UKF estimation for different noise uncertainties compared to EKF in pendulum systems

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***Abstract*—To develop new and reliable controls for autonomous systems, knowing the state of the system is required. The states can be modelled with dynamic models to estimate the state. The unscented Kalman filter (UKF) is a preferred method to estimate states over the Extended Kalman filter (EKF). No models are perfect, which requires the user to set uncertainty parameters for process noise and measurement noise for the UKF to account for. These parameters are often manually tuned and wrong values can affect the estimation negatively. In this study, different parameters for process respective measurement noises are tested for the UKF to see the estimations' accuracy compared to EKF. The non-linear systems in this study are a simple pendulum and a double pendulum. The comparison is done by studying the mean absolute error together with the standard deviation in the state estimation. Results showed that the UKF generally performed better in most cases. The difference between them was not significantly large when the uncertainty noises were set to the same value. The standard deviation for UKF was relatively larger than EKF. It was, however, noted that for highly non-linear systems like the double pendulum, the UKF was better than the EKF with a more stable standard deviation. The conclusion of this study was that choice of noise parameters is more important than the choice of estimator. When the uncertainties for process noise and measurement noise were not equal, the estimation was seen to be better. The closer the parameters were to the true uncertainty in the model and measurement.**

***Index Terms*—Unscented Kalman Filter (UKF), Extended Kalman filter (EKF), process noise, measurement noise, simple pendulum, double pendulum**

I. Introduction

A society that is slowly developing more and more autonomous systems requires effective and reliable controls for the system to accommodate uncertainties and disturbances that exist in our world. There are various uncertainties that can affect how a system will behave such as unpredictable environment, sensor uncertainties, uncertainties in actuation, and flaws and approximations in models [1]. Controls work by using various sensor signals and models using internal states to predict and control the system and make it autonomous. In general, engineers have access to deterministic models that can help to control systems. However, no model is perfect that is able to model disturbances acted upon the system. As mentioned, to control the system, the internal states are required which can be different from what is expected from the model due to disturbances [2].

For this purpose, there exist estimation algorithms that can estimate the state by using dynamic models and measurements from sensors whilst taking disturbances into account [1]. Two of these algorithms are called Extended Kalman Filter (EKF) and Unscented Kalman Filter (UKF). Both originate from the Kalman filter which is an optimal linear estimator. However, it works best for linear systems [2]. EKF solves this problem for non-linear systems by using a method of linearization and linearising around the previous state [3]. However, it has the disadvantage of introducing errors whose magnitude will depend on how non-linear the system is [4]. UKF was developed to reduce the error by using samples around the previous state and transforming them by using the non-linear model directly [1] [5]. Previous studies have that UKF performs significantly better than EKF and is today a preferred estimator [5] [6] [7].

As mentioned, these estimators account for uncertainties when estimating and rely on the input of what uncertainties exist in the system. This also includes that UKF's accuracy in estimation also relies on the correct input of uncertainties that exist in the measurements (called measurement noise) and acts on the system (process noise) [1]. Usually, the noises are manually tuned, which opens space for the estimations to be wrong if they are chosen arbitrarily [4]. As UKF is the preferred estimator, it would be desirable if the UKF is not sensitive to changes in these noise parameters and can estimate well.

In this study, UKF is tested in comparison with EKF by using different process noises and measurement noises to study how UKF reacts to the changes in comparison to EKF. Both estimators are applied on a simple pendulum and a double pendulum to also see if the degree of non-linearity in the system affects the results on both estimators with different noises. The comparison is measured by comparing the mean absolute error (MAE) of the estimation and the standard deviation as a measure of the estimations' uncertainty.

II. Background

A. *Extended Kalman Filter (EKF)*

One way of estimating states is using the Extended Kalman filter (EKF). EKF builds on the iterative algorithm of the linear Kalman filter where a prediction is done using a dynamic model and then performs a correction step using measurements [1]. To estimate states for non-linear state transition, the function for the dynamic model is linearized with first-order Taylor expansion. The linearization is done around the point representing the current estimated state to utilise the same algorithm as linear Kalman Filter [1]. The way Kalman-filter algorithms are designed, each state that is estimated can be represented with a Gaussian distribution using the estimated state as a mean and use the variance to represent the uncertainty for the estimation [1].

B. Unscented Kalman Filter (UKF)

A different way of estimating the state is to use unscented transformation in the prediction step which UKF does [5]. This is done by generating deterministic sigma points around the previous state which are propagated through the non-linear model. Each propagated point is then taken into account for predicting the state by using a weighted sum. Likewise is done for predicting the covariance by using the difference between the predicted state and the propagated sigma points to make matrices. The result is then summed with the uncertainty noise [5]. After the prediction step, the UKF also calculates the Kalman gain using the covariances to make the correction after a measurement. The method uses a sampled-based approach to estimate, which is the underlying reason why UKF generally perform well for non-linear models [1] [6]. However, like other Kalman filters, UKF assumes that the distribution for the estimated state is Gaussian [1] [5]. This assumption is relevant when choosing the purpose of the estimation as it can be unsuitable for multimodal distribution for the state.

C. UKF in comparison to EKF

As many non-linear problems can be linearized in one way or another, EKF will have satisfactory performance in more linear cases [1]. However, many phenomena have some non-linearity and the non-linearity can vary from case to case. Previous studies that compare EKF against UKF in a highly non-linear oscillating system, have seen results where UKF performs significantly better than EKF [7] [6]. Kandepu et al [7] have tested and compared the UKF with EKF for various systems. The results of their study showed that UKF can significantly perform better than EKF with smaller errors. The errors appear when linearizing the non-linear model around the previous state. The UKF does need any linearization and instead maps the sigma points through the dynamic model [5]. The covariance, also meaning the standard deviation, in UKF is also supposed to be at least better than EKF as the covariance. Julier and Uhlmann have presented in their paper [8] that the true covariance from UKF should be better than EKF. However, in practice, the estimated covariance underestimates the true covariance and can be a little larger and same order as the covariance from EKF [8]. The cases when UKF will perform equally well as EKF are usually linear cases. This is because the UKF will be identical to the linear Kalman Filter, which EKF also is when linearizing linear models [1]. Generally, the result from UKF will mainly depend on the non-linearity and the uncertainty of the prior state [1]. UKF also has the advantage of not requiring calculating derivatives compared to EKF [5]. For many non-linear models, finding the analytical derivatives can be complex with increasing order of the system. Therefore, from the perspective of implementation, UKF can be preferred if approximated differentiation and higher computational cost are not desired.

D. Process and measurement noises

In general, modelling dynamics is not always perfect due to non-linear dynamics that are introduced to the system for various physical reasons, which are difficult to model. Due to this limitation in modelling, an uncertainty term is added for a more accurate estimation of the states. Measurements that are used for correction can also have this issue which is taken into account for all Kalman filter methods. In the method, there are therefore two noises that are used for the uncertainty: process noise and measurement noise. In case when less correct models or measurements are used for estimating the states, increasing and tuning these noise terms can aid in better estimations [2] [4]. Ideally, the measurement noise is chosen according to the noise and uncertainty that appears in the measurement. In most cases, the uncertainty in measurement is given for the sensors' specifications [4]. Process noise, on the other hand, is harder to choose correctly. Often this parameter requires more tuning compared to the measurement. The tuning is done manually if no adaptive method for process noise is not implemented [4]. The process noise should ideally be chosen such that the value matches the covariance in the disturbance that the system is affected by. The uncertainty would into account physical phenomena that cannot be modelled, but also not accurate parameter values in the model [4].

E. Pendulum systems

In this study, both a simple pendulum and a double pendulum were used to study the EKF and UKF against each other. A simple pendulum consists of a massless rod with a mass at its end. The angle θ that can be estimated is the angle between the rod and the normal to the surface it's attached to. In figure 1 the simple pendulum would be illustrated with only θ_1 and m_1. The double pendulum has another rod attached to the first mass and has a second mass at its end as seen in figure 1. The second angle θ_2 is defined as seen in figure 1. The angular velocity $\dot{\theta}$ is the change in θ over time.

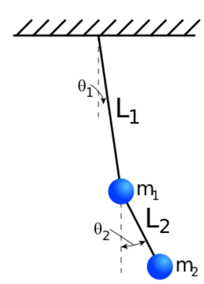

Fig. 1. The figure shows a double pendulum with two masses and its defined angles. Figure from [9].

III. Methodology

A. EKF

The Extended Kalman filter algorithm consisted of two main steps. One was the prediction step which used a dynamic model to predict, and lastly, a correction step which used measurement(s) to correct the prediction. The algorithm used in this study was based on the algorithm presented in [3]. The prediction step for estimating the predicted state $\bar{\mu}$ and uncertainty $\bar{\Sigma}$ used the following equations:

$$\bar{\mu}_t = g(\mu_{t-1}) + \varepsilon_E, \quad \varepsilon_E \sim \mathcal{N}(0,\, 0.1) \tag{1}$$
$$\bar{\Sigma}_t = G_t \Sigma_{t-1} G_t^T + R_t. \tag{2}$$

Here the dynamic model was $g(\mu_{t-1})$ that used the state from the previous time step to predict the state. To increase non-linearity, a noise term ε_E was added to the predicted state that followed the normal distribution according to (1). The noise term also had the purpose to create a slightly erroneous model to help give a better idea of how accurate the estimation could be. G_t was the Jacobian matrix derived and calculated from the dynamic model. Together with process noise R_t and the previous covariance matrix, the next covariance matrix is predicted.

Afterwards, a correction step was done to correct the estimation by using

$$K_t = \bar{\Sigma}_t H_t^T (H_t \bar{\Sigma}_t H_t^T + Q_t)^{-1} \tag{3}$$
$$\mu_t = \bar{\mu}_t + K_t(z_t - h(\bar{\mu}_t)) \tag{4}$$
$$\Sigma_t = (I - K_t H_t)\bar{\Sigma}_t. \tag{5}$$

K_t was the Kalman gain which was calculated from H_t, which was the Jacobian of the measurement model h, and a measurement noise Q_t. Lastly, the estimated state μ and the covariance matrix Σ were calculated with the help of the Kalman gain. The noises Q and R remained the same for all time steps during the simulation.

B. UKF

The Unscented Kalman filter algorithm was similar to EKF in the sense that it had a prediction step and a correction step. The algorithm in this study followed [6], but with different weights that are mentioned in [1] and [5]. The prediction for the estimated state relied on generating $2n+1$ sigma point, including the mean μ_{t-1}. The points were then propagated through the dynamic model and summed up with weights to predict the estimated state. n was the number of states that would be estimated. The formulas for generating the sigma points are

$$\lambda = \alpha^2(n+\kappa) - n \tag{6}$$
$$\mathcal{X}_{t-1}^0 = \mu_{t-1} \tag{7}$$
$$\mathcal{X}_{t-1}^i = \mu_{t-1} + (\sqrt{(n+\lambda)\Sigma_{t-1}})_i, \quad i = 1, ..., n \tag{8}$$
$$\chi_{t-1}^i = \mu_{t-1} - (\sqrt{(n+\lambda)\Sigma_{t-1}})_{i-n}, i = n+1, ..., 2n \tag{9}$$

where $\mathcal{X}^i$ were the sigma points. λ was a parameter that together with α and κ decided how far the points were from the mean. They were decided to be $\alpha = 1$ and $\kappa = 0$ similar to [6]. $\sqrt{\Sigma_{t-1}}$ was found from Cholesky decomposition where $\Sigma = LL^T$ and $\sqrt{\Sigma} = L$ that was the lower triangular matrix. The term added to μ_{t-1} was the i:s or $i-n$:s column of $\sqrt{\Sigma}$.

To calculate the weights used for estimating the state with w_m^i and the covariance with w_c^i, the following equations are were:

$$w_m^0 = \frac{\lambda}{n+\lambda} \tag{10}$$
$$w_c^0 = \frac{\lambda}{n+\lambda} + (1 - \alpha^2 + \beta) \tag{11}$$
$$w_m^i = w_c^i = \frac{1}{2(n+\lambda)}, \quad i = 1, ..., 2n. \tag{12}$$

$\beta = 2$ was the used parameter value in this study. Thereafter, the prediction is then done with

$$\mathcal{Y}_t^i = g(\mathcal{X}^i) + \varepsilon_U, \quad \varepsilon_U \sim \mathcal{N}(0,\, 0.1) \tag{13}$$
$$\bar{\mu}_t = \sum_{i=0}^{2n} w_m^i \mathcal{Y}_t^i \tag{14}$$
$$\bar{\Sigma}_t = R_t + \sum_{i=0}^{2n} w_c^i (\mathcal{Y}_t^i - \bar{\mu}_t)(\mathcal{Y}_t^i - \bar{\mu}_t)^T. \tag{15}$$

$\varepsilon_U = \varepsilon_E$ was the same as mentioned (1) and was also the same for all propagated sigma points at the same time step. This term was also added for the same reasons as it was done for EKF. g and R were the same model and process noise as in (1) and (2).

Lastly, a correction step was done with

$$\mathcal{Z}_t^i = h(\mathcal{Y}_t^i) \tag{16}$$
$$\hat{z}_t = \sum_{i=0}^{2n} w_m^i \mathcal{Z}_t^i \tag{17}$$
$$\bar{\Sigma}_t^{z,z} = Q_t + \sum_{i=0}^{2n} w_c^i (\mathcal{Z}_t^i - \hat{z}_t)(\mathcal{Z}_t^i - \hat{z}_t)^T \tag{18}$$
$$\bar{\Sigma}_t^{y,z} = \sum_{i=0}^{2n} w_c^i (\mathcal{Y}_t^i - \bar{\mu}_t)(\mathcal{Z}_t^i - \hat{z}_t)^T \tag{19}$$
$$K_t = \bar{\Sigma}_t^{y,z} (\bar{\Sigma}_t^{z,z})^{-1} \tag{20}$$
$$\mu_t = \bar{\mu}_t + K_t(z_t - \hat{z}_t) \tag{21}$$
$$\Sigma_t = \bar{\Sigma}_t - K_t \bar{\Sigma}_t^{z,z} K_t^T. \tag{22}$$

h was the same measurement model used for EKF. $\hat{z}_t$ was the estimated measurement calculated with a weighted sum. $\bar{\Sigma}_t^{z,z}$ and $\bar{\Sigma}_t^{y,z}$ were the predicted covariance and cross-covariance that were used to calculate the Kalman gain. The Kalman gain is then finally used to calculate the estimated state μ_t and covariance matrix Σ_t. The weights used are from (10)-(12).

C. Simple Pendulum

The simple pendulum used in this study followed the following system of differential equations:

$$\dot{\theta} = \omega \tag{23}$$
$$\ddot{\theta} = -\frac{g}{L} sin(\theta). \tag{24}$$

The g used here was the gravitational acceleration $9.82\ m/s^2$ and $L = 10\ m$. This is also the model the ground truth followed to compare the filters against a reference. To solve this system of differential equations, Runge-Kutta integrator of order 4 was used. The state vector used for the simple pendulum was

$$x_t = \begin{bmatrix} \theta & \dot{\theta} \end{bmatrix}^T. \quad (25)$$

The measurement model was only measuring the angle and would be

$$H = \begin{bmatrix} 1 & 0 \end{bmatrix} \quad (26)$$

$$h(x_t) = Hx_t. \quad (27)$$

The $h(x_t)$ would be used for UKF and H for EKF. In the simulations, the measurements were taken from the ground truth to correct the predicted estimation. The dynamic model $x_t = g(x_{t-1})$ that was used for the simple pendulum was

$$x_t = \begin{bmatrix} 1 & dt \\ 0 & 1 \end{bmatrix} x_{t-1} + \begin{bmatrix} 0 \\ dt \cdot \ddot{\theta}_{t-1} \end{bmatrix} \quad (28)$$

where $dt = 0.1$ was the time step used in this study and $\ddot{\theta}$ used (24) with θ_{t-1}. The EKF also needed the Jacobian to do the linearization which was derived from (23)-(24) using first-order Taylor approximation. The initial state $\mu_0 = \pi/4$ and $\mu_0 = \pi/2$ were used for both EKF and UKF. This was also the same for the ground truth. The initial covariance Σ_0 was the diagonal matrix with 4:s in the diagonal. The process noise R with size 2x2 in this study had variances of $\sigma^2 = 0.0001$ and 0.01, and measurement noise Q as scalar had also variances of $\sigma^2 = 0.0001$ and 0.01. 4 different combinations of these noises were used. The variances in the same noise matrix were never different from each other. Parameters that were needed for the UKF were $\alpha = 1, \beta = 2, \kappa = 0$, which was similar to [6]. To compare the methods against each other mean absolute error for the angle and the standard deviations were compared in all simulations. The simulation was run with 100 iterations.

D. Double Pendulum

The double pendulum followed the following system of equations [10]:

$$\dot{\theta}_1 = \omega_1 \quad (29)$$

$$\dot{\theta}_2 = \omega_2 \quad (30)$$

$$\ddot{\theta}_1 = \frac{A}{B} \quad (31)$$

$$\ddot{\theta}_2 = \frac{C}{D} \quad (32)$$

$$\begin{aligned} A &= -g(2m_1 + m_2)sin(\theta_1) - m_2 g sin(\theta_1 - 2\theta_2) \\ &\quad - 2sin(\theta_1 - \theta_2)m_2(\omega_2^2 L_2 + \omega_1^2 L_1 cos(\theta_1 - \theta_2)) \\ B &= L_1(2m_1 + m_2 - m_2 cos(2\theta_1 - 2\theta_2)) \\ C &= 2sin(\theta_1 - \theta_2)(\omega_1^2 L_1(m_1 + m_2) \\ &\quad + g(m_1 + m_2)cos(\theta_1) + \omega_2^2 L_2 m_2 cos(\theta_1 - \theta_2)) \\ D &= L_2(2m_1 + m_2 - m_2 cos(2\theta_1 - 2\theta_2)). \end{aligned}$$

The $g = 9.82\ m/s^2$, $L_1 = L_2 = 5\ m$ and $m_1 = m_2 = 1\ kg$. These equations were also used to calculate the ground truth using Runge-Kutta of order 4. The state vector was

$$x_t = \begin{bmatrix} \theta_1 & \theta_2 & \dot{\theta}_1 & \dot{\theta}_2 \end{bmatrix}^T. \quad (33)$$

Like the simple pendulum, only the angles from the ground truth were measured, and the measurement model was

$$H = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \quad (34)$$

$$h(x_t) = Hx_t. \quad (35)$$

The dynamic model $g(x_{t-1})$ for the double pendulum was

$$x_t = \begin{bmatrix} 1 & 0 & dt & 0 \\ 0 & 1 & 0 & dt \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} x_{t-1} + \begin{bmatrix} 0 \\ 0 \\ dt \cdot \ddot{\theta}_{1,t-1} \\ dt \cdot \ddot{\theta}_{2,t-1} \end{bmatrix}. \quad (36)$$

The time step $dt = 0.1$ was the same as the simple pendulum and the angular accelerations use the equations from (31) & (32). The Jacobian for EKF was derived from (29)-(32) using first-order Taylor approximation. Similar to the simulations with the simple pendulum, the initial states were $\mu_0 = \pi/4$ and $\pi/2$ for both EKF and UKF and for both angles. The initial covariance matrix Σ_0 was the diagonal matrix with 4:s in the diagonal. Process noise R and measurement noise Q were set with the variances $\sigma^2 = 0.0001$ and 0.1 with sizes of 4x4 and 2x2 respectively. 4 different combinations of these variances were tested. Like for the simple pendulum, the variances in the same noise matrix were never different from each other. The UKF had the parameters $\alpha = 1, \beta = 2, \kappa = 0$. The comparisons between EKF and UKF were also made with the mean absolute error for the angle, and the standard deviations were compared in all simulations. The simulation was run with 100 iterations.

IV. Results and Analysis

Four cases of the noises were tested for each starting angle. One case with both Q and R having variances of 0.0001 in the diagonal elements, the second with variances of 0.01, the third was with R having variances of 0.01 for both state estimations and Q having 0.0001 as variance, and lastly, for the fourth case, the two values for the variances in the noise matrices were switched. The starting angles were $\pi/4$ and $\pi/2$. Both methods were compared using the mean absolute error (MAE) and standard deviation. This procedure of testing EKF and UKF were applied for both the simple and double pendulum.

A. Simple Pendulum

In figure 2, it's seen in the MAE's plot that the UKF did perform better than EKF. Compared to previous studies this was an expected result. However, the standard deviation was larger for UKF compared to EKF. This behaviour with the standard deviation being larger for UKF was seen for all cases regardless of starting angle. When it came to MAE for the cases when the noises were $\sigma^2 = 0.0001$ for both Q and R or 0.01 for both, the difference between the MAE at the end of the simulation where similar with UKF generally staying consistently under EKF with good margin. However, the only case that had some slight difference was with a higher noise and smaller angle as seen in figure 3.

In figure 3 it is seen that the MAE were equal for both EKF and UKF which was different from previous studies.

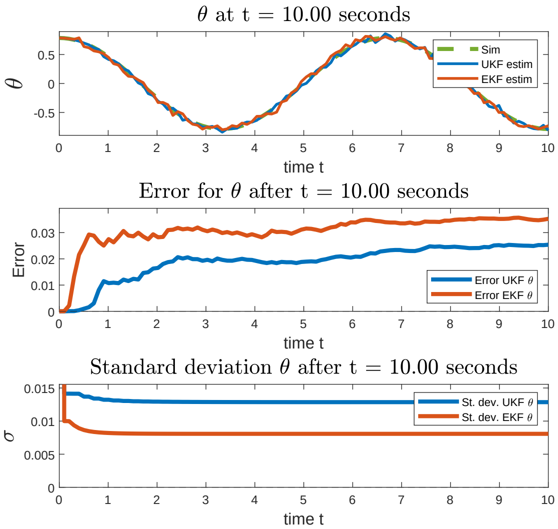

Fig. 2. The figure shows estimated angle, MAE and standard deviation of $\sigma^2 = 0.0001$ for both R and Q noises, starting with $\theta = \pi/4$.

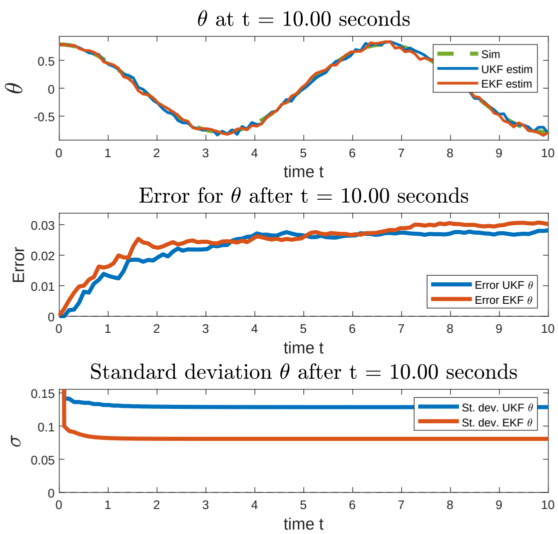

Fig. 3. The figure shows estimated angle, MAE and standard deviation of $\sigma^2 = 0.01$ for both R and Q noises, starting with $\theta = \pi/4$.

The cases when UKF performs equally to EKF are when the system is close to being linear as mentioned earlier. It would be a plausible explanation because a simple pendulum can be linearized to a linear expression and works well for small angles. On the other hand, a starting angle of $\theta = \pi/4$ is large where the errors from linearization are expected to have a significant impact. The performance of UKF did also depend on the noises in order to estimate better than EKF. It's possible that the noises were not suitable for this case for the UKF as they are usually tuned according to the system and filter.

In figure 4 with $\sigma^2 = 0.01$ for R and $\sigma^2 = 0.0001$ for Q, it's clearer that the ratio between process noise and measurement is important to achieve good estimations. For this simulation, it was seen that the MAE were of order 10^{-4}. On one hand, the performance of both EKF and UKF was similar with UKF generally being better. There was also a large difference in standard deviation with UKF being higher compared to

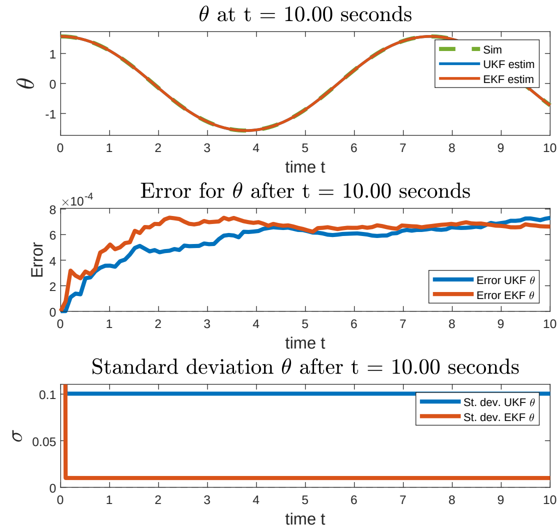

Fig. 4. The figure shows estimated angle, MAE and standard deviation of $\sigma^2 = 0.01$ for R and $\sigma^2 = 0.0001$ for Q, starting with $\theta = \pi/2$.

previous simulations in figure 2 and 3. This could be an expected behaviour as previously mentioned, but it looked like the UKF was affected by the choice of noise variance.

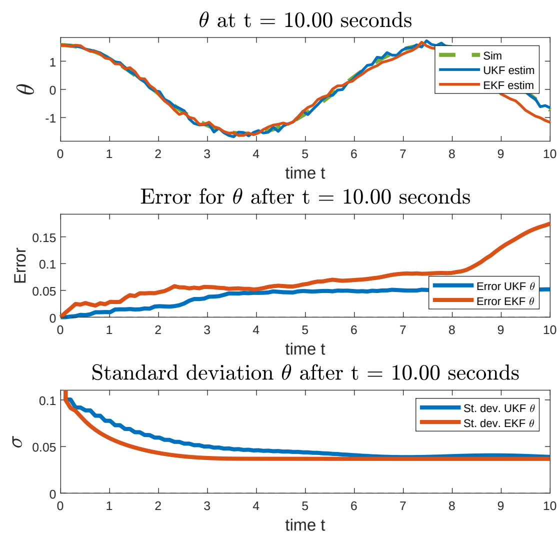

Fig. 5. The figure shows estimated angle, MAE and standard deviation of $\sigma^2 = 0.0001$ for R and $\sigma^2 = 0.01$ for Q, starting with $\theta = \pi/2$.

In figure 5, the noises had variances of $\sigma^2 = 0.0001$ for R and $\sigma^2 = 0.01$ for Q. Here, it was also shown that the ratio between the noises had an impact on the performance. Even if the estimation was not great as figure 4, the MAE for UKF was consistent, whilst EKF showed signs of divergence at the end. Similar behaviours as seen in figure 4 and 5 were also seen for smaller angles. A possible explanation as to maybe why this was not equally good as the case with the variances for Q and R switched, could depend on the measurement in this study. Generally, as previously mentioned, the choice of Q is supposed to match the uncertainty of the sensor/measurement. With measurements being taken from the ground truth would mean the uncertainty is none. For this reason, it makes sense to have a lower uncertainty noise for Q like in figure 4. Likewise, how Q should match the measurement uncertainty, R should ideally be chosen to be similar to the uncertainty in the model. In this study, we added a white noise of variance 0.1 to create a slightly erroneous model. The uncertainty that was seen for the case in figure 4 is closer to the variance of the white noise compared to the case in figure 5. The case with low process noise that did not match well with the model can be seen to impact the EKF's estimation considering how it performs for non-linear cases. This interpretation obviously assumed that process noise and the uncertainty in the model need to match. From the figures and analysis that have been presented, one should not draw the conclusion that changing the process and measurement noises, such that the ratio remains the same to favour one, will improve the estimation. To improve the estimation, both noises need to change with a suitable ratio between them for this to work, which the figures 2 and 3 seem to suggest.

B. Double Pendulum

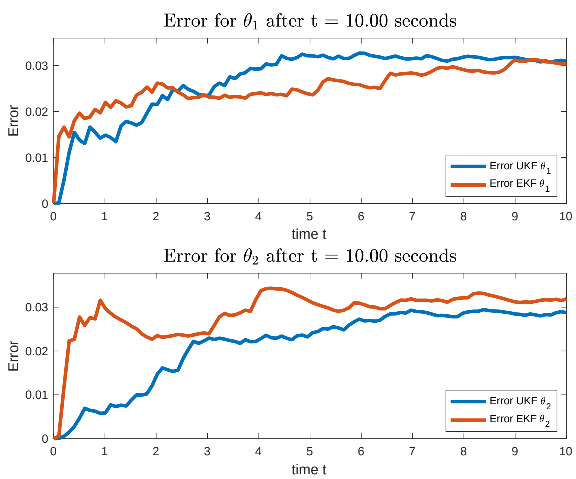

Fig. 6. The figure shows MAE for double pendulum of $\sigma^2 = 0.0001$ for both R and Q, starting with $\theta = \pi/2$.

In figure 6 and 7, the MAE and standard deviation are shown for $\theta = \pi/2$ where both Q and R have variances of 0.0001. The end result of this simulation was similar to the results in figure 2 where UKF in general estimated the states better. However, UKF's estimation for the first angle was not always the best and EKF's estimation for the second angle had large peaks of error as seen in figure 6. The standard deviation in figure 7 was similar to the previous simulation seen in figure 2, except that the estimated uncertainty was not completely stable. For the same noises in figure 6, but for a smaller starting angle, it was seen that the difference between UKF and EKF was similar. The EKF had moments where the EKF had small spikes and the standard deviation momentarily increased when the peaks in MAE happened. This situation with noises suggested that the EKF struggled to keep a consistent and good estimation for a non-linear system. Compared to previous studies, this seemed to be reasonable.

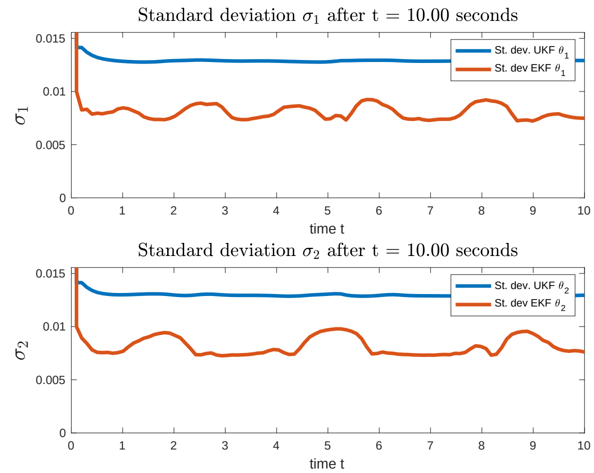

Fig. 7. The figure shows standard deviation for double pendulum of $\sigma^2 = 0.0001$ for both R and Q, starting with $\theta = \pi/2$.

When the variances for the noise matrices were increased to 0.01, almost the same results were seen with EKF struggling even though the difference in MAE after 10 seconds was small. One conclusion that can be drawn from these two variances is that the estimation will not be the best when the noises are the same for both process and measurement noises. This happened regardless of the system, starting angle, and if one of the noises was close to the ideal value.

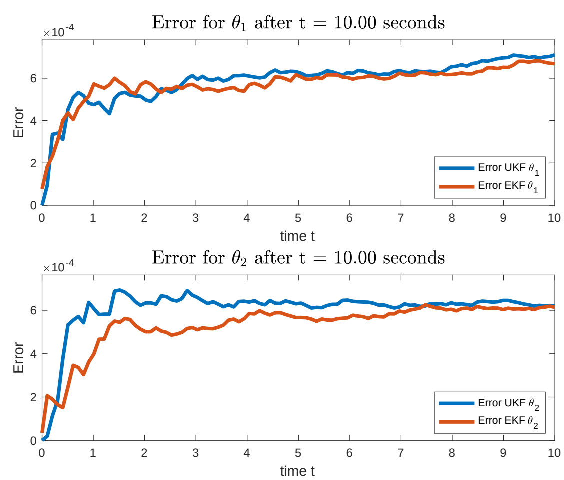

Fig. 8. The figure shows MAE for double pendulum of $\sigma^2 = 0.01$ for R and $\sigma^2 = 0.0001$ for Q, starting with $\theta = \pi/2$.

Figure 8 shows the simulation with variances $\sigma^2 = 0.01$ for R and $\sigma^2 = 0.0001$ for Q with starting angle $\theta = \pi/2$ for both masses. Just like figure 4, this was the best combination of noises with a significantly small MAE even if both estimations were equally good. The standard deviation for this situation was also similar to the simple pendulum system regardless of starting angle for the masses. This simulation for both systems could suggest that for a good estimation, it's important that the noises are correctly tuned for the system one is trying to model. The reason is that as both scenarios have significantly low MAE with stable standard deviation, both estimations were equally good with no large peaks of error.

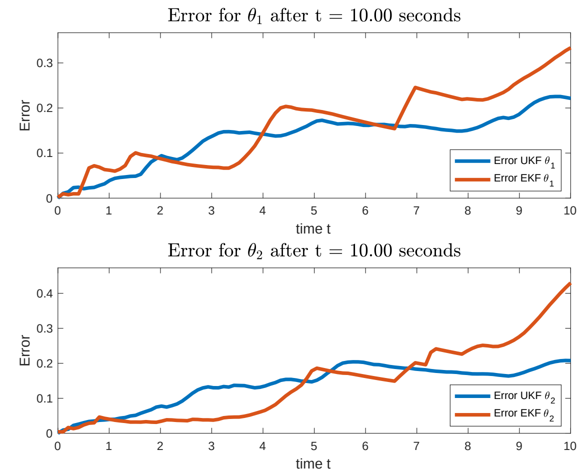

Fig. 9. The figure shows MAE for double pendulum of $\sigma^2 = 0.0001$ for R and $\sigma^2 = 0.01$ for Q, starting with $\theta = \pi/2$.

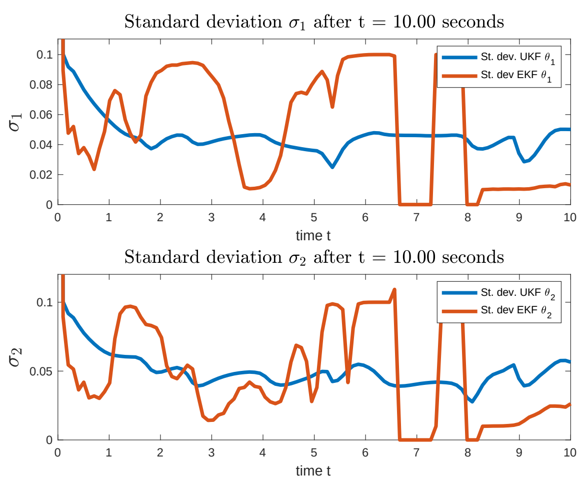

Fig. 10. The figure shows standard deviation for double pendulum of $\sigma^2 = 0.0001$ for R and $\sigma^2 = 0.01$ for Q, starting with $\theta = \pi/2$.

Figure 9 and 10 can clearly support what has previously been discussed. If the noises are set to bad values, the estimation will significantly be wrong and increase linearly. The standard deviation in figure 10 clearly showed that EKF has its clear disadvantages for non-linear systems. Although the UKF did not have the ideal noise parameters, it showed that the uncertainty was relatively more stable for non-linear systems. Generally, the measurement noises are easy to tune according to the specification of the sensor, whilst the process noise is commonly manually tuned to unknown uncertainties. Therefore, for general cases, it can be more suitable to use UKF when choosing the correct process noise is difficult. If the choice of process noise is wrong, the estimations' uncertainty will be more stable compared to EKF as seen in figure 10.

V. Conclusion

After comparing the UKF against the EKF when estimating the angles for a simple and double pendulum system using a model with noise, it was seen that UKF generally performed better than EKF for most cases. This conclusion also matches well with previous studies. However, it was noted that the difference between UKF and EKF was not always significantly large and could vary between situations. The simulations showed that the correct choice of process and measurement noises is more important than the choice of filter. The closer they are to the uncertainty in the model and measurement respectively, the better. When the noises are chosen correctly, UKF and EKF can estimate equally well. However, if the noises are chosen wrong, UKF's uncertainty will be more stable compared to EKF when working with highly non-linear systems.

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