Calculus I Lecture Notes

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Contents

Chapter 1

Precalculus

1.1 Sets

A **set** is a collection of elements.

$x \in A$ means x is an element of the set A. If x is not a member of A, we write $x \notin A$.

$\varnothing$ is the set which contains no element and is called the **empty set**.

There are finite sets (ex. $\{0,1,2\}$) and infinite sets (ex. $\{0,1,2,3,\ldots\}$).

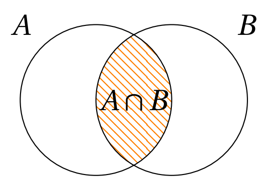
If every element of the set A is an element of the set B, we say that A is **subset** of B, and write $A \subset B$.

Example 1. *List all the subsets of* $\{0,1,2\}$.

For any set A, $A \subset A$ and $\varnothing \subset A$.

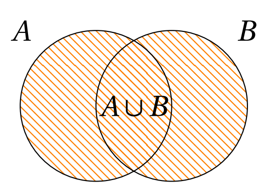
If $A \subset B$ and $B \subset A$, we write $A = B$.

$A \cap B = \{x : x \in A \text{ and } x \in B\}$ is called the **intersection** of A and B.



If the intersection of two sets is the empty set, those sets are called **disjoint**.

$A \cup B = \{x : x \in A \text{ or } x \in B\}$ is called the **union** of A and B.



Example 2. *For example if* $A = \{0,1,2,5,8\}$ *and* $B = \{1,3,5,6\}$ *then find* $A \cap B$ *and* $A \cup B$.

The set of all elements in A but not in B is denoted $A \setminus B = \{x \in A : x \notin B\}$ and is called the **complement** of B in A.

Example 3. $\{0,2,3,5\} \setminus \{2,5,7,8\} = \{0,3\}$

$A \times B = \{(a, b) : a \in A \text{ and } b \in B\}$ is called the **Cartesian** product of the sets A and B.

Example 4. *Write the cartesian product of* $A = \{0, 1, 2\}$ *and* $B = \{2, 3, 4\}$.

1.2 Real Numbers

The **integers** are $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$.

Integers come in a lot varieties:

- even integers that are of the form $2k$, for some $k \in \mathbb{Z}$,
- odd integers that are of the form $2k + 1$, for some $k \in \mathbb{Z}$
- positive and negative integers,
- primes, etc...

The **rational numbers** are $\mathbb{Q} = \left\{\frac{m}{n} : m, n \in \mathbb{Z} \text{ and } n \neq 0\right\}$.

Pythagoreans preached that all numbers could be expressed as the ratio of integers, and the discovery of irrational numbers is said to have shocked them.

Example 5. $\sqrt{2}$ *is not a rational number.*

Suppose that it is rational. Then $\sqrt{2} = m/n$, *where* $m, n \in \mathbb{Z}$ *and* $n \neq 0$. *Also assume* m *and* n *have no common divisor.*

$$m^2/n^2 = 2 \implies m^2 = 2n^2$$

Thus m *is even and we can write* $m = 2k$, *where* $k \in \mathbb{Z}$.

$$4k^2 = 2n^2 \implies n^2 = 2k^2$$

Thus n *is also even. But* m *and* n *cannot both be even. Accordingly, there can be no rational number whose square is 2.*

The set of irrational numbers is denoted by $\mathbb{I}$.
The set of real numbers is $\mathbb{R} = \mathbb{Q} \cup \mathbb{I}$.
Note that $\mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$.
The real numbers are ordered such that

1. $a < b \implies a + c < b + c$
2. $a < b$ and $c > 0$ implies $ac < bc$
3. $a < b$ and $c < 0$ implies $ac > bc$
4. $a > 0$ implies $\frac{1}{a} > 0$
5. $0 < a < b$ implies $\frac{1}{b} < \frac{1}{a}$

Intervals

The open interval $(a,b) = \{x \mid a < x < b\}$, closed interval ($[a,b]$), half open intervals $(a,b]$, $[a,b)$. It is possible that $a = -\infty$, $b = \infty$. Draw each interval on the real line.

Example 6. *Solve the following inequalities.*

1. $\frac{2}{x-1} \geq 5$.

   Solution. It is not right to multiply both sides by $x-1$ *and say* $5x-5 \leq 2$.

$$\frac{2}{x-1} \geq 5 \iff \frac{2}{x-1} - 5 \geq 0 \iff \frac{7-5x}{x-1} \geq 0.$$

   Now make a sign analysis to get interval $(1, 7/5]$

2. $3x - 1 \leq 5x + 3 \leq 2x + 15$.

   Solution. $-2 \leq x$ *and* $x \leq 4$.

The absolute value.

$$|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$$

ex. $|3| = |-3| = 3$

Geometrically, $|x|$ is the distance between x and 0 on the real line. And $|x - y|$ is the distance between x and y.

Properties *(can be proved from definition)*:

1. $|-x| = |x|$, (Do not fall into the trap $|-x| = x$, this is not always true!)
2. $|ab| = |a||b|$,
3. $|a+b| \leq |a| + |b|$, (triangle inequality).

From (2), for any x, $x^2 = |x^2| = |x|^2$

If D is a nonnegative number

$$|x| = D \implies x = -D \text{ or } x = D,$$

$$|x| < D \implies -D < x < D$$

$$|x| > D \implies x < -D \text{ or } x > D$$

More generally,

$$|x-a| = D \implies x = a - D \text{ or } x = a + D,$$

$$|x-a| < D \implies a - D < x < a + D$$

$$|x-a| > D \implies x < a - D \text{ or } x > a + D$$

Example 7. *Solve* $|3x-2| \leq 1$.

Solution.

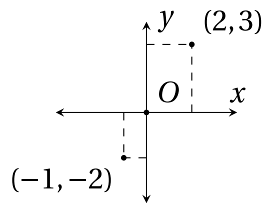
$$-1 \leq 3x - 2 \leq 1 \implies x \geq 1/3 \textit{ and } x \leq 1.$$

Example 8. *Solve the equation* $|x+1| > |x-3|$.

Solution. The distance between x and -1 *is greater than the distance between x and* 3*. So* $x > 1$.

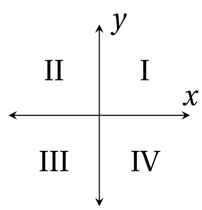
1.3 Cartesian Coordinates

Cartesian plane is $\mathbb{R} \times \mathbb{R} = \{(x, y) \mid x \in \mathbb{R} \text{ and } b \in \mathbb{R}\}$.



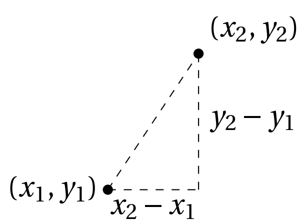
Horizontal axis is usually called the x axis, the vertical axis is called the y axis. Intersection of the axes is called the origin, denoted O.

The coordinate axes divide the Cartesian plane into four quadrants.



By the Pythogerean Theorem, the distance of two points (x_1, y_1) and (x_2, y_2) in the plane is

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$



The distance of (x, y) to the origin is $\sqrt{x^2 + y^2}$.

Example 9. *Find the distance between* $(-1, 1)$ *and* $(3, -4)$.

Equations of Lines

For any two points (x_1, y_1) and (x_2, y_2) on a non-vertical line L, the quantity $m = \frac{y_2 - y_1}{x_2 - x_1}$ is constant and is called the **slope** of the line L.

Let L be a nonvertical line. Let m be the slope of L and (x_1, y_1) be the coordinates of a point on L. If (x, y) is another point on L, then

$$\frac{y - y_1}{x - x_1} = m$$

Hence any (x, y) on L satisfies

$$y = m(x - x_1) + y_1$$

The above is known as an equation for the line L.

All points on a **vertical line** have their x coordinate equal to a constant a. So the equation of a vertical line is $x = a$. **Horizontal lines** have equations of the form $y = a$.

y-intercept of a nonvertical line L is the y-coordinate of the point where L intersects the y-axis. **x-intercept** of a nonhorizontal is defined similarly.

Example 10. *Find an equation of the line through the points* $(1,-1)$ *and* $(3,5)$. *Draw the line. Find the x and y intercepts.*

Example 11. *Find an equation of the line that passes through the point* $(-3,-4)$ *and has slope* 2. *Draw the line.*

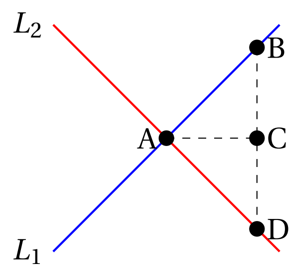
Example 12. *Find the slope and the two intercepts of the line with equation* $8x + 5y = 20$. *Draw the line.*

Parallel vs. perpendicular lines

We call two lines **parallel** if their slopes are equal.

We call two lines **perpendicular** if they intersect at right angles (90°).

Theorem 1. *Two nonvertical lines with slopes* m_1 *and* m_2 *are perpendicular if and only if* $m_1 m_2 = -1$.



Proof. Use the similarity of the triangles ABC and DAC to get

$$\frac{|BC|}{|AC|} = \frac{|AC|}{|CD|} \implies \frac{|BC||CD|}{|AC|^2} = 1$$

Slope of L_1 (m_1) is $|BC|/|AC| = 1$ and slope of L_2 (m_2) is $-|CD|/|AC|$. So $m_1 m_2 = -1$. □

Example 13. *Find an equation of the line through* $(1,-2)$ *that is parallel to the line L with equation* $3x - 2y = 1$. *Draw the lines.*

Example 14. *Find an equation of the line through* $(2,-3)$ *that is perpendicular to the line L with equation* $4x + y = 3$. *Draw the lines.*

1.4 Quadratic Equations

Circles and Disks

The circle is the set of all points that have the same distance (called radius of the circle) from a given point (called center of the circle).

If (x, y) is a point on a circle with center (a, b) and radius r then

$$\sqrt{(x-a)^2+(y-b)^2}=r \implies (x-a)^2+(y-b)^2=r^2$$

Example 15. *Find the center and radius of the circle $x^2+y^2-4x+6y=3$.*

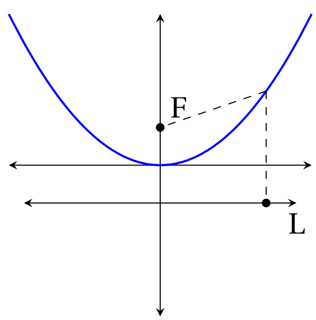
Solution. Complete to squares to get $(x-2)^2+(y+3)^2=16$.

The equation $(x-a)^2+(y-b)^2<r^2$ represents open disk and the equation $(x-a)^2+(y-b)^2 \le r^2$ represents closed disk or simply disk.

Example 16. *Draw $x^2+2x+y^2 \le 8$.*

Parabolas

A parabola P is the set of all points in the plane that are equidistant from a given line L (called directrix of P) and a point F (called the focus of P).



Example 17. *Find the equation of the parabola having the point $F(0, p)$ as focus and the line L with equation $y=-p$ as directrix.*

Solution. If $P(x, y)$ is any point on the parabola then squaring both sides of PF=PQ we get

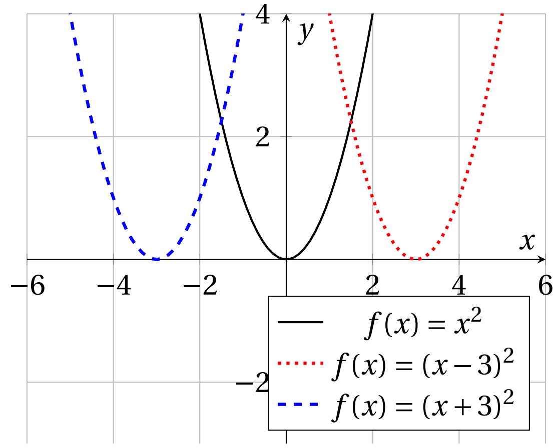
$$x^2+(y-p)^2=0^2+(y+p)^2$$

After simplifying, $y=x^2/4p$.

Shifting a Graph

Let $c>0$.

- To shift a graph c units to the right, replace x in its equation with $x-c$. To shift to left, replace x by $x+c$.
- To shift a graph c units up, replace y in its equation with $y-c$. To shift down, replace y by $y+c$.



1.5 Functions and Their Graphs

A **function** f on a set D into a set R is a rule that assigns a unique element $f(x)$ in R to each element x in D.

D is called the **domain** of f. R is called the target or **codomain** of f. The **range** of f is a subset of R containing of all possible values $f(x)$.

This definition is not mathematical as we did not define what a rule is. Formally one defines a function as a relation.

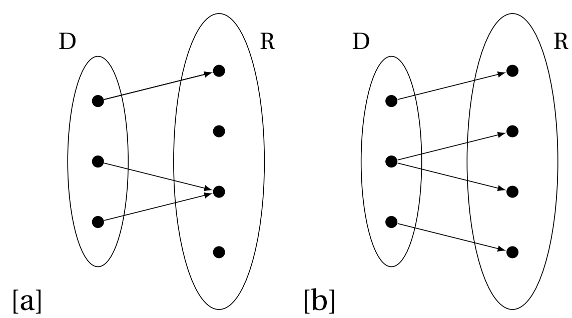


Figure 1.1: a) Not a function. b) A Function

Example 18. *Define a function on the set of all real numbers by* $f(x) = x^2 + 1$. *Find* $f(0)$, $f(2)$, $f(x+2)$.

$$f(x) = \frac{1}{x}, \qquad x > 0$$

means that the domain of f is the set $\{x \mid x > 0\}$.

Technically, this function is different from the function

$$f(x) = \frac{1}{x}, \qquad x < 0.$$

If we do not specify the domain of a function f, then the **domain convention** is to assume that the domain of f is the set of all real numbers for which f is defined.

So if we write

$$f(x) = \frac{1}{x},$$

we are assuming f is defined for all real numbers except 0.

Example 19. *Find the domain of* $f(x) = \sqrt{2-x}$.

Solution. *Its domain is all* x *for which* $2-x \geq 0$, *i.e. the interval* $(-\infty, 2]$.

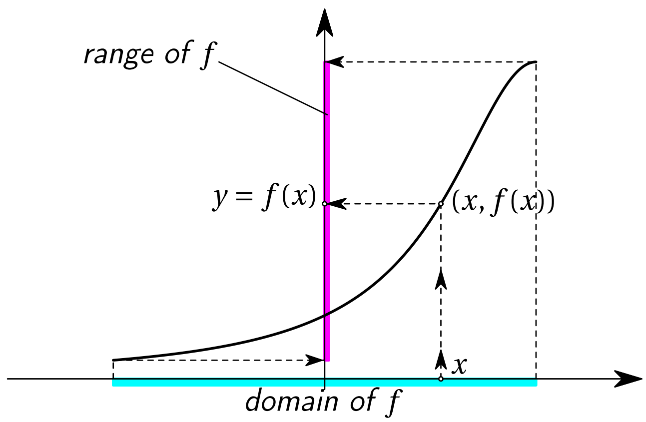
Example 20. *Find the domain of* $f(x) = \frac{1}{x^2-x}$.

A function $f : D \to R$ is **1-1** if $f(x_1) = f(x_2)$ then $x_1 = x_2$. A function $f : D \to R$ is **onto** if for every $y \in R$, there is an $x \in D$ such that $f(x) = y$.

Example 21. *Draw functions which are 1-1, onto, not 1-1 and not onto, similar to the Figure 1.1.*

Graph of a function

The *graph of a function* f is the set of all points whose coordinates are $(x, f(x))$ where x is in the domain of f.



Example 22. *A function which is given by the formula*

$$f(x) = mx + n$$

where m *and* n *are constants is called a* linear function. *Its graph is a straight line. The constants* m *and* n *are the* slope *and* y-intercept *of the line.*

Example 23. *The square root function* $f(x) = \sqrt{x}$ *has domain* $[0, \infty)$ *and takes* x *to its positive square root. Hence it has range* $[0, \infty)$.

Example 24. *The absolute value function* $f(x) = |x| = \sqrt{x^2}$ *has domain* $(-\infty, \infty)$ *and range* $[0, \infty)$.

Example 25. *Draw the graphs of some elementary functions*

$$c, x, x^2, \sqrt{x}, x^3, x^{1/3}, \frac{1}{x}, \frac{1}{x^2}, \sqrt{1-x^2}, |x|.$$

Example 26. *Sketch the graph of* $f(x) = 1 + \sqrt{x-4}$.

Solution: Shift the graph of $y = \sqrt{x}$ *1 unit up and 4 units to the right.*

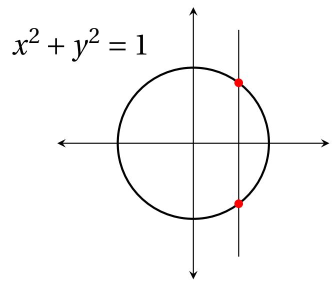
Example 27. *Sketch the graph of the function* $f(x) = \frac{2-x}{x-1}$.

Solution. $f(x) = \frac{2-x}{x-1} = -1 + \frac{1}{x-1}$. *So shift the graph of* $y = \frac{1}{x}$ *1 unit down and 1 unit to the right.*

Vertical Line Test

The graph of a function cannot intersect a vertical line "x = constant" in more than one point.

For example, the circle $x^2 + y^2 = 1$ is not a graph of a function.



Even and Odd Functions

Definition 1. *We say that f is an* ***even function*** *if $f(-x) = f(x)$ for every $x \in D$. We say that f is an* ***odd function*** *if $f(-x) = -f(x)$ for every $x \in D$.*

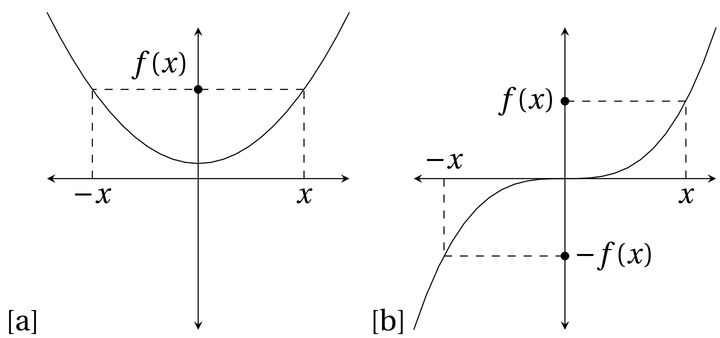


Figure 1.2: a) An even function, b) An odd function.

Odd functions are symmetric with respect to origin and even functions are symmetric with respect to the y-axis.

Example 28. *$f(x) = x$, $f(x) = x^3$ are odd and $f(x) = x^2$ and $f(x) = x^4$ are even and $f(x) = \frac{1}{x+1}$ is neither even or odd.*

Example 29. *$f(x) = x^3 + x$ is odd and $f(x) = \frac{1}{x^2-1}$ is even and $f(x) = x^2 + x$ is either even or odd.*

1.6 Operations on Functions

If f and g are functions, then for every x that belongs to the domains of both f and g we define functions

$(f+g)(x) = f(x) + g(x)$
$(f-g)(x) = f(x) - g(x)$
$(fg)(x) = f(x)g(x)$
$(f/g)(x) = f(x)/g(x)$ where $g(x) \neq 0$.

Example 30. *Let $f(x) = \frac{1}{x+2}$ and $g(x) = \frac{x}{x-1}$. Find $(f+g)(x)$, $(f-g)(x)$, $(fg)(x) = f(x)g(x)$ and $(f/g)(x)$ where $g(x) \neq 0$.*

Composition of Functions

If f and g are two functions, then

$$f \circ g(x) = f(g(x)).$$

The domain of $f \circ g$ consists of those numbers x in the domain of g for which $g(x)$ is in the domain of f.

Example 31. *Let $f(x) = \sqrt{x}$ and $g(x) = x+1$. Find $f \circ g$, $g \circ f$, $f \circ f$ and $g \circ g$. State the domains of each function.*

Function	Formula	Domain
f	$\sqrt{x}$	$[0,\infty)$
g	$x+1$	$\mathbb{R}$
$f \circ g$	$\sqrt{x+1}$	$[-1,\infty)$
$g \circ f$	$\sqrt{x}+1$	$[0,\infty)$
$f \circ f$	$x^{1/4}$	$[0,\infty)$
$g \circ g$	$x+2$	$\mathbb{R}$

Piecewise Defined Functions

Functions such as

$$g(x) = \begin{cases} 2x & \text{for } x < 0 \\ x^2 & \text{for } x \geq 0 \end{cases}$$

which are defined by different formulas on different intervals are sometimes called **piecewise defined functions.**

Inverse Functions

Remember that a function is **one-to-one** if for every value in the range, there is exactly one value in the domain.

A function is one-to-one if every horizontal line crosses its graph at most once, which is commonly known as the **horizontal line test**.

1.7 Polynomials and Rational Functions

Definition 2. *A **polynomial** is a function $P : \mathbb{R} \to \mathbb{R}$ such that*

$$P(x) = a_n x^n + \cdot + a_1 x + a_0.$$

*Here $a_n, \ldots, a_1$ are called the **coefficients** of the polynomial. We assume $a_n \neq 0$. The number n is called the **degree** of the polynomial.*

Example 32. *Write polynomials of degree 0, 1 and 2.*

Just as the quotient of two integers is called a rational number, the quotient of two polynomials is called a **rational function**. Give an example.

Let A_m be a polynomial of degree m, B_n be a polynomial of degree n with $m \geq n$. Then there are polynomial Q_{m-n} of degree $m-n$, R_k of degree $k<n$ such that

$$\frac{A_m}{B_n} = Q_{m-n} + \frac{R_k}{B_n}.$$

The quotient Q_{m-n} and the remainder R_k can be calculated by the "long division".

Example 33. *Using the long division algorithm, show that*

$$\frac{2x^3-3x^2+3x+4}{x^2+1} = 2x-3+\frac{x+7}{x^2+1}$$

If P is a polynomial and $P(r)=0$ then r is called a **root** of P.

The Fundamental Theorem of Algebra says every polynomial of degree greater than 0 must have a root. But these roots may be complex.

Example 34. x^2+1 *has no real roots. Its roots are* $i=\sqrt{-1}$ *and* $-i$.

Theorem 2. *If r is a root of the polynomial P then*

$$P(x) = (x-r)Q(x),$$

for some polynomial Q whose degree is 1 less than P.

The polynomial $x(x-7)^3$ has 4 roots: 0 and the other three are each equal to 7. We say that 7 is a root of **multiplicity** 3.

By the Fundamental Theorem of Algebra and the above theorem, every polynomial of degree n has exactly n (not necessarily distinct) roots.

Roots of Quadratic Polynomials

To obtain the solutions of

$$Ax^2+Bx+C=0, \qquad A \neq 0$$

Divide by A and complete to square

$$\left(x+\frac{B}{2A}\right)^2 = \frac{B^2}{4A^2} - \frac{C}{A} = \frac{B^2-4AC}{4A^2},$$

Taking the square root of both sides gives the quadratic formula

$$x = \frac{-B \pm \sqrt{B^2-4AC}}{2A}.$$

A form of this formula is known since B.C. 2000 by Babylonians.

The quantity $D=B^2-4AC$ is called the **discriminant** of the quadratic equation.

- If $D>0$ then there are two distinct real roots,
- If $D=0$ then there is 1 root of multiplicity 2,
- If $D<0$ then there are two complex conjugate roots.

Example 35. *Find the roots of the polynomials: (a)* x^2+x-1, *(b)* $9x^2-6x+1$, *(c)* $2x^2+x+1$.

Misc Factorings

- Difference of squares:
$$x^2 - a^2 = (x-a)(x+a)$$
- Difference of cubes:
$$x^3 - a^3 = (x-a)(x^2 + ax + a^2)$$
- Difference of nth powers
$$x^n - a^n = (x-a)(x^{n-1} + ax^{n-2} + a^2x^{n-3} + \cdots + a^{n-2}x + a^{n-1})$$
- If n is an odd integer then $x+a$ is a factor of $x^n + a^n$,
$$x^n + a^n = (x+a)(x^{n-1} - ax^{n-2} + a^2x^{n-3} - \cdots - a^{n-2}x + a^{n-1})$$

Chapter 2

Limits and Continuity

2.1 Informal definition of limits

Two main problems of calculus are

1. Derivative. Find the rate of change of f.

2. Integral. Find the area under a given curve.

Both are based on the concept of limit.

We say $\lim_{x\to a} f(x) = L$ to mean that $f(x)$ is "close enough" to L when x is "close enough" to *but not equal to a*. Hence $f(a)$ is unimportant for $\lim_{x\to a} f(x)$.

Example 36. *Which value is x close to when x is close to 2?* $\lim_{x\to 2} x = 2$.

Example 37. *Which value is 3 close to when x is close to 2?* $\lim_{x\to 2} 3 = 3$.

We can generalize these examples.

Theorem 3. *Let a and c be two real numbers. Then*

$$\lim_{x\to a} c = c, \qquad \lim_{x\to a} x = x.$$

The limit $\lim_{x\to a} f(x)$ may be different from $f(a)$ as the next example shows.

Example 38.

$$f(x) = \begin{cases} x, & \text{if } x \neq 2 \\ 1, & \text{if } x = 2 \end{cases}$$

Which value is $f(x)$ close to when x is close to (but not equal to) 2?

$\lim_{x\to 2} f(x) = \lim_{x\to 2} x = 2$ *although* $f(2) = 1$.

Informal definition of left and right limits

If $f(x)$ is close to L when $x < a$ and x is close enough to a then we say

$$\lim_{x\to a^-} f(x) = L$$

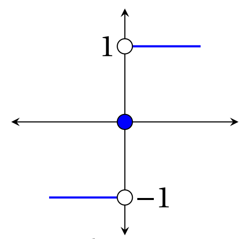
This is called the *left limit* of f at $x = a$.

Similarly we can define the right limit.

Theorem 4. $\lim_{x\to a} f(x) = L$ *if and only if both* $\lim_{x\to a^-} f(x) = L$ *and* $\lim_{x\to a^+} f(x) = L$.

Example 39. *Find the left and right limits of the signum function*

$$f(x) = \begin{cases} -1 & \text{for } x < 0 \\ 0 & \text{for } x = 0 \\ 1 & \text{for } x > 0 \end{cases}$$



In this example the one-sided limits exist, but are not equal

$$\lim_{x\searrow 0} f(x) = 1 \text{ and } \lim_{x\nearrow 0} f(x) = -1.$$

Hence $\lim_{x\to 0} f(x)$ *does not exist.*

Properties of Limits

Theorem 5. *Suppose*

$$\lim_{x\to a} f(x) = L, \qquad \lim_{x\to a} g(x) = M.$$

Then

$$\lim_{x\to a}\big(f(x) + g(x)\big) = L + M, \tag{2.1}$$

$$\lim_{x\to a}\big(f(x) - g(x)\big) = L - M, \tag{2.2}$$

$$\lim_{x\to a}\big(f(x) \cdot g(x)\big) = L \cdot M \tag{2.3}$$

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{L}{M}, \quad \text{if } M \neq 0. \tag{2.4}$$

Finally, if m and n are integers such that $L^{m/n}$ *is defined*

$$\lim_{x\to a} (f(x))^{m/n} = L^{m/n}. \tag{2.5}$$

Using the above properties we can evaluate the following limits.

Example 40. *Find* $\lim_{x\to 2} x^2 + 1$ *and* $\lim_{x\to 2} \dfrac{x^2+1}{6-x}$.

Solution. Using the product rule of limits and the Theorem 3,

$$\lim_{x\to 2} x^2 = \lim_{x\to 2} x \cdot \lim_{x\to 2} x = 2 \cdot 2 = 4$$

Using the sum rule of limits,

$$\lim_{x\to 2} x^2+1 = \lim_{x\to 2} x^2 + \lim_{x\to 2} 1 = 4+1 = 5$$

Using the division rule of limits,

$$\lim_{x\to 2} \frac{x^2+1}{6-x} = \frac{\lim_{x\to 2} x^2+1}{\lim_{x\to 2} 6-x} = \frac{5}{4}.$$

The above example is a special case of the following theorem.

Theorem 6. *If $P(x)$ is a polynomial then,*

$$\lim_{x\to a} P(x) = P(a)$$

If $Q(x)$ is another polynomial with $Q(a)\neq 0$ then

$$\lim_{x\to a} \frac{P(x)}{Q(x)} = \frac{P(a)}{Q(a)}.$$

The Squeeze Theorem

Theorem 7. *Suppose that $f(x)\leq g(x)\leq h(x)$ and $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$. Then $\lim_{x\to a} g(x) = L$.*

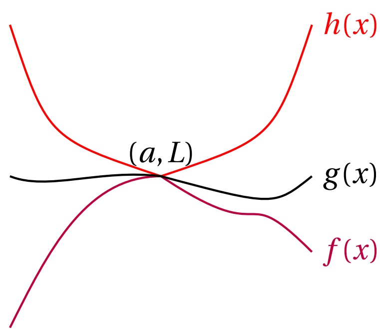


Figure 2.1: The Squeeze Theorem.

Example 41. *If $2-x^2\leq g(x)\leq 2\cos x$ for $-1\leq x\leq 1$, find $\lim_{x\to 0} f(x)$.*

Example 42. *Show that if $\lim_{x\to a}|f(x)| = 0$ then $\lim_{x\to a} f(x) = 0$.*
Note that $-|f(x)|\leq f(x)\leq |f(x)|$ and use the Squeeze Theorem.

More examples

Example 43. *Let*

$$f(x) = \frac{|x-2|}{x^2+x-6}.$$

Find $\lim_{x\to 2+} f(x)$, $\lim_{x\to 2-} f(x)$. Does $\lim_{x\to 2} f(x)$ exist?

In these example, we will compute $\lim_{x\to a} f(x)$ even when $f(a)$ does not exist.

Example 44. *Evaluate*

1. $\lim_{x\to-2} \dfrac{x^2+x-2}{x^2+5x+6}$,

   Remember that we consider x values close to but not equal to -2. Hence $x+2 \neq 0$ and we can make the simplification

$$\lim_{x\to-2} \frac{x^2+x-2}{x^2+5x+6} = \lim_{x\to-2} \frac{(x+2)(x-1)}{(x+2)(x+3)} = \lim_{x\to-2} \frac{x-1}{x+3} = \frac{-3}{1} = -3.$$

2. $\lim_{x\to5} \dfrac{\frac{1}{x}-\frac{1}{5}}{x-5}$,

3. $\lim_{x\to4} \dfrac{\sqrt{x}-2}{x^2-16}$,

   Trick is to multiply both sides by the conjugate expression.

4. $\lim_{x\to-2} \dfrac{x^2+2x}{x^2-4}$,

5. $\lim_{h\to0} \dfrac{\sqrt{4+h}-2}{h}$,

6. $\lim_{t\to0} \dfrac{t}{\sqrt{4+t}-\sqrt{4-t}}$,

7. $\lim_{x\to-1} \dfrac{x^3+1}{x+1}$,

8. $\lim_{x\to0} \dfrac{|3x-1|-|3x+1|}{x}$,

9. $\lim_{x\to2-} \dfrac{x^2-4}{|x+2|}$.

2.2 Limits at Infinity and Infinite Limits

Limits at Infinity

Definition 3. *We will say that $\lim_{x\to\infty} f(x) = L$ if $f(x)$ is "close enough" to L whenever $x > 0$ is "large enough".*

Similarly we define $\lim_{x\to-\infty} f(x) = L$ if $f(x)$ is "close enough" to L whenever $x < 0$ is "large enough".

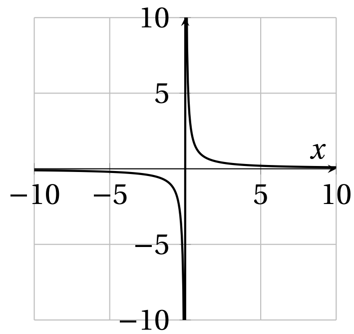
*If either $\lim_{x\to\infty} f(x) = L$ or $\lim_{x\to-\infty} f(x) = L$, we say that the line $y = L$ is an **horizontal asymptote** of the graph of f.*

Example 45. *Argue that*

$$\lim_{x\to\infty} 1/x = \lim_{x\to\infty} 1/x = 0.$$

by making a table of values of x and $1/x$.

$$f(x) = \frac{1}{x}$$



Recall that for ordinary limits, limit of product of functions is a product of limits of functions. Same is also true for limits at infinity. Hence

$$\lim_{x\to\infty}\frac{1}{x^2} = \lim_{x\to\infty}\frac{1}{x}\lim_{x\to\infty}\frac{1}{x} = 0\times 0 = 0.$$

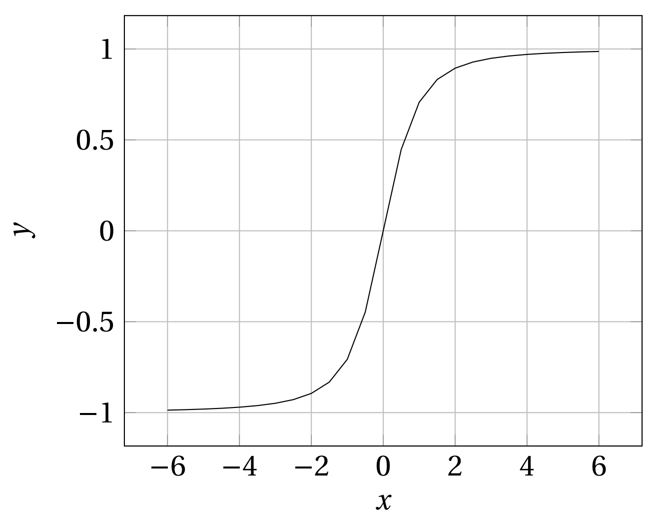
Similarly

$$\lim_{x\to-\infty}\frac{1}{x^2} = 0$$

Finally, for any positive integer n

$$\lim_{x\to\infty}\frac{1}{x^n} = \lim_{x\to-\infty}\frac{1}{x^n} = 0.$$

$$f(x) = \frac{x}{\sqrt{x^2+1}}$$



Example 46. *Let* $f(x) = \dfrac{x}{\sqrt{x^2+1}}$. *Find* $\lim_{x\to\infty} f(x)$, $\lim_{x\to-\infty} f(x)$.

$$\begin{aligned}\lim_{x\to\infty}\frac{x}{\sqrt{x^2+1}} &= \lim_{x\to\infty}\frac{x}{|x|\sqrt{1+1/x^2}} = \lim_{x\to\infty}\frac{x}{x\sqrt{1+1/x^2}} = \lim_{x\to\infty}\frac{1}{\sqrt{1+1/x^2}} = \frac{\lim_{x\to\infty}1}{\lim_{x\to\infty}\sqrt{1+1/x^2}}\\ &= \frac{1}{\sqrt{\lim_{x\to\infty}(1+1/x^2)}} = \frac{1}{1} = 1.\end{aligned}$$

Similarly,

$$\lim_{x\to-\infty}\frac{x}{\sqrt{x^2+1}}=-1$$

Limits of Rational Functions at Infinity

Recall that a rational function is a ratio of two polynomials.

Strategy. To find limits of rational functions at infinity, divide by the highest power of x appearing in the *denominator*.

Example 47.

$$\lim_{x\to\pm\infty}\frac{2x^2-x+3}{3x^2+5}=\lim_{x\to\pm\infty}\frac{2-\frac{1}{x}+\frac{3}{x^2}}{3+\frac{5}{x}}=\frac{2}{3}.$$

Example 48.

$$\lim_{x\to\pm\infty}\frac{x-5}{2x^2+4x+1}=\lim_{x\to\pm\infty}\frac{\frac{1}{x}-\frac{5}{x^2}}{2+\frac{4}{x}+\frac{1}{x^2}}=\frac{0}{2}=0.$$

We can generalize the above examples.

Theorem 8. *Let $P(x)=a_px^p+a_{p-1}x^{p-1}+\cdots+a_0$ be a polynomial of degree p and $Q(x)=b_qx^q+\cdots+b_0$ be a polynomial of degree q. If $p=q$, then*

$$\lim_{x\to\pm\infty}\frac{P(x)}{Q(x)}=\frac{a_p}{q_p},$$

If $p<q$, then

$$\lim_{x\to\pm\infty}\frac{P(x)}{Q(x)}=0,$$

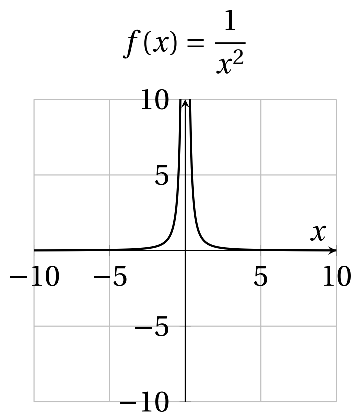
Example 49.

$$\lim_{x\to\infty}\sqrt{x^2+x}-x=\lim_{x\to\infty}\frac{(\sqrt{x^2+x}-x)(\sqrt{x^2+x}+x)}{\sqrt{x^2+x}+x}=\lim_{x\to\infty}\frac{x}{|x|\sqrt{1+\frac{1}{x}}+x}=\lim_{x\to\infty}\frac{1}{\sqrt{1+\frac{1}{x}}+1}=\frac{1}{2}.$$

Infinite Limits

Example 50. *The values of $\frac{1}{x^2}$ gets larger and larger as x approaches to 0. Thus $\lim_{x\to0}\frac{1}{x^2}$ does not exist. Although the limit does not exist, it is useful to state why it does not exist by writing*

$$\lim_{x\to0}\frac{1}{x^2}=\infty.$$



Example 51.

$$\lim_{x\to 0+} \frac{1}{x} = \infty.$$

$$\lim_{x\to 0-} \frac{1}{x} = -\infty.$$

$$\lim_{x\to 0} \frac{1}{x} \textit{ does not exist.}$$

Example 52.

$$\lim_{x\to -\infty} \sqrt{x^2+x} - x = \lim_{x\to -\infty} \frac{(\sqrt{x^2+x}-x)(\sqrt{x^2+x}+x)}{\sqrt{x^2+x}+x} = \lim_{x\to -\infty} \frac{x}{|x|\sqrt{1+\frac{1}{x}}+x} = \lim_{x\to -\infty} \frac{1}{-\sqrt{1+\frac{1}{x}}+1}$$

If $x<0$ then $\sqrt{1+\frac{1}{x}}<1$ and $-\sqrt{1+\frac{1}{x}}+1>0$. Hence the denominator is positive and approaches to zero. So

$$\lim_{x\to -\infty} \frac{1}{-\sqrt{1+\frac{1}{x}}+1} = \infty.$$

Behaviour of Polynomials at Infinity

Example 53.

$$\lim_{x\to\infty} 4x^3 - 2x + 1 = \lim_{x\to\infty} 4x^3 = \infty.$$

$$\lim_{x\to -\infty} -3x^5 + x^3 + 1 = \lim_{x\to -\infty} -3x^5 = \infty.$$

In general,

Theorem 9. *If $P(x) = a_n x^n + \cdots + a_0$ is a polynomial then*

$$\lim_{x\to\pm\infty} P(x) = \lim_{x\to\pm\infty} a_n x^n.$$

Example 54.

$$\lim_{x\to\infty} \frac{x^3+1}{x^2-2x} = \lim_{x\to\infty} \frac{x+\frac{1}{x^2}}{1-\frac{2}{x}} = \lim_{x\to\infty} \frac{x}{1} = \infty$$

Example 55. *1.* $\lim_{x\to 2} \dfrac{(x-2)^2}{x^2-4} = 0$

2. $\lim_{x\to 2+} \dfrac{x-3}{x^2-4} = -\infty$

3. $\lim_{x\to 2-} \dfrac{x-3}{x^2-4} = \infty$

4. $\lim_{x\to 2} \dfrac{x-3}{x^2-4}$ *does not exist.*

5. $\lim_{x\to\infty} \dfrac{2x-1}{\sqrt{3x^2+x+1}}$,

6. $\lim_{x\to 1+} \dfrac{\sqrt{x^2-x}}{x-x^2}$ *If* $x>1$ *then* $x-x^2 = x(1-x) < 0$. *So*

$$\lim_{x\to 1+} \frac{\sqrt{x^2-x}}{x-x^2} = \lim_{x\to 1+} \frac{-\sqrt{x^2-x}}{x^2-x} = \lim_{x\to 1+} \frac{-\sqrt{x^2-x}}{\sqrt{x^2-x}\sqrt{x^2-x}} = \lim_{x\to 1+} \frac{-1}{\sqrt{x^2-x}} = -\infty$$

2.3 Continuity

Let $f(x) = \sqrt{4-x^2}$. Domain of f is $[-2,2]$.

- $x=-2$ is the left end point of $\mathrm{Dom}(f)$.
- $x=2$ is the right end point of $\mathrm{Dom}(f)$.
- Any x with $-2<x<2$ is called an interior point of $\mathrm{Dom}(f)$.

Definition 4. *A function f is **continuous** at an interior point c of its domain if*

$$\lim_{x\to c} f(x) = f(c)$$

f is continuous at its left endpoint c if

$$\lim_{x\to c+} f(x) = f(c)$$

f is continuous at its right endpoint c if

$$\lim_{x\to c-} f(x) = f(c)$$

Note that f is discontinuous at c if

i) either $\lim_{x\to c} f(x)$ does not exist.

ii) or $\lim_{x\to c} f(x)$ exists but is not equal to $f(c)$.

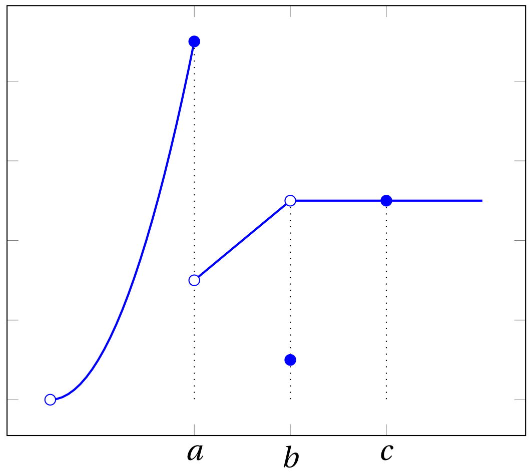
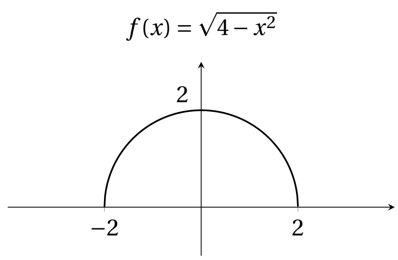


Figure 2.2: f is discontinuous at a because of (ii) and discontinuous at b because of (i). f is continuous at c.

Example 56. *$f(x) = \sqrt{4 - x^2}$ is continuous at every point of its domain.*



Definition 5. *f is called a continuous function if f is continuous at every pt of its domain.*

According to this definition $f(x) = \frac{1}{x}$ is continuous!!! 0 is not in domain of f. So we say f is undefined rather than discontinuous at 0.

There are lots of continuous functions:

- polynomials,
- rational functions,
- rational powers $x^{m/n}$
- trigonometric functions
- absolute value function $|x|$

Theorem 10. *If f and g are continuous at c then*

- *$f + g$, $f - g$, fg, are continuous at c,*
- *if k is constant then kf is continuous at c,*
- *$\dfrac{f}{g}$ continuous at c provided that $g(c) \neq 0$.*

- $f(x)^{1/n}$ *continuous at c provided that* $f(c) > 0$ *if n is even.*

Proof. Let's prove that if f and g are continuous at c then so is $f+g$. If f and g are continuous at c then

$$\lim_{x\to c} f(x) = f(c), \qquad \lim_{x\to c} g(x) = g(c),$$

By the limit rule,

$$\lim_{x\to c}(f(x)+g(x)) = \lim_{x\to c} f(x) + \lim_{x\to c} g(x) = f(c)+g(c).$$

The other proofs are similar. □

Composites of continuous funcs are continuous
If g is continuous at c and f is continuous at $g(c)$ then $f\circ g$ is continuous at c. In other words,

$$\lim_{x\to c} f(g(x)) = f(\lim_{x\to c} g(x)) = f(g(c)).$$

Example 57. *Find m so that*

$$g(x) = \begin{cases} x-m, & \text{if } x<3, \\ 1-mx, & \text{if } x\ge 3 \end{cases}$$

is continuous for all x.

Continuous Functions on Closed Intervals [a, b] are bounded

We call a function f to be **bounded** if there exists M and N such that $M \le f(x) \le N$ for all x in the domain of f.

Theorem 11. *If f is continuous on the closed interval $[a,b]$ then there exist numbers p and q in the interval $[a,b]$ s.t.*

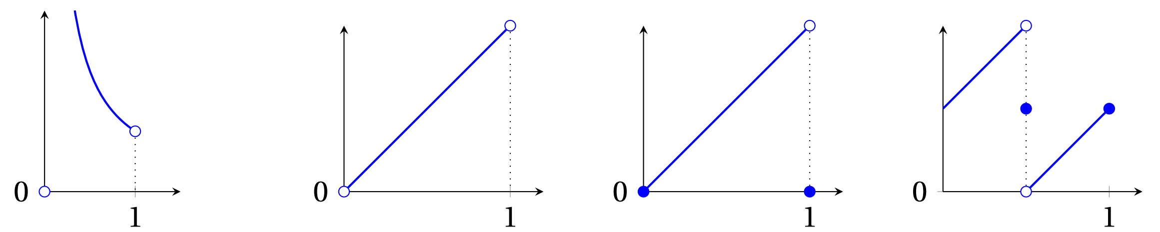
$$f(p) \le f(x) \le f(q)$$

*for all x in $[a,b]$. $f(p)$ is called the **aboslute minimum value** and $f(q)$ is called the **absolute maximum value**.*

This theorem is an existence theorem. It only guarantees the existence of p and q but does not tell how to actually find them.

The theorem says that continuous functions on closed intervals must be bounded.

Example 58. *The conclusions of the theorem may fail if the function f is not continuous or the interval is not closed.*



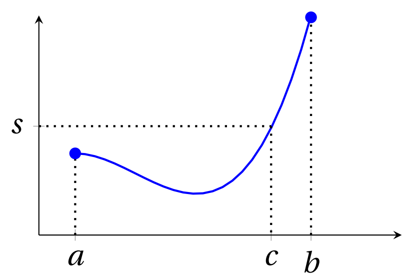
(a) The function $f(x) = 1/x$ on the open interval $(0,1)$ is continuous but unbounded and has no minimum and no maximum.

(b) The function $f(x) = x$ on $(0,1)$ is discontinuous, bounded and has no minimum and no maximum.

(c) This function is defined on the closed interval $[0,1]$, discontinuous, has a minimum but no maximum.

(d) This function is defined on the closed interval $[0,1]$, discontinuous, bounded, has no minimum but no maximum.

Theorem 12 (Intermediate Value Theorem)**.** *If f is continuous on $[a,b]$ and if s is between $f(a)$ and $f(b)$ then there exists c in $[a,b]$ s.t. $f(c) = s$.*



In particular, a continuous function on a closed interval takes every value between its minimum m and maximum M. Hence its range is a closed interval $[m,M]$.

Example 59. *Show that the equation $x^3 - x - 1 = 0$ has a solution in the interval* $[1,2]$.

Solution. *$f(x) = x^3 - x - 1$ is a polynomial and hence continuous. $f(1) = -1$ and $f(2) = 5$. Since 0 lies between -1 and 5, the intermediate value theorem assures us that there must be a number c in $[1,2]$ such that $f(c) = 0$.*

Bisection Algorithm

Intermediate Value Theorem is also an existence theorem. It does not say how to find c in its statement. Let's try to better estimate the root of previous example. Write $f(x) = x^3 - x - 1$ and try to find a smaller interval where a root lies of

$$f(x) = 0.$$

We know that a root lies in $[1,2]$, if say that the root is 1.5 the maximum error will be 0.5.

Now $f(1.5) = 0.875 > 0$. So a root lies in $[1,1.5]$, and if we say the root is 1.25 then the maximum error will be 0.25.

If this is not sufficient then compute $f(1.25) = -0.2969$, now if we say the root is 1.375 then the error is less than 0.125.

Next step is $f(1.1375) = 0.2246$. So a root must lie in $[1.25, 1375]$. The error is less than 0.0625 if we say the root is 1.315.

Going this way, we find the approximations, 1.3438, 1.3282, 1.3204. Hence the root must lie in $[1.3282, 1.3204]$. So the first two decimal digits of the root are 1.32.

In engineering, you almost never get exact results. All you can do is lower your error below an acceptable threshold.

2.4 Formal definition of Limit

The informal description of the limit uses phrases like “close enough” and “really very small”. “Fortunately” there is a good definition, i.e. one which is unambiguous and can be used to settle any dispute about the question of whether $\lim_{x\to a} f(x)$ equals some number L or not.

In this section we assume that f is defined in an open interval containing a except possibly at $x = a$.

Definition 6. *We say that*

$$\lim_{x\to a} f(x) = L$$

if for every $\epsilon > 0$ *there exists a* $\delta > 0$ *such that*

$$0 < |x-a| < \delta \textit{ implies } |f(x) - L| < \epsilon. \tag{2.6}$$

Why the absolute values? Recall that the quantity $|x - y|$ is the distance between the points x and y on the number line.

What are ϵ *and* δ*?* The quantity ϵ is how close you would like $f(x)$ to be to its limit L; the quantity δ is how close you have to choose x to a to achieve this. To prove that $\lim_{x\to a} f(x) = L$ you must assume that someone has given you an unknown $\epsilon > 0$, and then find a positive δ for which (2.6) holds. The δ you find will depend on ϵ.

When we first discussed the limit, say $\lim_{x\to 5} 2x + 1$, we made a table,

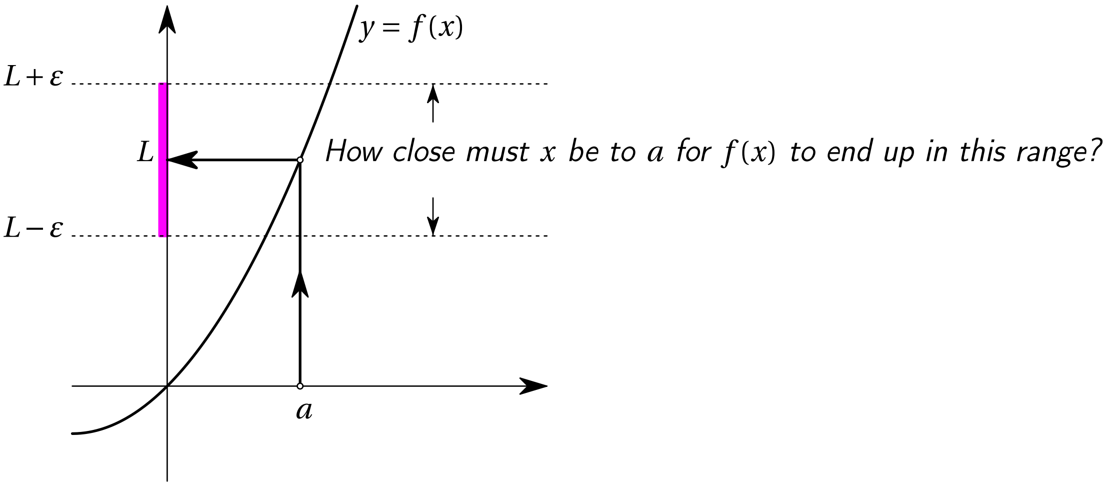
x	$f(x) = 2x+1$
5.1	11.2
5.01	11.02
5.001	11.002
4.9	10.8
4.99	10.98
4.999	10.998

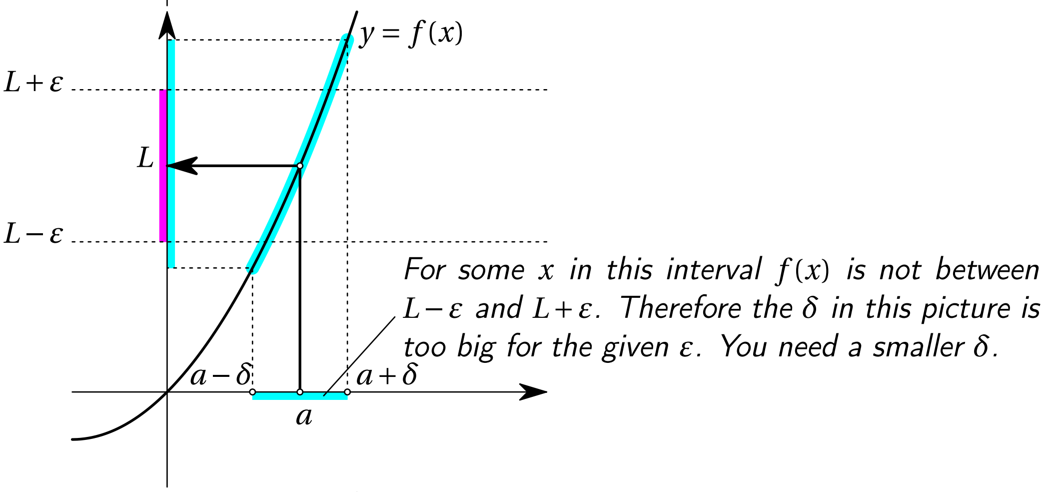
This table can be written also in this form.

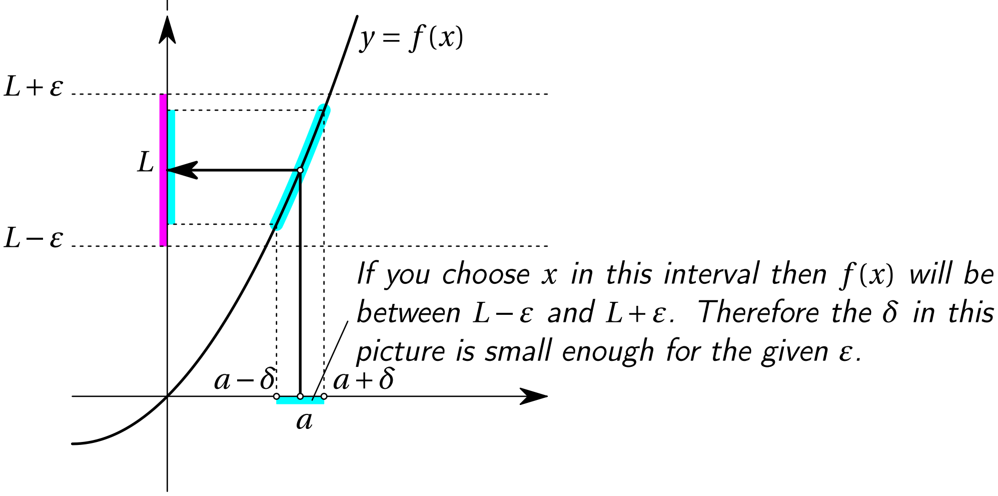
$\lvert x-5\rvert$	$\lvert f(x)-11\rvert$
0.1	0.2
0.01	0.02
0.001	0.002

It looks like for any $\epsilon > 0$, if $|x - 5| < \frac{\epsilon}{2}$ then $|f(x) - 11| < \epsilon$. Now let’s prove this!

Example 60. *Show that* $\lim_{x\to 5} 2x + 1 = 11$.







Solution. *We have $f(x) = 2x+1$, $a = 5$ and $L = 11$, and the question we must answer is "how close should x be to 5 if you want to be sure that $f(x) = 2x+1$ differs less than ϵ from $L = 11$?"*

$$|f(x)-L| = |(2x+1)-11| = |2x-10| = 2\cdot|x-5| = 2\cdot|x-a|.$$

So choose $\delta = \frac{\epsilon}{2}$. Then

$$|f(x)-L| < \epsilon \text{ whenever } 0 < |x-a| < \frac{\epsilon}{2}.$$

Example 61 ("Don't choose $\delta > 1$" trick). *Show that* $\lim_{x\to 3} x^2 = 9$.

Solution. *We have* $f(x) = x^2$, $a = 3$, $L = 9$, *and again the question is, "how small should* $|x-3|$ *be to guarantee* $|x^2 - 9| < \epsilon$*?"*

$$|x^2 - 9| = |(x-3)(x+3)| = |x+3| \cdot |x-3|.$$

Here is a trick that allows you to replace the factor $|x+3|$ *with a constant. We hereby agree that we always choose our* δ *so that* $\delta \le 1$. *If we do that, then we will always have*

$$|x-3| < \delta \le 1, \textit{i.e. } |x-3| < 1,$$

or $2 < x < 4$ *or* $|x+1| < 5$. *Therefore*

$$|x^2 - 1| = |x+1| \cdot |x-1| < 5|x-1|.$$

So choose

$$\delta = \min\{1, \frac{\epsilon}{5}\}.$$

2nd way: Note that $|x+3| = |x-3+6| < |x-3| + 6 < \delta + 6$

$$|f(x) - 9| = |x+3||x-3| < (\delta+6)\delta$$

So choose $(\delta+6)\delta < \epsilon$, *or*

$$(\delta+3)^2 < \epsilon + 9 \implies \delta < \sqrt{\epsilon+9} - 3$$

Example 62. *Show that* $\lim_{x\to 4} 1/x = 1/4$.

We apply the definition with $a = 4$, $L = 1/4$ *and* $f(x) = 1/x$. *Thus, for any* $\epsilon > 0$ *we try to show that if* $|x-4|$ *is small enough then one has* $|f(x) - 1/4| < \epsilon$.

We begin by estimating $|f(x) - \frac{1}{4}|$ *in terms of* $|x-4|$*:*

$$|f(x) - 1/4| = \left|\frac{1}{x} - \frac{1}{4}\right| = \left|\frac{4-x}{4x}\right| = \frac{|x-4|}{|4x|} = \frac{1}{|4x|}|x-4|.$$

As before, things would be easier if $1/|4x|$ *were a constant. To achieve that we again agree not to take* $\delta > 1$. *If we always have* $\delta \le 1$, *then we will always have* $|x-4| < 1$, *and hence* $3 < x < 5$. *How large can* $1/|4x|$ *be in this situation? Answer: the quantity* $1/|4x|$ *increases as you decrease* x, *so if* $3 < x < 5$ *then it will never be larger than* $1/|4 \cdot 3| = \frac{1}{12}$.

We see that if we never choose $\delta > 1$, *we will always have*

$$|f(x) - \frac{1}{4}| \le \frac{1}{12}|x-4| \quad \textit{for} \quad |x-4| < \delta.$$

To guarantee that $|f(x) - \frac{1}{4}| < \epsilon$ *we could threfore require*

$$\frac{1}{12}|x-4| < \epsilon, \quad \textit{i.e.} \quad |x-4| < 12\epsilon.$$

Hence if we choose $\delta = 12\epsilon$ *or any smaller number, then* $|x-4| < \delta$ *implies* $|f(x) - 4| < \epsilon$. *Of course we have to honor our agreement never to choose* $\delta > 1$, *so our choice of* δ *is*

$$\delta = \textit{the smaller of } 1 \textit{ and } 12\epsilon = \min\big(1, 12\epsilon\big).$$

Example 63. *Verify that* $\lim_{x\to 2} \dfrac{x-2}{1+x^2} = 0$.

Solution: Notice that $\dfrac{|x-2|}{|1+x^2|} < |x-2|$ *since* $1 + x^2 > 1$. *Hence choose* $\delta = \epsilon$.

2.5 Review Problems

Example 64. *Evaluate the limits if they exist. If they do not exist, state wheter they are ∞, $-\infty$ or just does not exist.*

1. $\lim_{x\to 2} \frac{x^2+1}{1-x^2}$,
2. $\lim_{x\to 1} \frac{x^2}{1-x^2}$,
3. $\lim_{x\to\infty} \frac{\cos x}{x}$, *(Hint: Use Sandwich Theorem)*
4. $\lim_{x\to-\infty} \frac{2x^3+2x-1}{-3x^3+x^2}$,
5. $\lim_{x\to-\infty} x+\sqrt{x^2-4x+1}$,
6. $\lim_{x\to 0} \frac{x}{|x-1|-|x+1|}$.