1 Easter Eggs

Note 14

You made the trek to Soda for a Spring Break-themed homework party, and every attendee gets to leave with a party favor. You're given a bag with 20 chocolate eggs and 40 (empty) plastic eggs. You pick 5 eggs (uniformly) without replacement.

(a) What is the probability that the first egg you drew was a chocolate egg?

(b) What is the probability that the second egg you drew was a chocolate egg?

(c) Given that the first egg you drew was an empty plastic one, what is the probability that the fifth egg you drew was also an empty plastic egg?

Solution:

(a) $\mathbb{P}[\text{chocolate egg}] = \frac{20}{60} = \frac{1}{3}$.

(b) Long calculation using Total Probability Rule: let C_i denote the event that the ith egg is chocolate, and P_i denote the event that the ith egg is plastic. We have

$$\begin{aligned}\mathbb{P}[C_2] &= \mathbb{P}[C_1 \cap C_2] + \mathbb{P}[P_1 \cap C_2] \\ &= \mathbb{P}[C_1]\mathbb{P}[C_2 \mid C_1] + \mathbb{P}[P_1]\mathbb{P}[C_2 \mid P_1] \\ &= \frac{1}{3} \cdot \frac{19}{59} + \frac{2}{3} \cdot \frac{20}{59} \\ &= \frac{1}{3}.\end{aligned}$$

Short calculation: By symmetry, this is the same probability as part (a), $1/3$. This is because we don't know what type of egg was picked on the first draw, so the distribution for the second egg is the same as that of the first. To see this rigorously observe that $\mathbb{P}[C_2 \cap P_1] = \mathbb{P}[P_2 \cap C_1]$ and, thus:

$$\begin{aligned}\mathbb{P}[C_2] &= \mathbb{P}[C_2 \cap C_1] + \mathbb{P}[C_2 \cap P_1] \\ &= \mathbb{P}[C_2 \cap C_1] + \mathbb{P}[P_2 \cap C_1] \\ &= \mathbb{P}[C_1]\end{aligned}$$

(c) By symmetry, since we don't know any information about the 2nd, 3rd, or 4th eggs, we have

$$\mathbb{P}[\text{5th egg} = \text{plastic} \mid \text{1st egg} = \text{plastic}] = \mathbb{P}[\text{2nd egg} = \text{plastic} \mid \text{1st egg} = \text{plastic}] = \frac{39}{59}.$$

Rigorously, notice that $\mathbb{P}[C_5 \cap P_2 \mid P_1] = \mathbb{P}[P_5 \cap C_2 \mid P_1]$ and therefore:

$$\begin{aligned}\mathbb{P}[P_5 \mid P_1] &= \mathbb{P}[P_5 \cap C_2 \mid P_1] + \mathbb{P}[P_5 \cap P_2 \mid P_1] \\ &= \mathbb{P}[C_5 \cap P_2 \mid P_1] + \mathbb{P}[P_5 \cap P_2 \mid P_1] \\ &= \mathbb{P}[P_2 \mid P_1]\end{aligned}$$

One could also brute force this with Total Probability Rule (like in the previous part), but the calculation is quite tedious.

2 Balls and Bins

Note 14

Suppose you throw n balls into n labeled bins one at a time.

(a) What is the probability that the first bin is empty?

(b) What is the probability that the first k bins are empty?

(c) Let A be the event that at least k bins are empty. Let m be the number of subsets of k bins out of the total n bins. If we assume A_i is the event that the ith set of k bins is empty. Then we can write A as the union of A_i's:

$$A = \bigcup_{i=1}^{m} A_i.$$

Compute m in terms of n and k, and use the union bound to give an upper bound on the probability $\mathbb{P}[A]$.

(d) What is the probability that the second bin is empty given that the first one is empty?

(e) Are the events that "the first bin is empty" and "the first two bins are empty" independent?

(f) Are the events that "the first bin is empty" and "the second bin is empty" independent?

Solution: Since the balls are thrown one at a time, there is an ordering, and so we are sampling with replacement where order matters rather than where it doesn't (which would correspond to each configuration in the stars and bars setup being equally likely).

(a) Note that this is a uniform sample space, with outcomes representing all possible ways to throw each ball individually into the bins. Here, $|\Omega| = n^n$, as each of the n balls has n possible bins to fall into, and out of these possibilities, $(n-1)^n$ of them leave the first bin empty—each ball would then have $n-1$ possible bins to fall into. This gives us an overall probability $\left(\frac{n-1}{n}\right)^n$ that the first bin is empty.

Equivalently, we can note that each throw is independent of all of the other throws. Since the probability that ball i does not land in the first bin is $\frac{n-1}{n}$, the probability that all of the balls do not land in the first bin is $\left(\frac{n-1}{n}\right)^n$.

(b) Similar to the previous part, we have the same uniform sample space of size n^n. Now, there are a total of $(n-k)^n$ possible ways to throw the balls into bins such that the first k bins are empty—each ball has $n-k$ possible bins to fall into.

Alternatively, we can similarly make use of independence. Since the probability that ball i does not land in the first k bins is $\frac{n-k}{n}$, the probability that all of the balls do not land in the first k bins is $\left(\frac{n-k}{n}\right)^n$.

(c) We use the union bound. Then

$$\mathbb{P}[A] = \mathbb{P}\left[\bigcup_{i=1}^{m} A_i\right] \leq \sum_{i=1}^{m} \mathbb{P}[A_i].$$

We know the probability of the first k bins being empty from part (b), and this is true for any set of k bins, so

$$\mathbb{P}[A_i] = \left(\frac{n-k}{n}\right)^n.$$

Then,

$$\mathbb{P}[A] \leq m \cdot \left(\frac{n-k}{n}\right)^n = \binom{n}{k}\left(\frac{n-k}{n}\right)^n.$$

(d) Using Bayes' Rule:

$$\begin{aligned}
\mathbb{P}[\text{2nd bin empty} \mid \text{1st bin empty}] &= \frac{\mathbb{P}[\text{2nd bin empty} \cap \text{1st bin empty}]}{\mathbb{P}[\text{1st bin empty}]} \\
&= \frac{(n-2)^n/n^n}{(n-1)^n/n^n} \\
&= \left(\frac{n-2}{n-1}\right)^n
\end{aligned}$$

Alternate solution: We know bin 1 is empty, so each ball that we throw can land in one of the remaining $n-1$ bins. We want the probability that bin 2 is empty, which means that each ball cannot land in bin 2 either, leaving $n-2$ bins. Thus for each ball, the probability that bin 2 is empty given that bin 1 is empty is $\frac{n-2}{n-1}$. For n total balls, this probability is $\left(\frac{n-2}{n-1}\right)^n$.

(e) They are dependent. Knowing the latter means the former happens with probability 1.

(f) In part (c) we calculated the probability that the second bin is empty given that the first bin is empty: $\left(\frac{n-2}{n-1}\right)^n$. The probability that the second bin is empty (without any prior information) is $\left(\frac{n-1}{n}\right)^n$. Since these probabilities are not equal, the events are dependent.

3 Mario's Coins

Note 14

Mario owns three identical-looking coins. One coin shows heads with probability $1/4$, another shows heads with probability $1/2$, and the last shows heads with probability $3/4$.

(a) Mario randomly picks a coin and flips it. He then picks one of the other two coins and flips it. Let X_1 and X_2 be the events of the 1st and 2nd flips showing heads, respectively. Are X_1 and X_2 independent? Please prove your answer.

(b) Mario randomly picks a single coin and flips it twice. Let Y_1 and Y_2 be the events of the 1st and 2nd flips showing heads, respectively. Are Y_1 and Y_2 independent? Please prove your answer.

(c) Mario arranges his three coins in a row. He flips the coin on the left, which shows heads. He then flips the coin in the middle, which shows heads. Finally, he flips the coin on the right. What is the probability that it also shows heads?

Solution:

(a) X_1 and X_2 are not independent. Intuitively, the fact that X_1 happened gives some information about the first coin that was chosen; this provides some information about the second coin that was chosen (since the first and second coins can't be the same coin), which directly affects whether X_2 happens or not.

To make this formal, we compute

$$\mathbb{P}[X_1] = \left(\frac{1}{3}\right)\left(\frac{1}{4}\right) + \left(\frac{1}{3}\right)\left(\frac{1}{2}\right) + \left(\frac{1}{3}\right)\left(\frac{3}{4}\right) = \frac{1}{2}$$

By symmetry, $\mathbb{P}[X_2] = \mathbb{P}[X_1]$, so

$$\mathbb{P}[X_1]\,\mathbb{P}[X_2] = \frac{1}{4}.$$

But if we consider the probability that both X_1 and X_2 happen, we have

$$\begin{aligned}\mathbb{P}[X_1 \cap X_2] &= \frac{1}{6}\Big[\left(\frac{1}{4}\right)\left(\frac{1}{2}\right) + \left(\frac{1}{4}\right)\left(\frac{3}{4}\right) + \left(\frac{1}{2}\right)\left(\frac{1}{4}\right) + \\ &\qquad \left(\frac{1}{2}\right)\left(\frac{3}{4}\right) + \left(\frac{3}{4}\right)\left(\frac{1}{4}\right) + \left(\frac{3}{4}\right)\left(\frac{1}{2}\right)\Big] \\ &= \frac{22}{96} = \frac{11}{48}\end{aligned}$$

which is not equal to $1/4$, violating the definition of independence.

(b) Y_1 and Y_2 are not independent. Intuitively, the fact that Y_1 happens gives some information about the coin that was picked, which directly influences whether Y_2 happens or not.

To make this formal, we compute

$$\mathbb{P}[Y_1] = \left(\frac{1}{3}\right)\left(\frac{1}{4}\right) + \left(\frac{1}{3}\right)\left(\frac{1}{2}\right) + \left(\frac{1}{3}\right)\left(\frac{3}{4}\right) = \frac{1}{2}$$

By symmetry, $\mathbb{P}[Y_2] = \mathbb{P}[Y_1]$, so

$$\mathbb{P}[Y_1]\,\mathbb{P}[Y_2] = \frac{1}{4}$$

But if we consider the probability that both Y_1 and Y_2 happen, we have

$$\mathbb{P}[Y_1 \cap Y_2] = \left(\frac{1}{3}\right)\left(\frac{1}{4}\right)^2 + \left(\frac{1}{3}\right)\left(\frac{1}{2}\right)^2 + \left(\frac{1}{3}\right)\left(\frac{3}{4}\right)^2 = \frac{14}{48} = \frac{7}{24}$$

which is not equal to $1/4$, violating the definition of independence.

(c) Let A be the coin with bias $1/4$, B be the fair coin, and C be the coin with bias $3/4$. There are six orderings, each with probability $1/6$: ABC, ACB, BAC, BCA, CAB, and CBA. Thus

$$\begin{aligned}
&\mathbb{P}[\text{Third coin shows heads} \mid \text{First two coins show heads}] \\
&= \frac{\mathbb{P}[\text{All three coins show heads}]}{\mathbb{P}[\text{First two coins show heads}]} \\
&= \frac{(\frac{1}{4})(\frac{1}{2})(\frac{3}{4})}{11/48} \\
&= \frac{3/32}{11/48} = \frac{9}{22}.
\end{aligned}$$

Note that the denominator was the probability calculated in part a, so we just plugged it in as $\frac{11}{48}$.