$= \cancel{= 6xyz}$

$= (-6yz + x^4z^2)\hat{i} - (4x^2 + 3x^3yz^2)\hat{j} + (-2x^5 - 9x^2y^2z)\hat{k}$

At the point $(1,-1,1)$;

$$\vec{A} \times \vec{\nabla}\phi = (6+1)\hat{i} - (4-3)\hat{j} + (-2-9)\hat{k}$$
$$= 7\hat{i} - \hat{j} - 11\hat{k} \quad \text{Ans:}$$

Q-45: Find $\vec{\nabla}|\vec{r}|^3$.

A: $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$|\vec{r}| = \sqrt{x^2+y^2+z^2} \quad \therefore |\vec{r}|^3 = (x^2+y^2+z^2)^{3/2}$.

Now $\vec{\nabla}|\vec{r}|^3 = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2+y^2+z^2)^{3/2}$ ——— (i)

$$\hat{i}\frac{\partial}{\partial x}(x^2+y^2+z^2)^{3/2} = \hat{i}\cdot\frac{3}{\cancel{2}}(x^2+y^2+z^2)^{\frac{1}{2}}\cdot\cancel{2}x$$
$$= \hat{i}\,3x(x^2+y^2+z^2)^{1/2}$$

Similarly $\hat{j}\frac{\partial}{\partial y}(x^2+y^2+z^2)^{3/2} = \hat{j}\,3y(x^2+y^2+z^2)^{1/2}$

& $\hat{k}\frac{\partial}{\partial z}(x^2+y^2+z^2)^{3/2} = \hat{k}\,3z(x^2+y^2+z^2)^{1/2}$.

From (i);

$$\therefore \vec{\nabla}|\vec{r}|^3 = 3(x^2+y^2+z^2)^{1/2}(x\hat{i} + y\hat{j} + z\hat{k})$$
$$= (3r)\vec{r} \quad \text{Ans:}$$

[illegible] : [illegible] [illegible].

gradient

Q.42 / F78 If $\phi = 2xz^4 - x^2y$, find $\vec{\nabla}\phi$ and $|\vec{\nabla}\phi|$ at the point $(2,-2,-1)$

A:- $\vec{\nabla}\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(2xz^4 - x^2y)$

$= \hat{i}(2z^4 - 2xy) + \hat{j}(-x^2) + \hat{k}(8xz^3)$

At the point $(2,-2,-1)$;

$\vec{\nabla}\phi = \hat{i}(2\cdot1 - 2\cdot2\cdot-2) + \hat{j}(-2)^2 + \hat{k}(8\cdot2\cdot-1)$

$= 10\hat{i} + 4\hat{j} - 16\hat{k}$ Ans;

$|\vec{\nabla}\phi| = \sqrt{10^2 + 4^2 + (-16)^2} = \sqrt{372} = \sqrt{4}\sqrt{93} = 2\sqrt{93}$ Ans;

Q.43: If $\vec{A} = 2x^2\hat{i} - 3yz\hat{j} + xz^2\hat{k}$ and $\phi = 2z - x^3y$,
find $\vec{A}\cdot\vec{\nabla}\phi$ and $\vec{A}\times\vec{\nabla}\phi$ at the point $(1,-1,1)$.

A:- $\vec{\nabla}\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(2z - x^3y)$

$= -3x^2y\hat{i} - x^3\hat{j} + 2\hat{k}$.

$\therefore \vec{A}\cdot\vec{\nabla}\phi = (2x^2\hat{i} - 3yz\hat{j} + xz^2\hat{k})\cdot(-3x^2y\hat{i} - x^3\hat{j} + 2\hat{k})$

$= -6x^4y + 3x^3yz + 2xz^2$

At the point $(1,-1,1)$; $\vec{A}\cdot\vec{\nabla}\phi = -6\cdot1\cdot-1 + 3\cdot1\cdot-1\cdot1 + 2\cdot1\cdot1$

$= 6 - 3 + 2 = 5$ Ans;

Now $\vec{A}\times\vec{\nabla}\phi = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cancel{\frac{\partial}{\partial x}\,2x^2} & \cancel{\frac{\partial}{\partial y}\,-3yz} & \cancel{\frac{\partial}{\partial z}} \\ 2x^2 & -3yz & xz^2 \\ -3x^2y & -x^3 & +2 \end{vmatrix}$

152. If $\vec{\nabla}\psi = (y^2 - 2xyz^3)\hat{i} + (3 + 2xy - x^2z^3)\hat{j} + (6z^3 - 3x^2yz^2)\hat{k}$; find ψ.

Ans:- $\vec{\nabla}\psi = (y^2 - 2xyz^3)\hat{i} + (3+2xy - x^2z^3)\hat{j} + (6z^3 - 3x^2yz^2)\hat{k}$

or, $\left(\hat{i}\frac{\partial\psi}{\partial x} + \hat{j}\frac{\partial\psi}{\partial y} + \hat{k}\frac{\partial\psi}{\partial z}\right) = (y^2 - 2xyz^3)\hat{i} + (3+2xy - x^2z^3)\hat{j} + (6z^3 - 3x^2yz^2)\hat{k}$.

$\therefore \frac{\partial\psi}{\partial x} = y^2 - 2xyz^3$; $\frac{\partial\psi}{\partial y} = 3 + 2xy - x^2z^3$; $\frac{\partial\psi}{\partial z} = 6z^3 - 3x^2y$

But $d\psi = \frac{\partial\psi}{\partial x}dx + \frac{\partial\psi}{\partial y}dy + \frac{\partial\psi}{\partial z}dz$

$$= (y^2 - 2xyz^3)dx + (3 + 2xy - x^2z^3)dy + (6z^3 - 3x^2yz^2)dz$$

Integrating;

$$\int d\psi = \int (y^2 - 2xyz^3)dx + \int (3 + 2xy - x^2z^3)dy + \int (6z^3 - 3x^2yz^2)dz$$

$$\psi = y^2x - \cancel{2}\cdot\frac{x^2}{\cancel{2}}yz^3 + 3y + \cancel{2}x\cdot\frac{y^2}{\cancel{2}} - x^2z^3y + \overset{3}{\cancel{6}}\cdot\frac{z^4}{\underset{2}{\cancel{4}}} - \cancel{3}x^2y\cdot\frac{z^3}{\cancel{3}} + \text{const}$$

$$\psi = xy^2 - x^2yz^3 + 3y + xy^2 - x^2yz^3 + \frac{3}{2}z^4 - x^2yz^3 + \text{const}$$

$$= 2xy^2 - 3x^2yz^3 + 3y + \frac{3}{2}z^4 + \text{constant. Ans.}$$

Gradient, Divergence & curl.

Vector Differential operator (Del): Denoted by $\vec{\nabla}$, where
$$\vec{\nabla} = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}.$$
It is an operator also known as Nabla which operates on three physical quantities Gradient, Divergence & curl.

Gradient: If $\varphi(x,y,z)$ is a defined and differentiable function in a scalar field, then gradient of φ is
$$\vec{\nabla}\varphi = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\varphi = \frac{\partial\varphi}{\partial x}\hat{i} + \frac{\partial\varphi}{\partial y}\hat{j} + \frac{\partial\varphi}{\partial z}\hat{k}.$$
which is a vector.

Note: The component of $\vec{\nabla}\varphi$ in the direction of a unit vector $\hat{a}$ is given by $\vec{\nabla}\varphi\cdot\hat{a}$ which is known as directional derivative in the direction of $\hat{a}$ (rate of change of φ in the direction of $\hat{a}$).

Divergence: If $\vec{v}$ defines a differentiable vector field in space, that is $\vec{v} = v_1\hat{i} + v_2\hat{j} + v_3\hat{k}$ is defined and differentiable at each point (x,y,z) in this space, then divergence of $\vec{v}$ is $\vec{\nabla}\cdot\vec{v} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\cdot\left(v_1\hat{i} + v_2\hat{j} + v_3\hat{k}\right)$
$$= \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}.$$
Note: If $\vec{\nabla}\cdot\vec{v} = 0$, then $\vec{v}$ is solenoidal.

curl (rotation): If $\vec{v}(x,y,z) = v_1\hat{i} + v_2\hat{j} + v_3\hat{k}$ is defined and differentiable at each point (x,y,z) in a certain space, then curl of $\vec{v}$ is
$$\vec{\nabla}\times\vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix}$$

Note: If $\vec{\nabla}\times\vec{v} \neq 0$, then $\vec{v}$ is rotational.
$\vec{\nabla}\times\vec{v} = 0$, then $\vec{v}$ is irrotational.
In case of force $\vec{F}$, if $\vec{\nabla}\times\vec{F} = 0$, then $\vec{F}$ is a conservative force field.

$\therefore$ At the point $(1,0,-2)$; $\frac{\partial^2}{\partial x \partial y}(\vec{A}\times\vec{B}) = -4\hat{i} - 8\hat{j}$. Ans.

Q.45/P-54. If $\vec{e}_1$ and $\vec{e}_2$ are constant vectors and λ is a constant scalar, show that $\vec{H} = e^{-\lambda x}(\vec{e}_1 \sin\lambda y + \vec{e}_2 \cos\lambda y)$ satisfies the partial differential equation $\frac{\partial^2 \vec{H}}{\partial x^2} + \frac{\partial^2 \vec{H}}{\partial y^2} = 0$.

Ans:- Given $\vec{H} = e^{-\lambda x}(\vec{e}_1 \sin\lambda y + \vec{e}_2 \cos\lambda y)$, then

LHS $$\frac{\partial^2 \vec{H}}{\partial x^2} + \frac{\partial^2 \vec{H}}{\partial y^2} = \frac{\partial}{\partial x}\left[-\lambda e^{-\lambda x}(\vec{e}_1 \sin\lambda y + \vec{e}_2 \cos\lambda y)\right]$$
$$+ \frac{\partial}{\partial y}\left[e^{-\lambda x}(\lambda \vec{e}_1 \cos\lambda y - \lambda \vec{e}_2 \sin\lambda y)\right]$$

$$= \lambda^2 e^{-\lambda x}(\vec{e}_1 \sin\lambda y + \vec{e}_2 \cos\lambda y)$$
$$+ e^{-\lambda x}(-\lambda^2 \vec{e}_1 \sin\lambda y - \lambda^2 \vec{e}_2 \cos\lambda y)$$

$$= \lambda^2 e^{-\lambda x}(\vec{e}_1 \sin\lambda y + \vec{e}_2 \cos\lambda y)$$
$$- \lambda^2 e^{-\lambda x}(\vec{e}_1 \sin\lambda y + \vec{e}_2 \cos\lambda y)$$

$$= \lambda^2 \vec{H} - \lambda^2 \vec{H}$$
$$= 0.$$ Proved