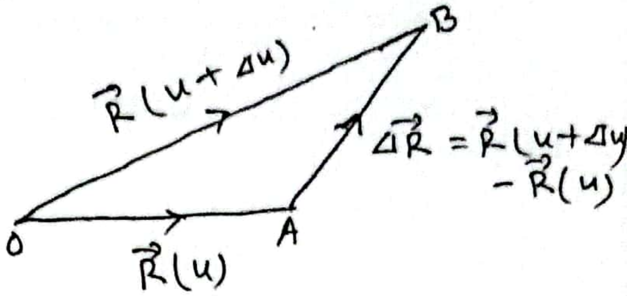
Vector Differentiation

Ordinary derivatives of vectors:

Let $\vec{R}(u)$ be a vector depending on a single scalar variable u, then in ΔOAB; $\Delta\vec{R} = \vec{R}(u+\Delta u) - \vec{R}(u)$



$$\frac{\Delta\vec{R}}{\Delta u} = \frac{\vec{R}(u+\Delta u) - \vec{R}(u)}{\Delta u}$$

$$\lim_{\Delta u \to 0} \frac{\Delta\vec{R}}{\Delta u} = \lim_{\Delta u \to 0} \frac{\vec{R}(u+\Delta u) - \vec{R}(u)}{\Delta u}$$

$$\frac{d\vec{R}}{du} = \lim_{\Delta u \to 0} \frac{\vec{R}(u+\Delta u) - \vec{R}(u)}{\Delta u}, \text{ if limit exists}$$

which is derivative of vector $\vec{R}$ w.r.to u.

Q: A particle moves along a curve whose parametric equations are $x = e^{-t}$, $y = 2\cos 3t$, $z = 2\sin 3t$, where t is the time.

(i) determine its velocity and accn at any time.

(ii) Find the magnitudes of velocity and accn at $t=0$.

A:- (i) Let $\vec{r}$ be the position vector of the particle,

then $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$.

$$= e^{-t}\hat{i} + 2\cos 3t\,\hat{j} + 2\sin 3t\,\hat{k}$$

Then velocity $\vec{v} = \frac{d\vec{r}}{dt} = -e^{-t}\hat{i} - 6\sin 3t\,\hat{j} + 6\cos 3t\,\hat{k}$.

& Acceleration $\vec{a} = \frac{d\vec{v}}{dt} = e^{-t}\hat{i} - 18\cos 3t\,\hat{j} - 18\sin 3t\,\hat{k}$.

(ii) At $t=0$; $\vec{v} = -\hat{i} - 0 + 6\hat{k} = -\hat{i} + 6\hat{k}$.

$\therefore |\vec{v}| = \sqrt{(-1)^2 + 6^2} = \sqrt{37}$.

At $t=0$; $\vec{a} = \hat{i} - 18\hat{j}$

$\therefore |\vec{a}| = \sqrt{1^2 + (-18)^2} = \sqrt{325}$

Ans:

Q. A particle moves along the curve $x = 2t^2$, $y = t^2 - 4t$, $z = 3t - 5$; where t is the time. Find the components of ~~a~~ its velocity and acc^n at $t = 1$ in the direction $\hat{i} - 3\hat{j} + 2\hat{k}$.

A:- $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$= 2t^2\hat{i} + (t^2 - 4t)\hat{j} + (3t - 5)\hat{k}$

velocity $\vec{v} = \dfrac{d\vec{r}}{dt} = 4t\hat{i} + (2t - 4)\hat{j} + 3\hat{k}$

at $t = 1$; $\vec{v} = 4\hat{i} - 2\hat{j} + 3\hat{k}$.

Now unit vector in the direction of $\hat{i} - 3\hat{j} + 2\hat{k}$ is

$$\hat{b} = \frac{\hat{i} - 3\hat{j} + 2\hat{k}}{\sqrt{1^2 + (-3)^2 + 2^2}} = \frac{1}{\sqrt{14}}(\hat{i} - 3\hat{j} + 2\hat{k})$$

Then the component of velocity $\vec{v}$ in the given direction

$$= \vec{v} \cdot \hat{b} = (4\hat{i} - 2\hat{j} + 3\hat{k}) \cdot \frac{1}{\sqrt{14}}(\hat{i} - 3\hat{j} + 2\hat{k})$$

$$= \frac{1}{\sqrt{14}}(4 + 6 + 6) = \frac{16}{\sqrt{14}} = \frac{8\sqrt{14}}{7}.$$

Ans:

Again, at $\vec{a} = \dfrac{d\vec{v}}{dt} = 4\hat{i} + 2\hat{j} + 0$

$\therefore$ component of $\vec{a}$ in the given direction is

$$= \vec{a} \cdot \hat{b} = (4\hat{i} + 2\hat{j}) \cdot \frac{1}{\sqrt{14}}(\hat{i} - 3\hat{j} + 2\hat{k})$$

$$= \frac{1}{\sqrt{14}}(4 - 6) = -\frac{2}{\sqrt{14}} = -\frac{\sqrt{14}}{7}$$

Ans:

Q.34 If $\vec{A} = t^2\hat{i} - t\hat{j} + (2t+1)\hat{k}$, $\vec{B} = (2t-3)\hat{i} + \hat{j} - t\hat{k}$.

Find (i) $\frac{d}{dt}(\vec{A} \times \vec{B})$ at $t=1$

(ii) $\frac{d}{dt}\left(\vec{A} \times \frac{d\vec{B}}{dt}\right)$ at $t=1$.

(i) $$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ t^2 & -t & 2t+1 \\ 2t-3 & 1 & -t \end{vmatrix}$$

$$= (t^2-2t-1)\hat{i} - \left[-t^3-(4t^2-4t-3)\right]\hat{j} + (t^2+2t^2-3t)\hat{k}$$

$$= (t^2-2t-1)\hat{i} + (t^3+4t^2-4t-3)\hat{j} + (3t^2-3t)\hat{k}$$

$$\therefore \frac{d}{dt}(\vec{A} \times \vec{B}) = (2t-2)\hat{i} + (3t^2+8t-4)\hat{j} + (6t-3)\hat{k}$$

At $t=1$; $$\frac{d}{dt}(\vec{A} \times \vec{B}) = 0 + (11-4)\hat{j} + (6-3)\hat{k}$$
$$= 7\hat{j} + 3\hat{k}$$ Ans.

Q.12. P-42. A particle moves so that its position vector is given by $\vec{r} = \cos\omega t\,\hat{i} + \sin\omega t\,\hat{j}$, where ω is constant. Show that

(i) the velocity $\vec{v}$ of the particle is perp. to $\vec{r}$.

(ii) the accn $\vec{a}$ is directed towards the origin and has magnitude proportional to the distance from the origin.

(i) $$\vec{r} = \cos\omega t\,\hat{i} + \sin\omega t\,\hat{j}$$
$$\vec{v} = \frac{d\vec{r}}{dt} = -\omega\sin\omega t\,\hat{i} + \omega\cos\omega t\,\hat{j}$$

Now $$\vec{r}\cdot\vec{v} = (\cos\omega t\,\hat{i} + \sin\omega t\,\hat{j})\cdot(-\omega\sin\omega t\,\hat{i} + \omega\cos\omega t\,\hat{j})$$
$$= -\omega\sin\omega t\cos\omega t + \omega\sin\omega t\cos\omega t$$
$$= 0$$ $\therefore \vec{v}$ is perp. to $\vec{r}$. Proved

(ii) Acceleration $\vec{a} = \frac{d\vec{v}}{dt}$

$$= \frac{d}{dt}(-\omega \sin\omega t\,\hat{i} + \omega \cos\omega t\,\hat{j})$$

$$= -\omega^2 \cos\omega t\,\hat{i} - \omega^2 \sin\omega t\,\hat{j}$$

$$= -\omega^2(\cos\omega t\,\hat{i} + \sin\omega t\,\hat{j})$$

$$= -\omega^2\vec{r}$$

$\therefore$ Acceleration is opposite to the direction of $\vec{r}$ i.e it is directed towards the origin and its magnitude is proportional to $\vec{r}$ which is the distance from the origin.

$$\left[\begin{array}{l} \vec{a} = -\omega^2\vec{r} \\ |\vec{a}| = |-\omega^2\vec{r}| \\ |\vec{a}| = |\omega^2\vec{r}| \\ |\vec{a}| \propto |\vec{r}| \end{array}\right.$$

Q44 / P-54. If $\vec{A} = x^2yz\,\hat{i} - 2xz^3\,\hat{j} + xz^2\,\hat{k}$; $\vec{B} = 2z\,\hat{i} + y\,\hat{j} - x^2\,\hat{k}$

find $\frac{\partial^2}{\partial x \partial y}(\vec{A}\times\vec{B})$ at the point $(1, 0, -2)$.

Ans :- $\vec{A} = x^2yz\,\hat{i} - 2xz^3\,\hat{j} + xz^2\,\hat{k}$; $\vec{B} = 2z\,\hat{i} + y\,\hat{j} - x^2\,\hat{k}$.

$$\therefore \vec{A}\times\vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x^2yz & -2xz^3 & xz^2 \\ 2z & y & -x^2 \end{vmatrix}$$

$$= (2x^3z^3 - xyz^2)\,\hat{i} - (-x^4yz - 2xz^3)\,\hat{j} + (x^2y^2z + 4xz^4)\,\hat{k}$$

$$= (2x^3z^3 - xyz^2)\,\hat{i} + (x^4yz + 2xz^3)\,\hat{j} + (x^2y^2z + 4xz^4)\,\hat{k}$$

Now $\frac{\partial^2}{\partial x \partial y}(\vec{A}\times\vec{B}) = \frac{\partial}{\partial x}\left[\frac{\partial}{\partial y}\left\{(2x^3z^3 - xyz^2)\,\hat{i} + (x^4yz + 2xz^3)\,\hat{j} + (x^2y^2z + 4xz^4)\,\hat{k}\right\}\right]$

$$= \frac{\partial}{\partial x}\left[-xz^2\,\hat{i} + x^4z\,\hat{j} + 2x^2yz\,\hat{k}\right)$$

$$= -z^2\,\hat{i} + 4x^3z\,\hat{j} + 4xyz\,\hat{k}.$$

$\therefore$ At the point $(1,0,-2)$; $\frac{\partial^2}{\partial x \partial y}(\vec{A}\times\vec{B}) = -4\hat{i}-8\hat{j}$. Ans.

Q.45 / P-54. If $\vec{e}_1$ and $\vec{e}_2$ are constant vectors and λ is a constant scalar, show that $\vec{H} = e^{-\lambda x}(\vec{e}_1 \sin\lambda y + \vec{e}_2 \cos\lambda y)$ satisfies the partial differential equation $\frac{\partial^2 \vec{H}}{\partial x^2} + \frac{\partial^2 \vec{H}}{\partial y^2} = 0$.

Ans:- Given $\vec{H} = e^{-\lambda x}(\vec{e}_1 \sin\lambda y + \vec{e}_2 \cos\lambda y)$, then

LHS
$$\frac{\partial^2 \vec{H}}{\partial x^2} + \frac{\partial^2 \vec{H}}{\partial y^2} = \frac{\partial}{\partial x}\left[-\lambda e^{-\lambda x}(\vec{e}_1 \sin\lambda y + \vec{e}_2 \cos\lambda y)\right]$$
$$+ \frac{\partial}{\partial y}\left[e^{-\lambda x}(\lambda \vec{e}_1 \cos\lambda y - \lambda \vec{e}_2 \sin\lambda y)\right]$$

$$= \lambda^2 e^{-\lambda x}(\vec{e}_1 \sin\lambda y + \vec{e}_2 \cos\lambda y)$$
$$+ e^{-\lambda x}(-\lambda^2 \vec{e}_1 \sin\lambda y - \lambda^2 \vec{e}_2 \cos\lambda y)$$

$$= \lambda^2 e^{-\lambda x}(\vec{e}_1 \sin\lambda y + \vec{e}_2 \cos\lambda y)$$
$$- \lambda^2 e^{-\lambda x}(\vec{e}_1 \sin\lambda y + \vec{e}_2 \cos\lambda y)$$

$$= \lambda^2 \vec{H} - \lambda^2 \vec{H}$$
$$= 0.$$ Proved.