Discrete Mathematics - Recurrence Relation

In this chapter, we will discuss how recursive techniques can derive sequences and be used for solving counting problems. The procedure for finding the terms of a sequence in a recursive manner is called **recurrence relation**. We study the theory of linear recurrence relations and their solutions. Finally, we introduce generating functions for solving recurrence relations.

Definition

A recurrence relation is an equation that recursively defines a sequence where the next term is a function of the previous terms (Expressing F_n as some combination of F_i with $i < n$).

Example − Fibonacci series − $F_n = F_{n-1} + F_{n-2}$, Tower of Hanoi − $F_n = 2F_{n-1} + 1$

Linear Recurrence Relations

A linear recurrence equation of degree k or order k is a recurrence equation which is in the format $x_n = A_1 x_{n-1} + A_2 x_{n-1} + A_3 x_{n-1} + \dots A_k x_{n-k}$ (A_n is a constant and $A_k \neq 0$) on a sequence of numbers as a first-degree polynomial.

These are some examples of linear recurrence equations −

Recurrence relations	Initial values	Solutions
$F_n = F_{n-1} + F_{n-2}$	$a_1 = a_2 = 1$	Fibonacci number
$F_n = F_{n-1} + F_{n-2}$	$a_1 = 1, a_2 = 3$	Lucas Number
$F_n = F_{n-2} + F_{n-3}$	$a_1 = a_2 = a_3 = 1$	Padovan sequence
$F_n = 2F_{n-1} + F_{n-2}$	$a_1 = 0, a_2 = 1$	Pell number

How to solve linear recurrence relation

Suppose, a two ordered linear recurrence relation is − $F_n = AF_{n-1} + BF_{n-2}$ where A and B are real numbers.

The characteristic equation for the above recurrence relation is −

$$x^2 - Ax - B = 0$$

Three cases may occur while finding the roots −

Case 1 − If this equation factors as $(x - x_1)(x - x_1) = 0$ and it produces two distinct real roots x_1 and x_2, then $F_n = ax_1^n + bx_2^n$ is the solution. [Here, a and b are constants]

Case 2 − If this equation factors as $(x - x_1)^2 = 0$ and it produces single real root x_1, then $F_n = ax_1^n + bnx_1^n$ is the solution.

Case 3 − If the equation produces two distinct complex roots, x_1 and x_2 in polar form $x_1 = r\angle\theta$ and $x_2 = r\angle(-\theta)$, then $F_n = r^n(acos(n\theta) + bsin(n\theta))$ is the solution.

Problem 1

Solve the recurrence relation $F_n = 5F_{n-1} - 6F_{n-2}$ where $F_0 = 1$ and $F_1 = 4$

Solution

The characteristic equation of the recurrence relation is −

$$x^2 - 5x + 6 = 0,$$

So, $(x - 3)(x - 2) = 0$

Hence, the roots are −

$x_1 = 3$ and $x_2 = 2$

The roots are real and distinct. So, this is in the form of case 1

Hence, the solution is −

$$F_n = ax_1^n + bx_2^n$$

Here, $F_n = a3^n + b2^n \ (As\ x_1 = 3\ and\ x_2 = 2)$

Therefore,

$$1 = F_0 = a3^0 + b2^0 = a + b$$

$$4 = F_1 = a3^1 + b2^1 = 3a + 2b$$

Solving these two equations, we get $a = 2$ and $b = -1$

Hence, the final solution is −

$$F_n = 2.3^n + (-1).\,2^n = 2.3^n - 2^n$$

Problem 2

Solve the recurrence relation − $F_n = 10F_{n-1} - 25F_{n-2}$ where $F_0 = 3$ and $F_1 = 17$

Solution

The characteristic equation of the recurrence relation is −

$$x^2 - 10x - 25 = 0$$

So $(x - 5)^2 = 0$

Hence, there is single real root $x_1 = 5$

As there is single real valued root, this is in the form of case 2

Hence, the solution is −

$$F_n = ax_1^n + bnx_1^n$$

$$3 = F_0 = a.5^0 + (b)(0.5)^0 = a$$

$$17 = F_1 = a.5^1 + b.1.5^1 = 5a + 5b$$

Solving these two equations, we get $a = 3$ and $b = 2/5$

Hence, the final solution is − $F_n = 3.5^n + (2/5).n.2^n$

Problem 3

Solve the recurrence relation $F_n = 2F_{n-1} - 2F_{n-2}$ where $F_0 = 1$ and $F_1 = 3$

Solution

The characteristic equation of the recurrence relation is −

$$x^2 - 2x - 2 = 0$$

Hence, the roots are −

$x_1 = 1 + i$ and $x_2 = 1 - i$

In polar form,

$x_1 = r\angle\theta$ and $x_2 = r\angle(-\theta),$ where $r = \sqrt{2}$ and $\theta = \frac{\pi}{4}$

The roots are imaginary. So, this is in the form of case 3.

Hence, the solution is −

$$F_n = (\sqrt{2})^n (acos(n.\sqcap/4) + bsin(n.\sqcap/4))$$

$$1 = F_0 = (\sqrt{2})^0 (acos(0.\sqcap/4) + bsin(0.\sqcap/4)) = a$$

$$3 = F_1 = (\sqrt{2})^1 (acos(1.\sqcap/4) + bsin(1.\sqcap/4)) = \sqrt{2}(a/\sqrt{2} + b/\sqrt{2})$$

Solving these two equations we get $a = 1$ and $b = 2$

Hence, the final solution is −

$F_n = (\sqrt{2})^n (cos(n.\pi/4) + 2sin(n.\pi/4))$

Non-Homogeneous Recurrence Relation and Particular Solutions

A recurrence relation is called non-homogeneous if it is in the form

$F_n = AF_{n-1} + BF_{n-2} + f(n)$ where $f(n) \neq 0$

Its associated homogeneous recurrence relation is $F_n = AF_{n-1} + BF_{n-2}$

The solution (a_n) of a non-homogeneous recurrence relation has two parts.

First part is the solution (a_h) of the associated homogeneous recurrence relation and the second part is the particular solution (a_t) .

$$a_n = a_h + a_t$$

Solution to the first part is done using the procedures discussed in the previous section.

To find the particular solution, we find an appropriate trial solution.

Let $f(n) = cx^n$; let $x^2 = Ax + B$ be the characteristic equation of the associated homogeneous recurrence relation and let x_1 and x_2 be its roots.

- If $x \neq x_1$ and $x \neq x_2$, then $a_t = Ax^n$
- If $x = x_1$, $x \neq x_2$, then $a_t = Anx^n$
- If $x = x_1 = x_2$, then $a_t = An^2x^n$

Example

Let a non-homogeneous recurrence relation be $F_n = AF_{n-1} + BF_{n-2} + f(n)$ with characteristic roots $x_1 = 2$ and $x_2 = 5$. Trial solutions for different possible values of

$f(n)$ are as follows −

f(n)	Trial solutions
4	A
5.2^n	$An2^n$
8.5^n	$An5^n$
4^n	$A4^n$
$2n^2+3n+1$	An^2+Bn+C

Problem

Solve the recurrence relation $F_n = 3F_{n-1} + 10F_{n-2} + 7.5^n$ where $F_0 = 4$ and $F_1 = 3$

Solution

This is a linear non-homogeneous relation, where the associated homogeneous equation is $F_n = 3F_{n-1} + 10F_{n-2}$ and $f(n) = 7.5^n$

The characteristic equation of its associated homogeneous relation is −

$$x^2 - 3x - 10 = 0$$

Or, $(x-5)(x+2) = 0$

Or, $x_1 = 5$ and $x_2 = -2$

Hence $a_h = a.5^n + b.(-2)^n$, where a and b are constants.

Since $f(n) = 7.5^n$, i.e. of the form $c.x^n$, a reasonable trial solution of at will be Anx^n

$a_t = Anx^n = An5^n$

After putting the solution in the recurrence relation, we get −

$$An5^n = 3A(n–1)5^{n-1} + 10A(n–2)5^{n-2} + 7.5^n$$

Dividing both sides by 5^{n-2}, we get

$$An5^2 = 3A(n-1)5 + 10A(n-2)5^0 + 7.5^2$$

Or, $25An = 15An - 15A + 10An - 20A + 175$

Or, $35A = 175$

Or, $A = 5$

So, $F_n = An5^n = 5n5^n = n5^{n+1}$

The solution of the recurrence relation can be written as −

$$F_n = a_h + a_t$$

$$= a.5^n + b.(-2)^n + n5^{n+1}$$

Putting values of $F_0 = 4$ and $F_1 = 3$, in the above equation, we get $a = -2$ and $b = 6$

Hence, the solution is −

$$F_n = n5^{n+1} + 6.(-2)^n - 2.5^n$$

Generating Functions

Generating Functions represents sequences where each term of a sequence is expressed as a coefficient of a variable x in a formal power series.

Mathematically, for an infinite sequence, say $a_0, a_1, a_2, \dots, a_k, \dots,$ the generating function will be −

$$G_x = a_0 + a_1 x + a_2 x^2 + \dots + a_k x^k + \dots = \sum_{k=0}^{\infty} a_k x^k$$

Some Areas of Application

Generating functions can be used for the following purposes −

- For solving a variety of counting problems. For example, the number of ways to make change for a Rs. 100 note with the notes of denominations Rs.1, Rs.2, Rs.5, Rs.10, Rs.20 and Rs.50
- For solving recurrence relations
- For proving some of the combinatorial identities
- For finding asymptotic formulae for terms of sequences

Problem 1

What are the generating functions for the sequences $\{a_k\}$ with $a_k = 2$ and $a_k = 3k$?

Solution

When $a_k = 2$, generating function, $G(x) = \sum_{k=0}^{\infty} 2x^k = 2 + 2x + 2x^2 + 2x^3 + \dots$

When $a_k = 3k, G(x) = \sum_{k=0}^{\infty} 3kx^k = 0 + 3x + 6x^2 + 9x^3 + \dots\dots$

Problem 2

What is the generating function of the infinite series; $1, 1, 1, 1, \dots$?

Solution

Here, $a_k = 1$, for $0 \le k \le \infty$

Hence, $G(x) = 1 + x + x^2 + x^3 + \dots\dots = \frac{1}{(1-x)}$

Some Useful Generating Functions

- For $a_k = a^k, G(x) = \sum_{k=0}^{\infty} a^k x^k = 1 + ax + a^2x^2 + \ldots\ldots\ldots\cdots = 1/(1 - ax)$

- For $a_k = (k+1), G(x) = \sum_{k=0}^{\infty}(k+1)x^k = 1 + 2x + 3x^2 \ldots\ldots\ldots\cdots = \frac{1}{(1-x)^2}$

- For $a_k = c_k^n, G(x) = \sum_{k=0}^{\infty} c_k^n x^k = 1 + c_1^n x + c_2^n x^2 + \ldots\ldots\ldots\cdots + x^2$
  $= (1+x)^n$

- For $a_k = \frac{1}{k!}, G(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} \ldots\ldots\ldots\cdots = e^x$