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| CS 70 | Discrete Mathematics and Probability Theory | |
| Fall 2023 | Tal, Rao | HW 09 |

# 1 Probability Potpourri

Note 13
Note 14

Provide brief justification for each part.

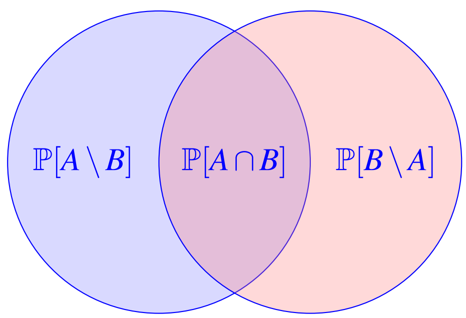
(a) For two events $A$ and $B$ in any probability space, show that $\mathbb{P}[A \setminus B] \geq \mathbb{P}[A] - \mathbb{P}[B]$.

(b) Suppose $\mathbb{P}[D \mid C] = \mathbb{P}[D \mid \overline{C}]$, where $\overline{C}$ is the complement of $C$. Prove that $D$ is independent of $C$.

(c) If $A$ and $B$ are disjoint, does that imply they're independent?

**Solution:**

(a) It can be helpful to first draw out a Venn diagram:




We can see here that $\mathbb{P}[A] = \mathbb{P}[A \cap B] + \mathbb{P}[A \setminus B]$, and that $\mathbb{P}[B] = \mathbb{P}[A \cap B] + \mathbb{P}[B \setminus A]$. Looking at the RHS, we have

$$\begin{aligned}\mathbb{P}[A] - \mathbb{P}[B] &= (\mathbb{P}[A \cap B] + \mathbb{P}[A \setminus B]) - (\mathbb{P}[A \cap B] + \mathbb{P}[B \setminus A]) \\ &= \mathbb{P}[A \setminus B] - \mathbb{P}[B \setminus A] \\ &\leq \mathbb{P}[A \setminus B]\end{aligned}$$

(b) Using the total probability rule, we have

$$\mathbb{P}[D] = \mathbb{P}[D \cap C] + \mathbb{P}[D \cap \overline{C}] = \mathbb{P}[D \mid C] \cdot \mathbb{P}[C] + \mathbb{P}[D \mid \overline{C}] \cdot \mathbb{P}[\overline{C}].$$

But we know that $\mathbb{P}[D \mid C] = \mathbb{P}[D \mid \overline{C}]$, so this simplifies to

$$\mathbb{P}[D] = \mathbb{P}[D \mid C] \cdot (\mathbb{P}[C] + \mathbb{P}[\overline{C}]) = \mathbb{P}[D \mid C] \cdot 1 = \mathbb{P}[D \mid C],$$

which defines independence.

(c) No; if two events are disjoint, we cannot conclude they are independent. Consider a roll of a fair six-sided die. Let $A$ be the event that we roll a 1, and let $B$ be the event that we roll a 2. Certainly $A$ and $B$ are disjoint, as $\mathbb{P}[A \cap B] = 0$. But these events are not independent: $\mathbb{P}[B \mid A] = 0$, but $\mathbb{P}[B] = 1/6$.

Since disjoint events have $\mathbb{P}[A \cap B] = 0$, we can see that the only time when disjoint $A$ and $B$ are independent is when either $\mathbb{P}[A] = 0$ or $\mathbb{P}[B] = 0$.

# 2 Independent Complements

Note 14

Let $\Omega$ be a sample space, and let $A, B \subseteq \Omega$ be two independent events.

(a) Prove or disprove: $\overline{A}$ and $\overline{B}$ must be independent.

(b) Prove or disprove: $A$ and $\overline{B}$ must be independent.

(c) Prove or disprove: $A$ and $\overline{A}$ must be independent.

(d) Prove or disprove: It is possible that $A = B$.

**Solution:**

(a) True. $\overline{A}$ and $\overline{B}$ must be independent:

$$\begin{aligned}
\mathbb{P}[\overline{A} \cap \overline{B}] &= \mathbb{P}[\overline{A \cup B}] && \text{(by De Morgan's law)} \\
&= 1 - \mathbb{P}[A \cup B] && \text{(since } \mathbb{P}[\overline{E}] = 1 - \mathbb{P}[E] \text{ for all } E\text{)} \\
&= 1 - (\mathbb{P}[A] + \mathbb{P}[B] - \mathbb{P}[A \cap B]) && \text{(union of overlapping events)} \\
&= 1 - \mathbb{P}[A] - \mathbb{P}[B] + \mathbb{P}[A]\mathbb{P}[B] && \text{(since } A \text{ and } B \text{ are independent)} \\
&= (1 - \mathbb{P}[A])(1 - \mathbb{P}[B]) \\
&= \mathbb{P}[\overline{A}]\mathbb{P}[\overline{B}] && \text{(since } \mathbb{P}[\overline{E}] = 1 - \mathbb{P}[E] \text{ for all } E\text{)}
\end{aligned}$$

(b) True. $A$ and $\overline{B}$ must be independent:

$$\begin{aligned}
\mathbb{P}[A \cap \overline{B}] &= \mathbb{P}[A - (A \cap B)] \\
&= \mathbb{P}[A] - \mathbb{P}[A \cap B] \\
&= \mathbb{P}[A] - \mathbb{P}[A]\mathbb{P}[B] \\
&= \mathbb{P}[A](1 - \mathbb{P}[B]) \\
&= \mathbb{P}[A]\mathbb{P}[\overline{B}]
\end{aligned}$$

(c) False in general. If $0 < \mathbb{P}[A] < 1$, then $\mathbb{P}[A \cap \overline{A}] = \mathbb{P}[\varnothing] = 0$ but $\mathbb{P}[A]\mathbb{P}[\overline{A}] > 0$, so $\mathbb{P}[A \cap \overline{A}] \neq \mathbb{P}[A]\mathbb{P}[\overline{A}]$; therefore $A$ and $\overline{A}$ are not independent in this case.

(d) True. To give one example, if $\mathbb{P}[A] = \mathbb{P}[B] = 0$, then $\mathbb{P}[A \cap B] = 0 = 0 \times 0 = \mathbb{P}[A]\mathbb{P}[B]$, so $A$ and $B$ are independent in this case. (Another example: If $A = B$ and $\mathbb{P}[A] = 1$, then $A$ and $B$ are independent.)

## 3 Monty Hall's Revenge

Note 13
Note 14

Due to a quirk of the television studio's recruitment process, Monty Hall has ended up drawing all the contestants for his game show from among the ranks of former CS70 students. Unfortunately for Monty, the former students' amazing probability skills have made his cars-and-goats gimmick unprofitable for the studio. Monty decides to up the stakes by asking his contestants to generalize to three new situations with a variable number of doors, goats, and cars:

(a) There are $n$ doors for some $n > 2$. One has a car behind it, and the remaining $n-1$ have goats. As in the ordinary Monty Hall problem, Monty will reveal one door with a goat behind it after you make your first selection. Compute the probability of winning if you switch, as well as the probability of winning if you don't switch, and compare the results.

(Hint: Think about the size of the sample space for the experiment where you *always* switch. How many of those outcomes are favorable?)

(b) Again there are $n > 2$ doors, one with a car and $n-1$ with goats, but this time Monty will reveal $n-2$ doors with goats behind them instead of just one. How does switching affect the probability of winning in this modified scenario?

(c) Finally, imagine there are $k < n-1$ cars and $n-k$ goats behind the $n > 2$ doors. After you make your first pick, Monty will reveal $j < n-k$ doors with goats. What values of $j,k$ maximize the relative improvement in your probability of winning if you choose to switch? (i.e. what $j,k$ maximizes the ratio between your probability of winning when you switch, and your probability of winning when you do not switch?)

**Solution:**

Throughout the solution, we will refer to $W$ as the event that the contestant wins, and $\mathbb{P}_S[W]$ and $\mathbb{P}_N[W]$ as the probabilities of this event happening if the contestant is (S)witching or (N)ot switching, respectively.

(a) $\mathbb{P}_N[W] = 1/n$ since only one out of $n$ initial choices gets us the car. Under the switching strategy two things can happen: Either the first choice hits the car, and so switching (to any of the remaining $n-2$ doors) will inevitably get us the goat, or our first choice picks a goat, leaving one of the remaining $n-2$ doors with the car. This sequence of choices—first choosing from one of $n$ doors, then switching to one of $n-2$ remaining doors—gives us a sample space of size $n(n-2)$. If we divide the number of favorable outcomes by the total number of outcomes, we get

$$\mathbb{P}_S[W] = \left( \underbrace{(n-1)}_{\text{first choice = goat}} \cdot \underbrace{1}_{\text{second choice = car}} \right) \Big/ \underbrace{n(n-2)}_{\text{total \# of choices}}$$
$$= \frac{n-1}{n(n-2)} = \frac{1}{n} \cdot \frac{n-1}{n-2}$$

which is larger than $\mathbb{P}_N[W] = 1/n$ (ever so slightly so the larger $n$ becomes, which demonstrates the intuitive fact that Monty's help gets decreasingly helpful the more doors there are), so switching doors is the better strategy.

(b) $\mathbb{P}_N[W] = 1/n$ remains unchanged. The same approach as in part (a) yields the same numerator as before. For the denominator, we need to figure out the size of the sample space for the experiment where we first pick a door at random, then switch. Again, there are $n$ ways of making the first choice. Once Monty reveals $n-2$ other doors, though, there is only one remaining option for us to switch to. Thus the denominator is much smaller:

$$\begin{aligned}\mathbb{P}_S[W] &= \left(\underbrace{(n-1)}_{\text{first choice = goat}} \cdot \underbrace{1}_{\text{second choice = car}}\right) \Big/ \underbrace{n \cdot 1}_{\text{total \# of choices}} \\ &= \frac{n-1}{n} = 1 - \frac{1}{n}\end{aligned}$$

so switching is again the better strategy.

(c) Now $\mathbb{P}_N[W] = k/n$ since $k$ doors hide a car. Reasoning about sample spaces in the same way we did in part (b) gives us a way to compute the denominator of $\mathbb{P}_S[W]$. However, now the numerator (number of favorable outcomes in the case where we switch) changes too:

$$\begin{aligned}\mathbb{P}_S[W] &= \left(\underbrace{k}_{\text{first choice = car}} \cdot \underbrace{k-1}_{\text{second choice = car}} + \underbrace{(n-k)}_{\text{first choice = goat}} \cdot \underbrace{k}_{\text{second choice = car}}\right) \Big/ \underbrace{n(n-j-1)}_{\text{total \# of choices}} \\ &= \frac{k(n-1)}{n(n-j-1)} = \frac{k}{n} \cdot \frac{n-1}{n-j-1}.\end{aligned}$$

From here we see that $\mathbb{P}_S[W]/\mathbb{P}_N[W] = \frac{n-1}{n-j-1}$, which is maximal if $j = n-k-1$. In other words, if Monty reveals all but one goat (which he does in the original show where $n = 3, k = 1$ and $j = 1 = n-k-1$), then the contestant can increase their chances of winning by a factor of $\frac{n-1}{k}$ (which is a factor of 2 in the original show). In particular, the largest relative advantage of switching is achieved when $k = 1$.

## 4 Cliques in Random Graphs

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Note 14

Consider the graph $G = (V, E)$ on $n$ vertices which is generated by the following random process: for each pair of vertices $u$ and $v$, we flip a fair coin and place an (undirected) edge between $u$ and $v$ if and only if the coin comes up heads.

(a) What is the size of the sample space?

(b) A $k$-clique in a graph is a set $S$ of $k$ vertices which are pairwise adjacent (every pair of vertices is connected by an edge). For example, a 3-clique is a triangle. Let $E_S$ be the event that a set $S$ forms a clique. What is the probability of $E_S$ for a particular set $S$ of $k$ vertices?

(c) Suppose that $V_1 = \{v_1, \dots, v_\ell\}$ and $V_2 = \{w_1, \dots, w_k\}$ are two arbitrary sets of vertices. What conditions must $V_1$ and $V_2$ satisfy in order for $E_{V_1}$ and $E_{V_2}$ to be independent? Prove your answer.

(d) Prove that $\binom{n}{k} \le n^k$. (You might find this useful in part (e))

(e) Prove that the probability that the graph contains a $k$-clique, for $k \ge 4\log_2 n + 1$, is at most $1/n$.

**Solution:**

(a) Between every pair of vertices, there is either an edge or there isn't. Since there are two choices for each of the $\binom{n}{2}$ pairs of vertices, the size of the sample space is $2^{\binom{n}{2}}$.

(b) For a fixed set of $k$ vertices to be a $k$-clique, all of the $\binom{k}{2}$ pairs of those vertices have to be connected by an edge. The probability of this event is $1/2^{\binom{k}{2}}$.

(c) $E_{V_1}$ and $E_{V_2}$ are independent if and only if $V_1$ and $V_2$ share at most one vertex: If $V_1$ and $V_2$ share at most one vertex, then since edges are added independently of each other, we have

$$\begin{aligned}
\mathbb{P}[E_{V_1} \cap E_{V_2}] &= \mathbb{P}[\text{all edges in } V_1 \text{ and all edges in } V_2 \text{ are present}] \\
&= \left(\frac{1}{2}\right)^{\binom{|V_1|}{2}} \cdot \left(\frac{1}{2}\right)^{\binom{|V_2|}{2}} \\
&= \mathbb{P}[E_{V_1}] \cdot \mathbb{P}[E_{V_2}].
\end{aligned}$$

Conversely, if $V_1$ and $V_2$ share at least two vertices, then their intersection $V_3 = V_1 \cap V_2$ has at least 2 elements, so we have

$$\begin{aligned}
\mathbb{P}[E_{V_1} \cap E_{V_2}] &= \left(\frac{1}{2}\right)^{\binom{|V_3|}{2}} \cdot \left(\frac{1}{2}\right)^{\binom{|V_1|}{2} - \binom{|V_3|}{2}} \cdot \left(\frac{1}{2}\right)^{\binom{|V_2|}{2} - \binom{|V_3|}{2}} \\
&= \left(\frac{1}{2}\right)^{\binom{|V_1|}{2} + \binom{|V_2|}{2} - \binom{|V_3|}{2}} \neq \mathbb{P}[E_{V_1}] \cdot \mathbb{P}[E_{V_2}].
\end{aligned}$$

(d) The algebraic solution is an application of the definition of $\binom{n}{k}$:

$$\begin{aligned}
\binom{n}{k} = \frac{n!}{(n-k)!k!} &= \frac{n \cdot (n-1) \cdots (n-k+1)}{k!} \\
&\le n \cdot (n-1) \cdots (n-k+1) \\
&\le n^k
\end{aligned}$$

(e) Let $A_S$ denote the event that $S$ is a $k$-clique, where $S \subseteq V$ is of size $k$. Then, the event that the graph contains a $k$-clique can be described as the union of $A_S$'s over all $S \subseteq V$ of size $k$. Using the union bound,

$$\mathbb{P}\left[\bigcup_{S \subseteq V, |S| = k} A_S\right] \le \sum_{S \subseteq V, |S| = k} \mathbb{P}[A_S] = \sum_{S \subseteq V, |S| = k} \frac{1}{2^{\binom{k}{2}}}.$$

Now, since there are $\binom{n}{k}$ ways of choosing a subset $S \subseteq V$ of size $k$, the right-hand side of the above equality is

$$\frac{\binom{n}{k}}{2^{\binom{k}{2}}} = \frac{\binom{n}{k}}{2^{k(k-1)/2}} \le \frac{n^k}{\left(2^{(k-1)/2}\right)^k} \le \frac{n^k}{\left(2^{(4\log n+1-1)/2}\right)^k} = \frac{n^k}{\left(2^{2\log n}\right)^k} = \frac{n^k}{n^{2k}} = \frac{1}{n^k} \le \frac{1}{n}.$$

# 5 Symmetric Marbles

Note 14 A bag contains 4 red marbles and 4 blue marbles. Rachel and Brooke play a game where they draw four marbles in total, one by one, uniformly at random, without replacement. Rachel wins if there are more red than blue marbles, and Brooke wins if there are more blue than red marbles. If there are an equal number of marbles, the game is tied.

(a) Let $A_1$ be the event that the first marble is red and let $A_2$ be the event that the second marble is red. Are $A_1$ and $A_2$ independent?

(b) What is the probability that Rachel wins the game?

(c) Given that Rachel wins the game, what is the probability that all of the marbles were red?

Now, suppose the bag contains 8 red marbles and 4 blue marbles. Moreover, if there are an equal number of red and blue marbles among the four drawn, Rachel wins if the third marble is red, and Brooke wins if the third marble is blue. All other rules stay the same.

(d) What is the probability that the third marble is red?

(e) Given that there are $k$ red marbles among the four drawn, where $0 \le k \le 4$, what is the probability that the third marble is red? Answer in terms of $k$.

(f) Given that the third marble is red, what is the probability that Rachel wins the game?

**Solution:**

(a) They are not independent; removing one red marble lowers the probability of the next marble being red.

(b) Let $p$ be the probability that Rachel wins. Since there are an equal number of red and blue marbles, by symmetry, the probability that Rachel wins and the probability that Brooke wins is the same. Thus, the probability that there is a tie is $1-p-p = 1-2p$.

We now compute the probability that there is a tie. For there to be a tie, two of the four marbles need to be red. There are $\binom{8}{4}$ ways to pick 4 marbles, and $\binom{4}{2}\binom{4}{2}$ to pick 2 red and blue marbles, respectively, giving a probability of

$$\frac{\binom{4}{2}\binom{4}{2}}{\binom{8}{4}} = \frac{36}{70} = \boxed{\frac{18}{35}}.$$

We conclude that $1-2p = \frac{18}{35}$. Solving for $p$ gives $p = \boxed{\frac{17}{70}}$.

(c) Let $A$ be the event that there are 3 red marbles drawn, and let $B$ be the event that there are 4 red marbles drawn. We wish to compute

$$\mathbb{P}[B \mid (A \cup B)] = \frac{\mathbb{P}[B \cap (A \cup B)]}{\mathbb{P}[A \cup B]} = \frac{\mathbb{P}[B]}{\mathbb{P}[A] + \mathbb{P}[B]}.$$

Similar to the calculation in part (b), the probability that there are 3 red marbles drawn is $\frac{\binom{4}{3}\binom{4}{1}}{\binom{8}{4}} = \frac{16}{70}$, and the probability that there are 4 red marbles drawn is $\frac{\binom{4}{4}\binom{4}{0}}{\binom{8}{4}} = \frac{1}{70}$, giving a final answer of $\frac{\frac{1}{70}}{\frac{16}{70}+\frac{1}{70}} = \boxed{\frac{1}{17}}$.

(d) By symmetry, the probability that the third marble is red is the same as the probability that the first marble is red, or the same as any marble being red. One way to see this is to imagine drawing the four marbles in order, then moving the first marble drawn to the third position. This is another way to draw four marbles that yields the same distribution.

There are 8 red marbles, and 12 marbles in total. Thus, the probability that the third marble is red is $\frac{8}{12} = \boxed{\frac{2}{3}}$.

(e) We are given that there are $k$ red marbles among the 4 drawn. By symmetry, each marble has the same probability of being red, so the probability that the third marble is red is $\boxed{\frac{k}{4}}$.

(f) The only way for Rachel to lose the game given that the third marble is red is if all the other marbles are blue. The probability that the third marble is red and all the other marbles are blue is $\frac{4}{12} \cdot \frac{3}{11} \cdot \frac{8}{10} \cdot \frac{2}{9} = \frac{8}{495}$, and the probability that the third marble is red is $\frac{8}{12} = \frac{2}{3}$, so the probability that Rachel loses given that the third marble is red is $\frac{\frac{8}{495}}{\frac{2}{3}} = \frac{4}{165}$, and the probability that Rachel wins given that the third marble is red is $\boxed{\frac{161}{165}}$.

## 6 Socks

Note 13
Note 14

Suppose you have $n$ different pairs of socks ($n$ left socks and $n$ right socks, for $2n$ individual socks total) in your dresser. You take the socks out of the dresser one by one without looking and lay them out in a row on the floor. In this question, we'll go through the computation of the probability that no two matching socks are next to each other.

(a) We can consider the sample space as the set of length $2n$ permutations. What is the size of the sample space $\Omega$, and what is the probability of a particular permutation $\omega \in \Omega$?

(b) Let $A_i$ be the event that the $i$th pair of matching socks are next to each other. Calculate $\mathbb{P}[A_i]$.

(c) Calculate $\mathbb{P}[A_1 \cap \ldots \cap A_k]$ for an arbitrary $k \geq 2$. (Hint: try using a counting based approach.)

(d) Putting these all together, calculate the probability that there is at least one pair of matching socks next to each other. Your answer can (and should) be expressed as a summation. (Hint: use Inclusion/Exclusion.)

(e) Using your answer from the previous part, what is the probability that no two matching socks are next to each other? (This should follow directly from your answer to the previous part, and also can be left as a summation.)

**Solution:**

(a) We have a uniform sample space of size $(2n)!$.

(b) Consider the $i$th matching pair as a single, condensed unit. As an example, in for $n = 3$, an original permutation could look like 132213. Let us condense both the 2's together, and label it as $B$. Then, a resulting string would look like $13B13$. Then, there are $2n - 1$ 'units' left that we can order, and thus $(2n-1)!$ ways to order them. Also, when we condensed them, either the left sock or the right sock could've came first, so there are 2 ways to condense this pair. Thus, the probability is $\frac{2(2n-1)!}{(2n)!}$.

(c) We will employ an analogous strategy to the previous part. We will consider all $k$ of these matching socks. There are $2^k$ ways to condense them. Once condensed, there are $(2n-k)!$ ways to order the remaining units. Thus, the probability is $2^k \frac{(2n-k)!}{(2n)!}$.

(d) We look for:

$$\begin{aligned}
\mathbb{P}[A] &= \mathbb{P}[A_1 \cup A_2 \cup \ldots \cup A_n] \\
&= \sum_{i=1}^{n} \mathbb{P}[A_i] - \sum_{1 \leq i < j \leq n} \mathbb{P}[A_i \cap A_j] + \cdots \\
&= \sum_{i=1}^{n} 2 \cdot \frac{(2n-1)!}{(2n)!} - \sum_{1 \leq i < j \leq n} 2^2 \cdot \frac{(2n-2)!}{(2n)!} + \cdots \\
&= \binom{n}{1} 2^1 \cdot \frac{(2n-1)!}{(2n)!} - \binom{n}{2} 2^2 \cdot \frac{(2n-2)!}{(2n)!} + \cdots \\
&= \sum_{k=1}^{n} (-1)^{k+1} \binom{n}{k} 2^k \cdot \frac{(2n-k)!}{(2n)!}
\end{aligned}$$

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(e) This is just the complement of the previous part, which becomes

$$1 - \mathbb{P}[A] = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{2^k (2n-k)!}{(2n)!}.$$