# Propositional Logic

Mathematics is assumed to be an exact science. Every statement in Mathematics must be precise. Also there can't be Mathematics without proofs and each proof needs proper reasoning. Proper reasoning involves logic. The dictionary meaning of 'Logic' is the science of reasoning. The rules of logic give precise meaning to mathematical statements. These rules are used to distinguish between valid & invalid mathematical arguments.
In addition to its importance in mathematical reasoning, logic has numerous applications in computer science to verify the correctness of programs & to prove the theorems in natural & physical sciences to draw conclusion from experiments, in social sciences & in our daily lives to solve a multitude of problems.
The area of logic that deals with propositions is called the propositional calculus or propositional logic. The mathematical approach to logic was first discussed by British mathematician George Boole; hence the mathematical logic is also called as Boolean logic.

**PROPOSITION (OR STATEMENT)**
A **proposition (**or a statement) is a declarative sentence (that is, a sentence that declares a fact) that is either true or false, but not both.
Mathematical identities are considered to be statements. Statements which are imperative, exclamatory, interrogative or open are not statements in logic.

**EXAMPLE:**
All the following declarative sentences are propositions.

| For Example consider the following sentences. | Comment |
|---|---|
| (i) MU is at Sylhet.<br>(ii) 2 + 3 = 5<br>(iii) The Sun rises in the east.<br>**(iv) Do your homework.**<br>**(v) What are you doing?**<br>(vi) 2 + 4 = 8<br>(vii) 5 < 4<br>(viii) The square of 5 is 15.<br>(ix) x +3=2<br>(x) May God Bless you! | All of them are propositions except (iv), (v),(ix) & (x) sentences ( i), (ii) are true, whereas (iii),(iv), (vii) & (viii) are false.<br>(iv) is command, hence not a proposition.<br>(v) is a question so not a statement.<br>(ix) is a declarative sentence but not a statement, since it is true or false depending on the value of x.<br>(x) is a exclamatory sentence and so it is not a statement. |
| 1. Washington, D.C., is the capital of the United States of America.<br>2. Sylhet is the capital of England.<br>3. 1 + 1 = 2.<br>4. 2 + 2 = 3. | Propositions 1 and 3 are true,<br>whereas 2 and 4 are false. |

**Compound statements:**
Many propositions are composites that are, composed of sub propositions and various connectives discussed subsequently. Such composite propositions are called compound propositions.
A proposition is said to be primitive if it cannot be broken down into simpler propositions, that is, if it is not composite.
**Example :**
Consider, for example following sentences.
a. “The sun is shining today and it is colder than yesterday”
b. “Sita is intelligent and she studies every night.”

**Logical Operations Or Logical Connectives:**
The phrases or words which combine simple statements are called logical connectives. There are five types of connectives. Namely, ‘not’, ‘and’, ‘or’, ‘if…then’, iff etc. The first one is a unitary operator whereas the other four are binary operators.
In the following table we list some possible connectives, their symbols & the nature of the compound statement formed by them.

| **Compound Proposition** | **Expression in English** | **Symbol** | **Compound statement** |
|---|---|---|---|
| **$\neg p$** | “It is not the case that p” | $\neg$ | Negation |
| **$p \wedge q$** | “Both p and q” | $\wedge$ | Conjunction |
| **$p \vee q$** | “p or q (or both)” | $\vee$ | Disjunction |
| **$p \rightarrow q$** | “if p then q” “p implies q” | $\rightarrow$ | Conditional or implication |
| **$p \leftrightarrow q$** | “p if and only if q” | $\leftrightarrow$ | Bi-conditional |
| **$p \oplus q$** | “p or q (but not both)” | $\oplus$ | Exclusive Or |

**Conjunction (AND):**
If two statements are combined by the word “and” to form a compound proposition (statement) then the resulting proposition is called the conjunction of two propositions.
Symbolically, if P & Q are two simple statements, then ‘$P \wedge Q$’ denotes the conjunction of P and Q and is read as ‘P and Q.
Since, $P \wedge Q$ is a proposition it has a truth value and this truth value depends only on the truth values of P and Q.
Specifically, if P & Q are true then $P \wedge Q$ is true; otherwise $P \wedge Q$ is false.

**Example:**
Let P: In this year monsoon is very good.
Q: The Rivers are flooded.
Then, $P \wedge Q$: In this year monsoon is very good and the rivers are flooded.

**Example:**
Find the conjunction of the propositions p and q where p is the proposition “Rebecca’s PC has more than 16 GB free hard disk space” and q is the proposition “The processor in Rebecca’s

PC runs faster than 1 GHz."

**Solution:** The conjunction of these propositions, $p \wedge q$, is the proposition "Rebecca's PC has more than 16 GB free hard disk space, and the processor in Rebecca's PC runs faster than 1 GHz." This conjunction can be expressed more simply as "Rebecca's PC has more than 16 GB free hard disk space, and its processor runs faster than 1 GHz." For this conjunction to be true, both conditions given must be true. It is false, when one or both of these conditions are false.

.

**The Truth Table for the Disjunction and Conjunction of Two Propositions.**

| P | Q | $P \wedge Q$ |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | F |
| F | F | F |

| P | Q | $P \vee Q$ |
|---|---|---|
| T | T | T |
| T | F | T |
| F | T | T |
| F | F | F |

**Disjunction (OR):**

Any two statements can be connected by the word 'or' to form a compound statement called disjunction.

Symbolically, if P and Q are two simple statements, then $P \vee Q$ denotes the disjunction of P and Q and read as ' P or Q '.

The truth value of $P \vee Q$ depends only on the truth values of P and Q.

Specifically if P and Q are false then $P \vee Q$ is false, otherwise $P \vee Q$ is true.

The truth table for disjunction is as follows.

**Example :**

P: Paris is in France

Q: $2 + 3 = 6$

Then $P \vee Q$: Paris is in France or $2 + 3 = 6$.

Here, $P \vee Q$

is true since P is true & Q is False.

Thus, the disjunction $P \vee Q$ is false only when P and Q are both false.

**Exclusive Or**

Let p and q be propositions. The exclusive or of p and q, denoted by $p \oplus q$, is the proposition that is true when exactly one of p and q is true and is false otherwise.

**The Truth Table for the Exclusive Or of Two Propositions.**

| P | Q | $P \oplus Q$ |
|---|---|---|
| T | T | F |
| T | F | T |
| F | T | T |
| F | F | F |

| P | $\neg Q$ |
|---|---|
| T | F |
| F | T |

**Negation (NOT)**

Given any proposition P, another proposition, called negation of P, can be formed by modifying it by "not". Also by using the phrase "It is not the case that or" "It is false that" before P we will able to find the negation.

Symbolically, $\neg$ P Read as "not P" denotes the negation of P. the truth value of $\neg$ P depends on the truth value of P

If P is true then $\neg$ P is false and if P is false then $\neg$ P is true. The truth table for Negation is as follows:

**EXAMPLE**

1. Let P: 3 is a factor of 12. Then Q = $\neg$ P: 3 is not a factor of 12.
   Here P is true & $\neg$ P is false.
2. Find the negation of the proposition "Vandana's smartphone has at least 32GB of memory" and express this in simple English.
   **Solution:** The negation is "It is not the case that Vandana's smartphone has at least 32GB of memory." This negation can also be expressed as "Vandana's smartphone does not have at least 32GB of memory" or even more simply as "Vandana's smartphone has less than 32GB of memory."
3. Find the negation of the proposition "Michael's PC runs Linux" and express this in simple English.
   **Solution:** The negation is "It is not the case that Michael's PC runs Linux."
   This negation can be more simply expressed as "Michael's PC does not run Linux."

**Conditional Statements**

Let p and q be propositions. The conditional statement p → q is the proposition "if p, then q." The conditional statement p → q is false when p is true and q is false, and true otherwise.

In the conditional statement p → q, p is called the hypothesis (or antecedent or premise) and q is called the conclusion (or consequence).

The statement p → q is called a conditional statement because p → q asserts that q is true on the condition that p holds. A conditional statement is also called an **implication**.

The truth table for the conditional statement p → q is shown in below. Note that the statement p → q is true when both p and q are true and when p is false (no matter what truth value q has).

| P | Q | P → Q |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | T |
| F | F | T |

**EXAMPLE**

Let p be the statement “Maria learns discrete mathematics” and
q the statement “Maria will find a good job.”
Express the statement $p \rightarrow q$ as a statement in English.

**Solution:**

From the definition of conditional statements, we see that when p is the statement “Maria learns discrete mathematics” and q is the statement “Maria will find a good job,” $p \rightarrow q$ represents the statement
“If Maria learns discrete mathematics, then she will find a good job.”
There are many other ways to express this conditional statement in English. Among the most Natural of these are:
“Maria will find a good job when she learns discrete mathematics.”
“For Maria to get a good job, it is sufficient for her to learn discrete mathematics.”
and “Maria will find a good job unless she does not learn discrete mathematics.”

**EXAMPLE**

| | | |
|---|---|---|
| Given: | p: $7^2 = 49$. | true |
| | q: A rectangle does not have 4 sides. | false |
| | r: Harrison Ford is an American actor. | true |
| | s: A square is not a quadrilateral. | false |
| **Problem:** | Write each conditional below as a sentence. Then indicate its truth value. | |

| | | | |
|---|---|---|---|
| 1. | $p \rightarrow q$ | If $7^2$ is equal to 49, then a rectangle does not have 4 sides. | false |
| 2. | $q \rightarrow r$ | If a rectangle does not have 4 sides, then Harrison Ford is an American actor. | true |
| 3. | $p \rightarrow r$ | If $7^2$ is equal to 49, then Harrison Ford is an American actor. | true |
| 4. | $q \rightarrow s$ | If a rectangle does not have 4 sides, then a square is not a quadrilateral. | true |
| 5. | $r \rightarrow \sim p$ | If Harrison Ford is an American actor, then $7^2$ is not equal to 49. | false |
| 6. | $\sim r \rightarrow p$ | If Harrison Ford is not an American actor, then $7^2$ is equal to 49. | true |

| | | |
|---|---|---|
| Given: | r: 8 is an odd number. | false |
| | s: 9 is composite. | true |
| **Problem:** | What is the truth value of $r \rightarrow s$? | |
| **Solution:** | Since hypothesis r is false and conclusion s is true, the conditional $r \rightarrow s$ is true. | |

| | | |
|---|---|---|
| Given: | r: 8 is an odd number. | false |

| | | |
|---|---|---|
| | s: 9 is composite. | true |
| **Problem:** | What is the truth value of $s \rightarrow r$? | |
| **Solution:** | Since hypothesis s is true and conclusion r is false, the conditional $s \rightarrow r$ is false. | |

**CONVERSE, CONTRAPOSITIVE, AND INVERSE**

We can form some new conditional statements starting with a conditional statement $p \rightarrow q$. In particular, there are three related conditional statements that occur so often that they have special names.

The proposition $q \rightarrow p$ is called the **converse** of $p \rightarrow q$.

The **contrapositive** of $p \rightarrow q$ is the proposition $\neg q \rightarrow \neg p$.

The proposition $\neg p \rightarrow \neg q$ is called the **inverse** of $p \rightarrow q$.

We will see that of these three conditional statements formed from $p \rightarrow q$, only the contrapositive always has the same truth value as $p \rightarrow q$.

We first show that the contrapositive, $\neg q \rightarrow \neg p$, of a conditional statement $p \rightarrow q$ always has the same truth value as $p \rightarrow q$. To see this, note that the contrapositive is false only When $\neg p$ is false and $\neg q$ is true, that is, only when p is true and q is false. We now show that neither the converse, $q \rightarrow p$, nor the inverse, $\neg p \rightarrow \neg q$, has the same truth value as $p \rightarrow q$ for all possible truth values of p and q. Note that when p is true and q is false, the original conditional statement is false, but the converse and the inverse are both true.

**EXAMPLE**

What are the contrapositive, the converse, and the inverse of the conditional statement? "The home team wins whenever it is raining?"

**Solution:** Because "q whenever p" is one of the ways to express the conditional statement $p \rightarrow q$, the original statement can be rewritten as
"If it is raining, then the home team wins."
Consequently, the contrapositive of this conditional statement is
"If the home team does not win, then it is not raining."
The converse is
"If the home team wins, then it is raining."
The inverse is
"If it is not raining, then the home team does not win."

Only the contrapositive is equivalent to the original statement.

**Example**: Let P: You are good in Mathematics. Q: You are good in Logic.
Then, $P \rightarrow Q$: If you are good in Mathematics then you are good in Logic.

1) Converse: $Q \rightarrow P$
If you are good in Logic then you are good in Mathematics.

2) Contra positive: $\neg Q \rightarrow \neg P$

If you are not good in Logic then you are not good in Mathematics.

3) Inverse: $\neg p \rightarrow \neg q$

If you are not good in Mathematics then you are not good in Logic.

**BICONDITIONALS**

Let p and q be propositions. The biconditional statement $p \leftrightarrow q$ is the proposition "p if and only if q." The biconditional statement $p \leftrightarrow q$ is true when p and q have the same truth values, and is false otherwise. Biconditional statements are also called bi-implications.
The truth table for $p \leftrightarrow q$ is shown in Table 6. Note that the statement $p \leftrightarrow q$ is true when both the conditional statements $p \rightarrow q$ and $q \rightarrow p$ are true and is false otherwise. That is why we use the words "if and only if" to express this logical connective and why it is symbolically written by combining the symbols→and←. There are some other common ways to express $p \leftrightarrow q$:
"p is necessary and sufficient for q"
"if p then q, and conversely"
"p iff q."
The last way of expressing the biconditional statement $p \leftrightarrow q$ uses the abbreviation "iff" for "if and only if." Note that $p \leftrightarrow q$ has exactly the same truth value as $(p \rightarrow q) \wedge (q \rightarrow p)$.

| P | Q | $P \leftrightarrow Q$ |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | F |
| F | F | T |

**EXAMPLE 10** Let p be the statement "You can take the flight," and let q be the statement "You buy a ticket."
Then $p \leftrightarrow q$ is the statement
"You can take the flight if and only if you buy a ticket."
This statement is true if p and q are either both true or both false, that is, if you buy a ticket and can take the flight or if you do not buy a ticket and you cannot take the flight. It is false when p and q have opposite truth values, that is, when you do not buy a ticket, but you can take the flight (such as when you get a free trip) and when you buy a ticket but you cannot take the flight (such as when the airline bumps you).

**Precedence of Logical Operators:**

| Operator | Precedence |
|---|---|
| $\neg$ | 1 |
| $\wedge$ | 2 |
| $\vee$ | 3 |
| $\rightarrow$ | 4 |
| $\leftrightarrow$ | 5 |

**EXAMPLE**

Construct the truth table of the compound proposition $(p \vee \neg q) \rightarrow (p \wedge q)$

**Solution**: Because this truth table involves two propositional variables p and q, there are four rows in this truth table, one for each of the pairs of truth values TT, TF, FT, and FF. The first two columns are used for the truth values of p and q, respectively. In the third column we find the truth value of $\neg q$, needed to find the truth value of $p \vee \neg q$, found in the fourth column.

The fifth column gives the truth value of $p \wedge q$. Finally, the truth value of $(p \vee \neg q) \rightarrow (p \wedge q)$ is found in the last column. The resulting truth table is shown in Table

| p | q | $p \wedge q$ | $\neg q$ | $(p \vee \neg q)$ | $(p \vee \neg q) \rightarrow (p \wedge q)$. |
|---|---|---|---|---|---|
| T | T | T | F | T | T |
| T | F | F | F | T | F |
| F | T | F | T | F | T |
| F | F | F | T | T | F |

**LOGICAL EQUIVALANCE:**
Compound propositions that have the same truth values in all possible cases are called logically equivalent.
**Definition:** The compound propositions P and Q are said to be logically equivalent if $P \leftrightarrow Q$ is a tautology. The notation $P \equiv Q$ denotes that P and Q are logically equivalent.

**Example 8:**
If P: “This book is good.” Q: “This book is costly.”
Write the following statements in symbolic form. a) This book is good & costly.
b) This book is not good but costly. c) This book is cheap but good.
d) This book is neither good nor costly. e) If this book is good then it is costly.
**Answers:**
a) $(p \vee q)$
b) $\neg p \vee q$
c) $\neg Q \vee P$
d) $\neg P \vee \neg Q$
e) $P \rightarrow Q$

Let p and q be the propositions
p :You drive over 65 miles per hour.
q :You get a speeding ticket.
Write these propositions using p and q and logical connectives (including negations).
**a)** You do not drive over 65 miles per hour.
**b)** You drive over 65 miles per hour, but you do not get a speeding ticket.
**c)** You will get a speeding ticket if you drive over 65 miles per hour.
**d)** If you do not drive over 65 miles per hour, then you will not get a speeding ticket.
**e)** Driving over 65 miles per hour is sufficient for getting
a speeding ticket.
**f )** You get a speeding ticket, but you do not drive over 65 miles per hour.
**g)** Whenever you get a speeding ticket, you are driving over 65 miles per hour.

**Example:**
Let p, q, and r be the propositions
p :You have the flu.
q :You miss the final examination.
r :You pass the course.
Express each of these propositions as an English sentence.
**a)** $p \rightarrow q$ **b)** $\neg q \leftrightarrow r$
**c)** $q \rightarrow \neg r$ **d)** $p \vee q \vee r$
**e)** $(p \rightarrow \neg r) \vee (q \rightarrow \neg r)$

**Tautology:** A tautology or universally true formula is a well formed formula, whose truth value is T for all possible assignments of truth values to the propositional variables. Consider $p \vee \neg p$, the truth table is as follows.

| $p$ | $\neg p$ | $p \vee \neg p$ |
|---|---|---|
| T | F | T |
| F | T | T |

$p \vee \neg p$ always takes value T for all possible truth value of P, it is a **tautology**

**Contradiction or fallacy:** A contradiction or (absurdity) is a well formed formula whose truth value is false (F) for all possible assignments of truth values to the propositional variables. Thus, in short a compound statement that is always false is a contradiction. Consider $p \wedge \neg p$, the truth table is as follows.

| $p$ | $\neg p$ | $p \wedge \neg p$ |
|---|---|---|
| T | F | F |
| F | T | F |

$p \wedge \neg p$ always takes value T for all possible truth value of P, it is a **fallacy**

**Contingency:**
A well-formed formula which is neither a tautology nor a contradiction is called a contingency. Thus, contingency is a statement pattern which is either true or false depending on the truth values of its component statement.

**Exercise:**

| Construct a truth table for each of these compound propositions. | Construct a truth table for each of these compound propositions. |
| --- | --- |
| **a)** $(p \lor q) \to (p \oplus q)$<br>**b)** $(p \oplus q) \to (p \land q)$<br>**c)** $(p \lor q) \oplus (p \land q)$<br>**d)** $(p \leftrightarrow q) \oplus (\neg p \leftrightarrow q)$<br>**e)** $(p \leftrightarrow q) \oplus (\neg p \leftrightarrow \neg r)$<br>**f )** $(p \oplus q) \to (p \oplus \neg q)$ | **a)** $p \to (\neg q \lor r)$<br>**b)** $\neg p \to (q \to r)$<br>**c)** $(p \to q) \lor (\neg p \to r)$<br>**d)** $(p \to q) \land (\neg p \to r)$<br>**e)** $(p \leftrightarrow q) \lor (\neg q \leftrightarrow r)$<br>**f )** $(\neg p \leftrightarrow \neg q) \leftrightarrow (q \leftrightarrow r)$ |

Show that $p \to q$ and $\neg p \lor q$ are logically equivalent.

Show that $\neg(p \to q)$ and $p \land \neg q$ are logically equivalent.

Show that $p \leftrightarrow q$ and $(p \land q) \lor (\neg p \land \neg q)$ are logically equivalent.

Show that $\neg(p \leftrightarrow q)$ and $p \leftrightarrow \neg q$ are logically equivalent.

Show that $p \to q$ and $\neg q \to \neg p$ are logically equivalent.

Show that $\neg p \leftrightarrow q$ and $p \leftrightarrow \neg q$ are logically equivalent.

Show that $\neg(p \oplus q)$ and $p \leftrightarrow q$ are logically equivalent.

Show that $\neg(p \leftrightarrow q)$ and $\neg p \leftrightarrow q$ are logically equivalent.

Show that $p \lor (q \land r)$ and $(p \lor q) \land (p \lor r)$ are logically equivalent

Show that $\neg(p \lor q)$ and $\neg p \land \neg q$ are logically equivalent

Show that $(p \to q) \land (p \to r)$ and $p \to (q \land r)$ are logically equivalent.

Show that $(p \to r) \land (q \to r)$ and $(p \lor q) \to r$ are logically equivalent.

Show that $(p \to q) \lor (p \to r)$ and $p \to (q \lor r)$ are logically equivalent.

Show that $(p \to r) \lor (q \to r)$ and $(p \land q) \to r$ are logically equivalent.

Show that $\neg p \to (q \to r)$ and $q \to (p \lor r)$ are logically equivalent.

Show that $p \leftrightarrow q$ and $(p \to q) \land (q \to p)$ are logically equivalent.

Show that $p \leftrightarrow q$ and $\neg p \leftrightarrow \neg q$ are logically equivalent.