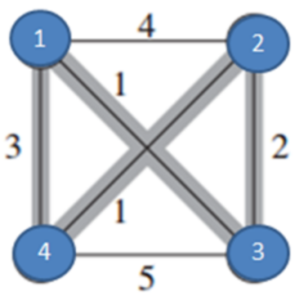
Travelling Salesman Problem (TSP) Using Dynamic Programming

Example Problem



$$\begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \left[\begin{array}{cccc} 0 & 4 & 1 & 3 \\ 4 & 0 & 2 & 1 \\ 1 & 2 & 0 & 5 \\ 3 & 1 & 5 & 0 \end{array} \right] \end{array}$$

Above we can see a complete directed graph and cost matrix which includes distance between each village. We can observe that cost matrix is symmetric that means distance between village 2 to 3 is same as distance between village 3 to 2.

Here problem is travelling salesman wants to find out his tour with minimum cost.

Say it is T (1,{2,3,4}), means, initially he is at village 1 and then he can go to any of {2,3,4}. From there to reach non-visited vertices (villages) becomes a new problem. Here we can observe that main problem spitted into sub-problem, this is property of dynamic programming.

Note: While calculating below right side values calculated in bottom-up manner. Red color values taken from below calculations.

T (1, {2,3,4}) = minimum of

= { (1,2) + T (2, {3,4}) 4+6=10
= { (1,3) + T (3, {2,4}) 1+3=4
= { (1,4) + T (4, {2,3}) 3+3=6

Here minimum of above 3 paths is answer but we know only values of (1,2) , (1,3) , (1,4) remaining thing which is T (2, {3,4}) …are new problems now. First we have to solve those and substitute here.

T (2, {3,4}) = minimum of

= { (2,3) + T (3, {4}) 2+5=7
= { (2,4) + T {4, {3}) 1+5=6

T (3, {2,4}) = minimum of

= { (3,2) + T (2, {4}) 2+1=3
= { (3,4) + T {4, {2}) 5+1=6

T (4, {2,3})   = minimum of

= { (4,2) + T (2, {3})   1+2=3
= { (4,3) + T {3, {2})   5+2=7

T (3, {4}) = (3,4) + T (4, {})   5+0=5

T (4, {3}) = (4,3) + T (3, {})   5+0=5

T (2, {4}) = (2,4) + T (4, {})   1+0=1

T (4, {2}) = (4,2) + T (2, {})   1+0 = 1

T (2, {3}) = (2,3) + T (3, {})   2+0 = 2

T (3, {2}) = (3,2) + T (2, {})   2+0=2

Here T (4, {}) is reaching base condition in recursion, which returns 0 (zero) distance.

This is where we can find final answer,

T (1, {2,3,4}) = minimum of

= { (1,2) + T (2, {3,4})   4+6=10 in this path we have to add +1 because this path ends with 3. From there we have to reach 1 so 3->1 distance 1 will be added total distance is 10+1=11
= { (1,3) + T (3, {2,4})   1+3=4 in this path we have to add +3 because this path ends with 3. From there we have to reach 1 so 4->1 distance 3 will be added total distance is 4+3=7
= { (1,4) + T (4, {2,3})   3+3=6 in this path we have to add +1 because this path ends with 3. From there we have to reach 1 so 3->1 distance 1 will be added total distance is 6+1=7

Time Complexity

Since we are solving this using Dynamic Programming, we know that Dynamic Programming approach contains sub-problems.

Here after reaching i^{th} node finding remaining minimum distance to that i^{th} node is a sub-problem.
If we solve recursive equation we will get total **$(n-1)\ 2^{(n-2)}$** sub-problems, which is **$O(n2^n)$**.
Each sub-problem will take **$O(n)$** time (finding path to remaining **$(n-1)$** nodes).
Therefore total time complexity is **$O(n2^n) * O(n) = O(n^2 2^n)$**
Space complexity is also number of sub-problems which is **$O(n2^n)$**

Program for Travelling Salesman Problem in C++

```
#include<iostream>

using namespace std;

int ary[10][10],completed[10],n,cost=0;

void takeInput()
{
	int i,j;

	cout<<"Enter the number of villages: ";
	cin>>n;

	cout<<"\nEnter the Cost Matrix\n";

	for(i=0;i < n;i++)
	{
		cout<<"\nEnter Elements of Row: "<<i+1<<"\n";

		for( j=0;j < n;j++)
			cin>>ary[i][j];

		completed[i]=0;
	}

	cout<<"\n\nThe cost list is:";

	for( i=0;i < n;i++)
	{
		cout<<"\n";

		for(j=0;j < n;j++)
			cout<<"\t"<<ary[i][j];
	}
}

int least(int c)
{
	int i,nc=999;
	int min=999,kmin;

	for(i=0;i < n;i++)
	{
		if((ary[c][i]!=0)&&(completed[i]==0))
			if(ary[c][i]+ary[i][c] < min)
			{
				min=ary[i][0]+ary[c][i];
				kmin=ary[c][i];
				nc=i;
			}
}

	if(min!=999)
		cost+=kmin;

	return nc;
}

void mincost(int city)
{
	int i,ncity;
```

```
		completed[city]=1;

		cout<<city+1<<"--->";
		ncity=least(city);

		if(ncity==999)
		{
			ncity=0;
			cout<<ncity+1;
			cost+=ary[city][ncity];

			return;
		}

		mincost(ncity);
}

int main()
{
		takeInput();

		cout<<"\n\nThe Path is:\n";
		mincost(0); //passing 0 because starting vertex

		cout<<"\n\nMinimum cost is "<<cost;

		return 0;
}
```

Output

Enter the number of villages: 4
Enter the Cost Matrix
Enter Elements of Row: 1
0 4 1 3
Enter Elements of Row: 2
4 0 2 1
Enter Elements of Row: 3
1 2 0 5
Enter Elements of Row: 4
3 1 5 0
The cost list is:
0 4 1 3
4 0 2 1
1 2 0 5
3 1 5 0
The Path is:
1—>3—>2—>4—>1
Minimum cost is 7