Advanced optimisation: solution for tutorial 1

Y.Hammam & H. Talbot

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1 Solutions

Very important : Try to find the solutions by yourself before using this solution.

1.1 Problem 1

1.1.1 Variables

Variables are labeled $x_1, x_2, \ldots$.

1.1.2 Constraints

Constraints on machine hours and materials allow us to write, in standard form

$$\begin{array}{ccccccc} 2x_1 & + & 3x_2 & + & x_3 & \leq 10 \\ x_1 & + & 4x_2 & + & 3x_3 & \leq 15 \end{array}$$

As usual with a simplex, the x_i are non-negative.

The cost function has the form:

$$\max\ z = 6x_1 + 4x_2 + 5x_3$$

1.1.3 Standard form

We need to introduce two slack variables x_4 et x_5. We also minimize $-z$ (to emulate maximizing a profit).

All the constraints are of the form $\leq$ so all the slack variables are positive.

Standard form is therefore

$$\begin{array}{cccccccccc} \min & -6x_1 & & -4x_2 & & -5x_3 & & & & \\ & 2x_1 & + & 3x_2 & + & x_3 & + & x_4 & & = 10 \\ & x_1 & + & 4x_2 & + & 3x_3 & & & + \quad x5 & = 15 \end{array}$$

$$x_1, x_2, \ldots, x_5 \geq 0$$

Which gives us:

$$A = \begin{bmatrix} 2 & 3 & 1 & 1 & 0 \\ 1 & 4 & 3 & 0 & 1 \end{bmatrix}, b = \begin{bmatrix} 10 \\ 15 \end{bmatrix}$$

$$C^\top = \begin{bmatrix} -6 & -4 & -5 & 0 & 0 \end{bmatrix}$$

1.1.4 Formula for the inverse of a 2×2 matrix

Always useful:

$$B^{-1} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^{-1} = \frac{1}{a_{11}a_{22} - a_{21}a_{12}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

1.1.5 Unfolding the algorithm

We use x_4 et x_5 as the initial basis because it is obviously feasible. In addition the initial matrix B is particularly simple.

1. iteration
   - $\text{IBV} = \{x_4 x_5\}, \text{NBV} = \{x_1 x_2 x_3\}$
   - $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$
   - $\bar{b} = \begin{bmatrix} 10 & 15 \end{bmatrix}$
   - $\bar{C}_e^\top = \begin{bmatrix} -6 & -4 & -5 \end{bmatrix}$
   - ratios$= \begin{bmatrix} 5 & 15 \end{bmatrix}$
   - so, x_1 enters, x_4 exits.
2. iteration
   - $\text{IBV} = \{x_1 x_5\}, \text{NBV} = \{x_2 x_3 x_4\}$
   - $B = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix},$
   - $B^{-1} = \begin{bmatrix} 1/2 & 0 \\ -1/2 & 1 \end{bmatrix},$
   - $\bar{b} = \begin{bmatrix} 5 & 10 \end{bmatrix}$
   - $\bar{C}_e^\top = \begin{bmatrix} 5 & -2 & 3 \end{bmatrix}$
   - ratios$= \begin{bmatrix} 10 & 4 \end{bmatrix}$
   - so, x_3 enters, x_5 exits.
3. iteration

- IBV $= \{x_3 x_5\}$, NBV $= \{x_2 x_4 x_5\}$
- $B = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$,
- $B^{-1} = \begin{bmatrix} 0.6 & -0.2 \\ -0.2 & 0.4 \end{bmatrix}$,
- $\bar{b} = [\ 3 \quad 4\ ]$
- $\bar{C}_e^\top = [\ 7 \quad 2.6 \quad 0.8\ ]$
- All the reduced costs are positive so this is the optimum with $z = -38$.

1.2 Problem 2

There are two ways to think about the problem. Both are correct but one is more efficient than the other.

1.2.1 Formulation

- We call
  - x_1 the number of A *produced*
  - x_2 the number of B produced
  - x_3 the number of C produced
- also:
  - x_1' the number of A sold
  - x_2' the number of B sold
  - x_3' the number of C sold
- With this, the objective function is: $\max z = 10x_1' + 56x_2' + 100x_3'$.
- The constraint on the work hours is: $x_1 + 2x_2 + 3x_3 \leq 35$.
- The fact that we need to "consume" 2 A to produce one B is written $x_1 \geq 2x_2$.
- The fact that we need to "consume"' 1 B to produce one C is written $x_2 \geq x_3$.
- to switch from x to x' requires a transition matrix:

$$
\begin{aligned}
x_1' &= x_1 - 2x_2 \\
x_2' &= x_2 - x_3 \\
x_3' &= x_3
\end{aligned}
$$

1.2.2 Resolution

There are two ways to go:

1. Express everything as a function of x (non prime). This requires expressing a new $\max z$ but is otherwise simple.

   Substituting via the transition matrix yields : $\max z = 10x_1 + 36x_2 + 44x_3$

   Yielding the following LP :

$$\begin{array}{llllll} \max z & = 10x_1 & +36x_2 & +44x_3 & \\ & x_1 & +2x_2 & +3x_3 & \leq 35 \\ & -x_1 & +2x_2 & & \leq 0 \\ & & -x_2 & +x_3 & \leq 0 \end{array}$$

   This is hard to solve by hand with 3×3 to invert, see here. optimum is achieved with the basis IBV $= \{x_1, x_2, x_3\}$:

$$\begin{array}{ll} x_1 & = 10 \\ x_2 & = 5 \\ x_3 & = 5 \end{array}$$

   Profit is 500 euros.

2. Alternatively, express everything as a function of the x'

   We need to invert the transition matrix $x \leftrightarrow x'$, This is easy via substitutions

$$\begin{array}{ll} x_3 & = x'_3 \\ x_2 & = x'_2 + x'_3 \\ x_1 & = x'_1 + 2x'_2 + 2x'_3 \end{array}$$

   The hours worked constraint is reexpressed ast $x'_1 + 4x'_2 + 7x'_3 \leq 35$. The "primed" cost function remains unchanged.

   The other relations reduce to x'_1 and x'_2 non-negative, which are standard LP, so we don't need to consider them. So we have a system with a single constraint, this is callled a "knapsack", the optimium is immediately found with $x'_3 = 5$, for a profit of 500 euros. Solving a knapsack involves trying all the variables in turn.

   This is exactly the same solution of course, only expressed differently.