Summary

EL 2805 – Reinforcement Learning

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Outline

1. Markov Decision Processes
2. RL problems
3. Stochastic Approximation and Stochastic Gradient Descent Algorithms
4. Policy evaluation and TD learning
5. Optimal control in RL
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1. Markov Decision Processes

1. MDPs

State space. S (finite). A state is available to the decision maker, and leads to Markovian dynamics.

Action space. For any $s \in S$, the set of available actions is A_s. $A = \cup_{s \in S} A_s$.

Dynamics. $\mathbb{P}[s_{t+1} = s' | s_t = s, a_t = a] = p_t(s'|s, a)$. Stationary dynamics if $p_t(s'|s, a) = p(s'|s, a)$.

Rewards. At time t: $r_t(s, a)$ when $(s_t, a_t) = (s, a)$ (deterministic). Stationary rewards if $r_t(s, a) = r(s, a)$.

Finite Horizon MDP

A policy $\pi = (\pi_1, \ldots, \pi_T)$ with $\pi_t : S \to A$ (or $\mathcal{P}(A)$).

Objective. Find a policy π (Markovian and deterministic) maximizing the expected reward accumulated in T rounds given the initial state s_1:
$\mathbb{E}[R(s_1, a_1^\pi, s_1^\pi, \ldots, s_T^\pi, a_T^\pi)] = \sum_{t=1}^T \mathbb{E}[r_t(s_t^\pi, a_t^\pi) | s_1^\pi = s_1]$.

Policy evaluation (Dynamic Programming). The state value function of π is
$V^\pi(s) = \sum_{t=1}^T \mathbb{E}[r_t(s_t^\pi, a_t^\pi) | s_1^\pi = s]$.
Define the rewards to go: $u_t^\pi(s) = \mathbb{E}\left[\sum_{u=t}^T r_u(s_u^\pi, a_u^\pi) \middle| s_t^\pi = s\right]$

- Start with: $u_T^\pi(s_T) = r_T(s_T, \pi(s_T))$ for all s_T
- Backward recursion to compute u_{t-1}^π from u_t^π

$$u_{t-1}^\pi(s_{t-1}) = r_{t-1}(s_{t-1}, a) + \sum_{j \in S} p_{t-1}(j | s_{t-1}, a) u_t^\pi(j)$$

We obtain: $V^\pi(s) = u_1^\pi(s)$ for any s.

Finite Horizon MDP

Optimal control. The value function $V^\star$ is the state value function of the best policy. It satisfies Bellamn's equations:

- For all s_T, $u_T^\star(s_T) = \max_a r_T(s_T, a)$
- For all $t = T-1, T-2, \ldots, 1$

$$u_t^\star(s_t) = \max_{a \in A_{s_t}} \underbrace{\left[r_t(s_t, a) + \sum_{j \in S} p_t(j|s_t, a) u_{t+1}^\star(j) \right]}_{Q_t(s_t,a) \text{ optimal reward from } t \text{ if } a \text{ selected}}$$

Value function: $V_T^\star(s) = u_1^\star(s)$, $\forall s \in S$.
An optimal policy π is obtained by selecting $\pi_t(s_t)$ at time t such that

$$Q_t(s_t, \pi_t(s_t)) = \max_{a \in A_{s_t}} Q_t(s_t, a)$$

Infinite Horizon Discounted MDP

Stationary dynamics and rewards. Stationary policy $\pi : S \to A$.

Objective. Find a policy π (Markovian and deterministic) maximizing the expected discounted reward given the initial state s_1: $\sum_{t=1}^{\infty} \mathbb{E}[\lambda^{t-1} r(s_t^\pi, a_t^\pi) | s_1^\pi = s_1]$.

Policy evaluation. the state value function V^π is the average reward under π given the initial state, $V^\pi(s)$. It satisfies:

$$\forall s, \quad V^\pi(s) = r(s, \pi(s)) + \lambda \sum_j p(j|s, \pi(s)) V^\pi(j).$$

(state, action)-value function $Q^\pi(s, a)$: the average reward under π given that the first action is a,

$$\forall (s, a), \quad Q^\pi(s, a) = r(s, a) + \lambda \sum_j p(j|s, a) Q^\pi(j, \pi(j)).$$

Infinite Horizon Discounted MDP

Optimal control. Value function and optimal policy: $V^\star(s) = \sup_{\pi \in MD} V^\pi(s)$ obtained by solving Bellman's equations through VI or PI algorithm:

$$\forall s, \quad V^\star(s) = \max_{a \in A_s} \quad \underbrace{\left[r(s,a) + \lambda \sum_{j \in S} p(j|s,a) V^\star(j) \right]}_{Q(s,a) \text{ optimal reward from state } s \text{ if } a \text{ selected}}$$

An optimal policy π is stationary $\pi = (\pi_1, \pi_1, \ldots)$ where $\pi_1 \in MD_1$ is defined by: for any s,

$$\pi_1(s) = \arg\max_{a \in A_s} Q(s,a)$$

Q is referred to as the Q-function.

Value Iteration

Algorithm: Value Iteration Input. Precision ϵ, discount factor λ

1. **Initialization.** Select a value function $V_0 \in \mathbb{R}^S$, $n = 0$, $\delta \gg 1$
2. **Value improvement.** While $(\delta > \frac{\epsilon(1-\lambda)}{\lambda})$ do
   (a) $V_{n+1} = \mathcal{L}(V_n)$, i.e., $\forall s \in S$, $V_{n+1}(s) = \max_{a \in A_s}(r(s,a) + \lambda \sum_j p(j|s,a)V_n(j))$
   (b) $\delta = \|V_{n+1} - V_n\|$, $n \leftarrow n+1$
3. **Output.** Policy π with $\forall s \in S$, $\pi(s) \in \arg\max_{a \in A_s}(r(s,a) + \lambda \sum_j p(j|s,a)V_n(j))$

The above VI algorithm returns an ϵ-optimal policy π, i.e., for any $s \in \mathcal{S}$, $V^\pi(s) \geq V^\star(s) - \epsilon$.

Policy Iteration

Algorithm: Policy Iteration

1. **Initialization.** Select a policy π_0 arbitrarily, $n = 0$
2. **Policy evaluation.** Evaluate the state value function V_n of π_n by solving:

$$\forall s \in S,\ V_n(s) = r(s, \pi_n(s)) + \lambda \sum_j p(j|s, \pi_n(s)) V_n(j)$$

3. **Policy improvement.** Update the policy:

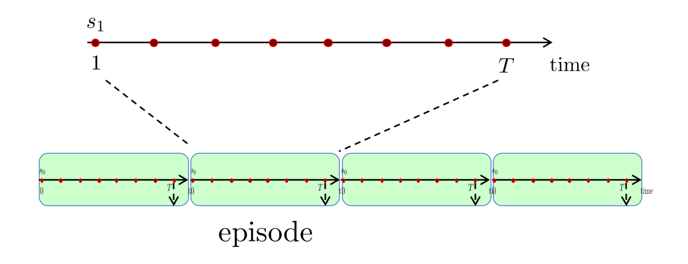
$$\forall s \in S,\ \pi_{n+1}(s) \in \arg\max_{a \in A_s} (r(s,a) + \lambda \sum_j p(j|s,a) V_n(j))$$

4. **Stopping criterion.** If $\pi_{n+1} = \pi_n$, return π_n.
   Otherwise $n \leftarrow n + 1$, and go to 2.

The **policy improvement theorem** states that under the PI algorithm, V_n is an increasing sequence, in the sense that for any $s \in \mathcal{S}$, $V_{n+1}(s) \geq V_n(s)$. In other words π_{n+1} is better than π_n. When the PI stops, it returns an optimal policy.

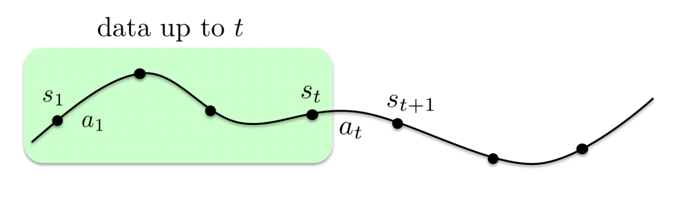
2. RL problems

Episodic RL problems



- Data: K episodes of length T (actions, states, rewards)
- Learning algorithm π : data $\mapsto \pi_K \in MD$
- Performance of π: how close π_K is from the optimal policy $\pi^\star$

Discounted RL problems



- Data: trajectory of the system up to time t (actions, states, rewards)
- Learning algorithm π : data $\mapsto \pi_t \in MD$
- Performance of π: how close π_t is from the optimal policy $\pi^\star$

On vs. Off-policy learning

An **off-policy** learner learns the value of the optimal policy independently of the agent's actions.
The policy used by the agent is often referred to as the **behavior** policy, and denoted by π_b.
Example: Q-learning.

An **on-policy** learner learns the value of the policy being carried out by the agent. The policy used by the agent is computed from the previous collected data. It is an *active learning* method as the gathered data is controlled.
Example: SARSA.

Exploration in RL problems

Exploration: RL is like trial-and-error: all actions in all states should be tested.
Off-policy learning: π_b should explore all actions.
On-policy learning: π_t should explore all actions, at **any** time t.

In RL, we use randomized policies.

ϵ-**soft policies.** Select actions uniformly at random with probability $\epsilon > 0$.

ϵ-**greedy policy w.r.t.** R**.** Let $R : S \times A \to \mathbb{R}$ a (state, action) value function. π is ϵ-greedy policy w.r.t. R iff in state s, it selects $\arg\max_a R(s, a)$ w.p. $1 - \epsilon$, and actions uniformly at random with probability ϵ.

3. Stochastic Approximation and Stochastic Gradient Descent Algorithms

Stochastic Approximation algorithm

Let $h : \mathbb{R}^d \to \mathbb{R}^d$ be a lipschitz continuous function. Root of h, i.e., $x^\star$ such that $h(x^\star) = 0$. To this aim, we have access to noisy estimates of h, i.e., for a given x, we can get from an Oracle a random variable $Y(x)$ with expectation $h(x)$.

Robbins-Monro Stochastic Approximation algorithm:

1. **Initialization:** $x^{(0)} \in \mathbb{R}^d$
2. **Iterations:** for $k \geq 0$, $x^{(k+1)} = x^{(k)} + \alpha_k [Y(x^{(k)})]$

Stochastic Approximation algorithm

Noise process: $M_{k+1} = Y(x^{(k)}) - h(x^{(k)})$.

A1. (Martingale difference) $\forall k$, $\mathbb{E}[M_{k+1}|x^{(k)}, \dots, x^{(1)}] = 0$ and $\forall k$, $\mathbb{E}[\|M_{k+1}\|_2^2 \mid x^{(k)}, \dots, x^{(1)}] \leq c_0(1 + \|x^{(k)}\|^2)$.

A2. (Stability) $\dot{x} = h(x)$ has a unique globally stable equilibrium $x^\star$. $\forall x$, $h_\infty(x) = \lim_{c \to \infty} \frac{h(cx)}{c}$ exists and 0 is the only globally stable point of $\dot{x} = h_\infty(x)$.

Convergence for decreasing learning rates. Under A1 and A2, if the learning rates α_k satisfy $\sum_{k=0}^{\infty} \alpha_k = \infty$ and $\sum_{k=0}^{\infty} \alpha_k^2 < \infty$, then $\lim_{k \to \infty} x^{(k)} = x^\star$ almost surely where $x^\star$ is the unique globally stable point of $\dot{x} = h(x)$.

Convergence for fixed learning rates. When $\alpha_k \alpha$ for all k, the algorithm does not converge, but is guaranteed to be in a neighborhood of $x^\star$ at the limit. The neighborhood is of size proportional to α.

SGD algorithm

Let $f : \mathcal{C} \to \mathbb{R}$ be a convex function defined over the convex set $\mathcal{C}$. We wish to find the minimizer of f. To this aim, we can not evaluate the gradients of f, but get only unbiased estimates of these gradients. Specifically, for any $x \in \mathcal{C}$, an Oracle can reveal $g(x)$, a r.v. such that $\nabla f(x) = \mathbb{E}[g(x)]$.

Kiefer-Wolfowitz SGD Algorithm:

1. **Initialization:** $x^{(0)}$
2. **Iterations:** for $k \geq 0$, $x^{(k+1)} = x^{(k)} - \alpha_k g(x^{(k)})$

SGD can enjoy the same convergence guarantees as those of the SA algorithm, if

$$\mathbb{E}[g(x^{(k)})|x^{(k)}, \dots, x^{(1)}] = 0.$$

4. Policy evaluation and TD learning

Finite RL problems

To evaluate a policy π, we can use Monte Carlo methods (just simulate). First visit MC algorithm:

Monte Carlo prediction algorithm:

1. **Initialization:** $\forall s$, $V^{(0)}(s) = 0$
2. **Iterations:** for episode $i = 1, \dots, n$
   generate $\tau_i = (s_1, a_{1,i}, r_{1,i}, \dots, s_{T,i}, a_{T,i}, r_{T,i})$ under π
   $G = 0$
   for $t = T, T-1, \dots, 1$:
   a. $G = r_{t,i} + G$
   b. Unless $s_{t,i}$ appears in $\{s_{1,i}, \dots, s_{t-1,i}\}$
      $V^{(i)}(s_{t,i}) = V^{(i-1)}(s_{t,i}) + \frac{1}{i}(G - V^{\pi}(s_{t,i}))$

Discounted RL problems: TD learning

Observe that the state value function of π satisfies $h(V^\pi) = 0$ where

$$\forall s, \quad h(V)(s) = r(s, \pi(s)) + \lambda \sum_j p(j|s, \pi(s)) V(j) - V(s).$$

TD(0) algorithm

1. **Initialization.**
   Select a value function $V^{(1)}$
   Initial state s_1
   Number of visits: $\forall s$, $n_s^{(1)} = 1_{(s=s_1)}$
2. **Value function updates.** For all $t \geq 1$, select action $\pi(s_t)$ and observe the new state s_{t+1} and reward r_t.
   Update the value function estimate: for all s,

$$V^{(t+1)}(s) = V^{(t)}(s) + 1_{(s_t=s)} \alpha_{n_s^{(t)}} \left(r_t + \lambda V^{(t)}(s_{t+1}) - V^{(t)}(s) \right)$$

   Update for all s, $n_s^{(t+1)} = n_s^{(t)} + 1_{(s=s_t)}$

Discounted RL problems: TD learning

Remarks.

1. TD learning is an *asynchronous* Stochastic Approximation algorithm.
2. The learning rates α_n are typically chosen as for the SA algorithm.
3. In TD methods, the term $r_t + \lambda V^{(t)}(s_{t+1})$ is often referred to as the **target**.
4. Note that TD learning is a **bootstrapping** method because the targeted value in each iteration depends on the current estimate of V^π.

5. Optimal control in RL

Q-learning

Off-policy learning of the optimal policy; behavior policy π_b. The Q-function solves:

$$\forall (s,a), \quad Q(s,a) = r(s,a) + \lambda \sum_j p(j|s,a) \max_{b \in \mathcal{A}_b} Q(j,b).$$

Q-learning algorithm

Parameter. Step sizes (α_t)

1. Initialization. Select a Q-function $Q^{(0)} \in \mathbb{R}^{S \times A}$

2. Observations. (s_t, a_t, r_t, s_{t+1}) under the behavior policy π_b

3. Q-function improvement. For $t \geq 0$. Update the estimated Q-function as follows: $\forall s, a,$

$$\begin{aligned} Q^{(t+1)}(s,a) &= Q^{(t)}(s,a) \\ &+ 1_{(s_t,a_t)=(s,a)} \alpha_{n^{(t)}(s_t,a_t)} \left[r_t + \lambda \max_{b \in \mathcal{A}} Q^{(t)}(s_{t+1}, b) - Q^{(t)}(s_t, a_t) \right] \end{aligned}$$

where $n^{(t)}(s,a) := \sum_{m=1}^{t} 1[(s,a) = (s_m, a_m)]$.

Q-learning

This is an asynchronous Stochastic Approximation algorithm because:

$$\mathbb{E}[r_t + \lambda \max_{b \in \mathcal{A}} Q^{(t)}(s_{t+1}, b) | s_t] = r(s_t, a_t) + \lambda \sum_j p(j|s_t, a_t) \max_{b \in \mathcal{A}_b} Q(j, b)$$

Convergence.

1. if the learning rates are appropriate;
2. the behavior policy explores enough: each (state, action) pairs should be visited infinitely often; example: under any ϵ-soft policies, e.g., π_b can be ϵ-greedy w.r.t. $Q^{(t)}$.

SARSA

SARSA is an on-policy learning algorithm. The algorithm maintains in step t both a policy π_t and its estimated (state, action) value function $Q^{(t)}$. In each iteration, SARSA

- updates π_t (the policy improvement step by taking the ϵ-greedy policy w.r.t. $Q^{(t)}$;
- updates $Q^{(t)}$ by applying a TD-learning step to evaluate the (state, action) value function of π_t.

The policy improvement step actually improves the policy (in average) according to the **policy improvement theorem**.

SARSA means (State, Action, Reward, State, Action), because the algorithm uses experiences of the type (s, a, r, s', a') generated under the policy π_t.

SARSA algorithm

Parameters. Step sizes (α_t), Exploration rate $\epsilon > 0$

1. Initialization. Select a Q-function $Q^{(0)} \in \mathbb{R}^{S \times A}$

2. Observations. $(s_t, a_t, r_t, s_{t+1}, a_{t+1})$ under π_t ϵ-greedy w.r.t. $Q^{(t)}$

3. Q-function improvement. For $t \geq 0$. Update the estimated Q-function as follows: $\forall s, a$,

$$\begin{aligned} Q^{(t+1)}(s,a) &= Q^{(t)}(s,a) \\ &+ 1_{(s_t,a_t)=(s,a)} \alpha_{n^{(t)}(s_t,a_t)} \left[r_t + \lambda Q^{(t)}(s_{t+1}, a_{t+1}) - Q^{(t)}(s_t, a_t) \right] \end{aligned}$$

where $n^{(t)}(s,a) := \sum_{m=1}^{t} 1[(s,a) = (s_m, a_m)]$.

Under SARSA, π_t does not converge towards an optimal policy, because π_t is ϵ-soft. When ϵ is very small, π_t approximates an optimal policy when t is large.

6. RL with function approximation

Function approximation

Beyond tabular MDPs. With large state and action spaces, Q-learning and SARSA converge very slowly (need to visit each (state, action) pairs a large number fo times).

Function approximation. Functions of interest such as the value function, the state value function of a policy, the Q-function belong to a set of parametrized functions.
For example, $V^\star \approx V_\theta \in \mathcal{V} = \{V_\mu : \mu \in \mathbb{R}^d\}$. One just needs to learn the parameter θ.

Examples.

1. Linear function approximation. In this case, we work with a basis of functions $\phi_1, \dots, \phi_d : \mathbb{R}^S \to \mathbb{R}$, and $V_\theta(s) = \sum_{i=1}^d \theta_i \phi_i(s)$.
2. Deep learning. Here the value of the function is the output of a neural network. The function is hence parametrized by the weights of the network: $V_w(s)$.

Policy evaluation with function approximation

Stationary policy π for an infinite horizon discounted MDP. The goal is then to identify the parameter θ such that V^π and V_θ are as close as possible. We minimize over θ the mean TD square error:

$$J(\theta) = \frac{1}{2}\mathbb{E}_{s\sim\mu}[(r(s,\pi(s)) + \lambda \sum_j p(j|s,\pi(s))V_\theta(j) - V_\theta(s))^2].$$

Semi-gradient method. The target is fixed and the gradient is taken w.r.t. $V_\theta(s)$ only.

TD(0) algorithm with function approximation

1. **Initialization.** θ, initial state s_1
2. **Iterations:** For every $t \geq 1$, observe s_t, a_t, r_t, s_{t+1} under π.
   Update θ as:
   $$\theta \leftarrow \theta + \alpha(r_t + \lambda V_\theta(s_{t+1}) - V_\theta(s_t))\nabla_\theta V_\theta(s_t)$$

SARSA with function approximation

The previous TD algorithms with function approximation can be applied to provide an approximation of the (state, action) value function of a given policy π. This can be implemented in the policy evaluation step of SARSA algorithm.

SARSA algorithm with function approximation

1. **Initialization.** θ, initial state s_1
2. **Iterations:** For every $t \geq 1$,
   compute π_t the ϵ-greedy policy w.r.t. Q_θ **(policy improvement)**
   take action a_t according to π_t, and observe r_t, s_{t+1}
   (alternative: select the "a_{t+1}" of the previous step as a_t)
   sample a_{t+1} according to π_t
   update θ as: **(policy evaluation)**

$$\theta \leftarrow \theta + \alpha(r_t + \lambda Q_\theta(s_{t+1}, a_{t+1}) - Q_\theta(s_t, a_t))\nabla_\theta Q_\theta(s_t, a_t)$$

Q-learning with function approximation

Function approximation in Q-learning: minimize the mean square Bellman error.

Bellman error. If $\tilde{Q}$ is an estimated Q-function, the corresponding Bellman error is defined as:

$$BE(s,a) = r(s,a) + \lambda \sum_j p(j|s,a) \max_b \tilde{Q}(j,b) - \tilde{Q}(s,a).$$

Objective function.

$$\begin{aligned} J(\theta) &= \frac{1}{2}\mathbb{E}_{(s,a)\sim\mu_b}[BE(s,a)^2] \\ &= \frac{1}{2}\mathbb{E}_{(s,a)\sim\mu_b}[(r(s,a) + \lambda \sum_j p(j|s,a) \max_b Q_\theta(j,b) - Q_\theta(s,a))^2], \end{aligned}$$

where μ_b is the stationary distribution of (s,a) under the behavior policy π_b.

Semi-gradient method. The semi-gradient is

$$-\mathbb{E}_{(s,a)\sim\mu_b}\Big[\underbrace{(r(s,a) + \lambda \sum_j p(j|s,a) \max_b Q_\theta(j,b)}_{\text{target}} - Q_\theta(s,a))\nabla_\theta Q_\theta(s,a)\Big]$$

Observing an experience (s, a, r, s') generated under π_b. The semi-gradient can be estimated (without bias) by:

$$-(r + \lambda \max_b Q_\theta(s', b) - Q_\theta(s, a))\nabla_\theta Q_\theta(s, a))$$

Q-learning with function approximation

1. **Initialization.** θ, initial state s_1
2. **Iterations:** For every $t \geq 1$,
   compute π_t the ϵ-greedy policy w.r.t. Q_θ
   take action a_t according to π_t, and observe r_t, s_{t+1}
   update θ as:

$$\theta \leftarrow \theta + \alpha(r_t + \lambda \max_b Q_\theta(s_{t+1}, b) - Q_\theta(s_t, a_t))\nabla_\theta Q_\theta(s_t, a_t)$$

Experience Replay and Fixing the Target

Issues of Q-learning with function approximation. (a) The successive updates are strongly correlated; (b) the target is not fixed, and the algorithm struggles to follow this moving target.

Experience replay. We maintain a buffer B of previous experiences (s, a, r, s'). At time t, we store the current experience in the buffer, but to update the Q-function parameter, we sample mini-batches of fixed size k from B uniformly at random. At time t Sample, we perform k updates: for $i = 1, \dots, k$, the experience (s_i, a_i, r_i, s'_i) is sampled uniformly at random from B, and we do:

$$\theta \leftarrow \theta + \alpha(r_i + \lambda \max_b Q_\theta(s'_i, b) - Q_\theta(s_i, a_i))\nabla_\theta Q_\theta(s_i, a_i)$$

Fixed target. We use a second parameter ϕ for the target. The target is fixed for C successive steps, and then aligned to θ.

Q-learning with function approximation, ER, and fixed targets

1. **Initialization.** θ and ϕ, replay buffer B, initial state s_1
2. **Iterations:** For every $t \geq 1$,
   compute π_t the ϵ-greedy policy w.r.t. Q_θ
   take action a_t according to π_t, and observe r_t, s_{t+1}
   store (s_t, a_t, r_t, s_{t+1}) in B
   sample k experiences (s_i, a_i, r_i, s'_i) from B
   for $i = 1, \dots, k$:

$$y_i = \begin{cases} r_i & \text{if episode stops in } s'_i \\ r_i + \lambda \max_b Q_\phi(s'_i, b) & \text{otherwise} \end{cases}$$

   update θ as:

$$\theta \leftarrow \theta + \alpha(y_i - Q_\theta(s_i, a_i))\nabla_\theta Q_\theta(s_i, a_i)$$

   every C steps: $\phi \leftarrow \theta$

7. Policy Gradient methods

Policy Gradient methods

Main principles.

1. Parametrize your (randomized) policy: $\pi \in \Pi = \{\pi_\theta : \theta \in \mathbb{R}^d\}$;
2. Formulate an objective to be maximized: $J(\theta) = \mathbb{E}_{s_1 \sim p}[V^{\pi_\theta}(s_1)]$;
3. Compute the gradient of $J(\theta)$ (the policy gradient theorem);
4. Implement a Stochastic Gradient Ascent algorithm.

Episodic RL problems

1. and 2. Fixed time horizon T. Initial state distribution $s_1 \sim p$.
Objective function:

$$J(\theta) = \mathbb{E}_{s_1 \sim p_1}[V^{\pi_\theta}(s_1)].$$

3. Policy gradient theorem. Rewrite $V^{\pi_\theta}(s_1)$ as:

$$V^{\pi_\theta}(s_1) = \mathbb{E}_{\pi_\theta}[\sum_{t=1} r(s_t, a_t)] = \sum_{\tau} \pi_\theta(\tau) R(\tau),$$

where τ denotes the random trajectory of an episode, $\pi_\theta(\tau)$ is the probability to observe this trajectory under π_θ, and $R(\tau)$ is the total reward collected during the episode τ. we have:

$$\nabla_\theta J(\theta) = \mathbb{E}_{\pi_\theta}[\nabla_\theta \log \pi_\theta(\tau) R(\tau)].$$

In the above expectations, $\mathbb{E}_{\pi_\theta}$ just indicates that the episode is generated under π_θ.

$\nabla \log \pi_\theta(\tau) = \sum_{t=1}^{T} \nabla \log \pi_\theta(s_t, a_t)$ is referred to as the **score function**.

Episodic RL problems

4. The policy gradient theorem states that when generating τ under π_θ, then $\nabla_\theta \log \pi_\theta(\tau) R(\tau)$ constitutes an unbiased estimator of $\nabla_\theta J(\theta)$. This naturally leads to REINFORCE algorithm:

REINFORCE Algorithm:

1. **Initialization:** select $\theta^{(0)}$ arbitrarily
2. **Iterations:** For all $k \geq 0$, for episode k, generate a trajectory under $\pi_{\theta^{(k)}}$:
   $(s_{1,k} = s, a_{1,k}, r_{1,k}, \ldots s_{T,k}, a_{T,k}, r_{T,k})$
   Update the parameter
   $$\theta^{(k+1)} = \theta^{(k)} + \alpha_k \left(\sum_{t=1}^{T} \nabla \log \pi_\theta(s_{t,k}, a_{t,k}) \right) \left(\sum_{t=1}^{T} r_{t,k} \right)$$

Episodic RL problems

REINFORCE is a SGD algorithm, and finds a local maxmizer of $J(\theta)$ (under the usual convergence conditions). In practice, the algorithm offers poor performance, because of the high variance of the gradient estimator $\nabla_\theta \log \pi_\theta(\tau) R(\tau)$. Three techniques can be used to reduce this variance:

- **Batches.** Generate n episodes (instead of 1) before updating θ; this divides the variance by $1/n$.
- **Reward to go.** Instead of $\nabla_\theta \log \pi_\theta(\tau) R(\tau)$, use $\sum_{t=1}^T \nabla \log \pi_\theta(s_t, a_t) \sum_{u=t}^T r(s_u, a_u)$; this estimator remains Unbiased.
- **Baseline.** Adding a baseline helps and does not introduce any bias. When n episodes are generated under the same θ, a natural baseline is the empirical rewards observed in the n episodes:
$$b = \frac{1}{n} \sum_{i=1}^n \sum_{t=1}^T r(s_{t,i}, a_{t,i})$$
The update is computed using $\nabla_\theta \log \pi_\theta(\tau)(R(\tau) - b)$.

In practice, you may consider combining the three above techniques.

Discounted RL problems

1. and 2. Infinite time horizon discounted MDP. The objective is to maximize $J(\theta) = \mathbb{E}_{s_1 \sim p}[V^{\pi_\theta}(s_1)]$ over all possible θ.

3. Policy gradient theorem. Introduce the discounted stationary distribution ρ_θ under π_θ:

$$\forall s \in \mathcal{S}, \quad \rho_\theta(s) = (1-\lambda) \sum_{s'} p(s') \sum_{k=1}^{\infty} \lambda^k \mathbb{P}_{\pi_\theta}[s_k = s | s_1 = s']$$

We have:

$$\nabla J(\theta) = \frac{1}{1-\lambda} \mathbb{E}_{s \sim \rho_\theta, a \sim \pi_\theta(s,\cdot)} \left[\nabla \log \pi_\theta(s,a) Q^{\pi_\theta}(s,a) \right]$$

There are two difficulties in using the above formula towards a SGD algorithm.
(a) We need a critic to estimate Q^{π_θ} (see the next subsection);
(b) Sampling s according to ρ_θ is not easy.

Sampling from the discounted stationary distribution

Existing literature. Most algorithms implicitly assume that ρ_θ is the stationary distribution of the state under π_θ: they are wrong! Indeed, ρ_θ depends on the discount factor.

With restarts. When you may restart the system when you wish, you can sample according to ρ_θ as follows. Generate $s_1 \sim p_1$; for $t \geq 1$, $a_t \sim \pi_\theta(s_t, \cdot)$, s_{t+1} drawn from $p(\cdot|s_t, a_t)$ with probability λ, and from p_1 with probability $1 - \lambda$. Then sampling a state randomly from the constructed trajectory corresponds to sampling from ρ_θ.

8. Actor-critic methods

Actor-critic algorithms

A policy gradient method with policy evaluation.

Policy gradient theorems.

Stationary MDPs with terminal state: $\nabla J(\theta) = \mathbb{E}_{(s,a)\sim\mu_\theta}\left[\nabla \log \pi_\theta(s,a) Q^{\pi_\theta}(s,a)\right];$

Discounted MDPs: $\nabla J(\theta) = \frac{1}{1-\lambda}\mathbb{E}_{s\sim\rho_\theta, a\sim\pi_\theta(s,\cdot)}\left[\nabla \log \pi_\theta(s,a) Q^{\pi_\theta}(s,a)\right].$

Policy evaluation. The (state, action) value function of a stationary policy π satisfies:

$$\forall (s,a), Q^{\pi_\theta}(s,a) = r(s,a) + \sum_j p(j|s,a) \sum_b \pi_\theta(j,b) Q^{\pi_\theta}(j,b).$$

To evaluate Q^{π_θ}, we can use TD learning and function approximation $Q^{\pi_\theta} \approx Q_\phi$: when the experience (s,a,r,s',a') is observed, we update ϕ following a semi-gradient descent algorithm:

$$\phi \leftarrow \phi + \beta(r + Q_\phi(s',a') - Q_\phi(s,a)) \nabla_\phi Q_\phi(s,a).$$

Q Actor-critic algorithms

QAC Algorithm:

1. **Initialization:** θ, ϕ, state $s \leftarrow s_1 \sim p_1$
2. **Iterations:** Loop
   If $s = \emptyset$, $s \leftarrow s_1 \sim p_1$
   Take action $a \sim \pi_\theta(s, \cdot)$ and observe r, s' (reward, next state)
   Sample the next action $a' \sim \pi_\theta(s', \cdot)$
   Update the parameters

$$\begin{aligned} \phi &\leftarrow \phi + \beta(r + Q_\phi(s', a') - Q_\phi(s, a))\nabla_\phi Q_\phi(s, a) \\ \theta &\leftarrow \theta + \alpha\left(\nabla_\theta \log \pi_\theta(s, a) Q_\phi(s, a)\right) \end{aligned}$$

   $s \leftarrow s'$, $a \leftarrow a'$

Note that the learning rates α and β at which θ and ϕ are updated may differ. Typically, we wish to keep the same policy π_θ for a period long enough so as to be able to estimate Q^{π_θ}. Hence, typically, α is chosen much smaller than β.

Actor-Critic algorithm with a baseline

Use a baseline to enhance the convergence properties of the algorithm.
The natural baseline is $V^{\pi_\theta}(s)$ since $V^{\pi_\theta}(s) = \mathbb{E}_{a\sim\pi_\theta(s,\cdot)}[Q^{\pi_\theta}(s,a)]$.

$$\text{Advantage function: } A^{\pi_\theta}(s,a) = Q^{\pi_\theta}(s,a) - V^{\pi_\theta}(s).$$

The policy gradient theorem states that:

$$\nabla_\theta J(\theta) = \mathbb{E}_{(s,a)\sim\mu_\theta}\left[\nabla \log \pi_\theta(s,a) A^{\pi_\theta}(s,a)\right].$$

Now when (s,a,r,s',a') is observed under π_θ, we get:

$$A^{\pi_\theta}(s,a) = r + \mathbb{E}[V^{\pi_\theta}(s')] - V^{\pi_\theta}(s)$$

Hence we can use and fit $V^{\pi_\theta} \approx V_\phi$ only! Using TD learning we get the following update:

$$\phi \leftarrow \phi + \beta(r + V_\phi(s') - V_\phi(s))\nabla_\phi V_\phi(s).$$

Actor-Critic algorithm with a baseline

The following two versions of the A2C (Advantage Actor-Critic) algorithms implement these ideas.

A2C Algorithm (TD version)

1. **Initialization:** θ, ϕ, state $s \leftarrow s_1 \sim p_1$
2. **Iterations:** Loop
   If $s = \emptyset$, $s \leftarrow s_1 \sim p_1$
   Take action $a \sim \pi_\theta(s, \cdot)$
   Observe r, s' (reward, next state)
   Sample the next action $a' \sim \pi_\theta(s', \cdot)$
   Update the parameters

$$\begin{aligned}\phi \leftarrow& \phi + \beta(r + V_\phi(s') - V_\phi(s))\nabla_\phi V_\phi(s) \\ \theta \leftarrow& \theta + \alpha\left(\nabla_\theta \log \pi_\theta(s, a)(r + V_\phi(s') - V_\phi(s))\right)\end{aligned}$$

   $s \leftarrow s'$, $a \leftarrow a'$

An interesting blog

Lilian Weng (Open AI)

RL overview: https://lilianweng.github.io/posts/2018-02-19-rl-overview/

Policy gradients: https://lilianweng.github.io/posts/2018-04-08-policy-gradient/