Chaos and Lyapunov Exponents Project

We will demonstrate, through simulation and analysis, how a simple deterministic mathematical system can become unpredictable due to exponential sensitivity to initial conditions, and to quantify this instability using Lyapunov exponents.

In this project, we will be studying a particular system, known as the **logistic map**. It is defined by the functions:

$$x_{n+1} = f(x_n), \qquad f(x) = rx(1-x) \tag{1}$$

This is the iteration of a non-linear function.

The system tries to go towards a fixed point. Using the equations above, we can define a fixed point x^* which satisfies:

$$x^* = f(x^*) \tag{2}$$

$$x = rx(1-x) \tag{3}$$

$$giving\ x^* = 0\ and\ x^* = 1 - \frac{1}{r}$$

This proves there are equilibrium states of the system, and that if the system is stable, the trajectory should approach one of these states. However, **the existence of these states does not imply stability**, as we will see later in both code and proof.

Since we want to analyse the response of the system under perturbation, we can define a new x_n where:

$$x_n = x^* + \varepsilon_n \tag{4}$$

Where ε_n is a small value.

We can now sub this into equation (1) to get:

$$x_{n+1} = f(x^* + \varepsilon_n) \tag{5}$$

Using the Taylor series, we can get...

$$x_{n+1} \approx f(x^*) + \varepsilon_n f'(x^*) \tag{6}$$

Since we have equation (2), we can sub it into equation (6) and use the form of equation (4) to declare:

$$\varepsilon_{n+1} \approx f'(x^*)\varepsilon_n \tag{7}$$

We can now assume a linear recurrence for this formula. By reusing this formula for each iteration of ε_n we can therefore deduce:

$$\varepsilon_n = \left(f'(x^*)\right)^n \varepsilon_0 \tag{8}$$

So therefore:

If $|f'(x^*)| > 1$, $\varepsilon_n \to 0$

If $|f'(x^*)| < 1$, ε_n grows exponentially

If $|f'(x^*)| = 1$, ε_n is at a marginal value.

Differentiating function (1) gives:

$$f'(x^*) = r(1 - 2x) \tag{9}$$

We can now sub in our non-zero fixed point value into the equation to give us:

$$f'(x^*) = 2 - r \tag{10}$$

This can now help us to justify what we saw when running our code. When r = 2.5, we saw that a fixed value was reached, as the derivative of the logistic map at the fixed point was equal to -0.5. This meant that the system simply tended to the fixed-point value. For r = 3.2, the derivative had an absolute value greater than 1, so the error caused oscillation of the value. And for r = 3.9, |-1.9| is significantly greater than 1, so no equilibrium is reached.

We can now try to quantify the change in behaviour of the system based on small perturbations in the initial conditions.

Consider two initial conditions that differ very slightly

$$x_0 \text{ and } \tilde{x}_0 = x_0 + \delta_0$$

Where $\delta_0 \ll 1$. After n iterations of our logistic map, we can call the trajectories that form based on the two different initial conditions as x_n and $\tilde{x}_n$, and we can define the separation between them as:

$$\delta_n = |x_n - \tilde{x}_n| \tag{11}$$

If the system is chaotic, chaos predicts that these small differences grow **exponentially**, ie:

$$\delta_n \approx \delta_0 e^{\lambda n} \tag{12}$$

Where λ is known as the **Lyapunov exponent**. This quantity measures the average rate at which the nearby paths diverge.

Taking natural logarithm of both sides of equation (12) gives:

$$\ln \delta_n \approx \ln \delta_0 + \lambda n \tag{13}$$

Thus, if the system is chaotic, a plot of $\ln \delta_n$ against n should give a line with a gradient of λ.

It is worth noting that in practice, tracking δ_n over many iterations is numerically unstable. Since the logistic map is bounded within [0,1], it is not reasonable for the separation to grow indefinitely and thus eventually saturates. Furthermore, when δ_n becomes extremely small, floating-point precision limits causes numerical underflow.

Thus, using the graphing method is a good way to understand it intuitively, however, it does not give a reliable output for Lyapunov exponent computation.

One may wonder how we can just introduce something known as the Lyapunov exponent with no mathematical basis. We will now try to prove its validity.

Near a trajectory x_n, a perturbation evolves according to the linearisation found in equation (7), written as:

$$\varepsilon_{n+1} \approx f'(x_n)\varepsilon_n \tag{14}$$

Iterating this relation yields:

$$|\varepsilon_n| = |\varepsilon_0| \prod_{k=0}^{N-1} |f'(x_k)| \tag{15}$$

Taking logarithms and averaging over many iterations leads to the Lyapunov exponent:

$$\lambda = \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \ln|f'(x_n)| \tag{16}$$

For the logistic map:

$$f(x) = rx(1-x) \text{ and } f'(x) = r(1-2x)$$

In numerical computation, the initial transient is discarded to ensure the system has settled onto the long-term attractor before averaging.

The sign of the Lyapunov exponent determines the qualitative behaviour of the system.

$\lambda < 0$ indicating convergence to a fixed value

$\lambda = 0$ indicates marginal stability

$\lambda > 0$ indicates chaotic behaviour of logistic map

We have moved beyond qualitative observations and quantified chaos using the Lyapunov exponent. This has helped us highlight why long-term prediction is almost impossible in chaotic systems despite being governed by a known set of equations.

It now only makes sense to examine how the behaviour of the system changes as r varies. As we can tell from equation (16), λ is a function of r. Thus, we can plot a graph of the Lyapunov exponent vs the parameter r to see values of r for which the system acts chaotically. By running a bit of code, we can obtain the following graph:

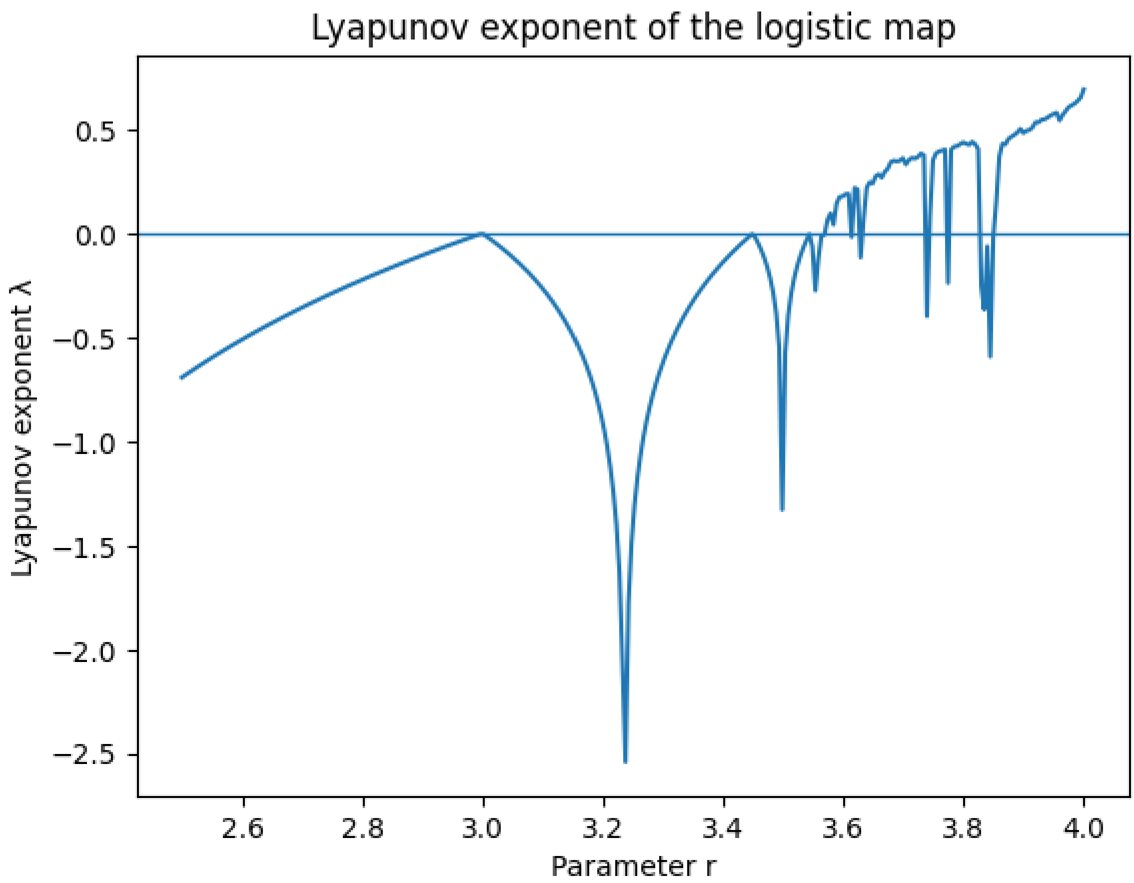

Figure 1: Graph of Lyapunov Exponent of logistic map with variation in parameter r

The logistic map behaves according to the characteristics of the Lyapunov exponent we mentioned earlier. A noteworthy feature of this graph is seen roughly in the region $r > 3.6$ where we see the graph dip below $\lambda = 0$. This signifies that for some parameters of r where generally the trend is unpredictable chaotic behaviour, you actually get periodic behaviour. This clearly suggests that chaotic and non-chaotic behaviour can coexist within the same deterministic system.

This project has shown how unpredictable behaviour can arise in a fully deterministic system due to exponential sensitivity to initial conditions. Using the logistic map, we demonstrated that while fixed points may exist, their stability depends on how small perturbations evolve under iteration.

By linearising the dynamics and computing the Lyapunov exponent, we quantified this sensitivity and showed that negative values correspond to stable or periodic behaviour, while positive values indicate chaos. Numerical difficulties encountered when directly comparing nearby trajectories further highlighted fundamental limits of long-term prediction in chaotic systems.