CS 70 Discrete Mathematics and Probability Theory
Fall 2023 Rao, Tal

HW 02

1 Airport

Note 3

Suppose that there are $2n+1$ airports, where n is a positive integer. The distances between any two airports are all different. For each airport, exactly one airplane departs from it and is destined for the closest airport. Prove by induction that there is an airport which has no airplanes destined for it.

Solution: We proceed by induction on n. For $n = 1$, let the 3 airports be A, B, C and without loss of generality suppose B, C is the closest pair of airports (which is well defined since all distances are different). Then the airplanes departing from B and C are flying towards each other. Since the airplane from A must fly to somewhere else, no airplanes are destined for airport A.

Now suppose the statement holds for $n = k$, i.e. when there are $2k+1$ airports. For $n = k+1$, i.e. when there are $2k+3$ airports, the airplanes departing from the closest two airports (say X and Y) must be destined for each other's starting airports. Removing these two airports reduce the problem to $2k+1$ airports.

From the inductive hypothesis, we know that among the $2k+1$ airports remaining, there is an airport with no incoming flights which we call airport Z. When we add back the two airports that we removed, there are two scenarios:

- Some of the flights get remapped to X or Y.
- None of the flights get remapped.

In either scenario, we conclude that the airport Z will continue to have no incoming flights when we add back the two airports, and so the statement holds for $n = k+1$. By induction, the claim holds for all $n \geq 1$.

2 Proving Inequality

Note 3

For all positive integers $n \geq 1$, prove with induction that

$$\frac{1}{3^1} + \frac{1}{3^2} + \ldots + \frac{1}{3^n} < \frac{1}{2}.$$

(Note: while you can use formula for an infinite geometric series to prove this, we require you to use induction. If direct induction seems difficult, consider strengthening the inductive hypothesis. Can you prove an equality statement instead of an inequality?)

Solution: Try a few cases and come up with a stronger inductive hypothesis. For example:

- $\frac{1}{3} = \frac{1}{2} - \frac{1}{6}$
- $\frac{1}{3} + \frac{1}{9} = \frac{1}{2} - \frac{1}{18}$
- $\frac{1}{3} + \frac{1}{9} + \frac{1}{27} = \frac{1}{2} - \frac{1}{54}$

One possible statement is

$$\frac{1}{3^1} + \frac{1}{3^2} + \ldots + \frac{1}{3^n} = \frac{1}{2} - \frac{1}{2 \cdot 3^n}$$

- *Base Case:* $n = 1$. $\frac{1}{3} = \frac{1}{2} - \frac{1}{6}$. True.
- *Inductive Hypothesis:* Assume the statement holds for $n \geq 1$.
- *Inductive Step:* Starting from the left hand side,

$$\begin{aligned}
\frac{1}{3^1} + \frac{1}{3^2} + \ldots + \frac{1}{3^n} + \frac{1}{3^{n+1}} &= \frac{1}{2} - \frac{1}{2 \cdot 3^n} + \frac{1}{3^{n+1}} \\
&= \frac{1}{2} - \frac{3-2}{2 \cdot 3^{n+1}} \\
&= \frac{1}{2} - \frac{1}{2 \cdot 3^{n+1}}.
\end{aligned}$$

Therefore, $\frac{1}{3^1} + \frac{1}{3^2} + \ldots + \frac{1}{3^n} = \frac{1}{2} - \frac{1}{2 \cdot 3^n} < \frac{1}{2}$.

Alternate Solution: Normal Induction without strengthening is viable for this problem.

Base Case: Suppose $n = 1$. We see that $\frac{1}{3^1} = \frac{1}{3} < \frac{1}{2}$.

Inductive Hypothesis: Suppose the statement is true for some arbitrary $n = k$.

Inductive Step: Utilizing the hypothesis we get

$$\begin{aligned}
\frac{1}{3^1} + \frac{1}{3^2} + \ldots + \frac{1}{3^k} + \frac{1}{3^{k+1}} &= \frac{1}{3} + \frac{1}{3}\left(\frac{1}{3^1} + \ldots + \frac{1}{3^k}\right) \\
&< \frac{1}{3} + \frac{1}{3} \cdot \frac{1}{2} \\
&= \frac{1}{2}
\end{aligned}$$

which completes the induction.

3 AM-GM

Note 3

For nonnegative real numbers $a_1, \cdots, a_n$, the arithmetic mean, or average, is defined by

$$\frac{a_1 + \cdots + a_n}{n},$$

and the geometric mean is defined by

$$\sqrt[n]{a_1 \cdots a_n}.$$

In this problem, we will prove the "AM-GM" inequality. More precisely, for all positive integers $n \geq 2$, given any nonnegative real numbers $a_1, \cdots, a_n$, we will show that

$$\frac{a_1 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 \cdots a_n}.$$

We will do so by induction on n, but in an unusual way.

(a) Prove that the inequality holds for $n = 2$. In other words, for nonnegative real numbers a_1 and a_2, show that

$$\frac{a_1 + a_2}{2} \geq \sqrt{a_1 a_2}.$$

(This equation might be of use: $(\sqrt{a} - \sqrt{b})^2 = a - 2\sqrt{ab} + b$)

(b) For some positive integer k, suppose that the AM-GM inequality holds for $n = 2^k$. Show that the AM-GM inequality holds for $n = 2^{k+1}$. (Hint: Think about how the AM-GM inequality for $n = 2$ could be used here.)

(c) For some positive integer $k \geq 2$, suppose that the AM-GM inequality holds for $n = k$. Show that the AM-GM inequality holds for $n = k - 1$. (Hint: In the AM-GM expression for $n = k$, consider substituting $a_k = \frac{a_1 + \cdots + a_{k-1}}{k-1}$.)

(d) Argue why parts (a) - (c) imply that the AM-GM inequality holds for all positive integers $n \geq 2$.

Solution:

(a) Since the a_1 and a_2 are nonnegative and real, $(\sqrt{a_1} - \sqrt{a_2})^2 \geq 0$ since the square of a real number is nonnegative. We can manipulate the equation:

$$\begin{aligned}
& (\sqrt{a_1} - \sqrt{a_2})^2 \geq 0 \\
\iff & a_1 - 2\sqrt{a_1 a_2} + a_2 \geq 0 \\
\iff & a_1 + a_2 \geq 2\sqrt{a_1 a_2} \\
\iff & \frac{a_1 + a_2}{2} \geq \sqrt{a_1 a_2}
\end{aligned}$$

Thus, our initial inequality holds. Note that since every step was bidirectional, performing the proof in the reverse direction is valid but in general you should not assume what you are proving to be true.

(b) Let $a_1, \cdots, a_{2^k}, a_{2^k+1}, \cdots, a_{2^{k+1}}$ be any nonnegative real numbers. By our hypothesis, we know that

$$l_1 = \frac{a_1 + \cdots + a_{2^k}}{2^k} \geq \sqrt[2^k]{a_1 \cdots a_{2^k}} = r_1$$

and

$$l_2 = \frac{a_{2^k+1} + \cdots + a_{2^{k+1}}}{2^k} \geq \sqrt[2^k]{a_{2^k+1} \cdots a_{2^{k+1}}} = r_2.$$

For ease of notation, we define l_1, l_2, r_1, and r_2 as above. Then applying part a), we know that

$$\frac{l_1 + l_2}{2} \geq \sqrt{l_1 l_2} \geq \sqrt{r_1 r_2}.$$

On the left hand side, we have that

$$\frac{l_1 + l_2}{2} = \frac{\frac{a_1 + \cdots + a_{2^k}}{2^k} + \frac{a_{2^k+1} + \cdots + a_{2^{k+1}}}{2^k}}{2} = \frac{a_1 + \cdots + a_{2^{k+1}}}{2^{k+1}},$$

and on the right hand side, we have that

$$\sqrt{r_1 r_2} = \sqrt{\sqrt[2^k]{a_1 \cdots a_{2^k}} \sqrt[2^k]{a_{2^k+1} \cdots a_{2^{k+1}}}} = \sqrt[2^{k+1}]{a_1 \cdots a_{2^{k+1}}},$$

so we conclude that

$$\frac{a_1 + \cdots + a_{2^{k+1}}}{2^{k+1}} \geq \sqrt[2^{k+1}]{a_1 \cdots a_{2^{k+1}}},$$

which is AM-GM for $n = 2^{k+1}$, as desired.

(c) Let $a_1, \cdots, a_k$ be any nonnegative real numbers. By our hypothesis, we know that

$$\frac{a_1 + \cdots + a_k}{k} \geq \sqrt[k]{a_1 \cdots a_k}.$$

Let $a_k = \frac{a_1 + \cdots + a_{k-1}}{k-1}$. Then we have that

$$\begin{aligned}
\frac{a_1 + \cdots + \frac{a_1 + \cdots + a_{k-1}}{k-1}}{k} &= \frac{(a_1 + \cdots + a_{k-1}) + \frac{a_1 + \cdots + a_{k-1}}{k-1}}{k} \\
&= \frac{k \frac{a_1 + \cdots + a_{k-1}}{k-1}}{k} \\
&= \frac{a_1 + \cdots + a_{k-1}}{k-1},
\end{aligned}$$

and

$$\sqrt[k]{a_1 \cdots a_k} = \sqrt[k]{a_1 \cdots a_{k-1} \frac{a_1 + \cdots + a_{k-1}}{k-1}},$$

so

$$\frac{a_1 + \cdots + a_{k-1}}{k-1} \geq \sqrt[k]{a_1 \cdots a_{k-1} \frac{a_1 + \cdots + a_{k-1}}{k-1}}.$$

Raising both sides to the k power and dividing both sides by $\frac{a_1 + \cdots + a_{k-1}}{k-1}$ gives

$$\left(\frac{a_1 + \cdots + a_{k-1}}{k-1} \right)^{k-1} \geq a_1 \cdots a_{k-1},$$

so taking the $k-1$th root of both sides gives

$$\frac{a_1+\cdots+a_{k-1}}{k-1} \geq \sqrt[k-1]{a_1\cdots a_{k-1}},$$

which is AM-GM for $n=k-1$, as desired.

(d) For any positive integer k, there exists a power of 2 2^ℓ greater or equal to k. Thus, starting at the base case in part a), we can use part b) to show that AM-GM applies for $n=2^\ell$, then use part c) to show that AM-GM applies to $n=k$.

4 A Coin Game

Note 3

Your "friend" Stanley Ford suggests you play the following game with him. You each start with a single stack of n coins. On each of your turns, you select one of your stacks of coins (that has at least two coins) and split it into two stacks, each with at least one coin. Your score for that turn is the product of the sizes of the two resulting stacks (for example, if you split a stack of 5 coins into a stack of 3 coins and a stack of 2 coins, your score would be $3\cdot 2=6$). You continue taking turns until all your stacks have only one coin in them. Stan then plays the same game with his stack of n coins, and whoever ends up with the largest total score over all their turns wins.

Prove that no matter how you choose to split the stacks, your total score will always be $\frac{n(n-1)}{2}$. (This means that you and Stan will end up with the same score no matter what happens, so the game is rather pointless.)

Solution:

We can prove this by strong induction on n.

Base Case: If $n=1$, you start with a stack of one coin, so the game immediately terminates. Your total score is zero–and indeed, $\frac{n(n-1)}{2}=\frac{1\cdot 0}{2}=0$.

Inductive Step: Suppose that if you start with i coins (for i between 1 and n inclusive), your score will be $\frac{i(i-1)}{2}$ no matter what strategy you employ. Now suppose you start with $n+1$ coins. In your first move, you must split your stack into two smaller stacks. Call the sizes of these stacks s_1 and s_2 (so $s_1+s_2=n+1$ and $s_1,s_2\geq 1$). Your end score comes from three sources: the points you get from making this first split, the points you get from future splits involving coins from stack 1, and the points you get from future splits involving coins from stack 2. From the rules of the game, we know you get s_1s_2 points from the first split. From the inductive hypothesis (which we can apply because s_1 and s_2 are between 1 and n), we know that the total number of points you get from future splits of stack 1 is $\frac{s_1(s_1-1)}{2}$ and similarly that the total number of points you get from future splits of stack 2 is $\frac{s_2(s_2-1)}{2}$, regardless of what strategy you employ in splitting them. Thus, the total number of points we score is

$$\begin{aligned}
s_1s_2 + \frac{s_1(s_1-1)}{2} + \frac{s_2(s_2-1)}{2} &= \frac{s_1(s_1-1) + 2s_1s_2 + s_2(s_2-1)}{2} \\
&= \frac{\big(s_1(s_1-1) + s_1s_2\big) + \big(s_2(s_2-1) + s_1s_2\big)}{2} \\
&= \frac{s_1(s_1+s_2-1) + s_2(s_1+s_2-1)}{2} \\
&= \frac{(s_1+s_2)(s_1+s_2-1)}{2}
\end{aligned}$$

Since $s_1 + s_2 = n+1$, this works out to $\frac{(n+1)(n+1-1)}{2}$, which is what we wanted to show your total number of points came out to. This completes our proof by induction.

5 Pairing Up

Note 4 Prove that for every even $n \geq 2$, there exists an instance of the stable matching problem with n jobs and n candidates such that the instance has at least $2^{n/2}$ distinct stable matchings.

Solution:

To prove that there exists such a stable matching instance for any even $n \geq 2$, it suffices to construct such an instance. But first, we look at the $n = 2$ case to generate some intuition. We can recognize that for the following preferences:

J_1	$C_1 > C_2$
J_2	$C_2 > C_1$

C_1	$J_2 > J_1$
C_2	$J_1 > J_2$

both $S = \{(J_1, C_1), (J_2, C_2)\}$ and $T = \{(C_1, J_2), (C_2, J_1)\}$ are stable pairings.

The $n/2$ in the exponent motivates us to consider pairing the n jobs into $n/2$ groups of 2 and likewise for the candidates. We pair up job $2k-1$ and $2k$ into a pair and candidate $2k-1$ and $2k$ into a pair, for $1 \leq k \leq n/2$.

From here, we recognize that for each pair (J_{2k-1}, J_{2k}) and (C_{2k-1}, C_{2k}), mirroring the preferences above would yield 2 stable matchings from the perspective of just these pairs. If we can extend this perspective to all $n/2$ pairs, this would be a total of $2^{n/2}$ stable matchings.

Our construction thus results in preference lists like follows:

J_1	$C_1 > C_2 > \dots$
J_2	$C_2 > C_1 > \dots$
$\vdots$	$\vdots$
J_{2k-1}	$C_{2k-1} > C_{2k} > \dots$
J_{2k}	$C_{2k} > C_{2k-1} > \dots$
$\vdots$	$\vdots$
J_{n-1}	$C_{n-1} > C_n > \dots$
J_n	$C_n > C_{n-1} > \dots$

C_1	$J_2 > J_1 > \dots$
C_2	$J_1 > J_2 > \dots$
$\vdots$	$\vdots$
C_{2k-1}	$J_{2k} > J_{2k-1} > \dots$
C_{2k}	$J_{2k-1} > J_{2k} > \dots$
$\vdots$	$\vdots$
C_{n-1}	$J_n > J_{n-1} > \dots$
C_n	$J_{n-1} > J_n > \dots$

Each match will have jobs in the kth pair paired to candidates in the kth pair for $1 \leq k \leq n/2$.

A job j in pair k will never form a rogue couple with any candidate c in pair $m \neq k$ since it always prefers the candidates in this pair over all candidates across other pairs. Since each job in pair k can be stably matched to either candidate in pair k, and there are $n/2$ total pairs, the number of stable matchings is $2^{n/2}$.

6 A Better Stable Pairing

Note 4

In this problem we examine a simple way to *merge* two different solutions to a stable matching problem. Let R, R' be two distinct stable pairings. Define the new pairing $R \wedge R'$ as follows:

> For every job j, j's partner in $R \wedge R'$ is whichever is better (according to j's preference list) of their partners in R and R'.

Also, we will say that a job/candidate *prefers* a pairing R to a pairing R' if they prefers their partner in R to their partner in R'.

(a) For this part only, consider the following example:

jobs	preferences	candidates	preferences
A	$1 > 2 > 3 > 4$	1	$D > C > B > A$
B	$2 > 1 > 4 > 3$	2	$C > D > A > B$
C	$3 > 4 > 1 > 2$	3	$B > A > D > C$
D	$4 > 3 > 2 > 1$	4	$A > B > D > C$

$R = \{(A,4),(B,3),(C,1),(D,2)\}$ and $R' = \{(A,3),(B,4),(C,2),(D,1)\}$ are stable pairings for the example given above. Calculate $R \wedge R'$ and show that it is also stable.

(b) Prove that, for any pairings R and R', no job prefers R or R' to $R \wedge R'$.

(c) Prove that, for any stable pairings R and R' where j and c are partners in R but not in R', one of the following holds:

- j prefers R to R' and c prefers R' to R; or

- j prefers R' to R and c prefers R to R'.

[*Hint*: Let J and C denote the sets of jobs and candidates respectively that prefer R to R', and J' and C' the sets of jobs and candidates that prefer R' to R. Note that $|J|+|J'|=|C|+|C'|$. (Why is this?) Show that $|J|\leq|C'|$ and that $|J'|\leq|C|$. Deduce that $|J'|=|C|$ and $|J|=|C'|$. The claim should now follow quite easily.]

(You may assume this result in the next part even if you don't prove it here.)

(d) Prove an interesting result: for any stable pairings R and R', (i) $R\wedge R'$ is a pairing, and (ii) it is also stable.

[*Hint*: for (i), use the results from part (c).]

Solution:

(a) $R\wedge R'=\{(A,3),(B,4),(C,1),(D,2)\}$. This pairing can be seen to be stable by considering the different combinations of jobs and candidates. For instance, A prefers 2 to their current partner 3. However, 2 prefers her current partner D to A. Similarly, A prefers 1 the most, but 1 prefers her current partner C to A. We can prove the stability of this pairing by considering the remaining pairs like this.

(b) Let j be a job, and let their partners in R and R' be c and c' respectively, and without loss of generality, let $c>c'$ in j's list. Then their partner in $R\wedge R'$ is c, whom they prefer over c'. However, for j to prefer R or R' over $R\wedge R'$, j must prefer c or c' over c, which is not possible (since $c>c'$ in their list).

(c) Let J and C denote the sets of jobs and candidates respectively that prefer R to R', and J' and C' the sets of jobs and candidates that prefer R' to R. Note that $|J|+|J'|=|C|+|C'|$, since the left-hand side is the number of jobs who have different partners in the two pairings, and the right-hand side is the number of candidates who have different partners.

Now, in R there cannot be a pair (j,c) such that $j\in J$ and $c\in C$, since this will be a rogue couple in R'. Hence the partner in R of every jobs in J must lie in C', and hence $|J|\leq|C'|$. A similar argument shows that every jobs in J' must have a partner in R' who lies in C, and hence $|J'|\leq|C|$.

Since $|J|+|J'|=|C|+|C'|$, both these inequalities must actually be tight, and hence we have $|J'|=|C|$ and $|J|=|C'|$. The result is now immediate: if the jobs j is partners with the candidate c in one but not both pairings, then

- either $j\in J$ and $c\in C'$, i.e., j prefers R to R' and c prefers R' to R,
- or $j\in J'$ and $c\in C$, i.e., j prefers R' to R and c prefers R to R'.

(d) (i) If $R\wedge R'$ is not a pairing, then it is because two jobs get the same candidate, or two candidates get the same job. Without loss of generality, assume it is the former case, with $(j,c)\in R$ and $(j',c)\in R'$ causing the problem. Hence j prefers R to R', and j' prefers R' to R. Using the results of the previous part would imply that c would prefer R' over R, and R over R' respectively, which is a contradiction.

(ii) Now suppose $R \wedge R'$ has a rogue couple (j,c). Then j strictly prefers c to their partners in both R and R'. Further, c prefers j to her partner in $R \wedge R'$. Let c's partners in R and R' be j_1 and j_2. If she is finally matched to j_1, then (j,c) is a rogue couple in R; on the other hand, if she is matched to j_2, then (j,c) is a rogue couple in R'. Since these are the only two choices for c's partner, we have a contradiction in either case.