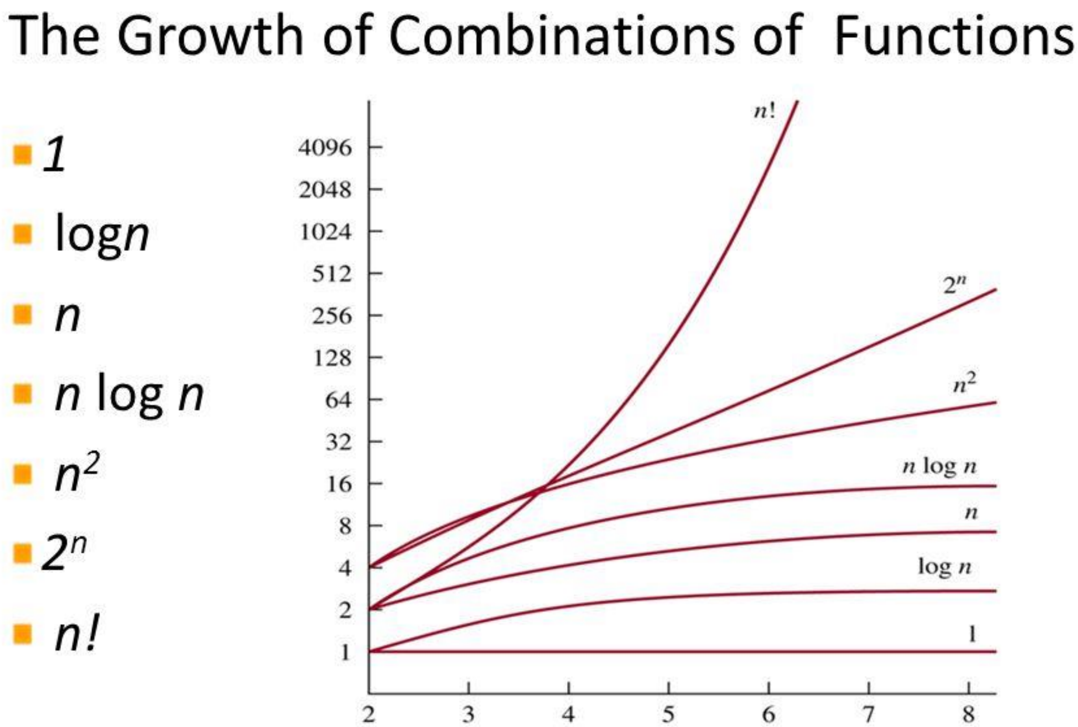


Growth of Functions.

The growth of a function is determined by the highest order term: if we add a bunch of terms, the function grows about as fast as the largest term (for large enough input values).

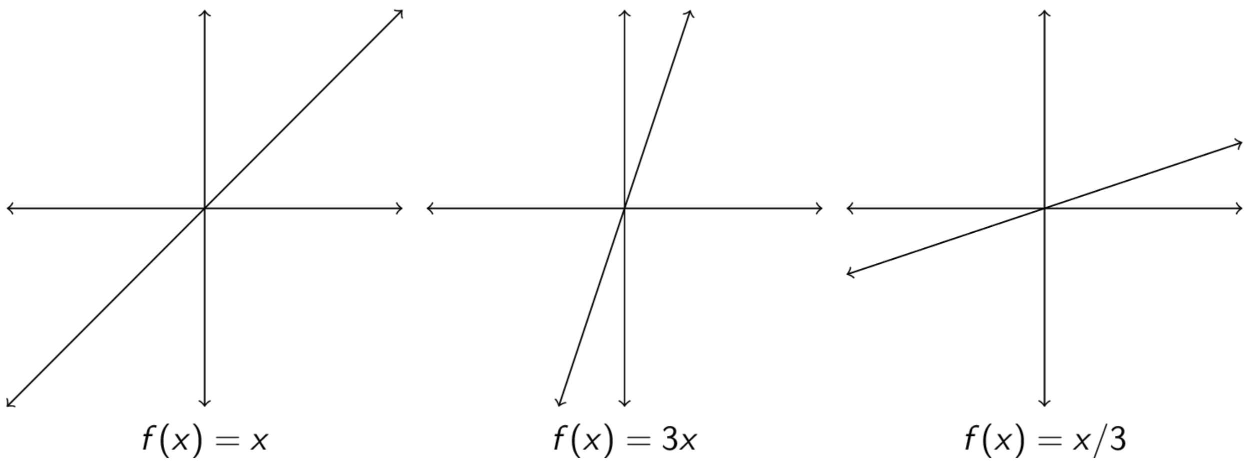
Given functions f and g, “g grows as fast as f”.

For example, $f(x) = x^2 + 1$ grows as fast as, $g(x) = x^2 + 2$ and, $h(x) = x^2 + x + 1$ because for large x, x^2 is much bigger than $1, 2$ or $x + 1$

Similarly, constant multiples don't matter that much:

$f(x) = x^2$, grows as fast as $g(x) = 2x^2$ and $h(x) = 100x^2$ because for large x, multiplying x^2 by a constant does not change it "too much" (at least not as much as increasing x).

Essentially, we are concerned with the shape of the curve:



All three of these functions are lines; their exact slope/y-intercept does not matter.

Big-Oh Notation

History : Big-*O* notation has been used in mathematics for more than a century. In computer science it is widely used in the analysis of algorithms. The German mathematician Paul Bachmann first introduced big-*O* notation in 1892 in an important book on number theory. The big-*O* symbol is sometimes called a **Landau symbol** after the German mathematician Edmund Landau, who used this notation throughout his work. The use of big-*O* notation in computer science was popularized by Donald

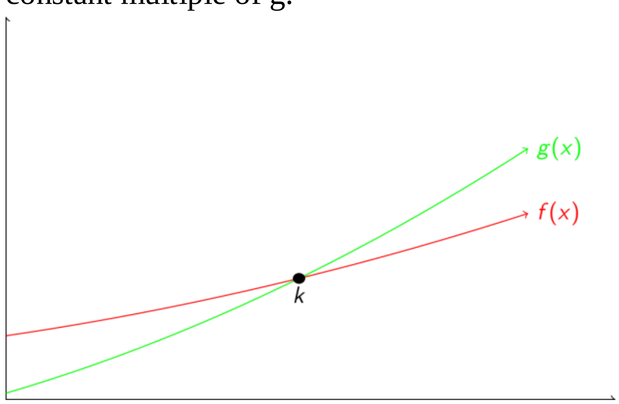
Definition:

Let f and g be functions from the set of integers or the set of real numbers to the set of real numbers. We say that f(x) is O(g(x)) if there are constants $c \geq 0$ and $k \geq 0$ such that

$|f(x)| \leq c\,|g(x)|$ whenever $x \geq k$.

This is read as f(x) is big O of g(x). In this definition, the constants C and k are called witnesses to the relationship f(x) is O(g(x))

Basically, f(x) is O(g(x)) means that, after a certain value of x, f is always smaller than some constant multiple of g.



To show that, $\mathbf{f(x) = 5x^2 + 10x + 3}$ is $\mathbf{O(x^2)}$

$$5x^2 + 10x + 3 \le 5x^2 + 10x^2 + 3x^2 = 18x^2$$

Each of the above steps is true for all x≥1, so take c=18, k=1 as witnesses.

To show that, $\mathbf{f(x) = 5x^2 - 10x + 3}$ is $\mathbf{O(x^2)}$

$$5x^2 + 10x + 3 \le 5x^2 + 3x^2 = 8x^2$$

The first step is true as long as 10x>0 (which is the same as x>0) and the second step is true as long as x≥1, so take c=8, k=1 as witnesses.

Is it true that $\boldsymbol{x^3}$ *is* $\mathbf{O(x^2)}$?

Suppose it is true. Then $x^3 \le cx^2$ for $x > k$.

Dividing through by x^2, we get that x≤c. This says that "x is always less than a constant", but this is not true: a line with positive slope is not bounded from above by any constant!

Therefore, x^3 is **not** $O(x^2)$.

Typically, we want the function inside the Oh to be as small and simple as possible.

Show that, $\boldsymbol{f(x) = x^3 + 3x - 2}$ *is* $\mathbf{O(x^3)}$

Solution

We notice that as long as x>1, $x^2 \le x^3$and $3x - 2 \le x^3$

Therefore, when x>1, we have that $|f(x)| = x^3 + 3x - 2 \le 2x^3$

So we choose k=1 and c=2. There are infinitely many other choices for pairs k, c that would work as well.

If two functions f and g are both big-O of the other one, we say that f and g have the *same order*.

We can also use the definition to show that a function is not big-O of another

Big-O notation is used extensively to describe the growth of functions, but it has limitations. In particular, when f (x) is O(g(x)), we have an upper bound, in terms of g(x), for the size of f (x) for large values of x. However, big-O notation does not provide a lower bound for the size of f (x) for large x. For this, we use **big-Omega (big- Ω) notation**. When we want to give both an upper and a lower bound on the size of a function f (x), relative to a reference function g(x), we use **big-Theta (big-Ω) notation**. Both big-Omega and big-Theta notation were introduced by Donald Knuth in the 1970s. His motivation for introducing these notations was the common misuse of big-O notation when both an upper and a lower bound on the size of a function are needed.

Let f and g be functions from the set of integers or the set of real numbers to the set of real numbers. We say that f (x) is Ω(g(x)) if there are positive constants C and k such that

|f (x)| ≥ C|g(x)|

whenever x > k. [This is read as “f (x) is big-Omega of g(x).”]

There is a strong connection between big-O and big-Omega notation. In particular, f (x) is Ω(g(x)) if and only if g(x) is O(f (x)).

$\mathbf{f(x) = 8x^3 + 5x^2 + 7}$ **is** $\Omega(g(x))$ where $g(x)$ is the function $g(x) = \mathbf{x^3}$ This is easy to see because $\mathbf{f(x) = 8x^3 + 5x^2 + 7 \geq 8x^3}$ for all positive real numbers x. This is equivalent to saying that $g(x) = \mathbf{x^3}$ **is** $\mathbf{O(8x^3 + 5x^2 + 7)}$ which can be established directly by turning the inequality around.

big-Theta

Let f and g be functions from the set of integers or the set of real numbers to the set of real numbers. We say that f (x) is $\Theta(g(x))$ if f (x) is $O(g(x))$ and f (x) is $\Omega(g(x))$. When f (x) is $\Theta(g(x))$ we say that f is big-Theta of g(x), that f (x) is of order g(x), and that f (x) and g(x) are of the same order.

f (x) is $\Theta(g(x))$ if and only if there are real numbers C1 and C2 and a positive real number k such that

$$C_1|g(x)| \leq f(x) \leq C_2\,|g(x)|$$

Whenever x > k. The existence of the constants C_1, C_2, and k tells us that f (x) is $\Omega(g(x))$ and that f (x) is $O(g(x))$, respectively.

Show that $\mathbf{3x^2 + 8x\log x}$ **is** $\Theta(x^2)$

Solution: $\mathbf{0 \leq 8x\log x \leq 8x^2}$, it follows that $\mathbf{3x^2 + 8x\log x \leq 11x^2}$ $\boldsymbol{for\ x > 1}$

Consequently, $\mathbf{3x^2 + 8x\log x}$ **is** $O(x^2)$

Clearly, x^2 is $O(\mathbf{3x^2 + 8x\log x})$

Consequently, $\mathbf{3x^2 + 8x\log x}$ **is** $\Theta(x^2)$

Example: Show that the sum of the first n positive integers is $\Theta(n^2)$.

Solution: Let $f(n) = 1 + 2 + \cdots + n$.

- We have already shown that $f(n)$ is $O(n^2)$.
- To show that $f(n)$ is $\Omega(n^2)$, we need a positive constant C such that $f(n) > Cn^2$ for sufficiently large n. Summing only the terms greater than $n/2$ we obtain the inequality

$$\begin{aligned} 1 + 2 + \cdots + n &\geq \lceil n/2 \rceil + (\lceil n/2 \rceil + 1) + \cdots + n \\ &\geq \lceil n/2 \rceil + \lceil n/2 \rceil + \cdots + \lceil n/2 \rceil \\ &= (n - \lceil n/2 \rceil + 1)\lceil n/2 \rceil \\ &\geq (n/2)(n/2) = n^2/4 \end{aligned}$$

- Taking $C = ¼$, $f(n) > Cn^2$ for all positive integers n. Hence, $f(n)$ is $\Omega(n^2)$, and we can conclude that $f(n)$ is $\Theta(n^2)$.