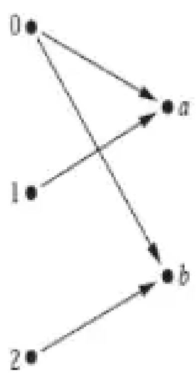
Relations

Relationship between elements of sets is represented using a mathematical structure called relation. The most intuitive way to describe the relationship is to represent in the form of ordered pair. In this section, we study the basic terminology and diagrammatic representation of relation. Let A and B be sets. A binary relation from A to B is a subset of A × B.

Note : If A, B and C are three sets, then a subset of A×B×C is known as ternary relation. Continuing this way a subset of $A_1 \times A_2 \times ... \times A_n$ is known as n – ary relation.
Let A and B be two sets. Suppose R is a relation from A to B (i.e. R is a subset of A × B). Then, R is a set of ordered pairs where each first element comes from A and each second element from B. Thus, we denote it with an ordered pair (a, b), where a ∈A and b ∈ B. We also denote the relationship with a R b, which is read as a related to b. The **domain** of R is the set of all first elements in the ordered pair and the **range** of R is the set of all second elements in the ordered pair.

Let A = {0, 1, 2} and B = {a, b}. Then {(0, a), (0, b), (1, a), (2, b)} is a relation from A to B. This means, for instance, that 0R a, but that 1_R b. Relations can be represented graphically, as shown in Figure, using arrows to represent ordered pairs. Another way to represent this relation is to use a table, which is also



R	a	b
0	×	×
1	×	
2		×

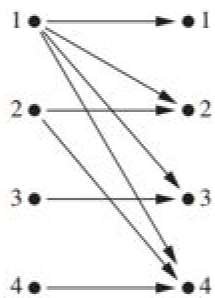
Functions as Relations

A function f from a set A to a set B assigns exactly one element of B to each element of A. The graph of f is the set of ordered pairs (a, b) such that b = f (a). Because the graph of f is a subset of A × B, it is a relation from A to B.
Moreover, the graph of a function has the property that every element of A is the first element of exactly one ordered pair of the graph. Conversely, if R is a relation from A to B such that every element in A is the first element of exactly one ordered pair of R, then a function can be defined with R as its graph. This can be done by assigning to an element a of A the unique element b ∈ B such that (a, b) ∈ R.

Relations on a Set

A relation on a set A is a relation from A to A.
In other words, a relation on a set A is a subset of A × A.

EXAMPLE: Let A be the set {1, 2, 3, 4}. Which ordered pairs are in the relation R = {(a, b) | a divides b}?
Solution: Because (a, b) is in R if and only if a and b are positive integers not exceeding 4 such that a divides b, we see that
R = {(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 4), (3, 3), (4, 4)}.



R	1	2	3	4
1	×	×	×	×
2		×		×
3			×	
4				×

Consider these relations on the set of integers:
R_1 = {(a, b) | a ≤ b},
R_2 = {(a, b) | a > b},
R_3 = {(a, b) | a = b or a = −b},
R_4= {(a, b) | a = b},
R_5= {(a, b) | a = b + 1},
R_6 = {(a, b) | a + b ≤ 3}.

Which of these relations contain each of the pairs (1, 1), (1, 2), (2, 1), (1,−1), and (2, 2)?
Solution: The pair (1, 1) is in R_1 , R_3 , R_4 and R_6 ; (1, 2) is in R_1 and R_6 ; (2, 1) is in R2, R5, and R_6; (1, −1) is in R_2, R_3 , and R_6 ; and finally, (2, 2) is in R1, R3, and R4.

How many relations are there on a set with n elements?
Solution: A relation on a set A is a subset of A×A. Because A ×A has n^2 elements when A has n elements, and a set with m elements has 2^m subsets, there are 2^{n^2}subsets of A × A. Thus, there are 2^{n^2}relations on a set with n elements. For example, there are $2^{3^2} = 2^9 = 512$ relations on the set {a, b, c}.

- A relation R on a set A is called **reflexive** if (a, a) ∈ R for every element a ∈ A.
- A relation R on a set A is called **symmetric** if (b, a) ∈ R whenever (a, b) ∈ R, for all a, b ∈ A.
- **Antisymmetric relation -** A relation R on a set A is said to be antisymmetric, if aRb and bRa hold if and only if when a = b. In other words, (a, b) ∉ R and (b, a) ∉ R if a ≠ b.
- A relation R on a set A is called **transitive** if whenever (a, b) ∈ R and (b, c) ∈ R, then (a, c) ∈ R, for all a, b, c ∈ A.

EXAMPLE: Consider the following relations on {1, 2, 3, 4}:

R_1 = {(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 1), (4, 4)},

R_2 = {(1, 1), (1, 2), (2, 1)},

R_3 = {(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (3, 3), (4, 1), (4, 4)},

R_4 = {(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3)},

R_5 = {(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)},

R_6 = {(3, 4)}.

*** Which of these relations are reflexive? Which of these relations are symmetric and which are antisymmetric? Which are transitive?

Solution: The relations R_3 and R_5 are reflexive because they both contain all pairs of the form (a, a), namely, (1, 1), (2, 2), (3, 3), and (4, 4). The other relations are not reflexive because they do not contain all of these ordered pairs. In particular, R_1 , R_2 , R_4 and R_6 are not reflexive because (3, 3) is not in any of these relations.

The relations R_2 and R_3 are symmetric, because in each case (b, a) belongs to the relation whenever (a, b) does. For R_2 the only thing to check is that both (2, 1) and (1, 2) are in the relation. For R3, it is necessary to check that both (1, 2) and (2, 1) belong to the relation, and (1, 4) and (4, 1) belong to the relation. The reader should verify that none of the other relations is symmetric. This is done by finding a pair (a, b) such that it is in the relation but (b, a) is not.
R_4 R_5 and R_6 are all antisymmetric. For each of these relations there is no pair of elements a and b with $a \neq b$ such that both (a, b) and (b, a) belong to the relation. The reader should verify that none of the other relations is antisymmetric. This is done by finding a pair (a, b) with $a \neq b$ such that (a, b) and (b, a) are both in the relation.

R_4 , R_5 and R_6 are transitive. For each of these relations, we can show that it is transitive by verifying that if (a, b) and (b, c) belong to this relation, then (a, c) also does. For instance, R4 is transitive, because (3, 2) and (2, 1), (4, 2) and (2, 1), (4, 3) and (3, 1), and (4, 3) and (3, 2) are the only such sets of pairs, and (3, 1), (4, 1), and (4, 2) belong to R_4. Similarly we verify that R_5 and R_6 are transitive.

R_1 is not transitive because (3, 4) and (4, 1) belong to R_1, but (3, 1) does not. R_2 is not transitive because (2, 1) and (1, 2) belong to R_2 but (2, 2) does not. R_3 is not transitive because (4, 1) and (1, 2) belong to R_3 , but (4, 2) does not.

Question : Which of the following are antisymmetric?

1. R = { (1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 1), (4, 4) }
2. R = { (1, 1), (1, 3), (3, 1) }
3. R = { (1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (3, 3),(4, 1), (4, 4) }

Solution: Rule of antisymmetric relation says that, if (a, b) ∈ R and (b, a) ∈ R, then it means a = b.

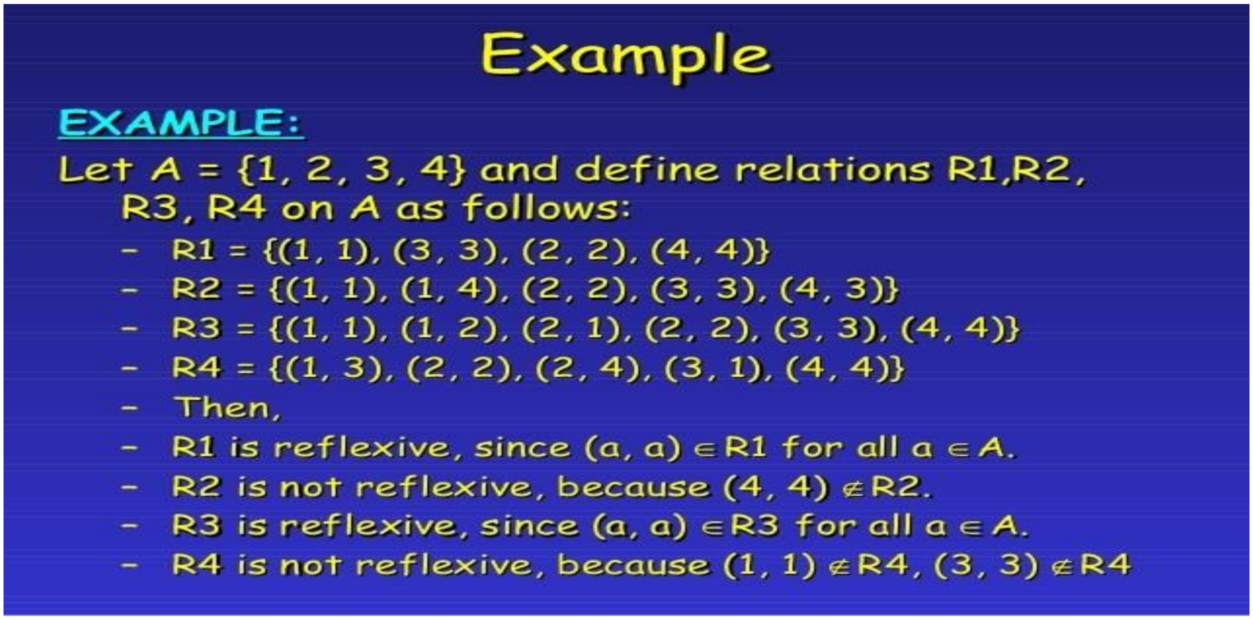
In case a ≠ b, then even if (a, b) ∈ R and (b, a) ∈ R holds, the relation cannot be antisymmetric.

Keeping that in mind, below are the final answers.

1. Here, R is not antisymmetric as (1, 2) ∈ R and (2, 1) ∈ R, but 1 ≠ 2.
2. R is not antisymmetric because of (1, 3) ∈ R and (3, 1) ∈ R, however, 1 ≠ 3.
3. Here, R is not antisymmetric because of (1, 2) ∈ R and (2, 1) ∈ R, but 1 ≠ 2. Also, (1, 4) ∈ R, and (4, 1) ∈ R, but 1 ≠ 4.

Question : R is the relation on set A and A = {1, 2, 3, 4}. Find the antisymmetric relation on set A.
Solution: The antisymmetric relation on set A = {1, 2, 3, 4} is;

R = { (1, 1), (2, 2), (3, 3), (4, 4) }.



Example

Example: Consider the following relations on $\{1, 2, 3, 4\}$

$R_1 = \{(1,1), (1,2), (2,1), (2,2), (3,4), (4,1), (4,4)\}$
$R_2 = \{(1,1), (1,2), (2,1)\}$
$R_3 = \{(1,1), (1,2), (1,4), (2,1), (2,2), (3,3), (3,4), (4,1), (4,4)\}$
$R_4 = \{(2,1), (3,1), (3,2), (4,1), (4,2), (4,3)\}$
$R_5 = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,3), (2,4), (3,3), (3,4), (4,4)\}$
$R_6 = \{(3,4)\}$

Which of these relations are symmetric and which are antisymmetric?

Solution

R_2 & R_3: symmetric $\Leftarrow$ each case (b, a) belongs to the relation whenever (a, b) does.
For R_2: only thing to check that both (1,2) & (2,1) belong to the relation
For R_3: it is necessary to check that both (1,2) & (2,1) belong to the relation.
None of the other relations is symmetric: find a pair (a, b) so that it is in the relation but (b, a) is not.

$R_1 = \{(1,1), (1,2), (2,1), (2,2), (3,4), (4,1), (4,4)\}$
$R_2 = \{(1,1), (1,2), (2,1)\}$
$R_3 = \{(1,1), (1,2), (1,4), (2,1), (2,2), (3,3), (4,1), (4,4)\}$
$R_4 = \{(2,1), (3,1), (3,2), (4,1), (4,2), (4,3)\}$
$R_5 = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,3), (2,4), (3,3), (3,4), (4,4)\}$
$R_6 = \{(3,4)\}$

Solution

R_4, R_5 and R_6: antisymmetric $\Leftarrow$for each of these relations there is no pair of elements a and b with $a \neq b$ such that both (a, b) and (b, a) belong to the relation.
None of the other relations is antisymmetric.: find a pair (a, b) with $a \neq b$ so that (a, b) and (b, a) are both in the relation.

$R_1 = \{(1,1), (1,2), (2,1), (2,2), (3,4), (4,1), (4,4)\}$
$R_2 = \{(1,1), (1,2), (2,1)\}$
$R_3 = \{(1,1), (1,2), (1,4), (2,1), (2,2), (3,3), (4,1), (4,4)\}$
$R_4 = \{(2,1), (3,1), (3,2), (4,1), (4,2), (4,3)\}$
$R_5 = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,3), (2,4), (3,3), (3,4), (4,4)\}$
$R_6 = \{(3,4)\}$

Symmetric

Let $A = \{1, 2, 3, 4\}$ and define relations R1, R2,

R3, and R4on A as follows.

- $R1 = \{(1, 1), (1, 3), (2, 4), (3, 1), (4,2)\}$
- $R2 = \{(1, 1), (2, 2), (3, 3), (4, 4)\}$
- $R3 = \{(2, 2), (2, 3), (3, 4)\}$
- $R4 = \{(1, 1), (2, 2), (3, 3), (4, 3), (4, 4)\}$
  - Then R1 is symmetric because for every order pair (a,b) in R1, we have (b,a) in R1, for example we have (1,3) in R1 then we have (3,1) in R1. similarly all other ordered pairs can be checked.
  - R2 is also symmetric.
  - R3 is not symmetric, because $(2,3) \in R3$ but $(3,2) \notin R3$.
  - R4 is not symmetric because $(4,3) \in R4$ but $(3,4) \notin R4$

Example

Example: Consider the following relations on $\{1, 2, 3, 4\}$

$R_1 = \{(1,1), (1,2), (2,1), (2,2), (3,4), (4,1), (4,4)\}$
$R_2 = \{(1,1), (1,2), (2,1)\}$
$R_3 = \{(1,1), (1,2), (1,4), (2,1), (2,2), (3,3), (3,4), (4,1), (4,4)\}$
$R_4 = \{(2,1), (3,1), (3,2), (4,1), (4,2), (4,3)\}$
$R_5 = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,3), (2,4), (3,3), (3,4), (4,4)\}$
$R_6 = \{(3,4)\}$

Which of these relations are transitive?

Solution

- R_4 , R_5 & R_6 : transitive $\Leftarrow$ verify that if (a, b) and (b, c) belong to this relation then (a, c) belongs also to the relation
  R_4 transitive since (3,2) and (2,1), (4,2) and (2,1), (4,3) and (3,1), and (4,3) and (3,2) are the only such sets of pairs, and (3,1) , (4,1) and (4,2) belong to R_4.
  Same reasoning for R_5 and R_6.
- R_1 : not transitive $\Leftarrow$ (3,4) and (4,1) belong to R_1, but (3,1) does not.
- R_2 : not transitive $\Leftarrow$ (2,1) and (1,2) belong to R_2, but (2,2) does not.
- R_3 : not transitive $\Leftarrow$ (4,1) and (1,2) belong to R_3, but (4,2) does not.

Transitive

EXAMPLE

- Let A = {1, 2, 3, 4} and define relations R1, R2 and R3 on A as follows:
- R1 = {(1, 1), (1, 2), (1, 3), (2, 3)}
- R2 = {(1, 2), (1, 4), (2, 3), (3, 4)}
- R3 = {(2, 1), (2, 4), (2, 3), (3,4)}
- Then R1 is transitive because (1, 1), (1, 2) are in R then to be transitive relation (1,2) must be there and it belongs to R Similarly for other order pairs.
- R2 is not transitive since (1,2) and (2,3) $\in$ R2 but (1,3) $\notin$ R2.
- R3 is transitive.

Anti-Symmetric Relation: Example

Let A = {1,2,3,4} and define the following relations on A.

R1 = {(1,1),(2,2),(3,3)}

R2 = {(1,2),(2,2), (2,3), (3,4), (4,1)}

R3={(1,3),(2,2), (2,4), (3,1), (4,2)}

R4={(1,3),(2,4), (3,1), (4,3)}

Which of above relations are Anti-Symmetric?

Combining Relations

EXAMPLE: Let A = {1, 2, 3} and B = {1, 2, 3, 4}. The relations
R_1 = {(1, 1), (2, 2), (3, 3)} and
R_2 = {(1, 1), (1, 2), (1, 3), (1, 4)} can be combined to obtain
$R_1 \cup R_2$= {(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (3, 3)},
$R_1 \cap R_2$ = {(1, 1)},
$R_1 - R_2$ = {(2, 2), (3, 3)},
$R_2 - R_1$= {(1, 2), (1, 3), (1, 4)}.

EXAMPLE: Let A and B be the set of all students and the set of all courses at a school, respectively.
Suppose that R_1 consists of all ordered pairs (a, b), where a is a student who has taken course b, and R_2 consists of all ordered pairs (a, b), where a is a student who requires course b to graduate.
What are the relations $R_1 \cup R_2$, $R_1 \cap R_2$, $R_1 \oplus R_2$, $R_1 - R_2$ and $R_2 - R_1$?
Solution: The relation R1 ∪ R2 consists of all ordered pairs (a, b), where a is a student who either has taken course b or needs course b to graduate, and $R_1 \cap R_2$ is the set of all ordered pairs (a, b), where a is a student who has taken course b and needs this course to graduate. Also, $R_1 \oplus R_2$ consists of all ordered pairs (a, b), where student a has taken course b but does not need it to graduate or needs course b to graduate but has not taken it. $R_1 - R_2$ is the set of ordered pairs (a, b), where a has taken course b but does not need it to graduate; that is, b is an elective course that a has taken. $R_2 - R_1$ is the set of all ordered pairs (a, b), where b is a course that a needs to graduate but has not taken.

Let R be a relation from a set A to a set B and S a relation from B to a set C. The composite of R and S is the relation consisting of ordered pairs (a, c), where a ∈ A, c ∈ C, and for which there exists an element b ∈ B such that (a, b) ∈ R and (b, c) ∈ S. We denote the composite of R and S by S ◦R.
EXAMPLE: What is the composite of the relations R and S, where R is the relation from {1, 2, 3} to {1, 2, 3, 4} with R = {(1, 1), (1, 4), (2, 3), (3, 1), (3, 4)} and S is the relation from {1, 2, 3, 4} to {0, 1, 2} with S = {(1, 0), (2, 0), (3, 1), (3, 2), (4, 1)}?
Solution: S ◦R is constructed using all ordered pairs in R and ordered pairs in S, where the second element of the ordered pair in R agrees with the first element of the ordered pair in S. For example, the ordered pairs (2, 3) in R and (3, 1) in S produce the ordered pair (2, 1) in S ◦R. Computing all the ordered pairs in the composite, we find S ◦R = {(1, 0), (1, 1), (2, 1), (2, 2), (3, 0), (3, 1)}.

Example : Determine whether the relation R on a set A is reflexive, irreflexire, symmetric, asymmetric antisymmetric or transitive.
A = set of all positive integers, a R b iff $a - b \leq 2$
Solution :

1) R is reflexive because $|a - a| = 0 < 2, \forall\, a \in A$
2) R is not irreflexive because $|1 - 1| = 0 < 2, \forall\, 1 \in A$ ($\therefore$ $A\ is\ the\ set\ of\ all\ positive\ integers.$)
3) R is symmetric because $|a - b| \leq 2 \;\Rightarrow |b - a| \leq 2 \;\therefore aRb \;\Rightarrow bRa$
4) R is not asymmetric because $|5 - 4| \leq 2\ and\ we\ have |4 - 5| \leq 2 \;\therefore 5R4 \;\Rightarrow 4R5$

5) R is not antisymmetric because $1R2 \; and \; 2R1$, $1R2 \Rightarrow |1-2| \leq 2 \;\; and$, $2R1 \Rightarrow |2-1| \leq 2. \; but \; 2 \neq 1$
6) R is not transitive because $5R4, 4R2$ but $5\not R2$

Representing Relations Using Matrices

A relation between finite sets can be represented using a zero–one matrix. Suppose that R is a relation from
$A = \{a_1, a_2, \dots, a_m\}$ to $B = \{b_1, b_2, \dots, b_n\}$. (Here the elements of the sets A and B have been listed in a particular, but arbitrary, order. Furthermore, when A = B we use the same ordering for A and B.) The relation R can be represented by the matrix $M_R = [m_{ij}]$, where

$$M_R = \begin{cases} 1, & if \; (a_i, b_j) \in R \\ 0, & if \; (a_i, b_j) \notin R. \end{cases} \quad A = \{a_1, a_2, \dots, a_m\}$$

In other words, the zero–one matrix representing R has a 1 as its (i, j) entry when a_i is related to b_j, and a 0 in this position if a_i is not related to b_j.

EXAMPLE: Suppose that A = {1, 2, 3} and B = {1, 2}. Let R be the relation from A to B containing (a, b)
if a ∈ A, b ∈ B, and a > b. What is the matrix representing R if $a_1 = 1, a_2 = 2$ and $a_3 = 3$, and $b_1 = 1$ and $b_2 = 2$?

Solution: Because R = {(2, 1), (3, 1), (3, 2)}, the matrix for R is

$$M_R = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$$

The 1s in M_R show that the pairs (2, 1), (3, 1), and (3, 2) belong to R. The 0s show that no other pairs belong to R.

EXAMPLE: Let $A = \{a_1, a_2, \dots, a_5\}$ to $B = \{b_1, b_2, \dots, b_5\}$ Which ordered pairs are in the relation R represented by the matrix

$$M_R = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

Solution: Because R consists of those ordered pairs (a_i, b_j) with $m_{ij} = 1$, it follows that
$R = \{(a_1, b_2), (a_2, b_1), (a_2, b_3), (a_2, b_4), (a_3, b_1), (a_3, b_3), (a_3, b_5)\}$

EXAMPLE: Suppose that the relation R on a set is represented by the matrix $M_R = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

Is R reflexive, symmetric, and/or antisymmetric?

Solution: Because all the diagonal elements of this matrix are equal to 1, R is reflexive. Moreover, because M_R is symmetric, it follows that R is symmetric. It is also easy to see that R is not antisymmetric.

$M_R = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$	$M_S = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$
$M_{S \circ R} = M_R \odot M_S =$	
$\begin{bmatrix} (1 \wedge 0) \vee (0 \wedge 0) \vee (1 \wedge 1) & (1 \wedge 1) \vee (0 \wedge 0) \vee (1 \wedge 0) & (1 \wedge 0) \vee (0 \wedge 1) \vee (1 \wedge 1) \\ (1 \wedge 0) \vee (1 \wedge 0) \vee (0 \wedge 1) & (1 \wedge 1) \vee (1 \wedge 0) \vee (0 \wedge 0) & (1 \wedge 0) \vee (1 \wedge 1) \vee (0 \wedge 1) \\ (0 \wedge 0) \vee (0 \wedge 0) \vee (0 \wedge 1) & (0 \wedge 1) \vee (0 \wedge 0) \vee (0 \wedge 0) & (0 \wedge 0) \vee (0 \wedge 1) \vee (0 \wedge 1) \end{bmatrix} =$	
$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$	

Suppose that the relations R_1 and R_2 on a set *A* are represented by the matrices

$$M_{R_1} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad M_{R_2} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

What are the matrices representing $R_1 \cup R_2$ and $R_1 \cap R_2$?

Solution: The matrices of these relations are

$$M_{R_1 \cup R_2} = M_{R_1} \vee M_{R_2} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$M_{R_1 \cap R_2} = M_{R_1} \wedge M_{R_2} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$