CENG 384 - Signals and Systems for Computer Engineers
Spring 2023
Homework 3

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1. Let's say x(t) $= \sum_{k=-\infty}^{\infty} a_k e^{jkw_0 t}$

   Integrate both side

   $\int_{-\infty}^{t} x(s)ds = \int_{-\infty}^{t} \sum_{k=-\infty}^{\infty} a_k e^{jkw_0 s} ds$

   $= \sum_{k=-\infty}^{\infty} \int_{-\infty}^{t} a_k e^{jkw_0 s} ds = \sum_{k=-\infty}^{\infty} a_k \int_{-\infty}^{t} e^{jkw_0 s} ds$

   $= \sum_{k=-\infty}^{\infty} a_k [\frac{e^{jkw_0 s}}{jkw_0} |_{-\infty}^{t}]$

   $= \sum_{k=-\infty}^{\infty} a_k [\frac{e^{jkw_0 t}}{jkw_0} - 0]$

   $= \sum_{k=-\infty}^{\infty} a_k \frac{e^{jkw_0 t}}{jkw_0}$

   So the coefficients become $(\frac{1}{jkw_0})a_k$. $w_0 = \frac{2*\pi}{T}$. Thus, the coefficients is $(\frac{1}{jk\frac{2*\pi}{T}})a_k$

2. (a) $\sum_{\forall l} a_l * a_{k-l}$ because of the multiplication property of fourier series.
   (b) Fourier series coefficient of even part of $x(t)$ is equal to real part of $a_k = \mathbb{R}\{a_k\}$.
   (c) $x(t-t_0) = a_k e^{-jk(w\pi/T)t_0}$, $x(t+t_0) = a_k e^{jk(2\pi/T)t_0}$. $x(t-t_0) + x(t+t_0) = a_k e^{-jk(w\pi/T)t_0} + a_k e^{jk(2\pi/T)t_0}$.

3. We know that $a_k = \frac{1}{T}\int_T x(t)e^{-jkw_0 t}dt$

   Let's take 1 period from -0.5 to 3.5. The period is 4 in this signal. We can write the a_k as

   $a_k = \frac{1}{T}\int_{-0.5}^{3.5}(2u(t) - 2u(t-1) - 2u(t-2) + 2u(t-3))e^{-jkw_0 t}dt$

   $a_k = \frac{1}{T}(\int_0^1 2e^{-jkw_0 t}dt + \int_1^2 0e^{-jkw_0 t}dt + \int_2^3 -2e^{-jkw_0 t}dt + \int_3^{3.5} 0e^{-jkw_0 t}dt)$

   $\frac{1}{4}[(\frac{2e^{-jkw_0 t}}{-jkw_0}|_0^1) + (\frac{-2e^{-jkw_0 t}}{-jkw_0}|_2^3)]$

   $= \frac{1}{4}[(\frac{2e^{-jkw_0}}{-jkw_0}) - \frac{2}{-jkw_0} + (\frac{2e^{-jkw_0 3}}{jkw_0}) - (\frac{2e^{-jkw_0 2}}{jkw_0})]$

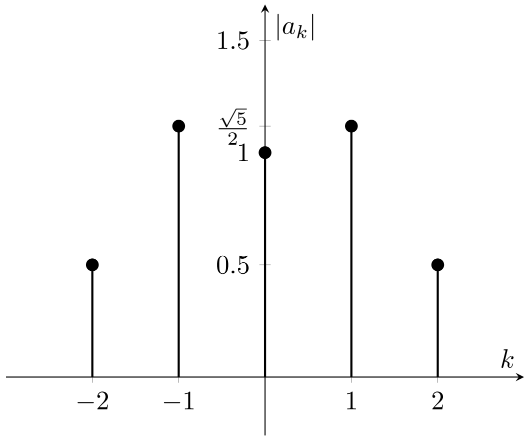
   $= \frac{1}{4}[\frac{-2e^{-jkw_0}+2+2e^{-3jkw_0}-2e^{-2jkw_0}}{jkw_0}]$

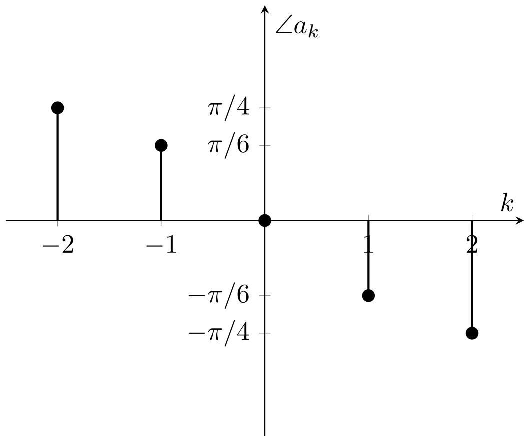
   $a_k = \frac{e^{-3jkw_0}-e^{-2jkw_0}-e^{jkw_0}+1}{2jkw_0}$

   $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jkw_0 t} = \sum_{k=-\infty}^{\infty} \frac{e^{-3jkw_0}-e^{-2jkw_0}-e^{-jkw_0}+1}{2jkw_0} e^{-jkw_0 t}$

   $= \sum_{k=-\infty}^{\infty} \frac{e^{-2jkw_0}-e^{-jkw_0}-e^{jkw_0}+1}{2jkw_0}$

4. (a) $sin(w_0t) = \frac{j}{2}(-e^{iw_0t} + e^{-iw_0t})$, $2 * cos(w_0t) = e^{iw_0t} + e^{-iw_0t}$, $cos(2w_0t + \pi/4) = (e^{j\pi/4} * e^{2iw_0t} + e^{-2iw0t}/e^{j\pi/4})/2$
$a_{-2} = \frac{1}{2\sqrt{j}}$, since $\sqrt{j} = (1+j)/\sqrt{2}$, $a{-}2 = \frac{1+j}{2\sqrt{2}}$
$a_{-1} = j/2 + 1$
$a_0 = 1$
$a_1 = 1 - j/2$
$a_2 = \frac{1}{2\sqrt{j}} = \frac{\sqrt{2}}{2+2j}$





(b) $\dot{y}(t) + y(t) = x(t)$, we should first find the particular solution of this system. We should write $x(t)$ as $e^{\lambda t}u(t)$ and $y(t)$ as $Kx(t)$ than solve the equation for K. $(\lambda K + K)e^{\lambda t}u(t) = e^{\lambda t}u(t)$, $\lambda K + K = 1$, $K = \frac{1}{1+\lambda}$. The pole of transfer function is the eigenvalue of system. Which is -1 for this question.

(c) $\frac{dy(t)}{dt} + y(t) = x(t)$

$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$

$= \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau = \int_{-\infty}^{\infty} h(\tau)(\sum_k a_k e^{jkw_0(t-\tau)})d\tau$

$= \sum_k a_k e^{jkw_0t} \int h(\tau)e^{-jkw_0\tau}d\tau$

$= \sum_k a_k H(jkw_0)e^{jkw_0t}$

$\sum_k a_k H(jkw_0)jkw_0 e^{jkw_0t} + \sum_k a_k H(jkw_0)e^{jkw_0t} = \sum_k a_k e^{jkw_0t}$

$\sum_k a_k H(jkw_0)jkw_0 e^{jkw_0t} + a_k H(jkw_0)e^{jkw_0t} - a_k e^{jkw_0t} = 0$

$\sum_k a_k e^{jkw_0t}(H(jkw_0)jkw_0 + H(jkw_0) - 1) = 0$

$H(jkw_0) = \frac{1}{jkw_0+1}$

$b_k = \frac{a_k}{jkw_0+1}$

(d) $y(t) = \sum_k b_k e^{jkw_0t}$

$y(t) = \sum_k \frac{a_k e^{jkw_0t}}{hkw_0+1}$

5. (a) $x[n] = sin(\frac{\pi}{2}n)$

$x[n] = \frac{1}{2j}(e^{j\frac{\pi}{2}n} - e^{-j\frac{\pi}{2}n})$ Take $w_0 = \frac{\pi}{2}$

So => $a_1 = \frac{1}{2j}$ and $a_{-1} = \frac{-1}{2j}$

$a_1 = \frac{1}{2j} = \frac{-j}{2}$

$a_{-1} = \frac{-1}{2j} = \frac{j}{2}$

(b) $y[n] = 1 + cos(\frac{\pi}{2}n)$

$= 1 + \frac{1}{2}(e^{j\frac{\pi}{2}n} + e^{-j\frac{\pi}{2}n})$ Take $w_0 = \frac{\pi}{2}$

$b_0 = 1,\ b_1 = b_{-1} = \frac{1}{2}$

(c) $x[n] \longleftrightarrow^{F.S.} a_k$

$y[n] \longleftrightarrow^{F.S.} b_k$

$z[n] = x[n]y[n] \longleftrightarrow^{F.S.} c_k = \sum_{l=<N>} a_l b_{k-l}$

$c_k = \sum_{l=0}^{3} a_l b_k - l$ since N = 4

$= a_0 b_k + a_1 b_{k-1} + a_2 b_{k-2} + a_3 b_{k-3} = a_1 b_{k-1} + a_3 b_{k-3}$

$c_0 = a_1 b_{-1} + a_3 b_{-3} = \frac{-j}{2}\frac{1}{2} + \frac{j}{2}\frac{1}{2} = 0$
$c_1 = a_1 b_0 + a_3 b_{-2} = \frac{-j}{2}1 + 0 = \frac{-j}{2}$
$c_2 = a_1 b_1 + a_3 b_{-1} = \frac{-j}{2}\frac{1}{2} + \frac{j}{2}\frac{1}{2} = 0$
$c_3 = a_1 b_2 + a_3 b_0 = \frac{-j}{2}0 + \frac{j}{2}1 = \frac{j}{2}$

(d) $x[n]y[n] = \sin(\frac{\pi}{2}n)(1 + cos(\frac{\pi}{2}n))$

$= sin(\frac{\pi}{2}n) + sin(\frac{\pi}{2}n)cos(\frac{\pi}{2}n) = sin(\frac{\pi}{2}n) + \frac{sin\pi n}{2}$ and $sin\pi n$ is zero. Therefore

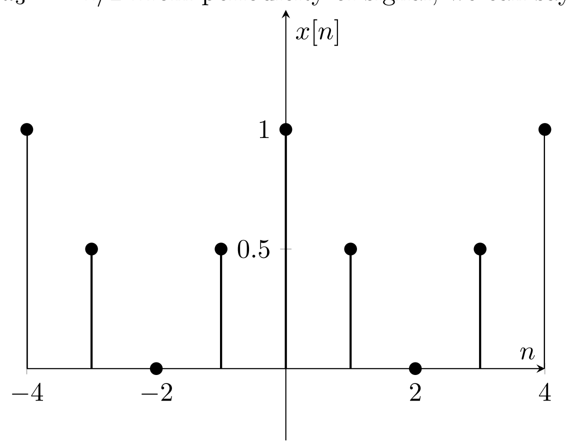
$= sin(\frac{\pi}{2}n)$

like part a

$z[n] = \frac{1}{2j}(e^{j\frac{\pi}{2}n} - e^{-j\frac{\pi}{2}n})$

$c_1 = \frac{-j}{2}$ and $c_{-1} = c_3 = \frac{j}{2}$. So as expected the result is the same with part c.

6. (a) We can see that the period of this signal is 4. We can then we can use the analysis formula to find fourier coefficients.
$a_k = \frac{1}{4}\sum_{n=0}^{3} x[n]e^{-jk(\pi/2)n}$
$a_k = \frac{1}{4}(0 + e^{-jk\pi/2} + 2e^{-jk\pi} + e^{-3jk\pi/2})$
$e^{jk\pi/2} = j$ so
$a_0 = 1$
$a_1 = -1/2$
$a_2 = 0$
$a_3 = -1/2$ From periodicity of signal, we can say that a_k does also have period 4.



(b) If we examine the graph, we can see that this signal is $y[n] = x[n]x[n-1]$.
We can use difference and multiplication properties to find the spectral coefficients of Fourier series.
$x[n+1] \leftrightarrow c_k = a_k e^{jk\pi/2} = \frac{1}{4}(e^{jk\pi} + 2e^{jk\pi/2} + 1)$
$x[n]x[n+1] \leftrightarrow b_k = \sum_{l=0}^{4} a_l c_{k-l}$
Or we can just use the analysis formula instead of calculating this sum.
$b_k = \frac{1}{4} \sum_{n=0}^{4} y[n] e^{-jk(\pi/2)(n-2)}$
$b_k = \frac{1}{4}(e^{jk\pi/2} + 2)$
$e^{j\pi/2} = j$
$b_0 = 3/4$
$b_1 = \frac{j+2}{4}$
$b_2 = 1/4$
$b_3 = \frac{4-j}{2}$.

7. (a) In CT LTI system

$e^{st} \rightarrow H(s)e^{st}$

$= e^{st} \int_{-\infty}^{\infty} h(\tau) e^{-s\tau} d\tau.$

$y(jw) = x(jw) * H(jw).$

If y(t) = x(t). We can say that y(jw) = x(jw). Then we can say that the coefficients should be identical. That implies that system does not change in any value. It can be possible if the transfer function H = 1 so that $|w| \leq 80$.

(b) In contrary to part a. If the $y(t) \neq x(t)$ then there should be some values w where the transfer function not equal to 1. As a result we can say that for w that there should be some values of w that $|w| > 80$.

8. (a)

```
def fourierCoefficients(signal, period, count):
    coefficients = []
    coefficients.append(np.mean(signal))

    for k in range(1, count + 1):
        value = 0
        for n in range(len(signal)):
            value += (signal[n] * np.cos(2 * np.pi * k * (n * period / len(signal)) / per
        coefficients.append(2 / len(signal) * value)

    for k in range(1, count + 1):
        value = 0
        for n in range(len(signal)):
            value += (signal[n] * np.sin(2 * np.pi * k * (n * period / len(signal)) / per
        coefficients.append(2 / len(signal) * value)
    return coefficients
```

Figure 1: Fourier Coefficients Calculate function

(b)

```python
def approximate(coefficients, period, count):
    result = np.empty(count)
    result.fill(coefficients[0])


    coef_num = (len(coefficients) - 1) // 2
    for k in range(1, coef_num + 1):
        for n in range(count):
            result[n] += coefficients[k] * np.cos(2 * np.pi * k * (n * period / count) /
            result[n] += coefficients[k + coef_num] * np.sin(2 * np.pi * k * (n * period

    return result
```

Figure 2: Approximation function

(c)

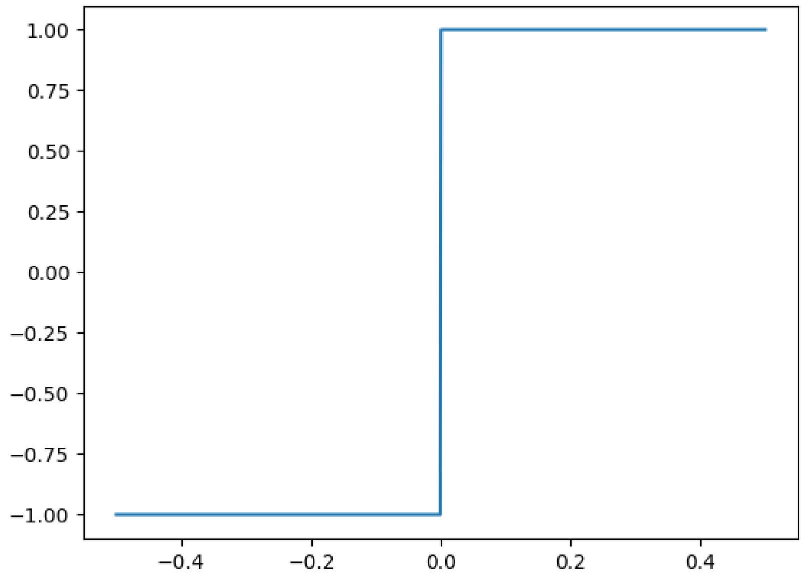


Figure 3: Original signal

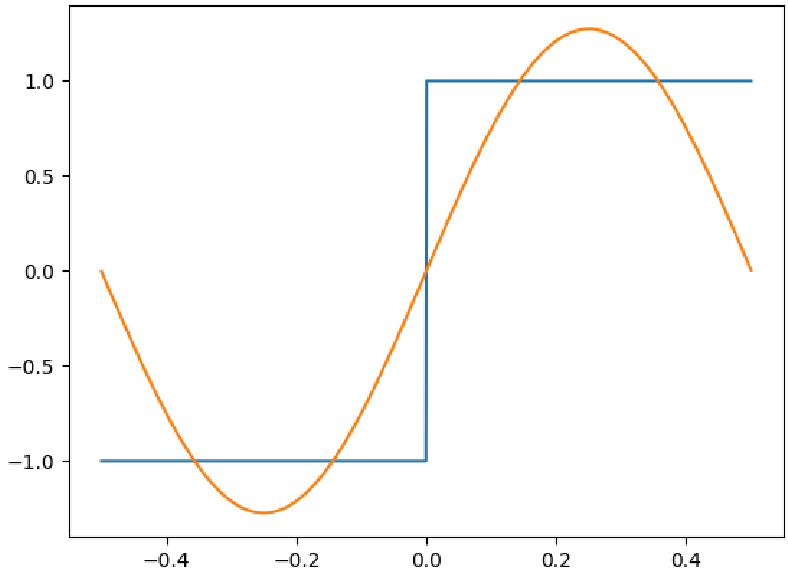


Figure 4: Approximated signal n = 1

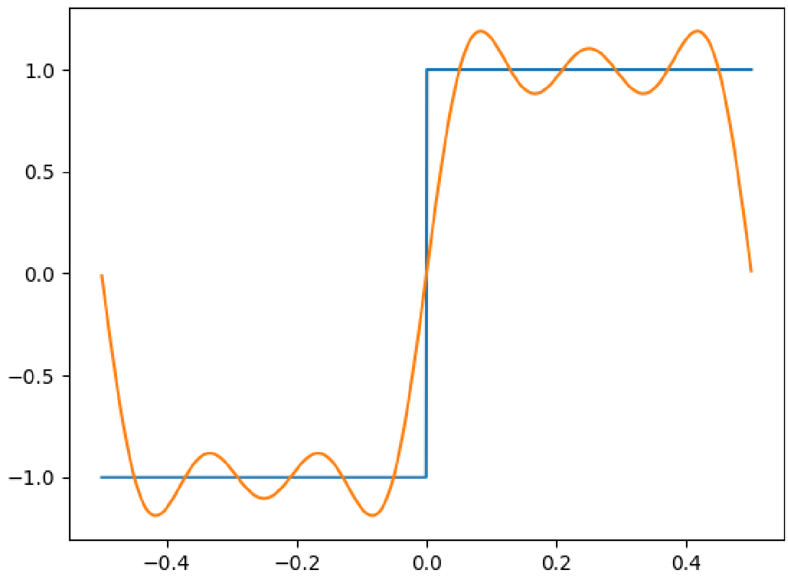


Figure 5: Approximated signal n = 5

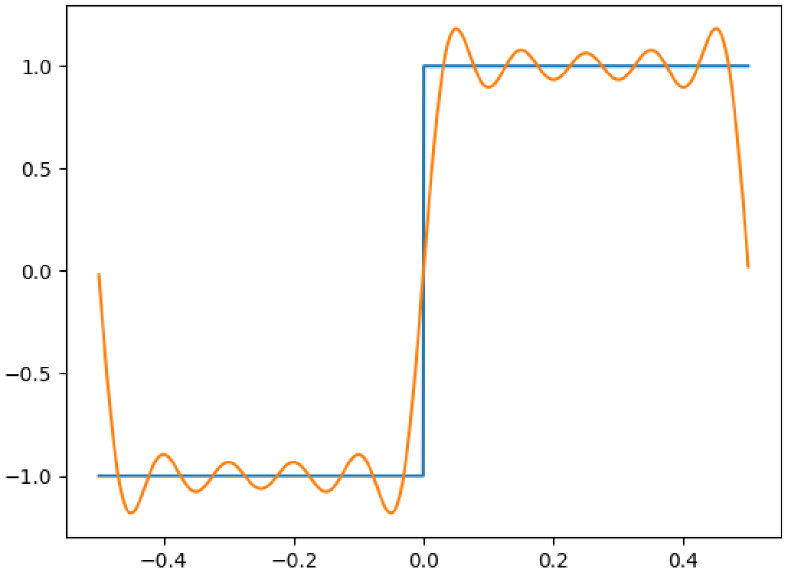


Figure 6: Approximated signal n = 10

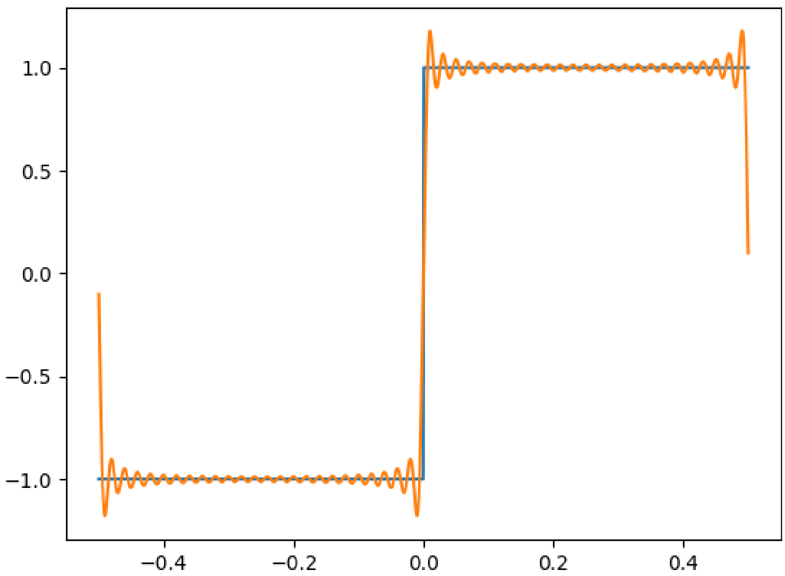


Figure 7: Approximated signal n = 50

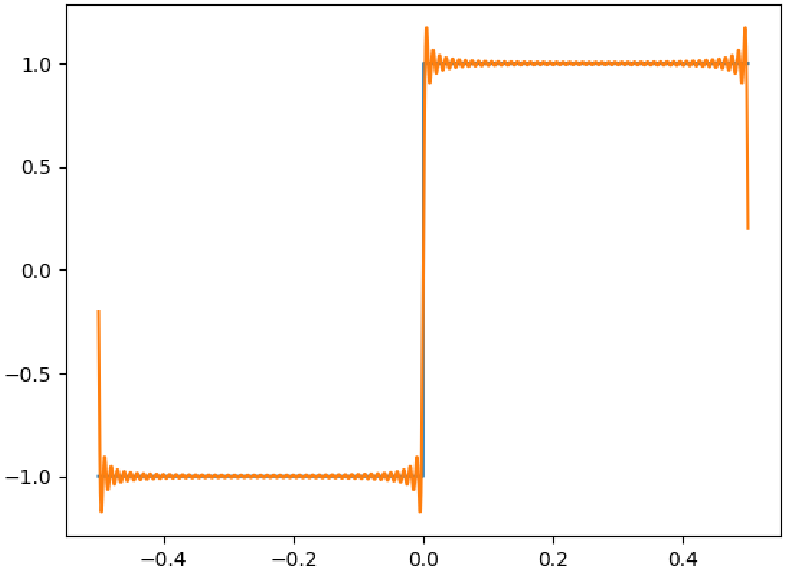


Figure 8: Approximated signal n = 100

(d)

```python
def sawtooth():
    result = np.empty(1000)
    t = np.linspace(-0.5, 0.5, 1000)
    for i in range(len(t)):
        if(t[i] < 0):
            result[i] = 1 + 2 * t[i]
        else:
            result[i] = -1 - 2 * t[i]
    return result
```

Figure 9: Sawtooth function

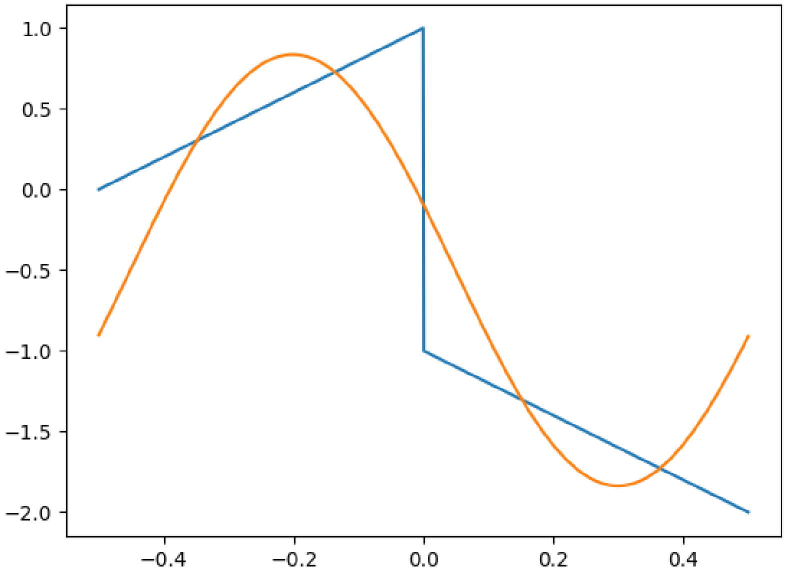


Figure 10: Approximated Saw signal n = 1

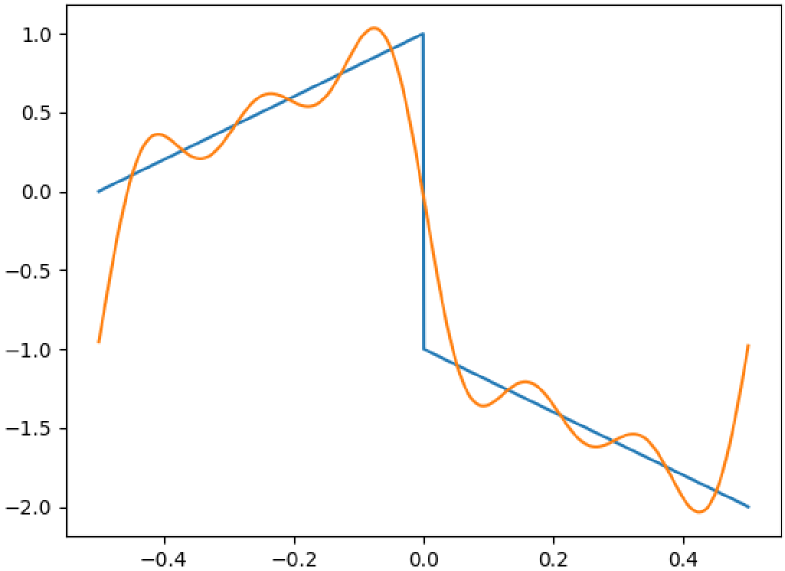


Figure 11: Approximated Saw signal n = 5

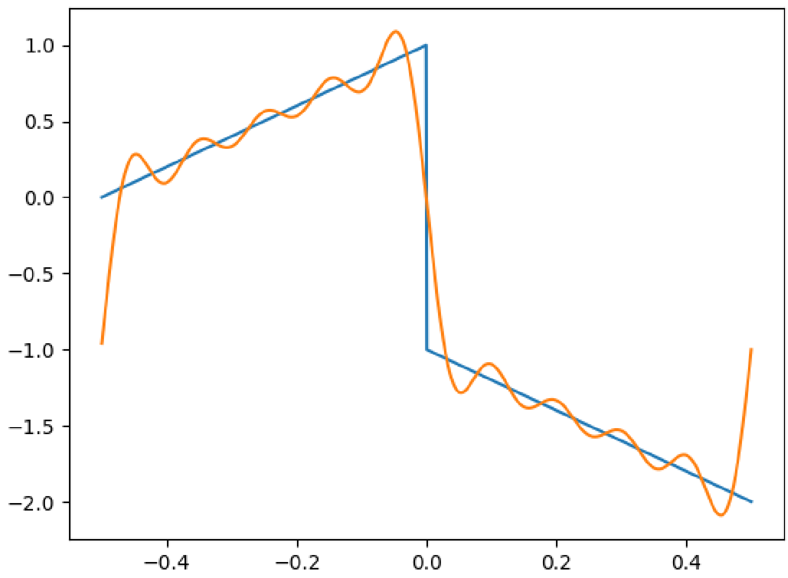


Figure 12: Approximated Saw signal n = 10

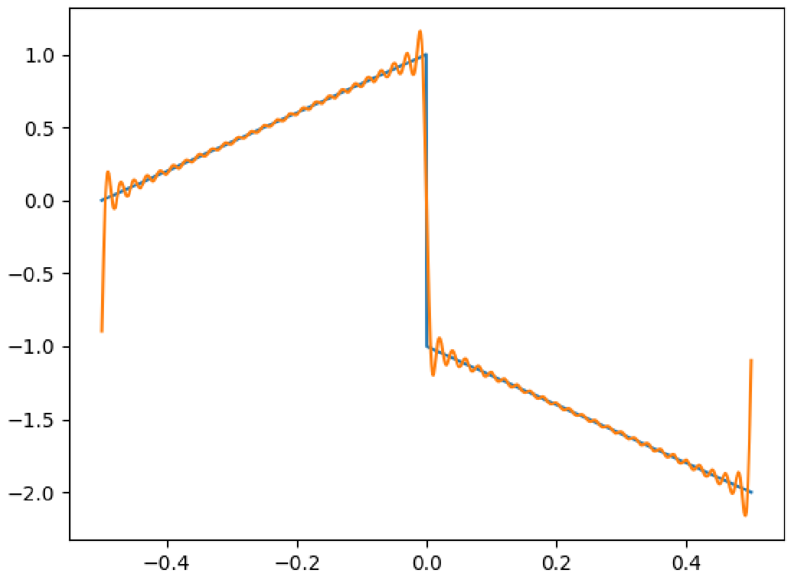


Figure 13: Approximated Saw signal n = 50

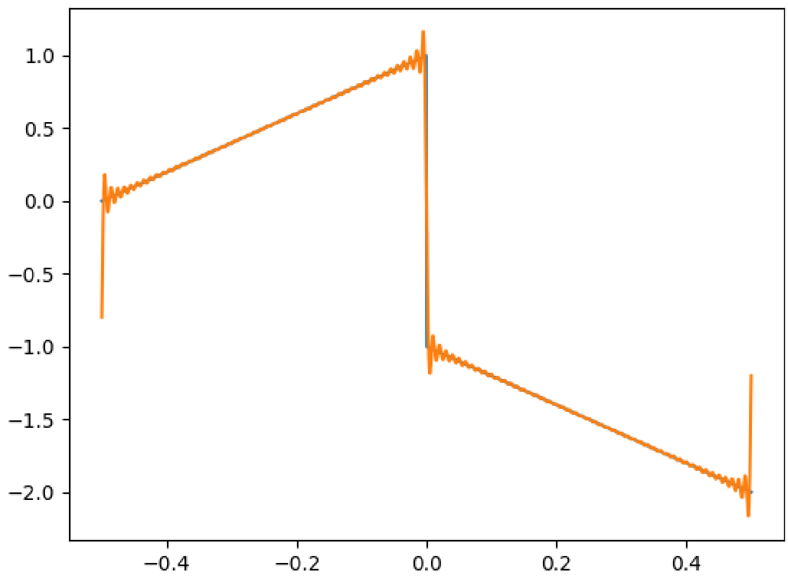


Figure 14: Approximated Saw signal n = 100

When we increase the n, the result will become more as expected signal.

Listing 1: Solution code

```
import matplotlib.pyplot as plt
import numpy as np
import scipy.signal as sp

def fourierCoefficients(signal, period, count):
    coefficients = []
    coefficients.append(np.mean(signal))

    for k in range(1, count + 1):
        value = 0
        for n in range(len(signal)):
            value += (signal[n] * np.cos(2 * np.pi * k * (n * period / len(signal)) / period))
        coefficients.append(2 / len(signal) * value)

    for k in range(1, count + 1):
        value = 0
        for n in range(len(signal)):
            value += (signal[n] * np.sin(2 * np.pi * k * (n * period / len(signal)) / period))
        coefficients.append(2 / len(signal) * value)
    return coefficients

def approximate(coefficients, period, count):
    result = np.empty(count)
    result.fill(coefficients[0])


    coef_num = (len(coefficients) - 1) // 2
    for k in range(1, coef_num + 1):
        for n in range(count):
            result[n] += coefficients[k] * np.cos(2 * np.pi * k * (n * period / count) / period)
            result[n] += coefficients[k + coef_num] * np.sin(2 * np.pi * k * (n * period / count) / per

    return result

def sawtooth():
    result = np.empty(1000)
    t = np.linspace(-0.5, 0.5, 1000)
    for i in range(len(t)):
        if(t[i] < 0):
            result[i] = 1 + 2 * t[i]
        else:
            result[i] = -1 - 2 * t[i]
    return result

def main():
    signal = []
    for i in range(500):
        signal.append(-1)
```

```
    for i in range(500):
        signal.append(1)

    plt.plot(np.linspace(-0.5, 0.5, 1000), signal, label='Original')
    plt.show()

    coef = fourierCoefficients(signal, 1, 1)
    approximation = approximate(coef, 1, 1000)
    plt.plot(np.linspace(-0.5, 0.5, 1000), signal, label='Original')
    plt.plot(np.linspace(-0.5, 0.5, 1000), approximation, label='Approximation')
    plt.show()

    coef = fourierCoefficients(signal, 1, 5)
    approximation = approximate(coef, 1, 1000)
    plt.plot(np.linspace(-0.5, 0.5, 1000), signal, label='Original')
    plt.plot(np.linspace(-0.5, 0.5, 1000), approximation, label='Approximation')
    plt.show()

    coef = fourierCoefficients(signal, 1, 10)
    approximation = approximate(coef, 1, 1000)
    plt.plot(np.linspace(-0.5, 0.5, 1000), signal, label='Original')
    plt.plot(np.linspace(-0.5, 0.5, 1000), approximation, label='Approximation')
    plt.show()


    coef = fourierCoefficients(signal, 1, 50)
    approximation = approximate(coef, 1, 1000)
    plt.plot(np.linspace(-0.5, 0.5, 1000), signal, label='Original')
    plt.plot(np.linspace(-0.5, 0.5, 1000), approximation, label='Approximation')
    plt.show()

    coef = fourierCoefficients(signal, 1, 100)
    approximation = approximate(coef, 1, 1000)
    plt.plot(np.linspace(-0.5, 0.5, 1000), signal, label='Original')
    plt.plot(np.linspace(-0.5, 0.5, 1000), approximation, label='Approximation')
    plt.show()

    signalSaw = sawtooth()

    coef = fourierCoefficients(signalSaw, 1, 1)
    approximation = approximate(coef, 1, 1000)
    plt.plot(np.linspace(-0.5, 0.5, 1000), signalSaw, label='Original')
    plt.plot(np.linspace(-0.5, 0.5, 1000), approximation, label='Approximation')
    plt.show()

    coef = fourierCoefficients(signalSaw, 1, 5)
    approximation = approximate(coef, 1, 1000)
    plt.plot(np.linspace(-0.5, 0.5, 1000), signalSaw, label='Original')
    plt.plot(np.linspace(-0.5, 0.5, 1000), approximation, label='Approximation')
    plt.show()

    coef = fourierCoefficients(signalSaw, 1, 10)
    approximation = approximate(coef, 1, 1000)
    plt.plot(np.linspace(-0.5, 0.5, 1000), signalSaw, label='Original')
    plt.plot(np.linspace(-0.5, 0.5, 1000), approximation, label='Approximation')
    plt.show()


    coef = fourierCoefficients(signalSaw, 1, 50)
    approximation = approximate(coef, 1, 1000)
    plt.plot(np.linspace(-0.5, 0.5, 1000), signalSaw, label='Original')
    plt.plot(np.linspace(-0.5, 0.5, 1000), approximation, label='Approximation')
    plt.show()

    coef = fourierCoefficients(signalSaw, 1, 100)
    approximation = approximate(coef, 1, 1000)
    plt.plot(np.linspace(-0.5, 0.5, 1000), signalSaw, label='Original')
    plt.plot(np.linspace(-0.5, 0.5, 1000), approximation, label='Approximation')
    plt.show()

if __name__ == "__main__":
    main()
```