HW 07 (Optional)

1 Functional Counting

Note 10

In each of the following questions, assume that n is a positive integer.

(a) How many strictly increasing functions are there from the set $\{1,2,\ldots,n-1,n\}$ to the set $\{1,2,\ldots,2n-1,2n\}$?

(b) How many surjective functions are there from the set $\{1,2,\ldots,2n-1,2n\}$ to the set $\{1,2,\ldots,n-1,n\}$? Leave your answer as a summation.

Solution:

(a) The answer is $\binom{2n}{n}$. The reason for this is that every strictly increasing function can be uniquely identified with its range. Since the domain of any such function has n elements, the range of any such function must be an n element subset of $\{1,2,\ldots,2n-1,2n\}$. There are $\binom{2n}{n}$ such subsets, so there are $\binom{2n}{n}$ strictly increasing functions from $\{1,2,\ldots,n-1,n\}$ to $\{1,2,\ldots,2n-1,2n\}$.

(b) The answer is

$$\sum_{k=0}^{n-1}(-1)^k\binom{n}{k}(n-k)^{2n}.$$

To see why, we use the Principle of Inclusion-Exclusion. In order to compute the number of surjective functions, we can equivalently compute the number of functions that aren't surjective. Let A denote the set of functions which aren't surjective. If we let A_j be the set of functions whose ranges' don't contain the number j, we can see that

$$A = A_1 \cup A_2 \cup \cdots \cup A_{n-1} \cup A_n.$$

Now, observe that

$$|A_j| = (n-1)^{2n},$$

as we have $n-1$ choices for the value of the function at each of our $2n$ points in the domain. Moreover, observe that $A_i \cap A_j$ is the set of functions which don't take on either of the values i or j, and similarly with 3 or more sets. In this way, we can see that the size of the intersection of k of these sets is $(n-k)^{2n}$, as we have $n-k$ choices for the values of the function. Now, applying the Principle of Inclusion-Exclusion, we get that

$$|A| = \binom{n}{1}(n-1)^{2n} - \binom{n}{2}(n-2)^{2n} + \cdots + (-1)^n\binom{n}{n-1}(n-(n-1))^{2n}.$$

Now, this is the number of functions that aren't surjective. To get the number of functions which are surjective, we need to subtract this from the total number of functions. The total number of functions is n^{2n} (n choices for the value of each of the $2n$ points in the domain), hence our final answer is

$$\begin{aligned} n^{2n} - |A| &= \binom{n}{0}(n-0)^{2n} - \binom{n}{1}(n-1)^{2n} + \cdots + (-1)^{n-1}\binom{n}{n-1}(n-(n-1))^{2n} \\ &= \sum_{k=0}^{n-1}(-1)^k\binom{n}{k}(n-k)^{2n}. \end{aligned}$$

2 Proofs of the Combinatorial Variety

Note 10

Prove each of the following identities using a combinatorial proof.

(a) For every positive integer $n > 1$,

$$\sum_{k=0}^{n} k \cdot \binom{n}{k} = n \cdot \sum_{k=0}^{n-1}\binom{n-1}{k}.$$

(b) For each positive integer m and each positive integer $n > m$,

$$\sum_{a+b+c=m}\binom{n}{a}\cdot\binom{n}{b}\cdot\binom{n}{c} = \binom{3n}{m}.$$

(Notation: the sum on the left is taken over all triples of nonnegative integers (a,b,c) such that $a+b+c=m$.)

Solution:

(a) Suppose we have n people and want to pick some of them to form a special committee. Moreover, suppose we want to pick a leader from among the committee members - how many ways can we do this?

We can do so by first picking the committee members, and then choosing the leader from among the chosen members. We can pick a committee of size k in $\binom{n}{k}$ ways, and once we have picked the committee, we have k choices for which member becomes the leader. In order to account for all possible committee sizes, we need to sum over all valid values of k, hence we get the expression

$$\sum_{k=0}^{n} k \cdot \binom{n}{k},$$

which is exactly the left hand side of the identity we want to prove.

Now, we can also count this set by first picking the leader for the committee, then choosing the rest of committee. We have n choices for the leader, and then among the remaining $n-1$

people, we can pick any subset to form the rest of the committee. Picking a subset of size k can be done in $\binom{n-1}{k}$ ways, hence summing over k, we get the expression

$$n \cdot \sum_{k=0}^{n-1} \binom{n-1}{k},$$

which is exactly the right hand side of the identity we want to prove.

(b) Suppose we have n distinguishable red pencils, n distinguishable blue pencils, and n distinguishable green pencils ($3n$ pencils total), and want to choose m of these pencils to bring to class. How many ways can be do this?

We can do so by just picking the m pencils without considering color, as they are all distinguishable. There are $\binom{3n}{m}$ ways of doing this, which is exactly the right hand side of the identity we want to prove.

We can also count this set by picking some red pencils, the picking some blue pencils, and then finally picking some green pencils. We can pick a red pencils, b blue pencils, and c green pencils (with the tacit assumption that $a+b+c=m$) in $\binom{n}{a}\cdot\binom{n}{b}\cdot\binom{n}{c}$ ways. Finally, in order to account for all possible distributions of pencils, we need to sum over all valid triples (a,b,c), which gives us the expression

$$\sum_{a+b+c=m} \binom{n}{a}\cdot\binom{n}{b}\cdot\binom{n}{c},$$

which is exactly the left hand side of the identity we want to prove.

3 Fibonacci Fashion

Note 10

You have n accessories in your wardrobe, and you'd like to plan which ones to wear each day for the next t days. As a student of the Elegant Etiquette Charm School, you know it isn't fashionable to wear the same accessories multiple days in a row. (Note that the same goes for clothing items in general). Therefore, you'd like to plan which accessories to wear each day represented by subsets $S_1, S_2, \dots, S_t$, where $S_1 \subseteq \{1,2,\dots,n\}$ and for $2 \le i \le t$, $S_i \subseteq \{1,2,\dots,n\}$ and S_i is disjoint from S_{i-1}.

(a) For $t \ge 1$, prove that there are F_{t+2} binary strings of length t with no consecutive zeros (assume the Fibonacci sequence starts with $F_0 = 0$ and $F_1 = 1$).

(b) Use a combinatorial proof to prove the following identity, which, for $t \ge 1$ and $n \ge 0$, gives the number of ways you can create subsets of your n accessories for the next t days such that no accessory is worn two days in a row:

$$\sum_{x_1 \ge 0} \sum_{x_2 \ge 0} \cdots \sum_{x_t \ge 0} \binom{n}{x_1}\binom{n-x_1}{x_2}\binom{n-x_2}{x_3}\cdots\binom{n-x_{t-1}}{x_t} = (F_{t+2})^n.$$

(You may assume that $\binom{a}{b} = 0$ whenever $a < b$.)

Solution:

(a) We will prove this by strong induction.

Base cases: For $k = 1$, the only binary strings possible are 0 and 1. Therefore, there are two possible binary strings, and $F_{k+2} = F_3 = 2$. For $k = 2$, the binary strings possible are 11, 01, and 10, and we have $F_{k+2} = F_4 = 3$, so the identity holds.

Inductive hypothesis: For $k \geq 2$, assume that for all $1 \leq x \leq k$, there are F_{x+2} binary strings of length x with no consecutive zeros.

Inductive step: Consider the set of binary strings of length $k+1$ with no consecutive zeros. We can group these into two sets: those which end with 0, and those which end with 1.
For those that end with a 0, these can be constructed by taking the set of binary strings of length $k-1$ with no consecutive zeros and appending 10 to the end of them. Then by the inductive hypothesis, this set is of size F_{k+1}. For those that end with a 1, these can be constructed by taking the set of binary strings of length k with no consecutive zeros and appending a 1 to the end of them. Then by the inductive hypothesis, this set is of size F_{k+2}.
Since the union of these two subsets (those which end with 0 and those which end with 1) cover all possible elements in the set of binary strings of length $k+1$ with no consecutive zeros, the size of this set will be $F_{k+1} + F_{k+2} = F_{k+3}$. This thus proves the inductive hypothesis.

(b) We first consider the left-hand-side of the identity. To create subsets of accessories that are consecutively disjoint with sizes $x_i = |S_i|$, $1 \leq i \leq n$, there are $\binom{n}{x_1}$ ways to create S_1, the subset of accessories you will wear on the first day. Then since S_2 must be disjoint from S_1, there are $\binom{n-x_1}{x_2}$ ways choose accessories to create S_2. Since S_3 must be disjoint from S_2, there are $\binom{n-x_2}{x_3}$ ways choose accessories to create S_3, and so on. Thus there are $\binom{n}{x_1}\binom{n-x_1}{x_2}\cdots\binom{n-x_{t-1}}{x_t}$ ways to create subsets of accessories $S_1, \ldots, S_t$ with respective sizes $x_1, \ldots, x_t$. Then altogether, $S_1, \ldots, S_t$ can be created in

$$\sum_{x_1 \geq 0}\sum_{x_2 \geq 0} \cdots \sum_{x_t \geq 0} \binom{n}{x_1}\binom{n-x_1}{x_2}\binom{n-x_2}{x_3}\cdots\binom{n-x_{t-1}}{x_t}$$

ways.

Now, consider the right-hand-side of the identity. Now for each accessory $i \in \{1, \ldots, n\}$, we will first decide which subsets $S_1, \ldots, S_t$ will contain accessory i, where we can't assign item i to consecutive subsets. For each accessory, we create a binary string of length t, where the leading digit represents S_1, the next digit represents S_2, and so on. We will say that a 0 in digit k means that we will wear the accessory on day k. Therefore, the number of ways we can assign accessory i to subsets $S_1, \ldots, S_t$ such that no two consecutive subsets both have accessory i is the same as the number of binary strings of length t with no consecutive zeros. Thus using the result in part (a), there are F_{t+2} ways to select the nonconsecutive subsets containing i among $S_1, \ldots, S_t$. Since we have n accessories, accessories $1, \ldots, n$ can be placed into subsets $S_1, \ldots, S_t$ in $(F_{t+2})^n$ ways.

This thus proves the identity.

4 Unions and Intersections

Note 11 Given:

- X is a countable, non-empty set. For all $i \in X$, A_i is an uncountable set.
- Y is an uncountable set. For all $i \in Y$, B_i is a countable set.

For each of the following, decide if the expression is "Always countable", "Always uncountable", "Sometimes countable, Sometimes uncountable".

For the "Always" cases, prove your claim. For the "Sometimes" case, provide two examples – one where the expression is countable, and one where the expression is uncountable.

(a) $X \cap Y$

(b) $X \cup Y$

(c) $\bigcup_{i \in X} A_i$

(d) $\bigcap_{i \in X} A_i$

(e) $\bigcup_{i \in Y} B_i$

(f) $\bigcap_{i \in Y} B_i$

Solution:

(a) Always countable. $X \cap Y$ is a subset of X, which is countable.

(b) Always uncountable. $X \cup Y$ is a superset of Y, which is uncountable.

(c) Always uncountable. Let x be any element of X. A_x is uncountable. Thus, $\bigcup_{i \in X} A_i$, a superset of A_x, is uncountable.

(d) Sometimes countable, sometimes uncountable.

Countable: When the A_i are disjoint, the intersection is empty, and thus countable. For example, let $X = \mathbb{N}$, let $A_i = \{i\} \times \mathbb{R} = \{(i,x) \mid x \in \mathbb{R}\}$. Then, $\bigcap_{i \in X} A_i = \varnothing$.

Uncountable: When the A_i are identical, the intersection is uncountable. Let $X = \mathbb{N}$, let $A_i = \mathbb{R}$ for all i. $\bigcap_{i \in X} A_i = \mathbb{R}$ is uncountable.

(e) Sometimes countable, sometimes uncountable.

Countable: Make all the B_i identical. For example, let $Y = \mathbb{R}$, and $B_i = \mathbb{N}$. Then, $\bigcup_{i \in Y} B_i = \mathbb{N}$ is countable.

Uncountable: Let $Y = \mathbb{R}$. Let $B_i = \{i\}$. Then, $\bigcup_{i \in Y} B_i = \mathbb{R}$ is uncountable.

(f) Always countable. Let y be any element of Y. B_y is countable. Thus, $\bigcap_{i \in Y} B_i$, a subset of B_y, is also countable.

5 Count It!

Note 11

For each of the following collections, determine and briefly explain whether it is finite, countably infinite (like the natural numbers), or uncountably infinite (like the reals):

(a) The integers which divide 8.

(b) The integers which 8 divides.

(c) The functions from $\mathbb{N}$ to $\mathbb{N}$.

(d) The set of strings over the English alphabet. (Note that the strings may be arbitrarily long, but each string has finite length. Also the strings need not be real English words.)

(e) The set of finite-length strings drawn from a countably infinite alphabet, $\mathscr{C}$.

(f) The set of infinite-length strings over the English alphabet.

Solution:

(a) Finite. They are $\{-8,-4,-2,-1,1,2,4,8\}$.

(b) Countably infinite. We know that there exists a bijective function $f:\mathbb{N}\to\mathbb{Z}$. Then the function $g(n)=8f(n)$ is a bijective mapping from $\mathbb{N}$ to integers which 8 divides.

(c) Uncountably infinite. We use Cantor's Diagonalization Proof:

Let $\mathscr{F}$ be the set of all functions from $\mathbb{N}$ to $\mathbb{N}$. We can represent a function $f\in\mathscr{F}$ as an infinite sequence $(f(0),f(1),\cdots)$, where the i-th element is $f(i)$. Suppose towards a contradiction that there is a bijection from $\mathbb{N}$ to $\mathscr{F}$:

$$\begin{aligned}
0 &\longleftrightarrow (f_0(0),f_0(1),f_0(2),f_0(3),\ldots)\\
1 &\longleftrightarrow (f_1(0),f_1(1),f_1(2),f_1(3),\ldots)\\
2 &\longleftrightarrow (f_2(0),f_2(1),f_2(2),f_2(3),\ldots)\\
3 &\longleftrightarrow (f_3(0),f_3(1),f_3(2),f_3(3),\ldots)\\
&\vdots
\end{aligned}$$

Consider the function $g:\mathbb{N}\to\mathbb{N}$ where $g(i)=f_i(i)+1$ for all $i\in\mathbb{N}$. We claim that the function g is not in our finite list of functions. Suppose for contradiction that it were, and that it was the n-th function $f_n(\cdot)$ in the list, i.e., $g(\cdot)=f_n(\cdot)$. However, $f_n(\cdot)$ and $g(\cdot)$ differ in the n-th argument, i.e. $f_n(n)\neq g(n)$, because by our construction $g(n)=f_n(n)+1$. Contradiction!

(d) Countably infinite. The English language has a finite alphabet (52 characters if you count only lower-case and upper-case letters, or more if you count special symbols – either way, the alphabet is finite).

We will now enumerate the strings in such a way that each string appears exactly once in the list. We will use the same trick as used in Lecture note 10 to enumerate the elements of $\{0,1\}^*$ We get our bijection by setting $f(n)$ to be the n-th string in the list. List all strings of length 1 in lexicographic order, and then all strings of length 2 in lexicographic order, and then strings of length 3 in lexicographic order, and so forth. Since at each step, there are only finitely many strings of a particular length ℓ, any string of finite length appears in the list. It is also clear that each string appears exactly once in this list.

(e) Countably infinite. Let $\mathscr{C} = \{a_1, a_2, \dots\}$ denote the alphabet. (We are making use of the fact that the alphabet is countably infinite when we assume there is such an enumeration.) We will provide two solutions:

Alternative 1: We will enumerate all the strings similar to that in part (b), although the enumeration requires a little more finesse. Notice that if we tried to list all strings of length 1, we would be stuck forever, since the alphabet is infinite! On the other hand, if we try to restrict our alphabet and only print out strings containing the first character $a \in \mathscr{C}$, we would also have a similar problem: the list

$$a, aa, aaa, \dots$$

also does not end.

The idea is to restrict *both* the length of the string and the characters we are allowed to use:

(a) List all strings containing only a_1 which are of length at most 1.

(b) List all strings containing only characters in $\{a_1, a_2\}$ which are of length at most 2 and have not been listed before.

(c) List all strings containing only characters in $\{a_1, a_2, a_3\}$ which are of length at most 3 and have not been listed before.

(d) Proceed onwards.

At each step, we have restricted ourselves to a finite alphabet with a finite length, so each step is guaranteed to terminate. To show that the enumeration is complete, consider any string s of length ℓ; since the length is finite, it can contain at most ℓ distinct a_i from the alphabet. Let k denote the largest index of any a_i which appears in s. Then, s will be listed in step $\max(k, \ell)$, so it appears in the enumeration. Further, since we are listing only those strings that have not appeared before, each string appears exactly once in the listing.

Alternative 2: We will encode the strings into ternary strings. Recall that we used a similar trick in Lecture note 10 to show that the set of all polynomials with natural coefficients is countable. Suppose, for example, we have a string: $S = a_5 a_2 a_7 a_4 a_6$. Corresponding to each of the characters in this string, we can write its index as a binary string: $(101, 10, 111, 100, 110)$. Now, we can construct a ternary string where "2" is inserted as a separator between each binary string. Thus we map the string S to a ternary string: 101210211121002110. It is clear that this mapping is injective, since the original string S can be uniquely recovered from this ternary string. Thus we have an injective map to $\{0,1,2\}^*$. From Lecture note 10, we know that the set $\{0,1,2\}^*$ is countable, and hence the set of all strings with finite length over $\mathscr{C}$ is countable.

(f) Uncountably infinite. We can use a diagonalization argument. First, for a string s, define $s[i]$ as the i-th character in the string (where the first character is position 0), where $i \in \mathbb{N}$ because the strings are infinite. Now suppose for contradiction that we have an enumeration of strings s_i for all $i \in \mathbb{N}$: then define the string s' as $s'[i] =$ (the next character in the alphabet after $s_i[i]$), where the character after z loops around back to a. Then s' differs at position i from s_i for all $i \in \mathbb{N}$, so it is not accounted for in the enumeration, which is a contradiction. Thus, the set is uncountable.

Alternative 1: The set of all infinite strings containing only as and bs is a subset of the set we're counting. We can show a bijection from this subset to the real interval $\mathbb{R}[0,1]$, which proves the uncountability of the subset and therefore entire set as well: given a string in $\{a,b\}^*$, replace the as with 0s and bs with 1s and prepend $'0.'$ to the string, which produces a unique binary number in $\mathbb{R}[0,1]$ corresponding to the string.

6 Countability Proof Practice

Note 11

(a) A disk is a 2D region of the form $\{(x,y) \in \mathbb{R}^2 : (x-x_0)^2 + (y-y_0)^2 \leq r^2\}$, for some $x_0, y_0, r \in \mathbb{R}$, $r > 0$. Say you have a set of disks in $\mathbb{R}^2$ such that none of the disks overlap (with possibly varying x_0, y_0, and r values). Is this set always countable, or potentially uncountable?

(*Hint*: Attempt to relate it to a set that we know is countable, such as $\mathbb{Q} \times \mathbb{Q}$.)

(b) A circle is a subset of the plane of the form $\{(x,y) \in \mathbb{R}^2 : (x-x_0)^2 + (y-y_0)^2 = r^2\}$ for some $x_0, y_0, r \in \mathbb{R}$, $r > 0$. Now say you have a set of circles in $\mathbb{R}^2$ such that none of the circles overlap (with possibly varying x_0, y_0, and r values). Is this set always countable, or potentially uncountable?

(*Hint*: The difference between a circle and a disk is that a disk contains all of the points in its interior, whereas a circle does not.)

Solution:

(a) Countable. Each disk must contain at least one rational point (an (x,y)-coordinate where $x, y \in \mathbb{Q}$) in its interior, and due to the fact that no two disks overlap, the cardinality of the set of disks can be no larger than the cardinality of $\mathbb{Q} \times \mathbb{Q}$, which we know to be countable.

(b) Possibly uncountable. Consider the circles $C_r = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = r\}$ for each $r \in \mathbb{R}$. For $r_1 \neq r_2$, C_{r_1} and C_{r_2} do not overlap, and there are uncountably many of these circles (one for each real number).