Curl :

$\frac{Q-89}{P-80}$. If $\vec{A} = 2xz^2\hat{i} - yz\hat{j} + 3xz^3\hat{k}$ & $\phi = x^2yz$

find (i) $\vec{\nabla} \times \vec{A}$ (ii) $\vec{\nabla} \times (\phi\vec{A})$ at the point (1,1,1).

Ans: $\vec{A} = 2xz^2\hat{i} - yz\hat{j} + 3xz^3\hat{k}$

$$\therefore \vec{\nabla} \times \vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xz^2 & -yz & 3xz^3 \end{vmatrix}$$

$$= \left\{\frac{\partial}{\partial y}(3xz^3) + \frac{\partial}{\partial z}(yz)\right\}\hat{i} - \left\{\frac{\partial}{\partial x}(3xz^3) - \frac{\partial}{\partial z}(2xz^2)\right\}\hat{j}$$
$$+ \left\{\frac{\partial}{\partial x}(-yz) - \frac{\partial}{\partial y}(2xz^2)\right\}\hat{k}$$

$$\vec{\nabla}\times\vec{A} = (0+y)\hat{i} - (3z^3 - 4xz)\hat{j} + (-0-0)\hat{k}$$
$$= y\hat{i} - (3z^3 - 4xz)\hat{j}$$

At the point $(1,1,1)$; $\vec{\nabla}\times\vec{A} = \hat{i} + \hat{j}$ Ans:

(ii) $\varphi = x^2yz$; $\therefore \varphi\vec{A} = 2x^3yz^3\hat{i} - x^2y^2z^2\hat{j} + 3x^3yz^4\hat{k}$.

$$\therefore \vec{\nabla}\times(\varphi\vec{A}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x^3yz^3 & -x^2y^2z^2 & 3x^3yz^4 \end{vmatrix}$$

After evaluation;

$$\vec{\nabla}\times(\varphi\vec{A}) = (3x^3z^4 + 2x^2y^2z)\hat{i} - (9x^2yz^4 - 6x^3yz^2)\hat{j} + (-2xy^2z^2 - 2x^3z^3)\hat{k}$$

Now $\vec{\nabla}\cdot\vec{\nabla}\times(\varphi\vec{A})$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\left[(3x^3z^4 + 2x^2y^2z)\hat{i} - (9x^2yz^4 - 6x^3yz^2)\hat{j} - (2xy^2z^2 + 2x^3z^3)\hat{k}\right]$$

$$= \frac{\partial}{\partial x}(3x^3z^4 + 2x^2y^2z) - \frac{\partial}{\partial y}(9x^2yz^4 - 6x^3yz^2) - \frac{\partial}{\partial z}(2xy^2z^2 + 2x^3z^3)$$

$$= \cancel{9x^2z^4} + \cancel{4xy^2z} - \cancel{9x^2z^4} + \cancel{6x^3z^2} - \cancel{4xy^2z} - \cancel{6x^3z^2}$$

$$= 0$$

[Note: Div (curl of any vector) $= 0$]

Q-84 Prove that $\vec{A} = 3y^4z^2\hat{i} + 4x^3z^2\hat{j} - 3x^2y^2\hat{k}$ is solenoidal.

Ans :- $\vec{\nabla}\cdot\vec{A} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\left(3y^4z^2\hat{i} + 4x^3z^2\hat{j} - 3x^2y^2\hat{k}\right)$

$= \frac{\partial}{\partial x}(3y^4z^2) + \frac{\partial}{\partial y}(4x^3z^2) + \frac{\partial}{\partial z}(-3x^2y^2)$

$= 0 + 0 - 0 = 0$

$\therefore \vec{A}$ is Solenoidal.

Q: Show that $\vec{F} = (4xy - z^3)\hat{i} + 2x^2\hat{j} - 3xz^2\hat{k}$ is irrotational.

Ans; $\vec{F} = (4xy - z^3)\hat{i} + 2x^2\hat{j} - 3xz^2\hat{k}$

Now $\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 4xy - z^3 & 2x^2 & -3xz^2 \end{vmatrix}$

$= \left\{\frac{\partial}{\partial y}(-3xz^2) - \frac{\partial}{\partial z}(2x^2)\right\}\hat{i} - \left\{\frac{\partial}{\partial x}(-3xz^2) - \frac{\partial}{\partial z}(4xy - z^3)\right\}\hat{j} + \left\{\frac{\partial}{\partial x}(2x^2) - \frac{\partial}{\partial y}(4xy - z^3)\right\}\hat{k}$

$= (-0 - 0)\hat{i} - (-3z^2 + 3z^2)\hat{j} + (4x - 4x)\hat{k}$

$= 0 - 0 + 0 = 0$

$\therefore \vec{F}$ is Irrotational.

Note: If $\vec{\nabla}\times\vec{F} = 0$; $\vec{F}$ is irrotational. In this case $\vec{F}$ is also conservative force ~~field~~.

<u>Conservative force</u>: A conservative force is that work done by it is independent of the path and depends only the initial & final position. In nature; gravitational force, magnetic force, electrostatic force etc.

Q.74 Prove that $\nabla^2 r^n = n(n+1) r^{n-2}$; where n is a ~~const~~ constant.

A:- $\nabla^2 r^n = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)(x^2+y^2+z^2)^{n/2}$ ——— (i)

Now $\frac{\partial^2}{\partial x^2}(x^2+y^2+z^2)^{n/2}$

$= \frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}(x^2+y^2+z^2)^{n/2}\right]$

$= \frac{\partial}{\partial x}\left[\frac{n}{\cancel{2}}(x^2+y^2+z^2)^{\frac{n}{2}-1} \cdot \cancel{2}x\right]$

$= \frac{\partial}{\partial x}\left[nx\,(x^2+y^2+z^2)^{\frac{n}{2}-1}\right]$

$= nx \cdot \left(\frac{n}{2}-1\right)(x^2+y^2+z^2)^{\frac{n}{2}-2} \cdot 2x + (x^2+y^2+z^2)^{\frac{n}{2}-1} \cdot n$

$= nx\left(\frac{n-2}{\cancel{2}}\right)(x^2+y^2+z^2)^{\frac{n}{2}-2} \cdot \cancel{2}x + n(x^2+y^2+z^2)^{\frac{n}{2}-1}$

$= nx^2(n-2)(x^2+y^2+z^2)^{\frac{n}{2}-2} + n(x^2+y^2+z^2)^{\frac{n}{2}-1}$

Similarly; $\frac{\partial^2}{\partial y^2}(x^2+y^2+z^2)^{n/2} = ny^2(n-2)(x^2+y^2+z^2)^{\frac{n}{2}-2} + n(x^2+y^2+z^2)^{\frac{n}{2}-1}$

& $\frac{\partial^2}{\partial z^2}(x^2+y^2+z^2)^{\frac{n}{2}} = nz^2(n-2)(x^2+y^2+z^2)^{\frac{n}{2}-2} + n(x^2+y^2+z^2)^{\frac{n}{2}-1}$

∴ From (i);

$$\nabla^2 r^n = n(n-2)(x^2+y^2+z^2)(x^2+y^2+z^2)^{\frac{n}{2}-2} + 3n(x^2+y^2+z^2)^{\frac{n}{2}}$$

$$= n(n-2)(x^2+y^2+z^2)^{\frac{n}{2}-1} + 3n(x^2+y^2+z^2)^{\frac{n}{2}-1}.$$

$$= \{n(n-2)+3n\}(x^2+y^2+z^2)^{\frac{n-2}{2}}$$

$$= (n^2+n)(x^2+y^2+z^2)^{\frac{n-2}{2}}$$

$$= n(n+1)\, r^{n-2}.$$

Proved