Q.4 Show that $\vec{\nabla} r^n = n r^{n-2} \vec{r}$; where $\vec{r}$ is a position vector.

A:- $\vec{\nabla} r^n = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2+y^2+z^2)^{n/2}$

$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$r = |\vec{r}| = (x^2+y^2+z^2)^{1/2}$

Now $\hat{i}\frac{\partial}{\partial x}(x^2+y^2+z^2)^{n/2}$

$= \hat{i}\,\frac{n}{2}(x^2+y^2+z^2)^{\frac{n}{2}-1} \cdot 2x$

$= \hat{i}\, n x (x^2+y^2+z^2)^{\frac{n}{2}-1}$

Similarly; $\hat{j}\frac{\partial}{\partial y}(x^2+y^2+z^2)^{n/2} = \hat{j}\, n y (x^2+y^2+z^2)^{\frac{n}{2}-1}$

$\hat{k}\frac{\partial}{\partial z}(x^2+y^2+z^2)^{n/2} = \hat{k}\, n z (x^2+y^2+z^2)^{\frac{n}{2}-1}$

$\therefore \vec{\nabla} r^n = n(x^2+y^2+z^2)^{\frac{n}{2}-1}(x\hat{i} + y\hat{j} + z\hat{k})$

$= n(x^2+y^2+z^2)^{\frac{n-2}{2}}\vec{r}$

$= \cancel{n(\sqrt{x}} = n r^{n-2}\vec{r}$ Proved

Q. Find the directional derivative of $\varphi = 4xz^3 - 3x^2y^2z$ at $(2,-1,2)$ in the direction $2\hat{i} - 3\hat{j} + 6\hat{k}$.

A:- $\vec{\nabla}\varphi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(4xz^3 - 3x^2y^2z)$

$= (4z^3 - 6xy^2z)\hat{i} + (-6x^2yz)\hat{j} + (12xz^2 - 3x^2y^2)\hat{k}$

At the point $(2,-1,2)$;

$\vec{\nabla}\varphi = (32-24)\hat{i} + 48\hat{j} + (96-12)\hat{k}$

$= \cancel{8\hat{i} + 48\hat{j} + 48\hat{k}} \cdot 8\hat{i} + 48\hat{j} + 84\hat{k}$.

Now unit vector in the direction of $2\hat{i}-3\hat{j}+6\hat{k}$ is

$$\hat{a} = \frac{2\hat{i}-3\hat{j}+6\hat{k}}{\sqrt{2^2+(-3)^2+6^2}} = \frac{1}{7}(2\hat{i}-3\hat{j}+6\hat{k})$$

Then the required directional derivative is

$$\vec{\nabla}\phi \cdot \hat{a} = (8\hat{i}+48\hat{j}+84\hat{k}) \cdot \frac{1}{7}(2\hat{i}-3\hat{j}+6\hat{k})$$

$$= \frac{1}{7}(16-144+504) = \frac{376}{7} \quad \text{Ans:}$$

Q.5. Show that $\vec{\nabla}\phi$ is a vector perpendicular to the surface $\phi(x,y,z)=c$, where c is a constant.

A:- Let $\vec{r} = x\hat{i}+y\hat{j}+z\hat{k}$ be a position vector to any point $P(x,y,z)$ on the surface.

$$\phi(x,y,z) = c$$
$$d\phi = 0.$$

$$\boxed{\begin{aligned}\vec{r} &= x\hat{i}+y\hat{j}+z\hat{k}\\ d\vec{r} &= dx\hat{i}+dy\hat{j}+dz\hat{k}\end{aligned}}$$

But $\quad d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz = 0$

or $\left(\hat{i}\frac{\partial\phi}{\partial x}+\hat{j}\frac{\partial\phi}{\partial y}+\hat{k}\frac{\partial\phi}{\partial z}\right)\cdot(dx\hat{i}+dy\hat{j}+dz\hat{k}) = 0$

or, $\vec{\nabla}\phi \cdot \vec{dr} = 0$

That is $\vec{\nabla}\phi$ is perpendicular to $\vec{dr}$, and therefore $\vec{\nabla}\phi$ is perpendicular to the surface.

Divergence:

Q.18 / P-65. Prove that $\vec{\nabla}\cdot(\varphi\vec{A}) = (\vec{\nabla}\varphi)\cdot\vec{A} + \varphi(\vec{\nabla}\cdot\vec{A})$

Proof:- $\vec{\nabla}\cdot(\varphi\vec{A}) = \vec{\nabla}\cdot(\varphi A_1\hat{i} + \varphi A_2\hat{j} + \varphi A_3\hat{k})$

$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot(\varphi A_1\hat{i} + \varphi A_2\hat{j} + \varphi A_3\hat{k})$

$= \frac{\partial}{\partial x}(\varphi A_1) + \frac{\partial}{\partial y}(\varphi A_2) + \frac{\partial}{\partial z}(\varphi A_3)$

$\vec{\nabla}\cdot(\varphi\vec{A}) = \frac{\partial\varphi}{\partial x}A_1 + \varphi\frac{\partial A_1}{\partial x} + \frac{\partial\varphi}{\partial y}A_2 + \varphi\frac{\partial A_2}{\partial y} + \frac{\partial\varphi}{\partial z}A_3 + \varphi\frac{\partial A_3}{\partial z}$.

$= \left(\frac{\partial\varphi}{\partial x}A_1 + \frac{\partial\varphi}{\partial y}A_2 + \frac{\partial\varphi}{\partial z}A_3\right) + \varphi\left(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}\right)$

$= \left(\hat{i}\frac{\partial\varphi}{\partial x} + \hat{j}\frac{\partial\varphi}{\partial y} + \hat{k}\frac{\partial\varphi}{\partial z}\right)\cdot(A_1\hat{i} + A_2\hat{j} + A_3\hat{k})$

$+ \varphi\left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot(A_1\hat{i} + A_2\hat{j} + A_3\hat{k})$

$= (\vec{\nabla}\varphi)\cdot\vec{A} + \varphi(\vec{\nabla}\cdot\vec{A})$ Proved.

Q.70 / P-79. If $\vec{A} = 3xyz^2\hat{i} + 2xy^3\hat{j} - x^2yz\hat{k}$, and $\varphi = 3x^2 - yz$, find (i) $\vec{A}\cdot(\vec{\nabla}\varphi)$, (ii) $\vec{\nabla}\cdot(\vec{\nabla}\varphi)$ at the point $(1,-1,1)$.

A:- (i) $\vec{\nabla}\varphi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(3x^2 - yz)$

$= 6x\hat{i} - z\hat{j} - y\hat{k}$

$\therefore \vec{A}\cdot(\vec{\nabla}\varphi)$

$= (3xyz^2\hat{i} + 2xy^3\hat{j} - x^2yz\hat{k})\cdot(6x\hat{i} - z\hat{j} - y\hat{k})$

$= 18x^2yz^2 - 2xy^3z + x^2y^2z$.

at the point $(1,-1,1)$; $\vec{A}\cdot(\vec{\nabla}\varphi) = -18 + 2 + 1 = -15$ Ans;

(ii) $\vec{\nabla}\cdot(\vec{\nabla}\varphi)$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\left(6x\hat{i} - z\hat{j} - y\hat{k}\right)$$

$$= \frac{\partial}{\partial x}(6x) - \frac{\partial}{\partial y}(z) - \frac{\partial}{\partial z}(y)$$

$$= 6 - 0 - 0 = 6 \quad \text{Ans:}$$

Q.73 Prove that $\nabla^2(\ln r) = \frac{1}{r^2}$; where $\vec{r}$ is a position vector.

A:- $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$$r = |\vec{r}| = (x^2+y^2+z^2)^{\frac{1}{2}}$$

$$\therefore \ln r = \ln(x^2+y^2+z^2)^{\frac{1}{2}} = \frac{1}{2}\ln(x^2+y^2+z^2)$$

$$\therefore \nabla^2(\ln r) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)(\ln r) \quad \text{--- (i)}$$

Now $\frac{\partial^2}{\partial x^2}(\ln r) = \frac{\partial^2}{\partial x^2}\left\{\frac{1}{2}\ln(x^2+y^2+z^2)\right\}$

$$= \frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}\left\{\frac{1}{2}\ln(x^2+y^2+z^2)\right\}\right]$$

$$= \frac{\partial}{\partial x}\left[\frac{1}{\cancel{2}}\cdot\frac{1}{x^2+y^2+z^2}\cdot\cancel{2}x\right]$$

$$= \frac{(x^2+y^2+z^2)\cdot 1 - x\cdot 2x}{(x^2+y^2+z^2)^2}$$

$$= \frac{y^2+z^2-x^2}{(x^2+y^2+z^2)^2}$$

Similarly $\frac{\partial^2}{\partial y^2}\ln(r) = \frac{x^2+z^2-y^2}{(x^2+y^2+z^2)^2}$

$$\frac{\partial^2}{\partial z^2}(\ln r) = \frac{x^2+y^2-z^2}{(x^2+y^2+z^2)^2}$$

$\therefore$ From (i);

$$\nabla^2 r^n = n(n-2)(x^2+y^2+z^2)(x^2+y^2+z^2)^{\frac{n}{2}-2} + 3n(x^2+y^2+z^2)^{\frac{n}{2}-1}.$$

$$= n(n-2)(x^2+y^2+z^2)^{\frac{n}{2}-1} + 3n(x^2+y^2+z^2)^{\frac{n}{2}-1}.$$

$$= \{n(n-2)+3n\}(x^2+y^2+z^2)^{\frac{n-2}{2}}$$

$$= (n^2+n)(x^2+y^2+z^2)^{\frac{n-2}{2}}$$

$$= n(n+1)\, r^{n-2}.$$ Proved