Mathematical Induction

Mathematical induction, is a technique for proving results or establishing statements for natural numbers. This part illustrates the method through a variety of examples.

Definition

Mathematical Induction is a mathematical technique which is used to prove a statement, a formula or a theorem is true for every natural number.

The technique involves two steps to prove a statement, as stated below −

Step 1(Base step) − It proves that a statement is true for the initial value.

Step 2(Inductive step) − It proves that if the statement is true for the n^{th} iteration (or number *n*), then it is also true for $(n+1)^{th}$ iteration (or number *n+1*).

How to Do It

Step 1 − Consider an initial value for which the statement is true. It is to be shown that the statement is true for n = initial value.

Step 2 − Assume the statement is true for any value of *n* = *k*. Then prove the statement is true for *n* = *k+1*. We actually break *n* = *k+1* into two parts, one part is *n* = *k* (which is already proved) and try to prove the other part.

Problem 1

$3^n - 1$ is a multiple of 2 for n = 1, 2, ...

Solution

Step 1 − For $n = 1, 3^1 - 1 = 3 - 1 = 2$ which is a multiple of 2

Step 2 − Let us assume $3^n - 1$ is true for $n = k$, Hence, $3^k - 1$ is true (It is an assumption)

We have to prove that $3^{k+1} - 1$ is also a multiple of 2

$$3^{k+1} - 1 = 3 \times 3^k - 1 = (2 \times 3^k) + (3^k - 1)$$

The first part $(2 \times 3k)$ is certain to be a multiple of 2 and the second part $(3k - 1)$ is also true as our previous assumption.

Hence, $3^{k+1} - 1$ is a multiple of 2.

So, it is proved that $3^n - 1$ is a multiple of 2.

Problem 2

$1 + 3 + 5 + \ldots + (2n - 1) = n^2$ for $n = 1, 2, \ldots$

Solution

Step 1 − For $n = 1, 1 = 1^2$, Hence, step 1 is satisfied.

Step 2 − Let us assume the statement is true for $n = k$.

Hence, $1 + 3 + 5 + \cdots + (2k - 1) = k^2$ is true (It is an assumption)

We have to prove that $1 + 3 + 5 + \ldots + (2(k + 1) - 1) = (k + 1)^2$ also holds

$$1 + 3 + 5 + \cdots + (2(k + 1) - 1)$$

$$= 1 + 3 + 5 + \cdots + (2k + 2 - 1)$$

$$= 1 + 3 + 5 + \cdots + (2k + 1)$$

$$= 1 + 3 + 5 + \cdots + (2k - 1) + (2k + 1)$$

$$= k^2 + (2k + 1)$$

$= (k+1)^2$

So, $1 + 3 + 5 + \cdots + (2(k+1) - 1) = (k+1)^2$ hold which satisfies the step 2.

Hence, $1 + 3 + 5 + \cdots + (2n-1) = n^2$ is proved.

Problem 3

Prove that $(ab)^n = a^n b^n$ is true for every natural number n

Solution

Step 1 − For $n = 1, (ab)^1 = a^1 b^1 = ab$, Hence, step 1 is satisfied.

Step 2 − Let us assume the statement is true for $n = k$, Hence, $(ab)^k = a^k b^k$ is true (It is an assumption).

We have to prove that $(ab)^{k+1} = a^{k+1} b^{k+1}$ also hold

Given, $(ab)^k = a^k b^k$

Or, $(ab)^k (ab) = (a^k b^k)(ab)$ [Multiplying both side by 'ab']

Or, $(ab)^{k+1} = (aa^k)(bb^k)$

Or, $(ab)^{k+1} = (a^{k+1} b^{k+1})$

Hence, step 2 is proved.

So, $(ab)^n = a^n b^n$ is true for every natural number n.

Strong Induction

Strong Induction is another form of mathematical induction. Through this induction technique, we can prove that a propositional function, $P(n)$ is true for all positive integers, n, using the following steps −

- **Step 1(Base step) −** It proves that the initial proposition $P(1)$ true.
- **Step 2(Inductive step) −** It proves that the conditional statement $[P(1) \land P(2) \land P(3) \land \cdots \land P(k)] \to P(k+1)$ is true for positive integers k.