1 CS70: The Musical

Edward, one of the previous head TA's, has been hard at work on his latest project, *CS70: The Musical*. It's now time for him to select a cast, crew, and directing team to help him make his dream a reality.

(a) First, Edward would like to select directors for his musical. He has received applications from $2n$ directors. Use this to provide a combinatorial argument that proves the following identity:

$$\binom{2n}{2} = 2\binom{n}{2} + n^2.$$

(b) Edward would now like to select a crew out of n people. Use this to provide a combinatorial argument that proves the following identity: (this is called Pascal's Identity)

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

(c) There are n actors lined up outside of Edward's office, and they would like a role in the musical (including a lead role). However, he is unsure of how many individuals he would like to cast. Use this to provide a combinatorial argument that proves the following identity:

$$\sum_{k=1}^{n} k\binom{n}{k} = n2^{n-1}$$

(d) Generalizing the previous part, provide a combinatorial argument that proves the following identity:

$$\sum_{k=j}^{n} \binom{n}{k}\binom{k}{j} = 2^{n-j}\binom{n}{j}.$$

Solution:

(a) Say that we would like to select 2 directors.

LHS: This is the number of ways to choose 2 directors out of the $2n$ candidates.

RHS: Split the $2n$ directors into two groups of n; one group consisting of experienced directors, or inexperienced directors (you can split arbitrarily). Then, we consider three cases: either we choose:

(a) Both directors from the group of experienced directors,

(b) Both directors from the group of inexperienced directors, or

(c) One experienced director and one inexperienced director.

The number of ways we can do each of these things is $\binom{n}{2}$, $\binom{n}{2}$, and n^2, respectively. Since these cases are mutually exclusive and cover all possibilities, it must also count the total number of ways to choose 2 directors out of the $2n$ candidates. This completes the proof.

(b) Say that we would like to select k crew members.

LHS: This is simply the number of ways to choose k crew members out of n candidates.

RHS: We select the k crew members in a different way. First, Edward looks at the first candidate he sees and decides whether or not he would like to choose the candidate. If he selects the first candidate, then Edward needs to choose $k-1$ more crew members from the remaining $n-1$ candidates. Otherwise, he needs to select all k crew members from the remaining $n-1$ candidates.

We are not double counting here - since in the first case, Edward takes the first candidate he encounters, and in the other case, we do not.

(c) In this part, Edward selects a subset of the n actors to be in his musical. Additionally, assume that he must select one individual as the lead for his musical.

LHS: Edward casts k actors in his musical, and then selects one lead among them (note that k $= \binom{k}{1}$). The summation is over all possible sizes for the cast - thus, the expression accounts for all subsets of the n actors.

RHS: From the n people, Edward selects one lead for his musical. Then, for the remaining $n-1$ actors, he decides whether or not he would like to include them in the cast. 2^{n-1} represents the amount of (possibly empty) subsets of the remaining actors. *(Note that for each actor, Edward has 2 choices: to include them, or to exclude them.)*

(d) In this part, Edward selects a subset of the n actors to be in the musical; additionally he must select j lead actors (instead of only 1 in the previous part).

LHS: Edward casts $k \geq j$ actors in his musical, then selects the j leads among them. Again, the summation is over all possible sizes for the cast (note that any cast that has $< j$ members is invalid, since Edward would be unable to select j lead actors) - thus, the expression accounts for all valid subsets of the n actors.

RHS: From the n people, Edward selects j leads for his musical. Then, for the remaining $n-j$ actors, he decides whether or not he would like to include them in the cast. Then, for the remaining $n-j$ actors, he decides whether or not he would like to include them in the cast. 2^{n-j} represents the amount of ways that Edward can do this.

2 Inclusion and Exclusion

Note 10

What is the total number of positive integers strictly less than 100 that are also coprime to 100?

Solution: It is sufficient to count the opposite: what is the total number of positive integers strictly less than 100 and *not* coprime to 100?

If a number is not coprime to 100, this means that the number is either a multiple of 2 or a multiple of 5. In this case, we have:

- 49 multiples of 2

- 19 multiples of 5

- 9 multiples of both 2 and 5

By inclusion-exclusion, the total number of positive integers not coprime to 100 is $49+19-9=59$, and there are 99 positive integers strictly less than 100.

As such, in total there are $99 - 59 = 40$ different positive integers strictly less than 100 that are coprime to 100.

3 Countability: True or False

(a) The set of all irrational numbers $\mathbb{R}\backslash\mathbb{Q}$ (i.e. real numbers that are not rational) is uncountable.

(b) The set of integers x that solve the equation $3x \equiv 2 \pmod{10}$ is countably infinite.

(c) The set of real solutions for the equation $x+y=1$ is countable.

For any two functions $f : Y \to Z$ and $g : X \to Y$, let their composition $f \circ g : X \to Z$ be given by $(f \circ g)(x) = f(g(x))$ for all $x \in X$. Determine if the following statements are true or false.

(d) f and g are injective (one-to-one) $\implies f \circ g$ is injective (one-to-one).

(e) f is surjective (onto) $\implies f \circ g$ is surjective (onto).

Solution:

(a) **True.** Proof by contradiction. Suppose the set of irrationals is countable. From Lecture note 10 we know that the set $\mathbb{Q}$ is countable. Since union of two countable sets is countable, this would imply that the set $\mathbb{R}$ is countable. But again from Lecture note 10 we know that this is not true. Contradiction!

(b) **True.** Multiplying both sides of the modular equation by 7 (the multiplicative inverse of 3 with respect to 10) we get $x \equiv 4 \pmod{10}$. The set of all intergers that solve this is $S = \{10k+4 : k \in \mathbb{Z}\}$ and it is clear that the mapping $k \in \mathbb{Z}$ to $10k+4 \in S$ is a bijection. Since the set $\mathbb{Z}$ is countably infinite, the set S is also countably infinite.

(c) **False.** Let $S \subset \mathbb{R} \times \mathbb{R}$ denote the set of all real solutions for the given equation. For any $x' \in \mathbb{R}$, the pair $(x',y') \in S$ if and only if $y' = 1-x'$. Thus $S = \{(x,1-x) : x \in \mathbb{R}\}$. Besides, the mapping x to $(x,1-x)$ is a bijection from $\mathbb{R}$ to S. Since $\mathbb{R}$ is uncountable, we have that S is uncountable too.

(d) **True.** Recall that a function $h : A \to B$ is injective iff $a_1 \neq a_2 \implies h(a_1) \neq h(a_2)$ for all $a_1, a_2 \in$ A. Let $x_1, x_2 \in X$ be arbitrary such that $x_1 \neq x_2$. Since g is injective, we have $g(x_1) \neq g(x_2)$. Now, since f is injective, we have $f(g(x_1)) \neq f(g(x_2))$. Hence $f \circ g$ is injective.

(e) **False.** Recall that a function $h : A \to B$ is surjective iff $\forall b \in B, \exists a \in A$ such that $h(a) = b$. Let $g : \{0,1\} \to \{0,1\}$ be given by $g(0) = g(1) = 0$. Let $f : \{0,1\} \to \{0,1\}$ be given by $f(0) = 0$ and $f(1) = 1$. Then $f \circ g : \{0,1\} \to \{0,1\}$ is given by $(f \circ g)(0) = (f \circ g)(1) = 0$. Here f is surjective but $f \circ g$ is not surjective.

4 Counting Cartesian Products

Note 11 For two sets A and B, define the cartesian product as $A \times B = \{(a,b) : a \in A, b \in B\}$.

(a) Given two countable sets A and B, prove that $A \times B$ is countable.

(b) Given a finite number of countable sets $A_1, A_2, \ldots, A_n$, prove that

$$A_1 \times A_2 \times \cdots \times A_n$$

is countable.

(c) Consider a countably infinite number of finite sets: $B_1, B_2, \ldots$ for which each set has at least 2 elements. Prove that $B_1 \times B_2 \times \cdots$ is uncountable.

Solution:

(a) As shown in lecture, $\mathbb{N} \times \mathbb{N}$ is countable by creating a zigzag map that enumerates through the pairs: $(0,0), (1,0), (0,1), (2,0), (1,1), \ldots$. Since A and B are both countable, there exists a bijection between each set and a subset of $\mathbb{N}$. Thus we know that $A \times B$ is countable because there is a bijection between a subset of $\mathbb{N} \times \mathbb{N}$ and $A \times B : f(i,j) = (A_i, B_j)$. We can enumerate the pairs (a,b) similarly.

(b) Proceed by induction.
Base Case: $n = 2$. We showed in part (a) that $A_1 \times A_2$ is countable since both A_1 and A_2 are countable.
Induction Hypothesis: Assume that for some $n \in \mathbb{N}$, $A_1 \times A_2 \times \cdots \times A_n$ is countable.
Induction Step: Consider $A_1 \times \cdots \times A_n \times A_{n+1}$. We know from our hypothesis that $A_1 \times \cdots \times A_n$ is countable, call it $C = A_1 \times \cdots \times A_n$. We proved in part (a) that since C is countable and A_{n+1} are countable, $C \times A_{n+1}$ is countable, which proves our claim.

(c) Let us assume that each B_i has size 2. If any of the sizes are greater than 2, that would only make the cartesian product larger. Notice that this is equivalent to the set of infinite length binary strings, which was proven to be uncountable in the notes.

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Alternatively, we could provide a diagonalization argument: Assuming for the sake of contradiction that $B_1 \times B_2 \times \cdots$ is countable and its elements can be enumerated in a list:

$$(b_{1,1}, b_{2,1}, b_{3,1}, b_{4,1}, \dots)$$
$$(b_{1,2}, b_{2,2}, b_{3,2}, b_{4,2}, \dots)$$
$$(b_{1,3}, b_{2,3}, b_{3,3}, b_{4,3}, \dots)$$
$$(b_{1,4}, b_{2,4}, b_{3,4}, b_{4,4}, \dots)$$
$$\vdots$$

where $b_{i,j}$ represents the item from set B_i that is included in the jth element of the Cartesian Product. Now consider the element $(\overline{b_{1,1}}, \overline{b_{2,2}}, \overline{b_{3,3}}, \overline{b_{4,4}}, \dots)$, where $\overline{b_{i,j}}$ represents any item from set B_i that differs from $b_{i,j}$ (i.e. any other element in the set). This is a valid element that should exist in the Cartesian Product $B_1, B_2, \dots$, yet it is not in the enumerated list. This is a contradiction, so $B_1 \times B_2 \times \cdots$ must be uncountable.