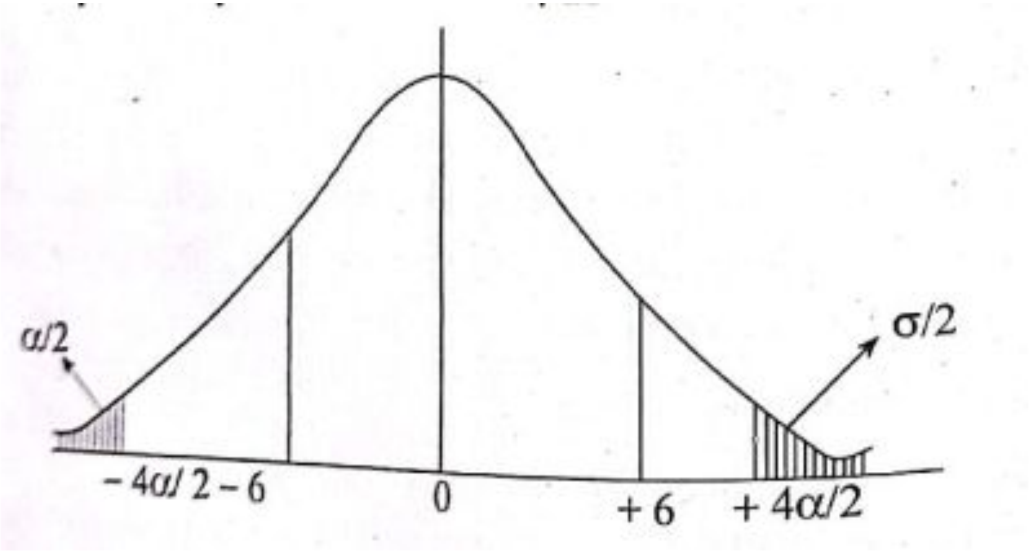
- Any normal distribution can be transform into a standard distribution that has a mean of 0 and variance of 1.
- Let x_i (i=1,2,3,.........n) be n IID random variables. Using **central limit theorem**, we have normal variate :

$$z = \frac{\sum_{i=1}^{n} x_i - n\mu}{\sqrt{n}.\sigma}$$

- If Sample Mean is $\bar{x} = \frac{1}{n}.\sum_{i=1}^{n} x_i$ Then, $z = \frac{\bar{x} - \mu}{\sigma / \sqrt{n}}$

Estimation Methods:

❖ The probability density function of the standard normal variable is shown in graph. The normal distribution is **symmetric about its mean**, so the probability that 2 is less than $-\mu_{\alpha/2}$ is also $\alpha/2$.



Estimation Methods:

- The probability that **μ** is lies between $-\mu_{\alpha/2}$ and $\mu_{\alpha/2}$ is 1 - α. That is: **[μ= z]**

$$\mathbf{Prob}[-\mu_{\alpha/2} \leq \mu \leq \mu_{\alpha/2}] = 1 - \alpha$$

- In terms of sample mean, this probability statement can be written as:

$$\mathbf{Prob}[\bar{x} + (\sigma/\sqrt{n}) * \mu_{\alpha/2} \leq \mu \leq \bar{x} - (\sigma/\sqrt{n}) * \mu_{\alpha/2}] = 1 - \alpha$$

- The constant 1 - α is the confidence level and the confidence interval is:

$$\bar{x} + (\sigma/\sqrt{n}) * \mu_{\alpha/2}$$

Estimation Methods:

- The **population variance σ^2** is usually not known, in which case it is replaced by an estimate calculated from the formula:

$$S^2 = \frac{1}{n-1} \sum_{i=1}^{n} (xi - \bar{x})^2$$

- The normalized random variable based on σ^2 is replaced by a normalized random variable based on σ^2. This has **student t - distribution, with n - 1 degree of freedom.**
- The student t - distribution is strictly accurate only when the population from which the samples are drawn is **normally distributed**. Expressed in terms of the estimated variance S^3, the confidence interval for $\bar{x}$ is defined by:

$$\bar{x} + \frac{s}{\sqrt{n}} t_{n-1,\, \alpha/2}$$

Estimation Methods:

Point Estimation for discrete time data:

- Point estimation is a fundamental concept in statistics where we aim **to estimate an unknown population parameter** (like θ) based on a sample of data ($y_1, y_2, \ldots, y_n$).
- Point estimators are statistical functions or formulas that provide a single numerical estimate for this parameter.

$$\theta` = \frac{1}{n} \sum_i^n y_i$$

- unbiased if its expected value is θ i.e. $E(\theta`) = \theta$
- biased if $E(\theta`) \neq \theta$ and
- difference $E(\theta`) - \theta$ is called bias of $\theta`$.

Estimation Methods:

❖ **Point estimation (Example):** Following are the random sample of height of people of the town. If the population mean is **6.1 ft.**, find the bias of the point estimator.

5.5	6.1	5.7	6.6	5.2	6.0	5.6	6.3	5.9	5.8

❖ We have θ = **6.1**

$E(\theta`) = (5.5 + 6.1 + 5.7 + 6.6 + 5.2 + 6.0 + 5.6 + 6.3 + 5.9 + 5.8)/10 = \mathbf{5.87}$

Now, bias of estimator = $\mathbf{E(\theta`) - \theta}$ = 5.87 – 6.1 = - 0.23

Simulation Run Statistics:

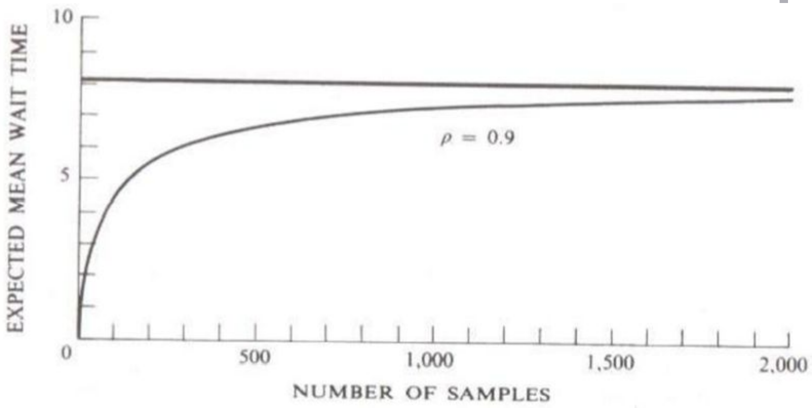
- In the estimation method, it is assumed that the observations a**re mutually independent and the distribution from which they are drawn is stationary.**
- Unfortunately many statistics of interest in simulation **do not meet** these conditions.
- Consider a single server system (M/M/1/FIFO) in which the arrival occurs with Poisson Distribution and service time has an exponential, **objective is to measure the mean waiting time.**
- **If X_i (i = 1, 2, ... , n) are the individual waiting time then**

$$X'(n) = \frac{1}{n} \sum_{i}^{n} x_i$$

- But the waiting time measured in this way is not independent.

Simulation Run Statistics:

- A simulation run is started with the system in some initial state, frequently the idle state in which no service is being given and entities are waiting.
- **Early arrivals has less waiting time.**
- Another problem that must be faced is that distribution is not stationary.
- Early arrivals get the service quickly, so a sample mean that include early arrivals **is biased.**
- Large number of simulation runs reduce the bias. The following figure show the mean waiting time for different sample sizes.



Replication of Runs:

- One way of obtaining **independent result is to repeat simulation**.
- The precision of results of a dynamic stochastic can be increased by repeating the experiment with different random numbers strings.
- **Repeating the experiment** with **different random numbers** for the **sample size n** gives a set of independent determination of **sample mean $\bar{x}(n)$** .
- For each simulation run, a **different** random numbers are used for same sample size ‘n’ and the simulation **gives a set of independent** determinations of sample mean.
- Even though the sample mean depends on **degree of auto-correlation**, the independent determinations can be used to **estimate the variance** of the distribution.

Replication of Runs:

- Suppose the experiment is **repeated p times** with independent random values of n sample sizes. Let x_{ij} be the ith observation in j_{th} run and let the sample mean and the variance for the j_{th} run is denoted by $\bar{x}_j$ **(n)** and s_j^2 **(n) respectively**. Then for j_{th} run, the estimates (sample mean and variance) are

$$\bar{x}_j(n) = \frac{1}{n}\sum_{i=1}^{n} x_{ij} \qquad s_j^2(n) = \frac{1}{n-1}\sum_{i=1}^{n}[x_{ij} - \bar{x}_j(n)]^2$$

- When we have similar mean and variances combining the result of **p independent measurement** gives the following estimate for the **mean x̄** and **variance s^2** of the populations as:

$$x = \frac{1}{p}\sum_{j=1}^{p} \bar{x}_j(n) \qquad s^2 = \frac{1}{p}\sum_{j=1}^{p} s_j^2(n)$$

Replication of Runs:

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- p = total number of independent measurements or experiments
- $\bar{X}_j$ (n) = the mean of the *jth* measurement
- $\bar{X}$ = the **overall mean** (average of all means)
- s_j^2(n) = variance from the *jth* measurement
- s^2 = **overall variance**, i.e., the average of all the individual variances

Replication of Runs:

Example:

Suppose you performed 3 independent experiments, and their means were:

- Experiment 1 → mean = 9
- Experiment 2 → mean = 11
- Experiment 3 → mean = 10

Then the combined mean is:

$\bar{X} = (9+11+10) / 3 = 10$

So the "overall mean" is simply the average of all individual means.

Example:

Suppose you again have 3 experiments with variances:

- Experiment 1 → variance = 2.0
- Experiment 2 → variance = 2.2
- Experiment 3 → variance = 1.8

Then combined variance:

$s^2 = (2.0+2.2+1.8) / 3 = 2.0$

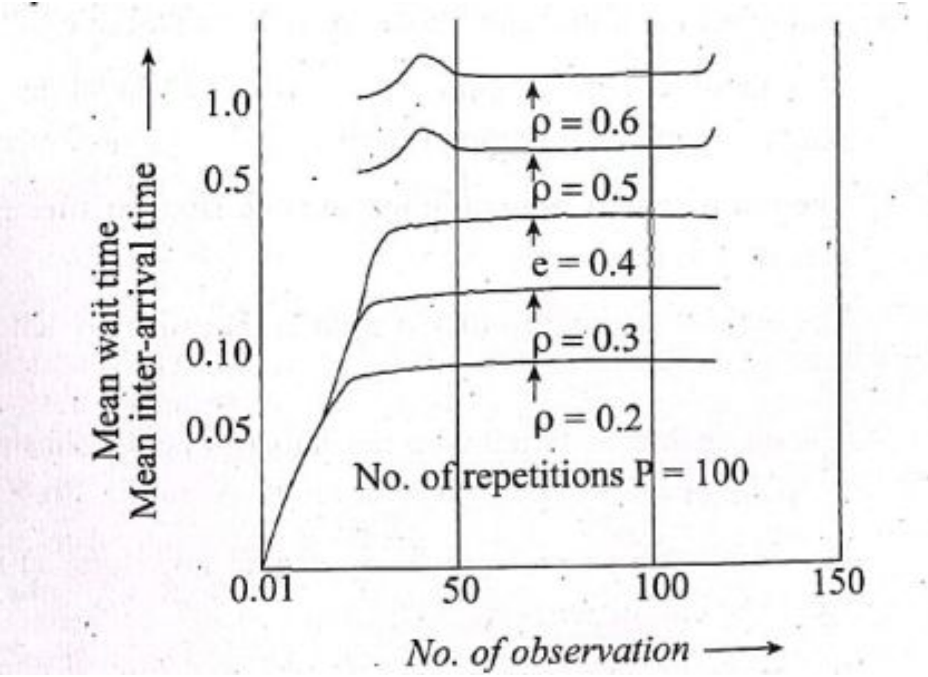
So the overall variance is 2.0.

Replication of Runs:

- The following figure shows the result of applying the procedure to experiment results for the $M/M/1$ system for different sample sizes.

System Utilization Rate:

$$\rho = \frac{\text{Arrival rate } (\lambda)}{\text{Service rate } (\mu)}$$



Replication of Runs:

Observations from the Graph

- **Early Stage (left side, few observations)**
  - The **values fluctuate** a lot because the simulation has **just started**.
  - Not enough samples have been collected, so the estimate of the **mean waiting time is unstable (random noise dominates**).
- **Middle Stage (as number of observations increases)**
  - The curve **begins to flatten.**
  - The estimate of the mean waiting time starts to **stabilize** — meaning the simulation is converging to a **true average value**.
- **Later Stage (after about 100 observations)**
  - The value becomes **nearly constant**.
  - This indicates the simulation has reached **steady-state** — the results are now statistically reliable.

Elimination of Initial Bias:

Two approaches to eliminate initial bias;

1. **The system can be started in a more representative state than the empty state.**
2. **The first part of simulation can be ignored.**

1. **The system can be started in a more representative state than the empty state**
   - ➢ The ideal situation is to know the steady-state distribution for the system and we then select the initial condition from that distribution.
   - ➢ During the study of the existing simulation system, there may be information about the expected outcome which makes it feasible to select better initial condition and thus eliminating the initial bias.

Elimination of Initial Bias:

2. **<u>The first part of simulation run can be ignored i.e. take care of those data that come only after steady-state</u>**
   - A more common approach to remove initial bias is to eliminate the initial section of the run.
     - The run is started from an idle state and is stopped after a certain period.
     - After this initial period, the run is restarted, and statistics are gathered starting from this point onward (or restart point).
     - Typically, simulations are programmed to gather statistics from the beginning, but the gathered data up to the restart point is discarded.
     - It is advisable to use pilot runs starting from idle state to judge how long the initial bias emains.

PYQs:

1. What are the different estimation methods used in analyzing simulation output? Elaborate the process of estimating internal bias in simulation. **2024 PU (8)**
2. Describe the different methods that can be used to minimize or eliminate internal bias during simulation analysis. How can using large sample sizes help in reducing bias? **2024 PU (8)**

References:

1. Jerry Banks, John S. Carson II, Barry L. Nelson, David M. Nicol – *Discrete-Event System Simulation* – Pearson (2013)
2. Averill M. Law – *Simulation Modeling and Analysis* – McGraw-Hill Education (2014)
3. Geoffrey Gordon – *System Simulation*

THANKS!

Do you have any questions?

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