1 Strings

Note 10

What is the number of strings you can construct given:

(a) n ones, and m zeroes?

(b) n_1 A's, n_2 B's and n_3 C's?

(c) $n_1, n_2, \ldots, n_k$ respectively of k different letters?

Solution:

(a) This is an $n+m$ length string. We choose n of those positions to be 1, and the rest will automatically be 0. Thus, the count is $\binom{n+m}{n}$. Another way of thinking about this is that there are $n+m$ positions, so we can consider $(n+m)!$ permutations. In this permutation, there are n ones, and the order of these ones doesn't actually matter. Every $n!$ way to order the ones is actually the exact same string, thus we divide by $n!$. Similarly, we divide by $m!$ to account for the zeros. Thus, we retrieve $\frac{(n+m)!}{n!m!}$.

(b) For this question, it is easier to consider the second method from the previous solution. There are $n_1+n_2+n_3$ positions, so we can consider $(n_1+n_2+n_3)!$ permutations. In this permutation, there are n_1 A's, and the order of these A's doesn't actually matter. Every $n_1!$ way to order the ones is actually the exact same string, thus we divide by $n_1!$. Similarly, we divide by $n_2!$ to account for the B's and also by $n_3!$ to account for the C's.
Alternatively, we could've used the counting positions strategy to approach this problem, though it is harder to generalize. We could consider an $n_1+n_2+n_3$ length string. First, we'll choose n_1 of those positions to be an A. Then, out of the n_2+n_3 positions left, we'll choose n_2 to be a B. Thus, the count becomes $\binom{n_1+n_2+n_3}{n_1}\binom{n_2+n_3}{n_2}$ which does evaluate to the same quantity.

(c) Using the same logic from the previous part, we generalize for a size k alphabet.

$$(n_1+n_2+\cdots+n_k)!/(n_1!\cdot n_2!\cdots n_k!)$$

.

2 You'll Never Count Alone

Note 10

(a) An anagram of LIVERPOOL is any re-ordering of the letters of LIVERPOOL, i.e., any string made up of the letters L, I, V, E, R, P, O, O, L in any order. For example, IVLERPOOL and POLIVOLRE are anagrams of LIVERPOOL but PIVEOLR and CHELSEA are not. The anagram does not have to be an English word.

How many different anagrams of LIVERPOOL are there?

(b) How many solutions does $y_0 + y_1 + \cdots + y_k = n$ have, if each y must be a non-negative integer?

(c) How many solutions does $y_0 + y_1 + \cdots + y_k = n$ have, if each y must be a positive integer?

Solution:

(a) In this 9 letter word, the letters L and O are each repeated 2 times while the other letters appear once. Hence, the number $9!$ overcounts the number of different anagrams by a factor of $2! \times 2!$ (one factor of $2!$ for the number of ways of permuting the 2 L's among themselves and another factor of $2!$ for the number of ways of permuting the 2 O's among themselves). Hence, there are $9!/(2!)^2$ different anagrams.

(b) $\binom{n+k}{k}$. We can imagine this as a sequence of n ones and k plus signs: y_0 is the number of ones before the first plus, y_1 is the number of ones between the first and second plus, etc. We can now count the number of sequences using the "balls and bins" method (also known as "stars and bars").

(c) $\binom{(n-(k+1))+k}{k} = \binom{n-1}{k}$. By subtracting 1 from all $k+1$ variables, and $k+1$ from the total required, we reduce it to problem with the same form as the previous problem. Once we have a solution to that we reverse the process, and adding 1 to all the non-negative variables gives us positive variables.

Alternatively, we can derive a method similar to stars and bars/balls and bins; here, the restriction to positive integers means that we cannot have any empty groups. In particular, instead of arranging all of the objects (i.e. all the stars and all the bars), we can instead choose where to place the bars.

Looking at the "gaps" between the stars (i.e. the 1's), we have a total of $n-1$ places to put the bars in between the n stars. Selecting k of these positions (we can't have two bars occupy the same gap, otherwise we'd have an empty group), we have a total of $\binom{n-1}{k}$ ways to group the 1's.

3 The Count

Note 10

(a) The Count is trying to choose his new 7-digit phone number. Since he is picky about his numbers, he wants it to have the property that the digits are non-increasing when read from left to right. For example, 9973220 is a valid phone number, but 9876545 is not. How many choices for a new phone number does he have?

(b) Now instead of non-increasing, they must be strictly decreasing. So 9983220 is no longer valid, while 9753210 is valid. How many choices for a new phone number does he have now?

(c) The Count now wants to make a password to secure his phone. His password must be exactly 10 digits long and can only contain the digits 0 and 1. On top of that, he also wants it to contain at least five consecutive 0's. How many possible passwords can he make?

Solution:

(a) This is actually a stars and bars problem in disguise! We have seven positions for digits, and nine dividers to partition these positions into places for nines, places for eights, etc. This is because we know that the digits are non-increasing, so all the nines (if any) must come first, then all the eights (if any), and so on. That means there are a total of 16 objects and dividers, and we are looking for where to put the nine dividers, so our answer is $\binom{16}{9}$.

(b) This can be found from just combinations. For any choice of 7 digits, there is exactly one arrangement of them that is strictly decreasing. Thus, the total number of strictly decreasing strings is exactly $\binom{10}{7}$.

(c) This problem is a bit trickier to approach, since there is a strong possibility of overcounting - it is not sufficient to just choose 5 consecutive positions to be 00000, and let the rest of the positions be arbitrary values.

One counting strategy is strategic casework - we will split up the problem into exhaustive cases based on where the run of 0's begins (we look at the leftmost zero of a run of at least 5 zeros). It can begin somewhere between the first digit and the sixth digit, inclusively.

If the run begins with the first digit, the first five digits are 0, and there are $2^5 = 32$ choices for the other 5 digits.

If the run begins after the i^{th} digit, then the $i-1^{th}$ digit must be a 1, and the other $(10-5-1=4)$ digits can be chosen arbitrarily. The other four digits can be freely chosen with $2^4 = 16$ possibilities. Thus the total number of valid passwords is $2^5 + 5 \cdot 2^4 = 112$. Note that, since there are only 10 digits, there can only be one occurence of the "100000" pattern.

4 Farmer's Market

Note 10

Suppose you want k items from the farmer's market. Count how many ways you can do this, assuming:

(a) There are pumpkins and apples at the market.

(b) There are pumpkins, apples, oranges, and pears at the market.

(c) There are n kinds of fruits at the market, and you want to end up with at least two different types of fruit.

Solution:

This is a classic "balls and bins" (also known as "stars and bars") problem.

(a) $k+1$. We can have 0 pumpkins and k apples, or 1 pumpkin and $k-1$ apples, etc. all the way to k pumpkins and 0 apples. We can equivalently think about this as k balls and 2 bins, or k stars and 1 bar, giving us $\binom{k+1}{1} = \binom{k+1}{k}$.

(b) $\binom{k+3}{3}$. We have k balls and 4 bins, or k stars and 3 bars.

(c) There are $\binom{n+k-1}{n-1} = \binom{n+k-1}{k}$ ways to choose k fruits of n types with no additional restrictions (i.e. k balls and n bins, or k stars and $n-1$ bars). n of these combinations, however, contain only one variety of fruit, so we subtract them for a total of $\binom{n+k-1}{n-1} - n = \binom{n+k-1}{k} - n$.