Calculus I Lecture Notes

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Chapter 1

Precalculus

1.1 Sets

A **set** is a collection of elements.

$x \in A$ means x is an element of the set A. If x is not a member of A, we write $x \notin A$.

$\varnothing$ is the set which contains no element and is called the **empty set**.

There are finite sets such as $\{0,1,2\}$ and infinite sets such as $\{0,1,2,3,...\}$.

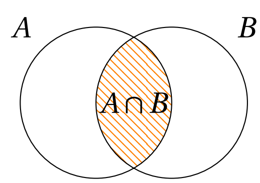
If every element of the set A is an element of the set B, we say that A is **subset** of B, and write $A \subset B$.

Example. *List all the subsets of* $\{0,1,2\}$.

For any set A, $A \subset A$ and $\varnothing \subset A$.

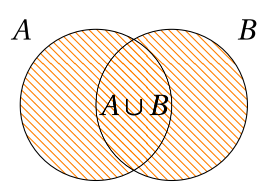
If $A \subset B$ and $B \subset A$, we write $A = B$.

$A \cap B = \{x : x \in A \text{ and } x \in B\}$ is called the **intersection** of A and B.



If the intersection of two sets is the empty set, those sets are called **disjoint**.

$A \cup B = \{x : x \in A \text{ or } x \in B\}$ is called the **union** of A and B.



Example. *For example if* $A = \{0,1,2,5,8\}$ *and* $B = \{1,3,5,6\}$ *then find* $A \cap B$ *and* $A \cup B$.

The set of all elements in A but not in B is denoted $A \setminus B = \{x \in A : x \notin B\}$ and is called the **complement** of B in A.

Example. $\{0,2,3,5\} \setminus \{2,5,7,8\} = \{0,3\}$

$A \times B = \{(a,b) : a \in A \text{ and } b \in B\}$ is called the **Cartesian** product of the sets A and B.

Example. *Write the cartesian product of* $A = \{0,1,2\}$ *and* $B = \{2,3,4\}$.

1.2 Real Numbers

The **integers** are $\mathbb{Z} = \{\ldots,-2,-1,0,1,2,\ldots\}$. Some important subsets of the set of integers are:

- even integers that are of the form $2k$, for some $k \in \mathbb{Z}$,
- odd integers that are of the form $2k+1$, for some $k \in \mathbb{Z}$
- positive and negative integers,
- primes, etc...

The **rational numbers** are $\mathbb{Q} = \left\{\frac{m}{n} : m,n \in \mathbb{Z} \text{ and } n \neq 0\right\}$.

Pythagoreans thought that all numbers are ratios of integers. The discovery of irrational numbers is said to have shocked them.

Example. $\sqrt{2}$ *is not a rational number.*

Suppose that it is rational. Then $\sqrt{2} = m/n$, *where* $m,n \in \mathbb{Z}$ *and* $n \neq 0$. *Also assume* m *and* n *have no common divisor.*

$$m^2/n^2 = 2 \implies m^2 = 2n^2$$

Thus m *is even and we can write* $m = 2k$, *where* $k \in \mathbb{Z}$.

$$4k^2 = 2n^2 \implies n^2 = 2k^2$$

Thus n *is also even. But* m *and* n *cannot both be even. Accordingly, there can be no rational number whose square is 2.*

The set of irrational numbers is denoted by $\mathbb{I}$.
The set of real numbers is $\mathbb{R} = \mathbb{Q} \cup \mathbb{I}$.
Note that $\mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$.
The real numbers are ordered such that

1. $a < b \implies a + c < b + c$
2. $a < b$ and $c > 0$ implies $ac < bc$
3. $a < b$ and $c < 0$ implies $ac > bc$
4. $a > 0$ implies $\frac{1}{a} > 0$
5. $0 < a < b$ implies $\frac{1}{b} < \frac{1}{a}$

Intervals

The open interval $(a,b) = \{x \mid a < x < b\}$, closed interval ($[a,b]$), half open intervals $(a,b]$, $[a,b)$. It is possible that $a = -\infty$, $b = \infty$. Draw each interval on the real line.

Example. *Solve the following inequalities.*

1. $\frac{2}{x-1} \geq 5$.

   Solution. *It is not right to multiply both sides by* $x-1$ *and say* $5x-5 \leq 2$.

   $$\frac{2}{x-1} \geq 5 \iff \frac{2}{x-1} - 5 \geq 0 \iff \frac{7-5x}{x-1} \geq 0.$$

   Now make a sign analysis to get interval $(1, 7/5]$

2. $3x-1 \leq 5x+3 \leq 2x+15$.

   Solution. $-2 \leq x$ *and* $x \leq 4$.

The absolute value.

$$|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$$

ex. $|3| = |-3| = 3$

Geometrically, $|x|$ is the distance between x and 0 on the real line. And $\left|x-y\right|$ is the distance between x and y.

Properties *(can be proved from definition)*:

1. $|-x| = |x|$, (Do not fall into the trap $|-x| = x$, this is not always true!)

2. $|ab| = |a|\,|b|$,

3. $|a+b| \leq |a| + |b|$, (triangle inequality).

From (2), for any x, $x^2 = \left|x^2\right| = |x|^2$

If D is a nonnegative number

$$|x| = D \implies x = -D \text{ or } x = D,$$
$$|x| < D \implies -D < x < D$$
$$|x| > D \implies x < -D \text{ or } x > D$$

More generally,

$$|x-a| = D \implies x = a-D \text{ or } x = a+D,$$
$$|x-a| < D \implies a-D < x < a+D$$
$$|x-a| > D \implies x < a-D \text{ or } x > a+D$$

Example. *Solve* $|3x-2| \leq 1$.

Solution.

$$-1 \leq 3x-2 \leq 1 \implies x \geq 1/3 \textit{ and } x \leq 1.$$

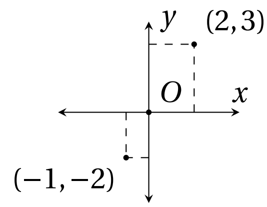
Example. *Solve the equation* $|x+1| > |x-3|$.

Solution. The distance between x *and* -1 *is greater than the distance between* x *and* 3*. So* $x > 1$.

1.3 Cartesian Coordinates

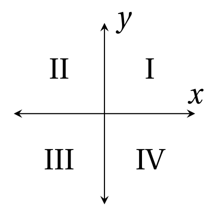
Cartesian plane is

$$\mathbb{R} \times \mathbb{R} = \{(x, y) \mid x \in \mathbb{R} \text{ and } b \in \mathbb{R}\}.$$



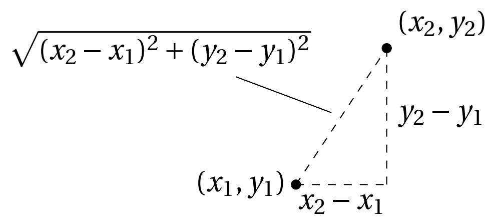
Horizontal axis is usually called the x axis, the vertical axis is called the y axis. Intersection of the axes is called the origin, denoted O.

The coordinate axes divide the Cartesian plane into four quadrants.



By the Pythagorean Theorem, the distance between two points (x_1, y_1) and (x_2, y_2) is

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$



The distance of (x, y) to the origin $(0,0)$ is $\sqrt{x^2 + y^2}$.

Example. *Find the distance between* $(-1,1)$ *and* $(3,-4)$.

Graphs of Equations

The set of all points (x, y) satisfying an equation in x and y is called the **graph of that equation**.

Example. *The graph of* $x^2 + y^2 = 4$ *is the set of all* (x, y) *whose distance to* $(0,0)$ *is 2, i.e. a circle with center at origin and radius 2.*

Equations of Lines

For any two points (x_1, y_1) and (x_2, y_2) on a non-vertical line L, the quantity $m = \dfrac{y_2 - y_1}{x_2 - x_1}$ is constant and is called the **slope** of the line L.

Let L be a nonvertical line. Let m be the slope of L and (x_1, y_1) be the coordinates of a point on L. If (x, y) is another point on L, then

$$\frac{y - y_1}{x - x_1} = m$$

Hence any (x, y) on L satisfies

$$y = m(x - x_1) + y_1$$

The above is known as an equation for the line L.

All points on a **vertical line** have their x coordinate equal to a constant a. So the equation of a vertical line is $x = a$. **Horizontal lines** have equations of the form $y = a$.

y-intercept of a nonvertical line L is the y-coordinate of the point where L intersects the y-axis. **x-intercept** of a nonhorizontal is defined similarly.

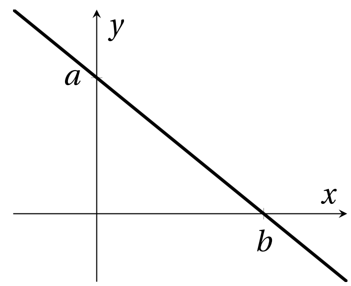


Figure 1.1: a is the x-intercept, b is the y-intercept.

Example. *Find an equation of the line through the points* $(1,-1)$ *and* $(3,5)$. *Draw the line. Find the x and y intercepts.*

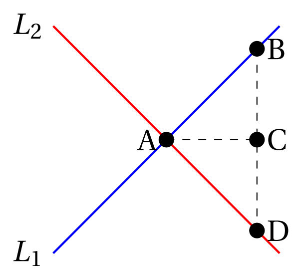
Example. *Find an equation of the line that passes through the point* $(-3,-4)$ *and has slope* 2. *Draw the line.*

Example. *Find the slope and the two intercepts of the line with equation* $8x+5y=20$. *Draw the line.*

Parallel vs. perpendicular lines

We call two lines **parallel** if their slopes are equal. We call two lines **perpendicular** if they intersect at right angles (90°).

Theorem 1. *Two nonvertical lines with slopes* m_1 *and* m_2 *are perpendicular if and only if* $m_1 m_2 = -1$.



Proof. Use the similarity of the triangles ABC and DAC to get

$$\frac{|BC|}{|AC|} = \frac{|AC|}{|CD|} \implies \frac{|BC||CD|}{|AC|^2} = 1$$

Slope of L_1 (m_1) is $|BC|\,/\,|AC| = 1$ and slope of L_2 (m_2) is $-|CD|\,/\,|AC|$. So $m_1 m_2 = -1$. □

Example. *Find an equation of the line through* $(1,-2)$ *that is parallel to the line L with equation* $3x-2y=1$. *Draw the lines.*

Example. *Find an equation of the line through* $(2,-3)$ *that is perpendicular to the line L with equation* $4x+y=3$. *Draw the lines.*

1.4 Quadratic Equations

Circles and Disks

The circle is the set of all points that have the same distance (called radius of the circle) from a given point (called center of the circle).

If (x, y) is a point on a circle with center (a, b) and radius r then

$$\sqrt{(x-a)^2+(y-b)^2} = r \implies (x-a)^2+(y-b)^2 = r^2$$

Example. *Find the center and radius of the circle* $x^2 + y^2 - 4x + 6y = 3$.

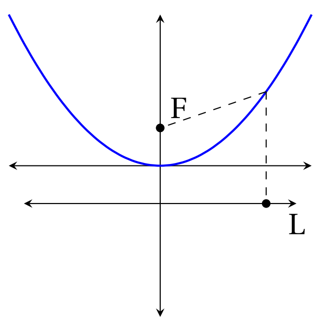
Solution. Complete to squares to get $(x-2)^2 + (y+3)^2 = 16$.

The equation $(x-a)^2+(y-b)^2 < r^2$ represents open disk and the equation $(x-a)^2+(y-b)^2 \leq r^2$ represents closed disk or simply disk.

Example. *Draw* $x^2 + 2x + y^2 \leq 8$.

Parabolas

A parabola P is the set of all points in the plane that are equidistant from a given line L (called directrix of P) and a point F (called the focus of P).



Example. *Find the equation of the parabola having the point* $F(0, p)$ *as focus and the line L with equation* $y = -p$ *as directrix.*

Solution. *If* $P(x, y)$ *is any point on the parabola then squaring both sides of PF=PQ we get*

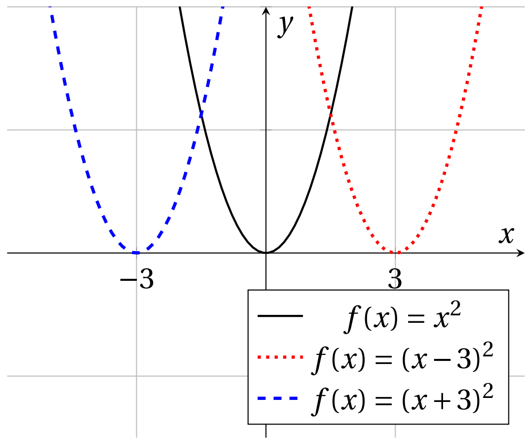
$$x^2 + (y-p)^2 = 0^2 + (y+p)^2$$

After simplifying, $y = x^2/4p$.

Shifting a Graph

Let $c > 0$.

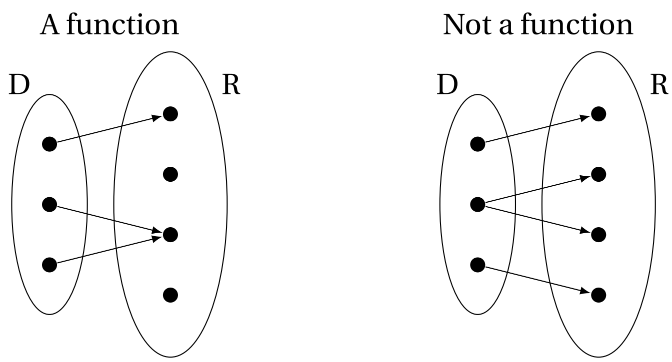
- To shift a graph c units to the right, replace x in its equation with $x - c$. To shift to left, replace x by $x + c$.
- To shift a graph c units up, replace y in its equation with $y - c$. To shift down, replace y by $y + c$.



1.5 Functions and Their Graphs

A **function** f on a set D into a set R is a rule that assigns a unique element $f(x)$ in R to each element x in D.

D is called the **domain** of f. R is called the target or **codomain** of f. The **range** of f is a subset of R containing of all possible values $f(x)$.



Example. *Define a function on the set of all real numbers by* $f(x) = x^2 + 1$. *Find* $f(0)$, $f(2)$, $f(x+2)$.

When defining a function, its domain should be defined. For example,

$$f(x) = \frac{1}{x}, \qquad x > 0$$

means that the domain of f is the set $\{x \mid x > 0\}$. This function is different from the function

$$f(x) = \frac{1}{x}, \qquad x < 0.$$

If we do not specify the domain of a function f, then the **domain convention** is to assume that the domain of f is the set of all real numbers for which f is defined.

So if we write

$$f(x) = \frac{1}{x},$$

we are assuming f is defined for all real numbers except 0.

Example. *Find the domain of* $f(x) = \sqrt{2-x}$.

Solution. *Its domain is all* x *for which* $2 - x \geq 0$, *i.e. the interval* $(-\infty, 2]$.

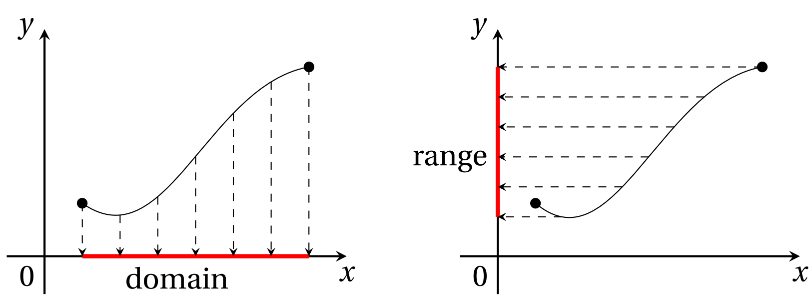
Example. *Find the domain of* $f(x) = \dfrac{1}{x^2 - x}$.

Graph of a function

> A picture is worth a thousand words.

The *graph of a function* f is the set of all points whose coordinates are $(x, f(x))$ where x is in the domain of f. We can visualize a function by plotting its graph set on the Cartesian plane.

> Why is visualization important? The answer probably lies in human psychology.



Some Elementary Functions

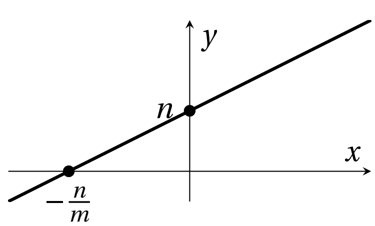
Linear Function

A function which is given by the formula

$$f(x) = mx + n$$

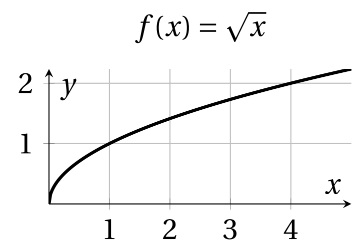
where m and n are constants is called a **linear function**. Its graph is a straight line. The constants m and n are the *slope* and y-*intercept* of the line. Its domain is all x and its range is all x.

$f(x) = mx + n$



Square Root Function

The square root function $f(x) = \sqrt{x}$ has domain $[0, \infty)$ and takes x to its positive square root. Hence it has range $[0, \infty)$.

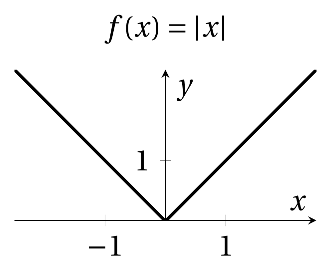


The absolute value function

The absolute value function is

$$|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$$

Its domain $(-\infty, \infty)$ and range $[0, \infty)$. We can only define the absolute value function by $f(x) = |x| = \sqrt{x^2}$.



Example. *Draw the graphs of some elementary functions*

$$c, x, x^2, x^3, x^{1/3}, \frac{1}{x}, \frac{1}{x^2}, \sqrt{1 - x^2}.$$

Example. *Sketch the graph of* $f(x) = 1 + \sqrt{x - 4}$.

Solution. *Shift the graph of* $y = \sqrt{x}$ *1 unit up and 4 units to the right.*

Example. *Sketch the graph of the function* $f(x) = \frac{2-x}{x-1}$.

Solution. $f(x) = \frac{2-x}{x-1} = -1 + \frac{1}{x-1}$. *So shift the graph of* $y = \frac{1}{x}$ *1 unit down and 1 unit to the right.*

Vertical Line Test

The graph of a function cannot intersect a vertical line "$x =$ constant*" in more than one point.*

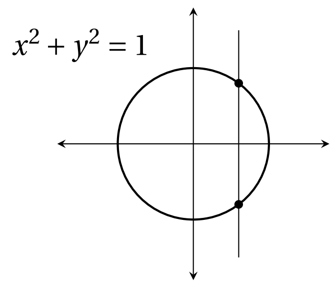
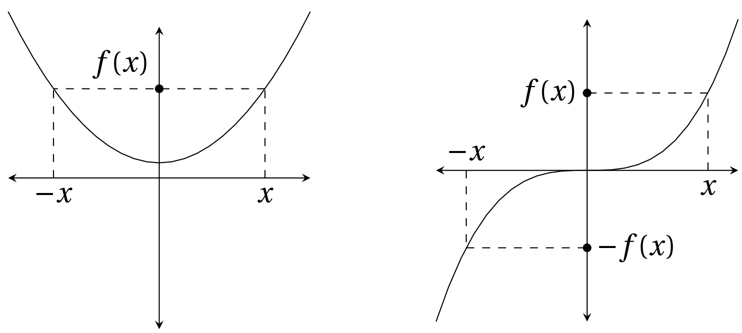


Figure 1.3: The circle $x^2 + y^2 = 1$ is not a graph of a function. It fails the vertical line test.

Even and Odd Functions

Definition 1. *We say that f is an **even function** if $f(-x) = f(x)$ for every $x \in D$. We say that f is an **odd function** if $f(-x) = -f(x)$ for every $x \in D$.*



(a) An even function

(b) An odd function.

Odd functions are symmetric with respect to origin and even functions are symmetric with respect to the y-axis.

Example. *$f(x) = x$, $f(x) = x^3$ are odd and $f(x) = x^2$ and $f(x) = x^4$ are even and $f(x) = \frac{1}{x+1}$ is neither even or odd.*

Example. *$f(x) = x^3 + x$ is odd and $f(x) = \frac{1}{x^2-1}$ is even and $f(x) = x^2 + x$ is either even or odd.*

1.6 Operations on Functions

If f and g are functions, then for every x that belongs to the domains of both f and g we define functions

- $(f + g)(x) = f(x) + g(x)$
- $(f - g)(x) = f(x) - g(x)$
- $(fg)(x) = f(x)g(x)$
- $(f/g)(x) = f(x)/g(x)$ where $g(x) \neq 0$.

Example. *Let $f(x) = \frac{1}{x+2}$ and $g(x) = \frac{x}{x-1}$. Find $(f+g)(x)$, $(f-g)(x)$, $(fg)(x) = f(x)g(x)$ and $(f/g)(x)$ where* $g(x) \neq 0$.

Composition of Functions

If f and g are two functions, then

$$f \circ g(x) = f(g(x)).$$

The domain of $f \circ g$ consists of those numbers x in the domain of g for which $g(x)$ is in the domain of f.

Example. *Let $f(x) = \sqrt{x}$ and $g(x) = x+1$. Find $f \circ g$, $g \circ f$, $f \circ f$ and $g \circ g$. State the domains of each function.*

Function	Formula	Domain
f	$\sqrt{x}$	$[0,\infty)$
g	$x+1$	$\mathbb{R}$
$f \circ g$	$\sqrt{x+1}$	$[-1,\infty)$
$g \circ f$	$\sqrt{x}+1$	$[0,\infty)$
$f \circ f$	$x^{1/4}$	$[0,\infty)$
$g \circ g$	$x+2$	$\mathbb{R}$

Piecewise Defined Functions

Functions such as

$$g(x) = \begin{cases} 2x & \text{for } x < 0 \\ x^2 & \text{for } x \geq 0 \end{cases}$$

which are defined by different formulas on different intervals are sometimes called **piecewise defined functions.**

1.7 Polynomials and Rational Functions

Definition 2. *A **polynomial** is a function $P : \mathbb{R} \to \mathbb{R}$ such that*

$$P(x) = a_n x^n + \cdot + a_1 x + a_0.$$

*Here $a_n, \dots, a_1$ are called the **coefficients** of the polynomial. We assume $a_n \neq 0$. The number n is called the **degree** of the polynomial.*

Example. *Write polynomials of degree 0, 1 and 2.*

Just as the quotient of two integers is called a rational number, the quotient of two polynomials is called a **rational function**. Give an example.

Let A_m be a polynomial of degree m, B_n be a polynomial of degree n with $m \geq n$. Then there are polynomial Q_{m-n} of degree $m-n$, R_k of degree $k < n$ such that

$$\frac{A_m}{B_n} = Q_{m-n} + \frac{R_k}{B_n}.$$

The quotient Q_{m-n} and the remainder R_k can be calculated by the "long division".

Example. *Using the long division algorithm, show that*

$$\frac{2x^3 - 3x^2 + 3x + 4}{x^2 + 1} = 2x - 3 + \frac{x+7}{x^2+1}$$

If P is a polynomial and $P(r) = 0$ then r is called a **root** of P.

The Fundamental Theorem of Algebra says every polynomial of degree greater than 0 must have a root. But these roots may be complex.

Example. *$x^2 + 1$ has no real roots. Its roots are $i = \sqrt{-1}$ and $-i$.*

Theorem 2. *If r is a root of the polynomial P then*

$$P(x) = (x - r)Q(x),$$

for some polynomial Q whose degree is 1 less than P.

The polynomial $x(x-7)^3$ has 4 roots: 0 and the other three are each equal to 7. We say that 7 is a root of **multiplicity** 3.

By the Fundamental Theorem of Algebra and the above theorem, every polynomial of degree n has exactly n (not necessarily distinct) roots.

Roots of Quadratic Polynomials

To obtain the solutions of

$$Ax^2 + Bx + C = 0, \qquad A \neq 0$$

Divide by A and complete to square

$$\left(x + \frac{B}{2A}\right)^2 = \frac{B^2}{4A^2} - \frac{C}{A} = \frac{B^2 - 4AC}{4A^2},$$

Taking the square root of both sides gives the quadratic formula

$$x = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}.$$

A form of this formula is known since B.C. 2000 by Babylonians.

The quantity $D = B^2 - 4AC$ is called the **discriminant** of the quadratic equation.

- If $D > 0$ then there are two distinct real roots,
- If $D = 0$ then there is 1 root of multiplicity 2,
- If $D < 0$ then there are two complex conjugate roots.

Example. *Find the roots of the polynomials: (a) $x^2 + x - 1$, (b) $9x^2 - 6x + 1$, (c) $2x^2 + x + 1$.*

Misc Factorings

- Difference of squares:
$$x^2 - a^2 = (x - a)(x + a)$$
- Difference of cubes:
$$x^3 - a^3 = (x - a)(x^2 + ax + a^2)$$

- Difference of nth powers

$$x^n - a^n = (x-a)(x^{n-1} + ax^{n-2} + a^2x^{n-3} + \cdots + a^{n-2}x + a^{n-1})$$

- If n is an odd integer then $x + a$ is a factor of $x^n + a^n$,

$$x^n + a^n = (x+a)(x^{n-1} - ax^{n-2} + a^2x^{n-3} - \cdots - a^{n-2}x + a^{n-1})$$

Chapter 2

Limits and Continuity

2.1 Informal definition of limits

Two main problems of calculus are

1. Derivative. Find the rate of change of f.

2. Integral. Find the area under a given curve.

Both are based on the concept of limit.

We say $\lim_{x \to a} f(x) = L$ to mean that $f(x)$ is "close enough" to L when x is "close enough" to *but not equal to* a. Hence $f(a)$ is unimportant for $\lim_{x \to a} f(x)$.

Example. *Which value is x close to when x is close to 2?*

$$\lim_{x \to 2} x = 2$$

Example. *Which value is 3 close to when x is close to 2?*

$$\lim_{x \to 2} 3 = 3$$

We can generalize these examples.

Theorem 3. *Let a and c be two real numbers. Then*

$$\lim_{x \to a} c = c, \qquad \lim_{x \to a} x = a.$$

The limit $\lim_{x \to a} f(x)$ may be different from $f(a)$ as the next example shows.

Example.

$$f(x) = \begin{cases} x, & \textit{if } x \neq 2 \\ 1, & \textit{if } x = 2 \end{cases}$$

Which value is $f(x)$ close to when x is close to (but not equal to) 2?

$\lim_{x \to 2} f(x) = \lim_{x \to 2} x = 2$ *although* $f(2) = 1$.

Informal definition of left and right limits

If $f(x)$ is close to L when $x < a$ and x is close enough to a then we say

$$\lim_{x \to a^-} f(x) = L$$

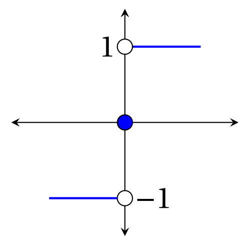
This is called the *left limit* of f at $x = a$.

Similarly we can define the right limit.

Theorem 4. $\lim_{x \to a} f(x) = L$ *if and only if both* $\lim_{x \to a^-} f(x) = L$ *and* $\lim_{x \to a^+} f(x) = L$.

Example. *Find the left and right limits of the signum function*

$$f(x) = \begin{cases} -1 & \text{for } x < 0 \\ 0 & \text{for } x = 0 \\ 1 & \text{for } x > 0 \end{cases}$$



Solution. *The one-sided limits exist, but are not equal*

$$\lim_{x \searrow 0} f(x) = 1 \text{ and } \lim_{x \nearrow 0} f(x) = -1.$$

Hence $\lim_{x \to 0} f(x)$ *does not exist.*

Properties of Limits

Theorem 5. *Suppose*

$$\lim_{x \to a} f(x) = L, \qquad \lim_{x \to a} g(x) = M.$$

Then

$$\lim_{x \to a} \big(f(x) + g(x)\big) = L + M, \tag{2.1}$$

$$\lim_{x \to a} \big(f(x) - g(x)\big) = L - M, \tag{2.2}$$

$$\lim_{x \to a} \big(f(x) \cdot g(x)\big) = L \cdot M \tag{2.3}$$

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{L}{M}, \quad \text{if } M \neq 0. \tag{2.4}$$

Finally, if m and n are integers such that $L^{m/n}$ *is defined*

$$\lim_{x \to a} (f(x))^{m/n} = L^{m/n}. \tag{2.5}$$

Using the above properties we can evaluate the following limits.

Example. *Find* $\lim_{x \to 2} x^2 + 1$ *and* $\lim_{x \to 2} \dfrac{x^2+1}{6-x}$.

Solution. *Using the product rule of limits and the Theorem 3,*

$$\lim_{x\to 2} x^2 = \lim_{x\to 2} x \cdot \lim_{x\to 2} x = 2\cdot 2 = 4$$

Using the sum rule of limits,

$$\lim_{x\to 2} x^2 + 1 = \lim_{x\to 2} x^2 + \lim_{x\to 2} 1 = 4 + 1 = 5$$

Using the division rule of limits,

$$\lim_{x\to 2} \frac{x^2+1}{6-x} = \frac{\lim_{x\to 2} x^2+1}{\lim_{x\to 2} 6-x} = \frac{5}{4}.$$

The above example is a special case of the following theorem.

Theorem 6. *If $P(x)$ is a polynomial then,*

$$\lim_{x\to a} P(x) = P(a)$$

If $Q(x)$ is another polynomial with $Q(a) \neq 0$ then

$$\lim_{x\to a} \frac{P(x)}{Q(x)} = \frac{P(a)}{Q(a)}.$$

The Squeeze Theorem

Theorem 7. *Suppose that $f(x) \le g(x) \le h(x)$ and $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$. Then $\lim_{x\to a} g(x) = L$.*

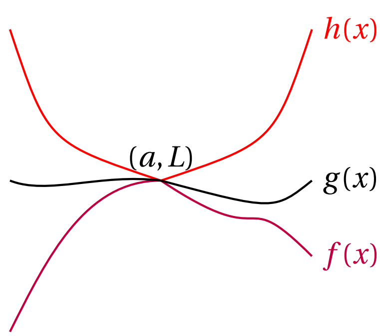


Figure 2.1: The Squeeze Theorem.

Example. *If $2 - x^2 \le g(x) \le 2\cos x$ for $-1 \le x \le 1$, find $\lim_{x\to 0} f(x)$.*

Example. *Show that if $\lim_{x\to a} |f(x)| = 0$ then $\lim_{x\to a} f(x) = 0$.*

Solution. *Note that $-|f(x)| \le f(x) \le |f(x)|$ and use the Squeeze Theorem.*

More examples

Example. *Let*

$$f(x) = \frac{|x-2|}{x^2+x-6}.$$

Find $\lim_{x\to 2+} f(x)$, $\lim_{x\to 2-} f(x)$. Does $\lim_{x\to 2} f(x)$ exist?

In these example, we will compute $\lim_{x\to a} f(x)$ even when $f(a)$ does not exist.

Example. *Evaluate*

1. $\lim_{x\to -2} \dfrac{x^2+x-2}{x^2+5x+6}$,

   Solution. *Remember that we consider x values close to but not equal to -2. Hence $x+2\neq 0$ and we can make the simplification*

$$\lim_{x\to -2} \frac{x^2+x-2}{x^2+5x+6} = \lim_{x\to -2} \frac{(x+2)(x-1)}{(x+2)(x+3)} = \lim_{x\to -2} \frac{x-1}{x+3} = \frac{-3}{1} = -3.$$

2. $\lim_{x\to 5} \dfrac{\frac{1}{x}-\frac{1}{5}}{x-5}$,
3. $\lim_{x\to 4} \dfrac{\sqrt{x}-2}{x^2-16}$, *Hint: multiply both sides by the conjugate expression.*
4. $\lim_{x\to -2} \dfrac{x^2+2x}{x^2-4}$,
5. $\lim_{h\to 0} \dfrac{\sqrt{4+h}-2}{h}$,
6. $\lim_{t\to 0} \dfrac{t}{\sqrt{4+t}-\sqrt{4-t}}$,
7. $\lim_{x\to -1} \dfrac{x^3+1}{x+1}$,
8. $\lim_{x\to 0} \dfrac{|3x-1|-|3x+1|}{x}$,
9. $\lim_{x\to 2-} \dfrac{x^2-4}{|x+2|}$.

2.2 Limits at Infinity and Infinite Limits

Limits at Infinity

Definition 3. *We will say that* $\lim_{x\to\infty} f(x) = L$ *if $f(x)$ is "close enough" to L whenever $x>0$ is "large enough".*

Similarly we define $\lim_{x\to -\infty} f(x) = L$ *if $f(x)$ is "close enough" to L whenever $x<0$ is "large enough".*

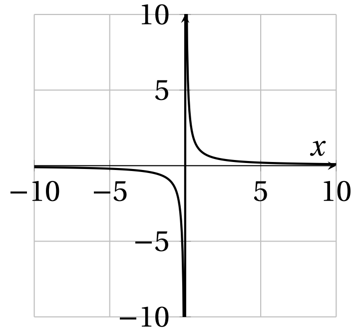
If either $\lim_{x\to\infty} f(x) = L$ *or* $\lim_{x\to -\infty} f(x) = L$*, we say that the line $y=L$ is an* ***horizontal asymptote*** *of the graph of f.*

Example. *Argue that*

$$\lim_{x\to\infty} 1/x = \lim_{x\to\infty} 1/x = 0.$$

by making a table of values of x and $1/x$.

$$f(x) = \frac{1}{x}$$



Recall that for ordinary limits, limit of product of functions is a product of limits of functions. Same is also true for limits at infinity. Hence

$$\lim_{x\to\infty} \frac{1}{x^2} = \lim_{x\to\infty} \frac{1}{x} \cdot \lim_{x\to\infty} \frac{1}{x} = 0 \times 0 = 0.$$

Similarly

$$\lim_{x\to-\infty} \frac{1}{x^2} = 0$$

Finally, for any positive integer n

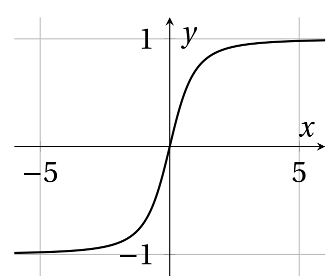
$$\lim_{x\to\infty} \frac{1}{x^n} = \lim_{x\to-\infty} \frac{1}{x^n} = 0.$$

Example. *Let* $f(x) = \dfrac{x}{\sqrt{x^2+1}}$. *Find* $\lim_{x\to\infty} f(x)$, $\lim_{x\to-\infty} f(x)$.

Solution.

$$\begin{aligned}
\lim_{x\to\infty} \frac{x}{\sqrt{x^2+1}} &= \lim_{x\to\infty} \frac{x}{|x|\sqrt{1+1/x^2}} \\
&= \lim_{x\to\infty} \frac{x}{x\sqrt{1+1/x^2}} \\
&= \lim_{x\to\infty} \frac{1}{\sqrt{1+1/x^2}} \\
&= \frac{\lim_{x\to\infty} 1}{\lim_{x\to\infty} \sqrt{1+1/x^2}} \\
&= \frac{1}{\sqrt{\lim_{x\to\infty}(1+1/x^2)}} = \frac{1}{1} = 1.
\end{aligned}$$

$$f(x) = \frac{x}{\sqrt{x^2+1}}$$



Similarly,

$$\lim_{x\to-\infty} \frac{x}{\sqrt{x^2+1}} = -1$$

Limits of Rational Functions at Infinity

Recall that a rational function is a ratio of two polynomials.

Strategy. To find limits of rational functions at infinity, divide by the highest power of x appearing in the *denominator.*

Example.

$$\lim_{x\to\pm\infty} \frac{2x^2 - x + 3}{3x^2 + 5} = \lim_{x\to\pm\infty} \frac{2 - \frac{1}{x} + \frac{3}{x^2}}{3 + \frac{5}{x}} = \frac{2}{3}.$$

Example.

$$\lim_{x\to\pm\infty} \frac{x - 5}{2x^2 + 4x + 1} = \lim_{x\to\pm\infty} \frac{\frac{1}{x} - \frac{5}{x^2}}{2 + \frac{4}{x} + \frac{1}{x^2}} = \frac{0}{2} = 0.$$

We can generalize the above examples.

Theorem 8. *Let $P(x) = a_p x^p + a_{p-1}x^{p-1} + \cdots + a_0$ be a polynomial of degree p and $Q(x) = b_q x^q + \cdots + b_0$ be a polynomial of degree q. If $p = q$, then*

$$\lim_{x\to\pm\infty} \frac{P(x)}{Q(x)} = \frac{a_p}{q_p},$$

If $p < q$, then

$$\lim_{x\to\pm\infty} \frac{P(x)}{Q(x)} = 0,$$

Example.

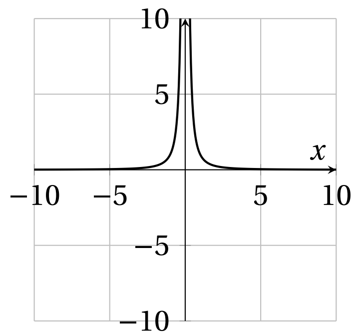
$$\lim_{x\to\infty} \sqrt{x^2 + x} - x = \lim_{x\to\infty} \frac{(\sqrt{x^2 + x} - x)(\sqrt{x^2 + x} + x)}{\sqrt{x^2 + x} + x} = \lim_{x\to\infty} \frac{x}{|x|\sqrt{1 + \frac{1}{x}} + x} = \lim_{x\to\infty} \frac{1}{\sqrt{1 + \frac{1}{x}} + 1} = \frac{1}{2}.$$

Infinite Limits

Example. *The values of $\frac{1}{x^2}$ gets larger and larger as x approaches to 0. Thus $\lim_{x\to 0} \frac{1}{x^2}$ does not exist. Although the limit does not exist, it is useful to state why it does not exist by writing*

$$\lim_{x\to 0} \frac{1}{x^2} = \infty.$$

$$f(x) = \frac{1}{x^2}$$



Example.

$$\lim_{x\to 0+} \frac{1}{x} = \infty.$$

$$\lim_{x\to 0-}\frac{1}{x}=-\infty.$$

$$\lim_{x\to 0}\frac{1}{x}\ \textit{does not exist.}$$

Example.

$$\lim_{x\to-\infty}\sqrt{x^2+x}-x=\lim_{x\to-\infty}\frac{(\sqrt{x^2+x}-x)(\sqrt{x^2+x}+x)}{\sqrt{x^2+x}+x}=\lim_{x\to-\infty}\frac{x}{|x|\sqrt{1+\frac{1}{x}}+x}=\lim_{x\to-\infty}\frac{1}{-\sqrt{1+\frac{1}{x}}+1}$$

If $x<0$ then $\sqrt{1+\frac{1}{x}}<1$ and $-\sqrt{1+\frac{1}{x}}+1>0$. Hence the denominator is positive and approaches to zero. So

$$\lim_{x\to-\infty}\frac{1}{-\sqrt{1+\frac{1}{x}}+1}=\infty.$$

Behaviour of Polynomials at Infinity

Example.

$$\lim_{x\to\infty}4x^3-2x+1=\lim_{x\to\infty}4x^3=\infty.$$

$$\lim_{x\to-\infty}-3x^5+x^3+1=\lim_{x\to-\infty}-3x^5=\infty.$$

In general,

Theorem 9. *If $P(x)=a_nx^n+\cdots+a_0$ is a polynomial then*

$$\lim_{x\to\pm\infty}P(x)=\lim_{x\to\pm\infty}a_nx^n.$$

Example.

$$\lim_{x\to\infty}\frac{x^3+1}{x^2-2x}=\lim_{x\to\infty}\frac{x+\frac{1}{x^2}}{1-\frac{2}{x}}=\lim_{x\to\infty}\frac{x}{1}=\infty$$

Example.

1. $\lim_{x\to 2}\dfrac{(x-2)^2}{x^2-4}=0$
2. $\lim_{x\to 2+}\dfrac{x-3}{x^2-4}=-\infty$
3. $\lim_{x\to 2-}\dfrac{x-3}{x^2-4}=\infty$
4. $\lim_{x\to 2}\dfrac{x-3}{x^2-4}$ *does not exist.*
5. $\lim_{x\to\infty}\dfrac{2x-1}{\sqrt{3x^2+x+1}}$,
6. $\lim_{x\to 1+}\dfrac{\sqrt{x^2-x}}{x-x^2}$

Solution. *If $x>1$ then $x-x^2=x(1-x)<0$. So*

$$\lim_{x\to 1+}\frac{\sqrt{x^2-x}}{x-x^2}=\lim_{x\to 1+}\frac{-\sqrt{x^2-x}}{x^2-x}=\lim_{x\to 1+}\frac{-\sqrt{x^2-x}}{\sqrt{x^2-x}\sqrt{x^2-x}}=\lim_{x\to 1+}\frac{-1}{\sqrt{x^2-x}}=-\infty$$

2.3 Continuity

Let $f(x) = \sqrt{4-x^2}$. Domain of f is $[-2,2]$.

- $x = -2$ is the left end point of $\mathrm{Dom}(f)$.
- $x = 2$ is the right end point of $\mathrm{Dom}(f)$.
- Any x with $-2 < x < 2$ is called an interior point of $\mathrm{Dom}(f)$.

Definition 4. *A function f is **continuous** at an interior point c of its domain if*

$$\lim_{x\to c} f(x) = f(c)$$

f is continuous at its left endpoint c if

$$\lim_{x\to c+} f(x) = f(c)$$

f is continuous at its right endpoint c if

$$\lim_{x\to c-} f(x) = f(c)$$

The following theorem gives an alternative definition of continuity which is sometimes useful.

Theorem 10. *A function f is **continuous** at an interior point c of its domain if and only if*

$$\lim_{h\to 0} f(c+h) = f(c)$$

f is continuous at its left endpoint c if

$$\lim_{h\to 0+} f(c+h) = f(c)$$

f is continuous at its right endpoint c if

$$\lim_{h\to 0-} f(c+h) = f(c)$$

Proof. Let $h = x-c$. Then $x \to c$ if and only if $h \to 0$. So $\lim_{h\to 0} f(c+h) = f(c)$ is the same as $\lim_{h\to 0} f(c+h) = f(c)$. □

Note that f is discontinuous at c if

i) either $\lim_{x\to c} f(x)$ does not exist.

ii) or $\lim_{x\to c} f(x)$ exists but is not equal to $f(c)$.

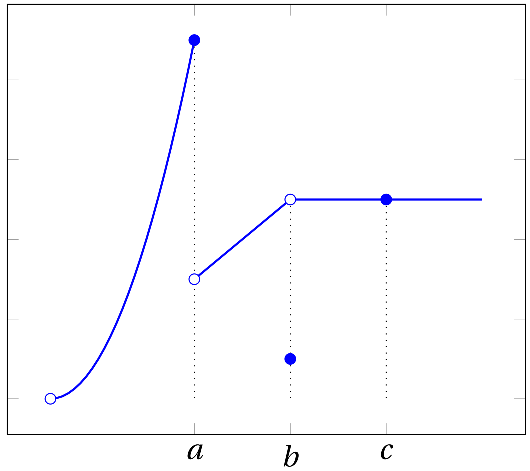
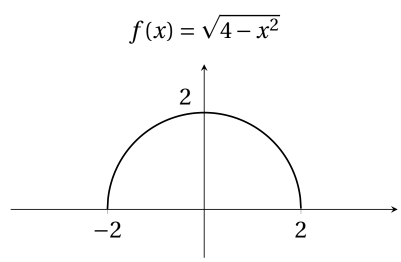


Figure 2.2: f is discontinuous at a because of (ii) and discontinuous at b because of (i). f is continuous at c.

Example. *$f(x) = \sqrt{4-x^2}$ is continuous at every point of its domain.*



Definition 5. *f is called a continuous function if f is continuous at every pt of its domain.*

According to this definition $f(x) = \frac{1}{x}$ is continuous!!! 0 is not in domain of f. So we say f is undefined rather than discontinuous at 0.

There are lots of continuous functions:

- polynomials,
- rational functions,
- rational powers $x^{m/n}$
- trigonometric functions
- absolute value function $|x|$

Theorem 11. *If f and g are continuous at c then*

- *$f+g$, $f-g$, fg, are continuous at c,*
- *if k is constant then kf is continuous at c,*
- *$\dfrac{f}{g}$ continuous at c provided that $g(c) \neq 0$.*

- *$f(x)^{1/n}$ continuous at c provided that $f(c) > 0$ if n is even.*

Proof. Let's prove that if f and g are continuous at c then so is $f+g$. If f and g are continuous at c then

$$\lim_{x\to c} f(x) = f(c), \qquad \lim_{x\to c} g(x) = g(c),$$

By the limit rule,

$$\lim_{x\to c}(f(x)+g(x)) = \lim_{x\to c} f(x) + \lim_{x\to c} g(x) = f(c)+g(c).$$

The other proofs are similar. □

Composites of continuous functions are continuous
If g is continuous at c and f is continuous at $g(c)$ then $f\circ g$ is continuous at c. In other words,

$$\lim_{x\to c} f(g(x)) = f(\lim_{x\to c} g(x)) = f(g(c)).$$

Example. *Find m so that*

$$g(x) = \begin{cases} x-m, & \textit{if } x<3, \\ 1-mx, & \textit{if } x\ge 3 \end{cases}$$

is continuous for all x.

Continuity of Trigonometric Functions

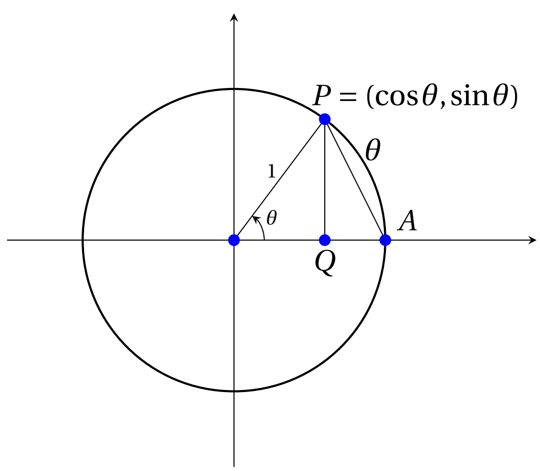
Theorem 12. $\sin x$ *and* $\cos x$ *are continuous at $x=0$, i.e.*

$$\lim_{x\to 0}\sin x = \sin 0 = 0, \qquad \lim_{x\to 0}\cos x = \cos 0 = 1.$$

Proof.

$$|1-\cos\theta| = |AQ| \le |AP| \le \theta,$$
$$|\sin\theta| = |PQ| \le |AP| \le \theta$$

In other words, $-\theta \le \sin\theta \le \theta$ and using the squeeze theorem we get $\lim_{\theta\to 0}\sin\theta = 0$. Similarly, we get $\lim_{\theta\to 0} 1-\cos\theta = 0$ or $\lim_{\theta\to 0}\cos\theta = 1$.



□

Theorem 13. $\sin x$ *and* $\cos x$ *are continuous for all x.*

Proof. By Theorem 10, we need to prove $\lim_{h\to 0}\sin(x+h) = \sin x$ for any x.

$$\lim_{h\to 0}\sin(x+h) = \lim_{h\to 0}\sin x\cos h + \cos x\sin h = \sin x \lim_{h\to 0}\cos h + \cos x \lim_{h\to 0}\sin h = \sin x.$$

Prove the continuity of $\cos x$ as an exercise. □

Continuous Functions on Closed Intervals [a, b] are bounded

We say a function f is **bounded** if there exists M and N such that $M \le f(x) \le N$ for all x in the domain of f.

Theorem 14. *If f is continuous on the closed interval $[a,b]$ then there exist numbers p and q in the interval $[a,b]$ s.t.*

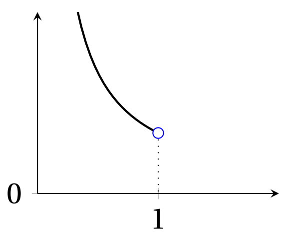
$$f(p) \le f(x) \le f(q)$$

*for all x in $[a,b]$. $f(p)$ is called the **absolute minimum value** and $f(q)$ is called the **absolute maximum value**.*

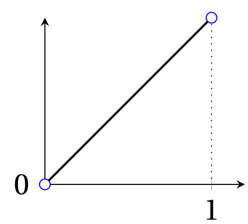
This theorem is an existence theorem. It only guarantees the existence of p and q but does not tell how to actually find them.

Also the theorem says that continuous functions on closed intervals must be bounded.

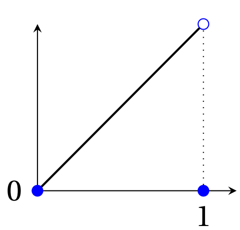
Example. *The conclusions of the theorem may fail if the function f is not continuous or the interval is not closed.*



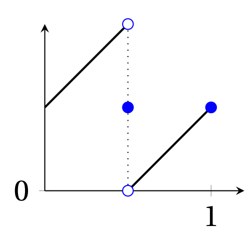
(a) The function $f(x) = 1/x$ on the open interval $(0,1)$ is continuous but unbounded and has no minimum and no maximum.



(b) The function $f(x) = x$ on $(0,1)$ is discontinuous, bounded and has no minimum and no maximum.



(a) This function is defined on the closed interval $[0,1]$, discontinuous, has a minimum but no maximum.



(b) This function is defined on the closed interval $[0,1]$, discontinuous, bounded, has no minimum but no maximum.

Theorem 15 (Intermediate Value Theorem)**.** *If f is continuous on $[a,b]$ and if s is between $f(a)$ and $f(b)$ then there exists c in $[a,b]$ s.t. $f(c) = s$.*

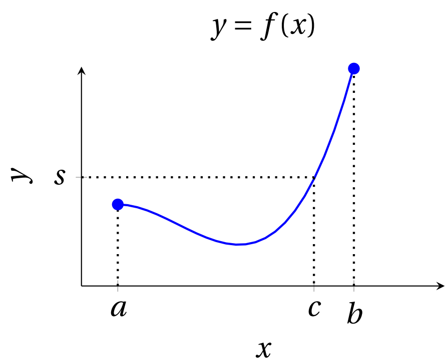


Figure 2.5: Illustration of the intermediate value theorem.

In particular, a continuous function on a closed interval takes every value between its minimum m and maximum M. Hence its range is a closed interval $[m, M]$.

Example. *Show that the equation $x^3 - x - 1 = 0$ has a solution in the interval $[1,2]$.*

Solution. *$f(x) = x^3 - x - 1$ is a polynomial and hence continuous. $f(1) = -1$ and $f(2) = 5$. Since 0 lies between -1 and 5, the intermediate value theorem assures us that there must be a number c in $[1,2]$ such that $f(c) = 0$.*

Bisection Algorithm

Intermediate Value Theorem is also an existence theorem. It does not say how to find c in its statement. Let's try to better estimate the root of previous example. Write $f(x) = x^3 - x - 1$ and try to find a smaller interval where a root lies of

$$f(x) = 0.$$

We know that a root lies in $[1,2]$, if say that the root is 1.5 the maximum error will be 0.5.

Now $f(1.5) = 0.875 > 0$. So a root lies in $[1,1.5]$, and if we say the root is 1.25 then the maximum error will be 0.25.

If this is not sufficient then compute $f(1.25) = -0.2969$, now if we say the root is 1.375 then the error is less than 0.125.

Next step is $f(1.1375) = 0.2246$. So a root must lie in $[1.25, 1375]$. The error is less than 0.0625 if we say the root is 1.315.

Going this way, we find the approximations, 1.3438, 1.3282, 1.3204. Hence the root must lie in $[1.3204, 1.3282]$. So the first two decimal digits of the root are 1.32.

In engineering, you almost never get exact results. All you can do is lower your error below an acceptable threshold.

2.4 Formal definition of Limit

The informal description of the limit uses phrases like "close enough" and "really very small". "Fortunately" there is a good definition, i.e. one which is unambiguous and can be used to settle any dispute about the question of whether $\lim_{x\to a} f(x)$ equals some number L or not.

In this section we assume that f is defined in an open interval containing a except possibly at $x = a$.

Definition 6. *We say that*

$$\lim_{x \to a} f(x) = L$$

if for every $\epsilon > 0$ *there exists a* $\delta > 0$ *such that*

$$0 < |x - a| < \delta \textit{ implies } |f(x) - L| < \epsilon. \tag{2.6}$$

Why the absolute values? Recall that the quantity $|x - y|$ is the distance between the points x and y on the number line.

What are ϵ *and* δ*?* The quantity ϵ is how close you would like $f(x)$ to be to its limit L; the quantity δ is how close you have to choose x to a to achieve this. To prove that $\lim_{x \to a} f(x) = L$ you must assume that someone has given you an unknown $\epsilon > 0$, and then find a positive δ for which (2.6) holds. The δ you find will depend on ϵ.

When we first discussed the limit, say $\lim_{x \to 5} 2x + 1$, we made a table,

x	$f(x) = 2x + 1$
5.1	11.2
5.01	11.02
5.001	11.002
4.9	10.8
4.99	10.98
4.999	10.998

This table can be written also in this form.

$\lvert x - 5\rvert$	$\lvert f(x) - 11\rvert$
0.1	0.2
0.01	0.02
0.001	0.002

It looks like for any $\epsilon > 0$, if $|x - 5| < \frac{\epsilon}{2}$ then $\left|f(x) - 11\right| < \epsilon$. Now let's prove this!

Example. *Show that* $\lim_{x \to 5} 2x + 1 = 11$.

Solution. *We have* $f(x) = 2x + 1$, $a = 5$ *and* $L = 11$, *and the question we must answer is "how close should* x *be to* 5 *if you want to be sure that* $f(x) = 2x + 1$ *differs less than* ϵ *from* $L = 11$*?"*

$$|f(x) - L| = |(2x + 1) - 11| = |2x - 10| = 2 \cdot |x - 5| = 2 \cdot |x - a|.$$

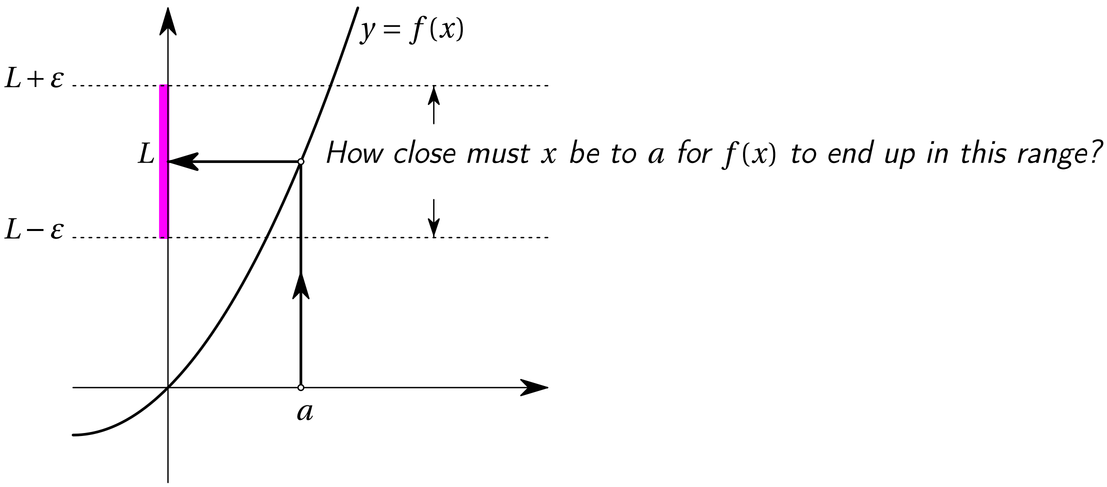
So choose $\delta = \frac{\epsilon}{2}$. *Then*

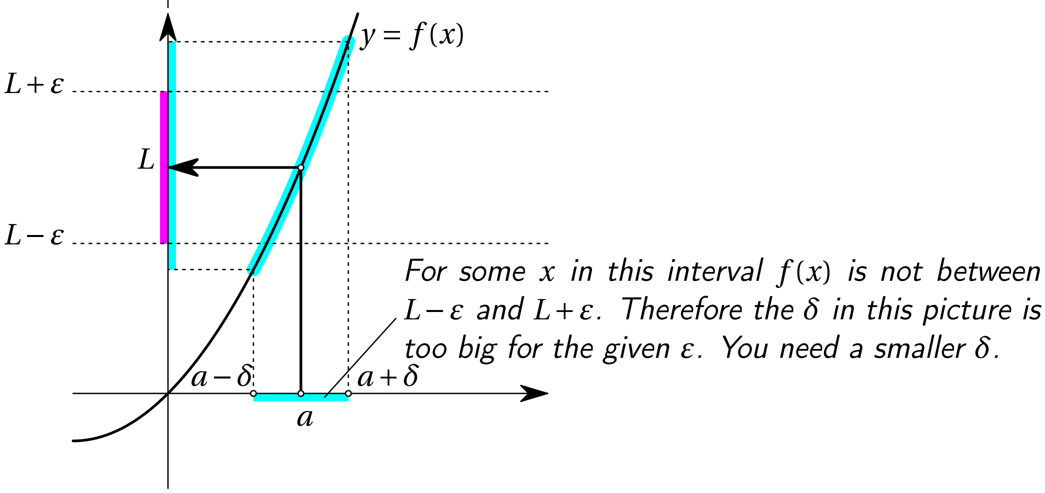
$$|f(x) - L| < \epsilon \textit{ whenever } 0 < |x - a| < \frac{\epsilon}{2}.$$

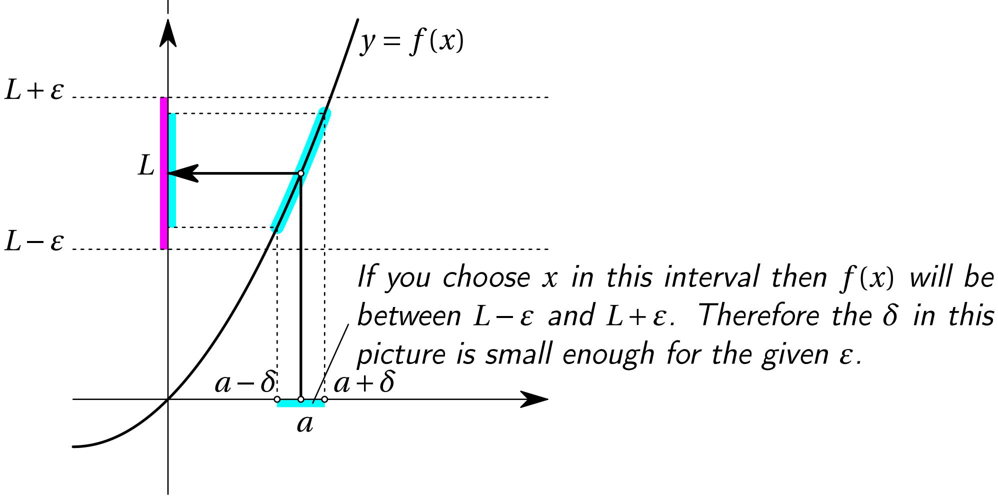
Example ("Don't choose $\delta > 1$" trick)**.** *Show that* $\lim_{x \to 3} x^2 = 9$.

Solution. *We have* $f(x) = x^2$, $a = 3$, $L = 9$, *and again the question is, "how small should* $|x - 3|$ *be to guarantee* $|x^2 - 9| < \epsilon$*?"*

$$|x^2 - 9| = |(x - 3)(x + 3)| = |x + 3| \cdot |x - 3|.$$







Here is a trick that allows you to replace the factor $|x+3|$ with a constant. We hereby agree that we always choose our δ so that $\delta \le 1$. If we do that, then we will always have

$$|x-3| < \delta \le 1, \textit{i.e. } |x-3| < 1,$$

or $2 < x < 4$ or $|x+1| < 5$. Therefore

$$|x^2 - 1| = |x+1| \cdot |x-1| < 5|x-1|.$$

So choose

$$\delta = \min\{1, \frac{\epsilon}{5}\}.$$

2nd way: Note that $|x+3| = |x-3+6| < |x-3|+6 < \delta+6$

$$\left|f(x)-9\right| = |x+3|\,|x-3| < (\delta+6)\delta$$

So choose $(\delta+6)\delta < \epsilon$*, or*

$$(\delta+3)^2 < \epsilon+9 \implies \delta < \sqrt{\epsilon+9}-3$$

Example. *Show that* $\lim_{x\to 4} 1/x = 1/4$.

Solution. *We apply the definition with* $a=4$*,* $L=1/4$ *and* $f(x)=1/x$*. Thus, for any* $\epsilon>0$ *we try to show that if* $|x-4|$ *is small enough then one has* $|f(x)-1/4|<\epsilon$*.*

We begin by estimating $|f(x)-\frac{1}{4}|$ *in terms of* $|x-4|$*:*

$$|f(x)-1/4| = \left|\frac{1}{x}-\frac{1}{4}\right| = \left|\frac{4-x}{4x}\right| = \frac{|x-4|}{|4x|} = \frac{1}{|4x|}\,|x-4|.$$

As before, things would be easier if $1/|4x|$ *were a constant. To achieve that we again agree not to take* $\delta>1$*. If we always have* $\delta\le 1$*, then we will always have* $|x-4|<1$*, and hence* $3<x<5$*. How large can* $1/|4x|$ *be in this situation? Answer: the quantity* $1/|4x|$ *increases as you decrease x, so if* $3<x<5$ *then it will never be larger than* $1/|4\cdot 3| = \frac{1}{12}$*.*

We see that if we never choose $\delta>1$*, we will always have*

$$|f(x)-\frac{1}{4}| \le \frac{1}{12}|x-4| \quad \textit{for} \quad |x-4|<\delta.$$

To guarantee that $|f(x)-\frac{1}{4}|<\epsilon$ *we could threfore require*

$$\frac{1}{12}|x-4|<\epsilon, \quad \textit{i.e.} \quad |x-4|<12\epsilon.$$

Hence if we choose $\delta=12\epsilon$ *or any smaller number, then* $|x-4|<\delta$ *implies* $|f(x)-4|<\epsilon$*. Of course we have to honor our agreement never to choose* $\delta>1$*, so our choice of* δ *is*

$$\delta = \textit{the smaller of } 1 \textit{ and } 12\epsilon = \min\big(1, 12\epsilon\big).$$

Example. *Verify that* $\lim_{x\to 2}\dfrac{x-2}{1+x^2} = 0$.

Solution. *Notice that* $\dfrac{|x-2|}{\left|1+x^2\right|} < |x-2|$ *since* $1+x^2>1$*. Hence choose* $\delta=\epsilon$*.*

2.5 Review Problems

Example. *Evaluate the limits if they exist. If they do not exist, state wheter they are* ∞*,* $-\infty$ *or just does not exist.*

1. $\lim_{x\to 2}\dfrac{x^2+1}{1-x^2}$,

2. $\lim_{x\to 1} \dfrac{x^2}{1-x^2}$,

3. $\lim_{x\to\infty} \dfrac{\cos x}{x}$, *(Hint: Use Sandwich Theorem)*

4. $\lim_{x\to-\infty} \dfrac{2x^3+2x-1}{-3x^3+x^2}$,

5. $\lim_{x\to-\infty} x+\sqrt{x^2-4x+1}$,

Solution.

$$\begin{aligned}
\lim_{x\to-\infty} x+\sqrt{x^2-4x+1} &= \lim_{x\to-\infty} x+|x|\sqrt{1-\frac{4}{x}+\frac{1}{x^2}} = \lim_{x\to-\infty} x\left(1-\sqrt{1-\frac{4}{x}+\frac{1}{x^2}}\right) \\
&= \lim_{x\to-\infty} x\left(1-\sqrt{1-\frac{4}{x}+\frac{1}{x^2}}\right)\frac{\left(1+\sqrt{1-\frac{4}{x}+\frac{1}{x^2}}\right)}{\left(1+\sqrt{1-\frac{4}{x}+\frac{1}{x^2}}\right)} \\
&= \lim_{x\to-\infty} x\left(1-\left(1-\frac{4}{x}+\frac{1}{x^2}\right)\right)\lim_{x\to-\infty}\frac{1}{\left(1+\sqrt{1-\frac{4}{x}+\frac{1}{x^2}}\right)} \\
&= \lim_{x\to-\infty} x\left(\frac{4}{x}-\frac{1}{x^2}\right)\frac{1}{2} = \lim_{x\to-\infty}\left(4-\frac{1}{x}\right)\frac{1}{2} = 2.
\end{aligned}$$

6. $\lim_{x\to 0} \dfrac{x}{|x-1|-|x+1|}$.

Chapter 3

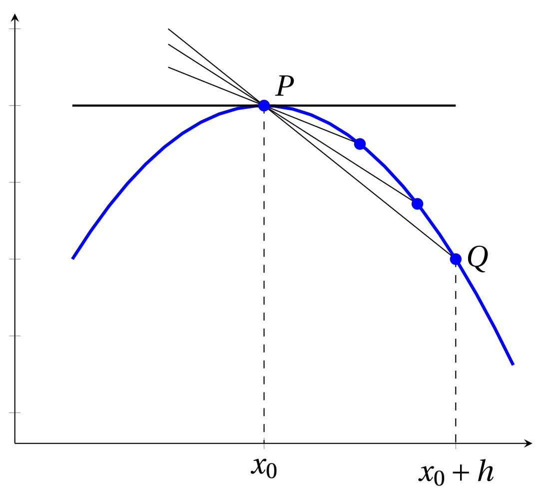
Differentiation

3.1 Tangent Lines and Their Slopes

Problem: Find a straight line L that is tangent to a curve C at a point P.

"For simplicity, restrict ourselves to curves which are graphs of functions."

How do we define the tangent line to a curve?



The slope of the line PQ is

$$\frac{f(x_0+h)-f(x_0)}{h}.$$

Definition 7. *Suppose f is cts at $x = x_0$ and*

$$\lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h} = m$$

If the limit exists, then the line with equation

$$y = m(x-x_0)+f(x_0)$$

*is called **the tangent line** to the graph of $y = f(x)$ at $P = (x_0, f(x_0))$. If the limit does not exist and $m = \infty$ or $m = -\infty$ then the tangent line is the vertical line $x = x_0$. If the limit does not exist and is not $\pm\infty$ then there is no tangent line at P.*

Example. *Find an equation of the tangent line to the curve* $y = x^2$ *at* $(1,1)$.

Solution. *The slope is*

$$m = \lim_{h\to 0} \frac{f(1+h) - f(1)}{h} = 2.$$

And an equation is $y = 2(x-1)+1$.

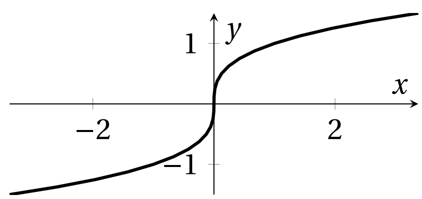
Example. *Find an equation of the tangent line to the curve* $y = x^{1/3} = \sqrt[3]{x}$ *at the origin.*

Solution. *The slope of the tangent line is*

$$m = \lim_{h\to 0} \frac{h^{1/3}}{h} = \infty.$$

So the tangent line is a vertical line $x = 0$ *(in other words the y-axis).*

$f(x) = x^{1/3}$

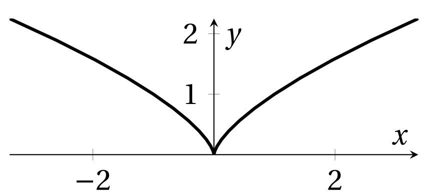


Remark. *Tangent lines to curves such as circles and parabolas do not cross these curves, they just touch at a single point. However, for graphs of functions tangent lines may cross the curve such as above. In fact at inflection points (which we will define later) they always do! For example the tangent line to the graph of* $f(x) = x^3$ *at* $x = 0$ *is the y-axis.*

Example. *Does* $f(x) = x^{2/3}$ *have a tangent line at* $(0,0)$*?*

Solution. *The limit of the difference quotient is undefined at* 0 *since the right limit is* ∞ *while the left limit is* $-\infty$. *Hence the graph has no tangent line at* $(0,0)$.

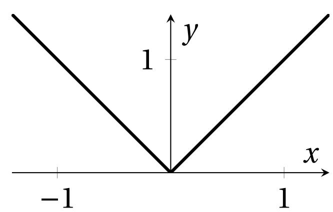
$f(x) = x^{2/3}$



"We say that this curve has a cusp at the origin. A cusp is an infinitely sharp point. If you were traveling along the curve, you would have to stop and turn 180° at the origin."

Example. *Does* $f(x) = |x|$ *have a tangent line at* $(0,0)$*?*

Solution. *The difference quotient is* $\dfrac{|h|}{h}$ *which has right limit* 1 *and left limit* -1 *at* $h = 0$.



3.2 Derivative

Definition 8. *The **derivative** of a function* f *at* x *is*

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}.$$

whenever the limit exists. If $f'(x)$ *exists,* f *is called **differentiable** at* x.

`f'(x) is the slope of the tangent line to the graph of f at (x, f(x)).`

We will regard f' as a function whose domain is those x at which f is differentiable.
Another way of defining derivative is

$$f'(x_0) = \lim_{x\to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{h}$$

Two limits are equivalent. This can be seen by letting $x = x_0 + h$.

Example. *Show that the derivative of the linear function* $f(x) = ax + b$ *is* $f'(x) = a$. *In particular the derivative of a constant function is zero.*

Example. *Use the definition of the derivative to calculate the derivatives of a)* $f(x) = x^2$, *b)* $f(x) = \frac{1}{x}$, *c)* $f(x) = \sqrt{x}$.

The previous three formulas are special cases of the following **Power Rule for Derivative**:

$$f(x) = x^r \implies f'(x) = r x^{r-1}$$

whenever x^{r-1} makes sense.

Example.

$$f(x) = x^{5/3} \implies f'(x) = x^{2/3},$$

for all x. *How about* $f'(-1/8)$*?*

$$f(x) = \frac{1}{\sqrt{x}} \implies f'(x) = -\frac{1}{2} x^{-3/2}$$

for $x > 0$.

Example. *Differentiate the absolute value function* $f(x) = |x|$ *to get*

$$f'(x) = sgn(x) = \begin{cases} -1, & if\, x < 0 \\ 1, & if\, x > 0 \end{cases}$$

Note that f *is not differentiable at* 0.

Example. *How should the function* $f(x) = x sgn(x)$ *be defined at* $x = 0$ *so that it is continuous there? Is it then differentiable there?*

Notations for Derivative

Let $y = f(x)$. We denote the derivative by

$$y' = f'(x) = \frac{dy}{dx} = \frac{d}{dx} f(x).$$

If we want to evaluate the derivative at point x_0

$$y' \mid_{x=x_0} = f'(x_0) = \frac{dy}{dx} \mid_{x=x_0} = \frac{d}{dx} f(x) \mid_{x=x_0} .$$

The notations y' and $f'(x)$ are *Lagrange notations* for the derivative. The notations $\frac{dy}{dx}$ and $\frac{d}{dx} f(x)$ are called *Leibniz notations* for the derivative.

The Leibniz notation is suggested by the definition of the derivative. Let $\Delta y = f(x+h) - f(x)$ be the increment in y and $\Delta x = x + h - x = h$ be the increment in x. Then

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

3.3 Differentiation Rules

Differentiability is stronger than continuity.

Theorem 16. *If* f *is differentiable at* x *then* f *is cts at* x.

Proof.

$$\lim_{h\to 0} (f(x+h) - f(x)) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \lim_{h\to 0} h = f'(x)0 = 0$$

This means

$$0 = \lim_{h\to 0} f(x+h) - \lim_{h\to 0} f(x) = \lim_{h\to 0} f(x+h) - f(x)$$

Hence

$$\lim_{h\to 0} f(x+h) = f(x)$$

□

Theorem 17. *If f and g are differentiable at x then*

$$(f+g)'(x) = f'(x) + g'(x),$$

$$(f-g)'(x) = f'(x) - g'(x),$$

and for any constant c

$$(cf)'(x) = cf'(x).$$

Proof. Let's prove the derivative of sums is sum of derivatives. The others are similar.

$$\begin{aligned}(f+g)'(x) &= \lim_{h\to 0} \frac{(f+g)(x+h)-(f+g)(x)}{h} = \lim_{h\to 0} \frac{f(x+h)+g(x+h)-f(x)+g(x)}{h} \\ &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} + \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} = f'(x)+g'(x),\end{aligned}$$

□

The sum rule extends to any number of functions.

$$(f_1 + \cdots + f_n)'(x) = f_1'(x) + \cdots + f_n'(x).$$

Example. *Take the derivative of*

$$f(x) = 5\sqrt{x} + \frac{3}{x} - 19$$

It is NOT true that derivative of product of functions is a product of their derivatives. Usually $(fg)'(x) \neq f(x)g(x)$.

Theorem 18. *If f and g are differentiable at x then*

$$(fg)'(x) = f'(x)g(x) + f(x)g'(x).$$

Example. *Find the derivative of $f(x) = (x^2+x+1)(2x+\frac{1}{x})$.*

The product rule can be extended to any number of functions

$$(f_1 f_2 f_3)' = f_1' f_2 f_3 + f_1 f_2' f_3 + f_1 f_2 f_3'$$

$$(f_1 \cdots f_n)' = f_1' f_2 \cdots f_3 + f_1 f_2' f_3 \cdots f_n + \cdots + f_1 \cdots f_{n-1} f_n'.$$

Theorem 19. *If f is differentiable at x and $f(x) \neq 0$ then $1/f$ is diff at x, and*

$$\left(\frac{1}{f}\right)'(x) = \frac{-f'(x)}{f(x)^2}.$$

Proof.

$$\frac{d}{dx}\frac{1}{f(x)} = \lim_{h\to 0} \frac{\dfrac{1}{f(x+h)} - \dfrac{1}{f(x)}}{h} = \lim_{h\to 0} \frac{-1}{f(x+h)f(x)} \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

The result follows by limit rules and continuity of f. □

Example. *Differentiate* $y = \dfrac{x^5}{x^{2/3}+1}$.

Theorem 20. *If f and g are differentiable at x and $g(x) \neq 0$ then*

$$\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}$$

Proof. Using the product rule and reciprocal rule,

$$\left(\frac{f}{g}\right)'(x) = \left(\frac{1}{g}(x)f(x)\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}$$

□

Example. *Find the derivative of* $f(x) = \dfrac{a+bx}{m+cx}$.

Example. *Find an equation of the tangent line to* $y = \dfrac{2}{3-4\sqrt{x}}$ *at the point* $(1,-2)$.

Solution. *Let us define* $g(x) = 3-4\sqrt{x}$. *Then* $g'(x) = -4\frac{1}{2\sqrt{x}} = -\frac{2}{\sqrt{x}}$ *and*

$$y' = 2\frac{-g'(x)}{g(x)^2} = 2\frac{\frac{2}{\sqrt{x}}}{(3-4\sqrt{x})^2} = \frac{4}{\sqrt{x}(3-4\sqrt{x})^2}$$

Hence $y'(1) = 4$. *And the equation of the tangent line is* $y = 4(x-1)-2$.

Example. *Find the x-coordinates of points on the curve* $y = \frac{x+1}{x+2}$ *where the tangent line is parallel to the line* $y = 4x$.

Solution. *Solving* $y' = 4$, *we find* $x = -3/2$ *and* $x = -5/2$.

Example. *If* $f(2) = 2$ *and* $f'(2) = 3$, *calculate*

$$\frac{d}{dx}\left(\frac{x^2}{f(x)}\right)\bigg|_{x=2}$$

Solution. *Answer is*

$$\frac{2\cdot 2f(2) - 2^2 f'(2)}{f(2)^2} = \frac{8-12}{4} = -1.$$

3.4 Chain Rule

The following theorem is known as the chain rule.

Theorem 21. *If $f(u)$ is differentiable at $u = g(x)$ and $g(x)$ is differentiable at x, then*

$$(f \circ g)'(x) = f'(g(x))g'(x)$$

Derivative of the composite function = derivative of outer function evaluated at inner function times derivative of inner function.

In Leibniz notation, if $y = f(u)$ where $u = g(x)$ then

$$y = f(g(x)) = (f \circ g)(x)$$

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

where $\frac{dy}{du}$ is evaluated at $u = g(x)$.

Example. *Find the derivative of* $y = \sqrt{x^2+1}$.

Solution. *Here* $y = f(g(x))$ *where* $f(u) = \sqrt{u}$ *and* $u = x^2+1$.

$$\frac{dy}{dx} = f'(g(x))g'(x) = \frac{1}{2\sqrt{g(x)}}g'(x) = \frac{1}{2\sqrt{x^2+1}}2x = \frac{x}{\sqrt{x^2+1}}.$$

Example. *Differentiate* $y = (x^3-1)^{1000}$.

Solution. *Let* $u = (x^3-1)$ *then* $y = u^{1000}$. $y' = 1000u^{999}u' = 1000(x^3-1)^{999}3x^2$.

Example. *Take the derivative* $f(t) = \left|t^2-1\right|$.

Solution.

$$f'(t) = (sgn(t^2-1))(2t) = \begin{cases} 2t, & \text{if } t<-1, t>1 \\ -2t, & \text{if } -1<t<1 \\ \text{undefined} & \text{if } t \pm 1 \end{cases}$$

Example. *Express in terms of* f *and* f'.

a) $\frac{d}{dx}f(x^2)$,

b) $\frac{d}{dx}(f(\pi-2f(x)))^4$.

Solution. *For (a)*

$$\frac{d}{dx}f(x^2) = f'(x^2)2x$$

For (b)

$$\frac{d}{dx}[f(\pi-2f(x))]^4 = 4[f(\pi-2f(x))]^3 f'(\pi-2f(x))(-2f'(x)).$$

Example. *For*

$$f(x) = \left(1+\sqrt{2x+1}\right)^{-4/3}$$

evaluate $f'(0)$.

Solution.

$$\begin{aligned} f'(x) &= \frac{-4}{3}(1+\sqrt{2x+1})^{-7/3}\frac{d}{dx}\sqrt{2x+1} = \frac{-4}{3}(1+\sqrt{2x+1})^{-7/3}\frac{1}{2\sqrt{2x+1}}\frac{d}{dx}(2x+1) \\ &= \frac{-4}{3}(1+\sqrt{2x+1})^{-7/3}\frac{1}{2\sqrt{2x+1}}2 \end{aligned}$$

Hence

$$f'(0) = -\frac{1}{2^{1/3}3}.$$

Example. *Find an equation of the tangent line to the graph of*

$$y = (1+x^{2/3})^{3/2}$$

at $x=-1$.

Solution.

$$y' = \frac{3}{2}(1+x^{2/3})^{1/2}\frac{2}{3}x^{-1/3}.$$

$$y'(-1) = \frac{3}{2}(1+1)^{1/2}\frac{2}{3}(-1) = -\sqrt{2}.$$

3.5 Derivatives of Trigonometric Functions

The radian measure of an angle is defined to be the length of the arc of a unit circle corresponding to that angle.

$$\text{angle in degrees} = \text{angle in radians}\cdot\frac{180^\circ}{\pi}.$$

In calculus all angles are measured in radians. By an angle of $\pi/3$ we mean $\pi/3$ radians or 60° not $(\pi/3)^\circ \approx 1.04^\circ$.

Theorem 22. $\lim_{\theta\to 0}\dfrac{\sin\theta}{\theta} = 1$.

Proof.

Suppose $0 < \theta < \frac{\pi}{2}$.
Area of OQP triangle is $\frac{1}{2}\sin\theta\cos\theta$.
Area of OAP arc is $\frac{\theta}{2\pi}\pi 1^2$.
Area of OAT triangle is $\frac{1}{2}\tan\theta = \frac{\sin\theta}{2\cos\theta}$.

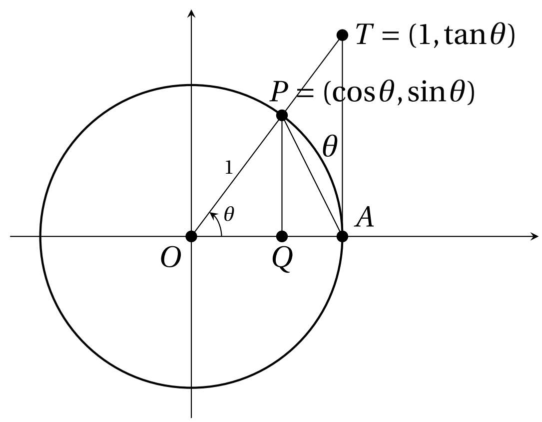
$$\frac{1}{2}\sin\theta\cos\theta \le \frac{\theta}{2} \le \frac{\sin\theta}{2\cos\theta}$$

Multiply by $\frac{2}{\sin\theta} > 0$

$$\cos\theta \le \frac{\theta}{\sin\theta} \le \frac{1}{\cos\theta}$$

Take reciprocal to get

$$\cos\theta \le \frac{\sin\theta}{\theta} \le \frac{1}{\cos\theta}, \tag{3.1}$$



for $0 < \theta < \frac{\pi}{2}$.
Use the squeeze theorem to show that

$$\lim_{\theta\to 0+} \frac{\sin\theta}{\theta} = 1$$

Similarly, we can show that (3.1) holds for $-\frac{\pi}{2} < \theta < 0$ and hence

$$\lim_{\theta\to 0-} \frac{\sin\theta}{\theta} = 1$$

□

Example. *Show that* $\lim_{h\to 0} \frac{\cos h - 1}{h} = 0$.

Solution.

$$\lim_{h\to 0} \frac{\cos h - 1}{h} = \lim_{h\to 0} \frac{(\cos h - 1)(\cos h + 1)}{h(\cos h + 1)} = \lim_{h\to 0} \frac{\cos^2 h - 1}{h(\cos h + 1)}$$
$$= \lim_{h\to 0} \frac{-\sin^2 h}{h(\cos h + 1)} = -\lim_{h\to 0} \frac{\sin h}{h} \frac{\sin h}{\cos h + 1} = -1 \cdot 0 = 0$$

Theorem 23. $\sin x$ *is differentiable for every* x *and*

$$\frac{d}{dx}\sin x = \cos x$$

Proof.

$$\frac{d}{dx}\sin x = \lim_{h\to 0} \frac{\sin(x+h) + \sin x}{h} = \lim_{h\to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h}$$
$$= \lim_{h\to 0} \frac{\sin x(\cos h - 1)}{h} + \lim_{h\to 0} \frac{\cos x \sin h}{h} = \sin x \lim_{h\to 0} \frac{(\cos h - 1)}{h} + \cos x \lim_{h\to 0} \frac{\sin h}{h} = \cos x$$

□

Theorem 24. $\cos x$ *is differentiable for every* x *and*

$$\frac{d}{dx}\cos x = -\sin x.$$

Proof.

$$\frac{d}{dx}\cos x = \frac{d}{dx}\sin\left(\frac{\pi}{2}-x\right) = -\cos\left(\frac{\pi}{2}-x\right) = -\sin x.$$

□

Example. *Evaluate the derivative of*

a) $\sin(\pi x)+\cos(3x)$,

b) $x^2\cos(\sqrt{x})$,

c) $\dfrac{\cos x}{1-\sin x}$,

d) $\sin(\cos(\tan t))$

The derivatives of the other trigonometric functions

$$\tan x = \frac{\sin x}{\cos x}, \quad \sec x = \frac{1}{\cos x}, \quad \cot x = \frac{\cos x}{\sin x}, \quad \csc x = \frac{1}{\sin x}.$$

Since cos and sin are eveywhere differentiable, the above functions are differentiable everywhere except where their denominators are zero. The derivatives of these functions can be derived by using quotient and reciprocal rules.

$$\frac{d}{dx}\tan x = \sec^2 x, \quad \frac{d}{dx}\sec x\tan x, \quad \frac{d}{dx}\cot x = -\csc^2 x, \quad \frac{d}{dx}\csc = -\csc x\cot x.$$

Example. *Verify the derivative formulas for* $\tan x$ *and* $\sec x$.

Example. *Find the points on the curve* $y=\tan(2x)$, $-\pi/4 < x < \pi/4$, *where the normal is parallel to the line* $y=-x/8$.

3.6 Higher Order Derivatives

Derivative of derivative is called **second derivative**. If $y=f(x)$ then

$$y'' = f''(x) = \frac{d}{dx}\frac{d}{dx}y = \frac{d^2}{dx^2}y = \frac{d^2}{dx^2}f(x).$$

Similar notations can be used for third, fourth, etc. derivatives. For n-th derivative, we write

$$y^{(n)} = f^{(n)}(x) = \frac{d^n y}{dx^n}$$

Example. *Calculate all the derivatives of* $y=x^3$.

Example. *Calculate all the derivatives of* $y=x^n$ *where* n *is a positive integer.*

Solution.

$$y^{(k)} = \begin{cases} \frac{n!}{(n-k)!}x^{n-k} & if 0 \le k \le n \\ 0 & if k > n \end{cases}$$

Example. *Show that if A, B and k are constants, then the function $y = A\cos(kt) + B\sin(kt)$ is a solution of the second order differential equation*

$$\frac{d^2y}{dx^2} + k^2y = 0.$$

Example. *If $y = \tan kx$ show that $y'' = 2k^2y(1+y^2)$.*

Example. *If f and g are twice differentiable functions, show that*

$$(fg)'' = f''g + 2f'g' + fg''.$$

What do you think about the general formula for $\frac{d^n}{dx^n}(fg)$?

3.7 Mean Value Theorem

Suppose you drive in 2 hours from city A to city B which are 200km apart. That means your average speed was 100km/h. Even if you did not travel constant speed, there was at least one instant where your speed was exactly 100km/h. This is called **mean value theorem**.

Theorem 25 (The Mean-Value Theorem)**.** *Suppose that f is continuous on the interval $[a,b]$ and that it is differentiable on the open interval (a,b). Then there exists a point c in the open interval (a,b) s.t.*

$$\frac{f(b)-f(a)}{b-a} = f'(c).$$

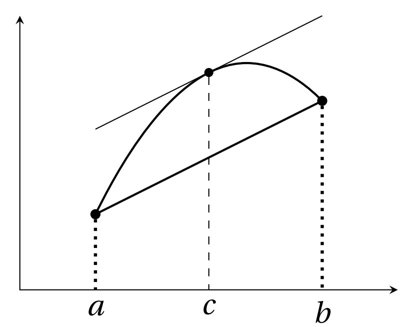


Figure 3.1: Mean Value Theorem says that the slope of the secant line joining two points on the graph of of $f(x)$ is equal to the slope of the tangent line at some point $x = c$ between a and b.

Let $f(t)$ denote the distance from city A. Then $f(0) = 0$ and $f(2) = 200$. Mean Value Theorem says there is a time $t = c$ s.t. $f'(c) = 100$.

Example. *Let $f(x) = |x|$ on $[-1,1]$. Show that there is no $c \in [-1,1]$ satisfying the conclusion of the Mean Value Theorem. Why?*

The Mean Value Theorem is an *existence theorem* like Intermediate Value Theorem. In particular

- We don't know how to find c.
- We don't know how many different c can be found satisfying Mean Value Theorem (there is at least one).

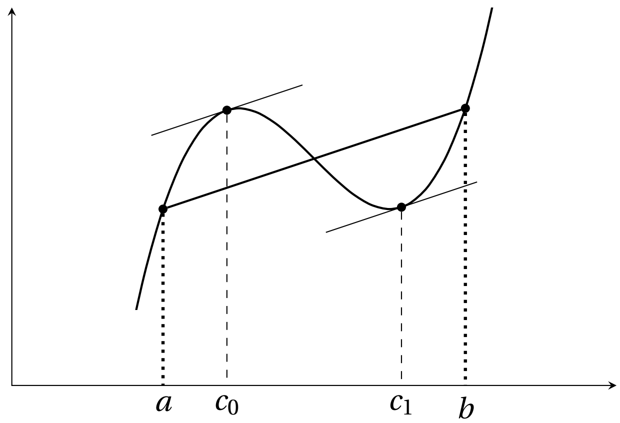


Figure 3.2: There may be more than one c satisfying the conclusion of the Mean Value Theorem.

Example. *Show that* $\sin x < x$ *for all* $x > 0$.

Solution. *For* $x > 2\pi$, *we have* $\sin x < 1 < 2\pi < x$. *Now assume* $0 < x < 2\pi$. *By the Mean Value Theorem there exists* c, $0 < c < x$ *such that*

$$\frac{\sin x - \sin 0}{x - 0} = \cos c.$$

Hence $\sin x = x\cos c$. *Since* $0 < c < 2\pi$, $\cos c < 1$. *Since also* $x > 0$, *we have* $x\cos c < x$. *So* $\sin x = x\cos c < x$.

Example. *Show that* $\sqrt{1+x} < 1 + \frac{x}{2}$ *for all* $x > 0$.

Solution. *Let* $f(x) = \sqrt{1+x}$. *Then* $f'(c) < \frac{1}{2}$ *for* $c > 0$. *Use Mean Value Theorem.*

Example. *Determine all the numbers* c *which satisfy the conclusions of the Mean Value Theorem for*

$$f(x) = x^3 + 2x^2 - x, \qquad x \in [-1, 2]$$

Solution. *Solve*

$$3c^2 + 4c - 1 = f'(c) = \frac{f(2) - f(-1)}{2 - (-1)} = \frac{14 - 2}{3} = 4$$

Solutions of $3c^2 + 4c - 5 = 0$ *are*

$$c_{\pm} = \frac{-4 \pm \sqrt{76}}{6}.$$

Notice that only $\frac{-4+\sqrt{76}}{6}$ *lies in* $[-1, 2]$.

Example. *Suppose* f *is continuous and differentiable on* $[3, 9]$. *Suppose* $f(3) = -4$, *and* $f'(x) \le 10$ *for all* x. *What is the largest value possible for* $f(9)$?

Solution. *By Mean Value Theorem, there exists* $c \in (3, 9)$ *such that*

$$f(9) - f(3) = f'(c)(9 - 3) \le 10 \times 6 = 60.$$

So $f(9) \le 60 + f(3) = 56$.

Definition 9. *Suppose f is defined on an interval I. If for all x_1, x_2 in I s.t. $x_2 > x_1$,*

If	Then on I, f is
$f(x_2) > f(x_1)$	increasing
$f(x_2) < f(x_1)$	decreasing
$f(x_2) \geq f(x_1)$	non-decreasing
$f(x_2) \leq f(x_1)$	non-increasing

Theorem 26. *Suppose f is differentiable on an open interval I. If for all $x \in I$,*

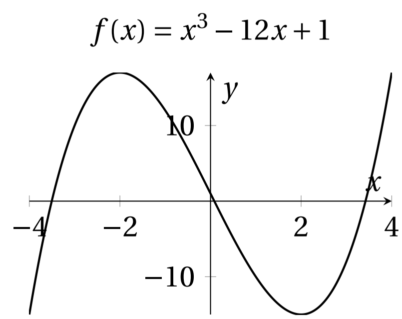
If	Then on I, f is
$f'(x) > 0$	increasing
$f'(x) < 0$	decreasing
$f'(x) \geq 0$	non-decreasing
$f'(x) \leq 0$	non-increasing

Proof. Let's prove the first statement. Let $x_2 > x_1$ in I. By the Mean Value Theorem, there exists c, $x_1 < c < x_2$, such that $f(x_2) - f(x_1) = f'(c)(x_2 - x_1)$. Since $f'(c) > 0$ and $x_2 - x_1 > 0$, we have $f(x_2) > f_{(}x_1)$. So f is increasing. □

Example. *On what intervals is $f(x) = x^3 - 12x + 1$ increasing or decreasing?*

Solution.

$f'(x) = 3(x-2)(x+2)$. *So f is decreasing on $(-2, 2)$ and increasing otherwise.*



We know that if f is a constant function then its derivative is zero. The converse is also true.

Theorem 27. *If $f'(x) = 0$ on an interval I then $f(x) = C$, a constant on I.*

Proof. Choose x_0 in I. Let $C = f(x_0)$. If x is any other point in I then by Mean Value Theorem, $f(x) - f(x_0) = f'(c)(x - x_0) = 0$. □

3.8 Implicit Differentiation

We learned to find the slope of a curve that is the graph of a function. But not all curves are graphs of functions, for example the circle $x^2 + y^2 = 1$.

Curves are graphs of equations in two variables

$$F(x, y) = 0.$$

For the circle $F(x, y) = x^2 + y^2 - 1$.

Example. *Find the slope of the circle* $x^2 + y^2 = 25$ *at the point* $(3, -4)$.

Solution. *1st method. Solve the equation* $x^2 + y^2 = 1$ *for* y*. There are two solutions* $y_{1,2} = \pm\sqrt{25 - x^2}$*. The point lies on the graph of* y_2*. Take derivative of* y_2*.*

2nd method. To differentiate with respect to x *treat* y *as a function of* x *and use Chain Rule.*

$$\frac{d}{dx}\left(x^2 + y(x)^2\right) = \frac{d}{dx}0 = 0.$$

This gives

$$2x + 2y(x)\frac{dy(x)}{dx} = 0$$

or

$$\frac{dy}{dx} = -\frac{2x}{2y}$$

Plug in $x = 3$, $y = -4$ *to find* $\frac{dy}{dx} = 3/4$.

This second method is known as the ***implicit differentiation****.*

Example. *Find an equation of the tangent line to the curve* $x \sin(xy - y^2) = 0$ *at* $(1, 1)$

Example. *Find* y'' *in terms of* x *and* y *if* $xy + y^2 = 2x$.

The General Power Rule for Derivative

So far, we proved the following rule

$$\frac{dx^r}{dx} = rx^{r-1}$$

for integer exponents r and a few special exponents such as $r = 1/2$. Using the implicit differentiation, we can give a proof for any rational exponent $r = m/n$ where $n \neq 0$.

If $y = x^{m/n}$ then $y^n = x^m$. Differentiating implicitly

$$ny^{n-1}\frac{dy}{dx} = \frac{dy^n}{dx} = \frac{dx^m}{dx} = mx^{m-1}$$

Hence

$$\frac{dy}{dx} = \frac{m}{n}x^{m-1}y^{1-n} = rx^{m-1}x^{(1-n)m/n} = rx^{m-1+r-m} = rx^r.$$

3.9 Exam 1 Review

Section	Exercises
1.2	7-36, 37-42, 43-46, 49-60, 74, 75
1.3	1-10, 11-34
1.4	17, 18, 29, 31
1.5	7-10
2.1	1-12, 13-17, 18-24
2.2	11-24 (ignore differentials), 30-33, 34-39, 40-49
2.3	1-50
2.4	1-16, 30-34, 36-39
2.5	1-36, 39-42, 45-46
2.6	1-12
2.8	1-3, 5-7, 8-15
2.9	1-8, 9-16

Table 3.1: Exam 1 Review Problems from Adams & Essex Calculus: A Complete Course 7th Edition

Sample Exam 1

1. Find the following limits if they exist.

   a)
   $$\lim_{x\to\infty}\left(\frac{x^2}{x+1}-\frac{x^2}{x-1}\right)$$

   b)
   $$\lim_{x\to 0}\frac{x}{|x-1|-|x+1|}$$

   c)
   $$\lim_{x\to-\infty}\left(x+\sqrt{x^2-4x+1}\right)$$

2. Show that $f(x)=x^3+x-1$ has a zero between $x=0$ and $x=1$.

3. Find the slope of the tangent line to the curve
   $$\tan(xy^2)=\frac{2xy}{\pi}$$
   at the point $(-\pi,1/2)$.

4. Calculate the derivatives.

   a)
   $$y=\frac{1+\sqrt{x}}{x^4+1}$$

   b)
   $$y=\left(\sin(\sqrt{x})+1\right)^3+7x\cos x$$

5. Is the function $y=|x-2|$ differentiable at $x=2$? Show your work using the limit definition of the derivative.

Chapter 4

Transcendental Functions

4.1 Inverse Functions

Definition 10. *f is called **one-to-one** if $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$ or equivalently*

$$f(x_1) = f(x_2) \implies x_1 = x_2$$

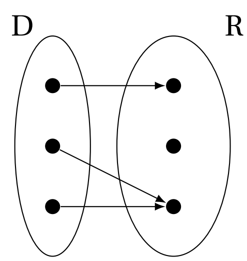


Figure 4.1: A function which is not 1-1.

Horizontal Line Test. Let $f : \mathbb{R} \to \mathbb{R}$. By definition of a function any vertical line intersects the graph at one point. f is 1-1 if its graph is never intersected by any horizontal line more than once.

Theorem 28. *Increasing or decreasing functions are 1-1.*

Definition 11. *If f is one-to-one then it has an inverse function f^{-1} defined as follows: If x is in the range of f then it is in the domain of f^{-1} and*

$$f^{-1}(x) = y \iff x = f(y).$$

If f is not 1-1 then it is not invertible.

Given $y = f(x)$, to find the inverse function, we solve x in terms of y.

Example. *Show that $f(x) = 2x - 1$ is one-to-one and find its inverse $f^{-1}(x)$.*

Solution. *Since $f'(x) = 2 > 0$, f is increasing on $\mathbb{R}$ and therefore one-to-one for all x. Let $y = f^{-1}(x)$, solve for y to get*

$$f^{-1}(x) = \frac{x+1}{2}.$$

Usually, we can not solve $y = f(x)$, for example for $y = x + x^3$.

Properties of inverse functions

1. The domain of f^{-1} is the range of f.
2. The range of f^{-1} is the domain of f.
3. $f(f^{-1}(x)) = x$ for all x in the domain of f^{-1}.

   Proof. If $f^{-1}(x) = y$ then $x = f(y)$ and $f(f^{-1}(x)) = f(y) = x$. □

4. $f^{-1}(f(x)) = x$ for all x in the domain of f.
5. $(f^{-1})^{-1}(x) = f(x)$ for all x in the domain of f. (The inverse of inverse of f is f.)

   Proof.
   $$(f^{-1})^{-1}(x) = y \iff f^{-1}(y) = x \iff y = f(x).$$
   □

6. The graph of f^{-1} is the reflection of the graph of f in the line $x = y$. (Because if (a, b) is a point on the graph of $y = f(x)$ then (b, a) is a point on the graph of $y = f^{-1}(x)$).

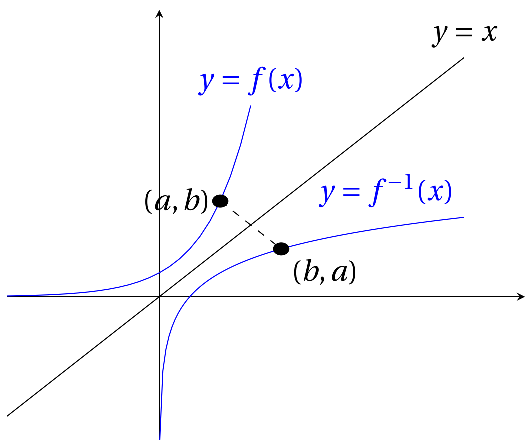


Figure 4.2: The graph of the inverse function is a reflection along $y = x$.

Inverting Non One-to-one Functions by Restricting the Domain of Definition

The function $f(x) = x^2$ is not one-to-one because $(-a)^2 = a^2$ for any a. Hence f is not invertible. Let us define a new function F by restricting the domain of f,

$$F(x) = x^2, \quad x \geq 0.$$

Then $F^{-1}(x) = \sqrt{x}$.

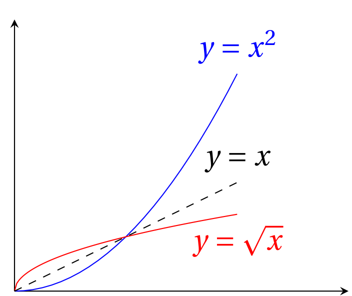


Figure 4.3: The restriction of x^2 to $[0, \infty)$ and its inverse.

Conversely, since the range of the 1-1 function $\sqrt{x}$ is $[0,\infty)$, the domain of its inverse $g(x) = x^2$ is $x \geq 0$.

Derivatives of Inverse Functions

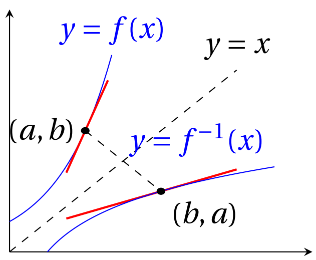
Let $y = f^{-1}(x)$. Then $f(y) = x$. Considering y as a function of x and using implicit differentiation,

$$\frac{d}{dx}f(y) = \frac{d}{dx}x \implies f'(y)\frac{dy}{dx} = 1 \implies \frac{dy}{dx} = \frac{1}{f'(y)}$$

In short if $y = f^{-1}(x)$ then

$$(f^{-1})'(x) = \frac{1}{f'(y)} = \frac{1}{f'(f^{-1}(x))}$$

This formula says if $f'(y) \neq 0$ then f^{-1} is differentiable at x.



In Leibniz notation, $\frac{dy}{dx} = (f^{-1})'(x)$ while $\frac{dx}{dy} = f'(y)$, the above formula reads

$$\frac{dy}{dx}\frac{dx}{dy} = 1.$$

For example, if $y = x^2, x \geq 0$, then $x = \sqrt{y}$ and $\frac{dy}{dx} = 2x$ and $\frac{dx}{dy} = \frac{1}{2\sqrt{y}}$. So

$$\frac{dy}{dx}\frac{dx}{dy} = 2x\frac{1}{2\sqrt{y}} = 2x\frac{1}{2x} = 1.$$

Example. *Show that $f(x) = x^3 + x$ is one-to-one on the whole real line and find $(f^{-1})'(10)$. Hint: $2^3 + 2 = 10$.*

Solution. *First $f'(x) = 3x^2 + 1 > 0$. Hence f is 1-1. Let $y = f^{-1}(x)$.*

$$(f^{-1})'(x) = \frac{1}{3y^2 + 1} = \frac{1}{3(f^{-1}(x))^2 + 1}.$$

$$(f^{-1})'(10) = \frac{1}{3(f^{-1}(10))^2 + 1}.$$

$$y = f^{-1}(10) \implies f(y) = 10 \implies y = 2 \implies f^{-1}(10) = 2.$$

Thus

$$(f^{-1})'(10) = \frac{1}{3 \times 2^2 + 1} = \frac{1}{13}.$$

Example. *If $f(x) = 3x + x^3$, show that f has an inverse and find the slope of $y = f^{-1}(x)$ at $x = 0$.*

4.2 Exponential and Logarithmic Functions

An **exponential function** is a function of the form $f(x) = a^x$ where the **base** a is a positive constant and the **exponent** x is the variable. Let's define this function.

- $a^0 = 1$.
- $a^n = a \cdot a \cdots a$ (n -times) if $n = 1, 2, 3, \ldots$.
- $a^{-n} = \frac{1}{a^n}$ if $n = 1, 2, 3, \ldots$.
- $a^{m/n} = \sqrt[n]{a^m}$ if $n = 1, 2, \ldots$ and $m = \pm 1, \pm 2, \ldots$.

How should we define a^x if x is not rational? What does 2^π mean?

Example. *Since the irrational number $\pi = 3.141592\ldots$ is the limit of the sequence of rational numbers*

$$r_1 = 3 \quad r_2 = 3.1 \quad r_3 = 3.14 \quad \ldots$$

we can calculate 2^π as the limit of the sequence

$$2^3 = 8 \quad 2^{3.1} = 8.5741877\ldots \quad 2^{3.14} = 8.8152409\ldots$$

This gives

$$2^\pi = \lim_{n\to\infty} 2^{r_n} = 8.824977\ldots$$

If x is irrational, then we define a^x as the limit values a^r for rational numbers r approaching x

$$a^x = \lim_{\substack{r\to x \\ \text{r is rational}}} a^r.$$

Laws of Exponents
If $a > 0$ and $b > 0$ and x, y are real numbers then

1. $a^0 = 1$,
2. $a^{x+y} = a^x a^y$,
3. $a^{-x} = \dfrac{1}{a^x}$,
4. $a^{x-y} = \dfrac{a^x}{a^y}$,
5. $(a^x)^y = a^{xy}$,
6. $(ab)^x = a^x b^x$.

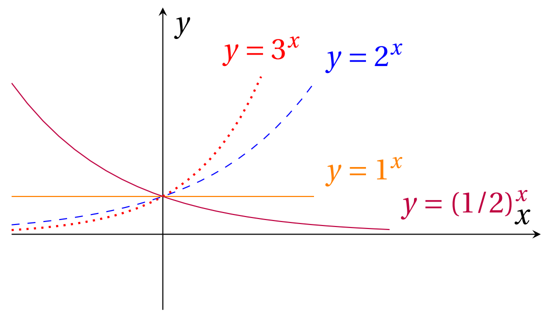
If $a > 1$ then

$$\lim_{x\to\infty} a^x = \infty, \qquad \lim_{x\to-\infty} a^x = 0.$$

If $0 < a < 1$ then

$$\lim_{x\to\infty} a^x = 0, \qquad \lim_{x\to-\infty} a^x = \infty.$$

The domain of a^x is $(-\infty, \infty)$ and its range is $(0, \infty)$.



Logarithm

If $a > 0$ and $a \neq 1$ then the function a^x is 1-1 (1^x has no inverse). The inverse function of a^x is $\log_a x$, called the **logarithm of x base a**.

$$y = \log_a x \iff x = a^y$$

Since a^x has domain $(-\infty, \infty)$, and range $(0, \infty)$, $\log_a x$ has domain $(0, \infty)$ and range $(-\infty, \infty)$.
Since a^x and $\log_a x$ are inverse functions

$$\log_a a^x = x \quad \forall x, \qquad a^{\log_a x} = x, \quad x > 0$$

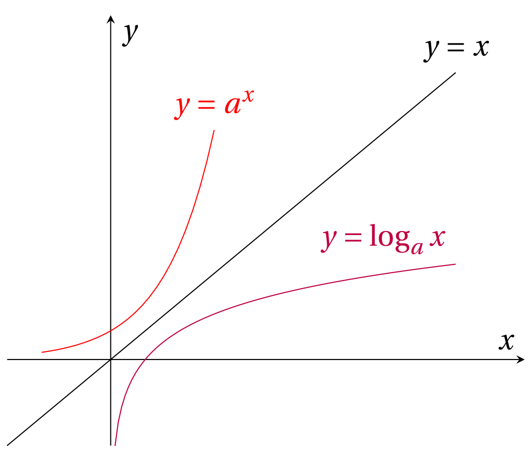


Figure 4.4: The graph of logarithmic function is a reflection of the graph of the exponential function in the line $y = x$.

Laws of Logarithm If $x > 0$, $y > 0$, $a > 0$, $b > 0$, $a \neq 1$, $b \neq 1$, then

1. $\log_a 1 = 0$
2. $\log_a(xy) = \log_a x + \log_a y$
3. $\log_a(\frac{1}{x}) = -\log_a x$
4. $\log_a(\frac{x}{y}) = \log_a x - \log_a y$
5. $\log_a x^y = y \log_a x$
6. $\log_a x = \dfrac{\log_b x}{\log_b a}$

Example. *Prove* $\log_a(xy) = \log_a x + \log_a y$ *using laws of exponent.*

Solution. *Take* $u = \log_a x$, $v = \log_a y$ *then* $x = a^u$, $y = a^v$ *and*

$$xy = a^{u+v} \iff u + v = \log(xy)$$

Example. *Simplify*

1. $\log_2 10 + \log_2 12 - \log_2 15$.

$$\log_2 10 + \log_2 12 - \log_2 15 = \log_2 \frac{10 \times 12}{15} = \log_2 8 = 3$$

2. $\log_{a^2} a^3$.

$$\log_{a^2} a^3 = \frac{\log_a a^3}{\log_a a^2} = \frac{3}{2}$$

3. $3^{\log_9 4}$.

$$3^{\log_9 4} = 3^{\frac{1}{2}\log_3 4} = 3^{\log_3 2} = 2$$

Example. *Solve*

$$3^{x-1} = 2^x,$$

in terms of $a = \log 2$ *and* $b = \log 3$.

Solution. *Take logarithm base 3 of both sides.*

$$(x-1)\log_3 3 = x\log_3 2 \iff x - 1 = x\log_3 2 \iff x = \frac{1}{1-\log_3 2} = \frac{1}{1-a/b}$$

Numerically $x \approx 2.70951$.

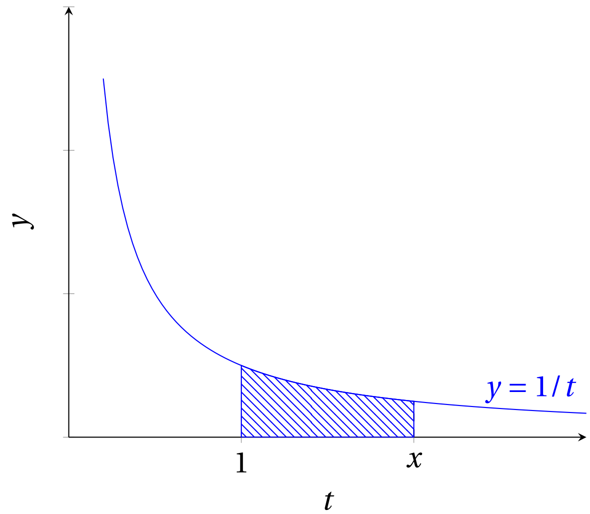
4.3 The Natural Logarithm and Exponential

$f(x)$	$f'(x)$
$x^3/3$	x^2
$x^2/2$	x
x	1
x^0	0
$-x^{-1}$	x^{-2}
$-x^{-2}/2$	x^{-3}

Table 4.1: What is the mysterious function whose derivative is x^{-1}?

Definition 12. *For* $x > 0$, *let* A_x *be the area bounded by the curve* $y = 1/t$, *the t-axis and the vertical lines* $t = 1$ *and* $t = x$. *The* ***natural logarithm*** *function is defined by*

$$\ln x = \begin{cases} A_x & x \geq 1 \\ -A_x & 0 < x < 1 \end{cases}$$



- Domain of $\ln x$ is $(0, \infty)$,
- $\ln 1 = 0$,
- $\ln x > 0$ if $x > 1$,
- $\ln x < 0$ if $0 < x < 1$,

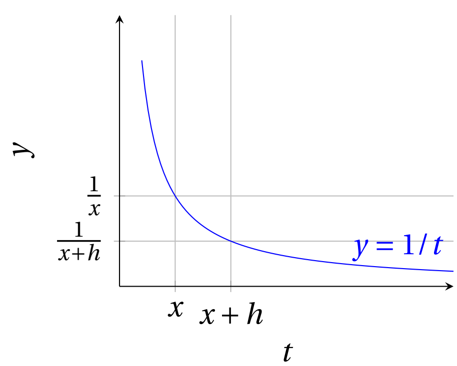
Theorem 29. *If $x > 0$ then* $\dfrac{d}{dx}\ln x = \dfrac{1}{x}$

Proof. For $h > 0$, $\ln(x+h) - \ln x$ is the area under $1/t$ between $t = x$ and $t = x + h$. Thus

$$\frac{h}{x+h} < \ln(x+h) - \ln x < \frac{h}{x}$$

Thus

$$\frac{1}{x+h} < \frac{\ln(x+h) - \ln x}{h} < \frac{1}{x}$$



Now use Squeeze Theorem to get

$$\lim_{h \to 0+} \frac{\ln(x+h) - \ln x}{h} = \frac{1}{x}$$

Similar argument holds for $h < 0$. □

Theorem 30. *If $x \neq 0$ then*

$$\frac{d}{dx}\ln|x| = \frac{1}{x},$$

and

$$\int \frac{1}{x}\,dx = \ln|x| + C.$$

Proof. If $x < 0$, then by Chain Rule,

$$\frac{d}{dx}\ln|x| = \frac{d}{dx}\ln(-x) = \frac{1}{-x}(-1) = \frac{1}{x}.$$

□

This also shows that $\ln x$ is an **increasing** function for all $x > 0$.

Example. *Find the derivatives of*

1. $y = \ln|\cos x|$

2. $y = \ln(x + \sqrt{x^2 + 1})$

Solution. *For (1),*

$$y' = \frac{1}{\cos x}(-\sin x) = -\tan x.$$

For (2),

$$y' = \frac{1}{\sqrt{x^2 + 1}}.$$

The natural logarithm function $\ln x$ satisfies all the rules that the regular logarithms satisfy, that's why we call it natural log after all!

Theorem 31. 1. $\ln(xy) = \ln x + \ln y$.

2. $\ln(1/x) = -\ln x$.

3. $\ln(x/y) = \ln x - \ln y$.

4. $\ln x^r = r \ln x$.

Proof. For (i), if y is constant, then for all $x > 0$

$$\frac{d}{dx}(\ln(xy) - \ln x) = \frac{y}{xy} - \frac{1}{x} = 0$$

Thus for each $y > 0$, $\ln(xy) - \ln x = C$ (a constant depending on y) for $x > 0$. Setting $x = 1$ we get $C = \ln y$. The others can be done similarly (homework). □

Also note that

$$\ln 2^n = n \ln 2 \to \infty \text{ as } n \to \infty.$$

$$\ln 2^{-n} = -n \ln 2 \to -\infty \text{ as } n \to \infty.$$

This, combined with $\ln x$ is increasing shows that

$$\lim_{x\to\infty} \ln x = \infty, \qquad \lim_{x\to 0+} \ln x = -\infty.$$

Thus domain of $\ln x$ is $(0,\infty)$ and the range of $\ln x$ is $(-\infty,\infty)$.

The Exponential Function

Let $f(x) = \ln x$. Since $f'(x) = 1/x > 0 \implies f$ is increasing $\implies f$ is 1-1 $\implies f$ has an inverse. Call its inverse $\exp x$. Thus

$$\exp x = y \iff x = \ln y$$

- $\exp 0 = 1$ (since $\ln 1 = 0$),
- Domain of exp is $(-\infty, \infty)$ (since range of ln is $(-\infty, \infty)$),
- Range of exp = Domain of ln = $(0, \infty)$,
- Cancellation identities

$$\begin{aligned} \exp \ln x &= x, \quad x > 0 \\ \ln \exp x &= x, \quad -\infty < x < \infty. \end{aligned}$$

Definition 13. $e = \exp(1) \approx 2.718\ldots.$

Thus $\ln e = 1$. Hence e is the number for which the area bounded by $y = 1/x$, the x-axis and the lines $x = 1$, $x = e$ is 1.

$$e^x = \exp(\ln(e^x)) = \exp(x \ln e) = \exp(x).$$

Since exp is actually an exponential function, its inverse must be a logarithm

$$\ln x = \log_e x$$

The derivative of $y = e^x$ is calculated by implicit differentiation:

$$y = e^x \iff x = \ln y \iff 1 = \frac{y'}{y} \iff y' = y = e^x$$

This is a remarkable property:

$$\frac{d}{dx} e^x = e^x, \qquad \int e^x \, dx = e^x + C$$

Example. *Find the derivatives of*

1. $e^{x^2 - 3x}$,
2. $\sqrt{1 + e^{2x}}$

General Exponentials and Logaritms

Definition 14. *If $a > 0$ then for all real x, we define*

$$a^x = e^{x \ln a}$$

This coincides with our previous definition that a^x is the limit of a^{r_n} where r_n are rational numbers tending to x.

Example. $2^\pi = e^{\pi \ln 2} \approx 8.825$.

Derivative of $y = a^x$.

$$\frac{d}{dx}a^x = \frac{d}{dx}e^{x\ln a} = e^{x\ln a}\ln a = a^x \ln a.$$

Example. *Show that the graph of $f(x) = x^\pi - \pi^x$ has negative slope at $x = \pi$.*

Solution. $f'(\pi) = \pi^\pi(1 - \ln\pi)$. *Note that* $\ln\pi > \ln e = 1$

Definition 15. *Let $y = a^x$. Then $\frac{dy}{dx} = a^x \ln a$ which is negative if $0 < a < 1$ and positive if $a > 1$. Thus a^x is 1-1 and has an inverse function. We define its inverse as $\log_a x$.*

Derivative of $y = \log_a x$.

$$\frac{d}{dx}\log_a x = \frac{d}{dx}\frac{\ln x}{\ln a} = \frac{1}{ax}.$$

Logarithmic Differentiation

Example. *Let $y = x^x$, $x > 0$. Find y'.*

Solution. *Neither the power rule $d/dx(x^a) = ax^{a-1}$ nor the exponential rule $d/dx(a^x) = \ln a a^x$ works.*

$$\ln y = x\ln x \implies \frac{y'}{y} = 1\ln x + x\frac{1}{x} \implies y' = x^x(\ln x + 1)$$

This technique is called **logarithmic differentiation** and is used to differentiate functions of the form $y = (f(x))^{g(x)}$ $(f(x) > 0)$.

Example. *Find dy/dt if $y = (\sin t)^{\ln t}$ where $0 < t < \pi$.*

Solution.

$$y' = (\sin t)^{\ln t}\left(\frac{\ln\sin t}{t} + \ln t \cot t\right).$$

Example. *If $y = \dfrac{(x+1)(x+2)(x+3)}{\sqrt{x+4}}$, find y'.*

Solution. *Since $(x+1)$ is not necessarily positive, $\ln(x+1)$ may or may not be defined. So we take the absolute value and then logarithm.*

$$\ln|y| = \ln|x+1| + \ln|x+2| + \ln|x+3| - \frac{1}{2}\ln|x+4|$$

$$\frac{y'}{y} = \frac{1}{x+1} + \cdots$$

4.4 The Inverse Trigonometric Functions

The six trigonometric functions are periodic and hence not 1-1. However we can restrict their domains in such a way that the restricted functions are 1-1.

The $\sin^{-1} x$ or $\arcsin x$ is the inverse of the $\sin x$ restricted to $[-\pi/2, \pi/2]$,

$$\sin(\sin^{-1} y) = y, \qquad -1 \le y \le 1,$$
$$\sin^{-1}(\sin x) = x, \qquad -\pi/2 \le x \le \pi/2.$$

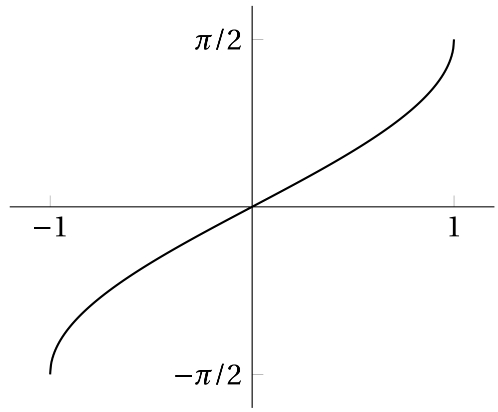


Figure 4.5: $f(x) = \arcsin x$.

Example. *Simplify*

1. $\sin^{-1} \frac{1}{2} = \frac{\pi}{6}$,
2. $\sin^{-1} \frac{-\sqrt{2}}{2} = -\frac{\pi}{4}$,
3. $\sin^{-1} 2$ *is undefined since* 2 *is not in the range of sine.*

Example. *Simplify*

1. $\sin(\sin^{-1} 0.7) = 0.7$,
2. $\sin^{-1}(\sin 3\pi/4) = \pi/4$,
3. $\cos(\sin^{-1} 0.6)$.

   Solution. *Let* $\theta = \sin^{-1} 0.6$. *By the Pythagorean Theorem,* $\cos\theta = 0.8$.

4. *Similarly* $\cos(\sin^{-1} x) = \frac{1}{1-x^2}$.

Let $y = \sin^{-1} x$ so that $x = \sin y$. Then

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-x^2}}.$$

Thus

$$\frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1-x^2}}.$$

The Inverse Tangent (or Arctangent) Function

Define the $\tan^{-1} x = \arctan x$ to be the inverse of $(-\pi/2, \pi/2)$.

$$\begin{aligned} \tan(\tan^{-1} x) &= x, \qquad -\infty < x < \infty, \\ \tan^{-1}(\tan x) &= x, \qquad -\pi/2 \le x \le \pi/2. \end{aligned}$$

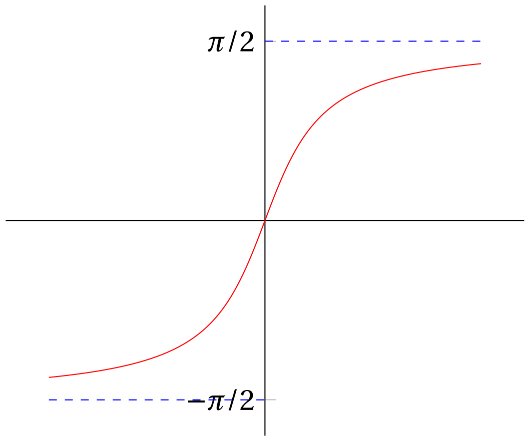


Figure 4.6: $f(x) = \arcsin x$.

Example. *1.* $\tan(\tan^{-1} 3) = 3$,

2. $\tan^{-1}(\tan \frac{3\pi}{4}) = \tan^{-1} -1 = -\frac{\pi}{4}$

3. $\cos(\tan^{-1} x) = \frac{1}{\sqrt{1+x^2}}$

Let $y = \tan^{-1} x$ so that $x = \tan y$,

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{\sec^2 y} = \frac{1}{1+\tan^2 y} = \frac{1}{1+x^2}.$$

Example. *Find the slope of the curve* $\tan^{-1}\left(\frac{2x}{y}\right) = \frac{\pi x}{y^2}$ *at the point* $(1,2)$.

Solution. *Taking* $\frac{d}{dx}$ *of both sides*

$$\frac{1}{1+\left(\frac{2x}{y}\right)^2} 2\left(\frac{y - xy'}{y^2}\right) = \pi\left(\frac{y^2 - 2xyy'}{y^4}\right)$$

Plugging $x = 1$, $y = 2$,

$$2 - y' = \pi(1 - y') \implies y' = \frac{\pi - 2}{\pi - 1}.$$

Other inverse trigonometric functions

$\cos x$ is 1-1 on $[0,\pi]$ so we define $\cos^{-1} x$ for $\cos x$ restricted to $[0,\pi]$.

$$y = \cos^{-1} x \iff x = \cos y \qquad 0 \le y \le \pi.$$

Note that $\sin(\cos^{-1} x)$ For the derivative,

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{-\sin y} = -\frac{1}{\sqrt{1-x^2}}$$

Note that

$$\frac{d}{dx}\cos^{-1} x = -\frac{d}{dx}\sin^{-1} x$$

The inverse and the derivative of other trigonometric functions can be defined similarly.

Quiz Problems

Example. *Simplify*

1. $\cos(\tan^{-1} \frac{1}{2})$
2. $\tan(\cos^{-1} x)$

Example. *Differentiate* $(\sin^{-1} x^2)^{1/2}$.

Chapter 5

Applications of Derivatives

5.1 Related Rates

Example. *How fast is the area of a rectangle changing if one side is 10cm long and is increasing at a rate of 2cm/s and the other side is 8cm long and is decreasing at a rate of 3cm/s?*

Solution. *The area A, and the lengths of sides x and y are functions of time t. Also $A = xy$. We are given $\frac{dx}{dt} = 2$, $\frac{dy}{dt} = -3$ when $x = 10$, $y = 8$.*

Then

$$\frac{dA}{dt} = \frac{dx}{dt}y + x\frac{dy}{dt}$$

gives $\frac{dA}{dt} = -14$.

In the previous problem, if $\frac{dx}{dt}$ and $\frac{dy}{dt}$ are constant, then $x(t) = 10 + 2t$, $y(t) = 8 - 3t$ and $A(t) = (10 + 2t)(8 - 3t)$. So $A(1) = 60$. But we found that A changes -14cm^2/s so we expect to find $A(1) = -66$. How is this possible?

This is possible because $A'(t) = -14 - 6t$. Hence A changes in a non-constant fashion event though x and y changes constantly. What we are computing in this problem is $A'(0) = -14$. And this result holds even if $\frac{dx}{dt}$ and $\frac{dy}{dt}$ are not constant. (maybe velocity of x is not constant and it accelerates according to $x(t) = 10 + 2t + t^2$)

Example. *How fast is the surface area of a ball changing when the volume of the ball is $32\pi/3$ cm^3 and is increasing at 2cm^3/s? (The surface are of the ball is $A = 4\pi r^2$ and the volume is $V = \frac{4}{3}\pi r^3$. Note that $V(r) = \int_0^r A(r)dr$)*

Solution. *When $V = 32\pi/3$, $r = 2$, $\frac{dV}{dt} = 2$*

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

gives $\frac{dr}{dt} = \frac{1}{8\pi}$.

Now

$$\frac{dA}{dt} = 8\pi r \frac{dr}{dt} = 2$$

Example. *A point is moving to the right along the first quadrant portion of the curve $x^2 y^3 = 72$. When the point has coordinates $(3, 2)$, its horizontal velocity is 2 units/s. What is its vertical velocity?*

Solution. *Taking d/dt of both sides*

$$2x\frac{dx}{dt}y^3 + x^2 3y^2\frac{dy}{dt} = 0$$

At $x = 3$, $y = 2$, $\frac{dx}{dt} = 2$,

$$\frac{dy}{dt} = -\frac{8}{3}.$$

Example. *The area of a circle is decreasing at a rate of 2 cm^2/min. How fast is the radius of the circle changing when the area is 100 cm^2?*

5.2 Indeterminate Forms

To evaluate the limit $\lim_{x\to 0}\frac{\sin x}{x}$ we can not plug in $x = 0$. We call $\sin x/x$ an **indeterminate form** of $[0/0]$ at $x = 0$.

The limit of such an indeterminate form can be any number.

$$\lim_{x\to 0}\frac{x}{x} = 1, \qquad \lim_{x\to 0}\frac{x}{x^3} = \infty, \qquad \lim_{x\to 0}\frac{x^3}{x^2} = 0.$$

There are other types of indeterminate forms $[\infty/\infty]$, $[0\cdot\infty]$, $[\infty-\infty]$, $[0^\infty]$, $[\infty^0]$, $[1^\infty]$.

Indeterminate forms can usually be brought to the form $[0/0]$ or $[\infty/\infty]$. To evaluate limits of the form $[0/0]$ a very useful method is the following.

Theorem 32 (l'Hopital's Rules)**.** *Suppose that f and g are differentiable on an interval containing a. Suppose also that $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$ and $\lim_{x\to a}\frac{f'(x)}{g'(x)} = L$. Then*

$$\lim_{x\to a}\frac{f(x)}{g(x)} = L.$$

Similar results hold for $\lim_{x\to a+}$ and $\lim_{x\to a-}$ and for the cases $a = \pm\infty$.

Proof. Proof follows from generalized mean value theorem. □

`Note that in applying l'Hopital's rule we calculate the quotient of the derivatives, not the derivative of the quotients.`

Example. *Evaluate*

$$\lim_{x\to 1}\frac{\ln x}{x^2-1}$$

Solution.

$$\lim_{x\to 1}\frac{\ln x}{x^2-1} \qquad \left[\frac{0}{0}\right]$$

$$\lim_{x\to 1}\frac{\ln x}{x^2-1} = \lim_{x\to 1}\frac{\frac{1}{x}}{2x} = \lim_{x\to 1}\frac{1}{2x^2} = \frac{1}{2}.$$

If one application of the l'Hopital's rule again gives an indeterminate form, we can apply it again.

Example. *Evaluate*

$$\lim_{x\to 0}\frac{2\sin x - \sin(2x)}{2e^x - 2 - 2x - x^2}$$

Solution. *Applying l'Hopital's rule three times we get the answer* 3.

Example.

$$\lim_{x\to 1+}\frac{x}{\ln x}$$

Solution. *If you apply the l'Hopital's rule, you get the wrong answer of 1. This is not an indeterminate form, and you can't use l'Hopital's rule. The real answer is* ∞.

Example.

$$\lim_{x\to 0+}\frac{1}{x}-\frac{1}{\sin x}$$

Solution. *This is an indeterminate form of type* $[\infty-\infty]$ *which can be brought to the form* $[0/0]$.

$$\lim_{x\to 0+}\frac{1}{x}-\frac{1}{\sin x}=\lim_{x\to 0+}\frac{\sin x-x}{x\sin x}=\lim_{x\to 0+}\frac{\cos x-1}{\sin x+x\cos x}=\lim_{x\to 0+}\frac{-\sin x}{\cos x+\cos x-x\sin x}=\frac{0}{-2}=0.$$

where we use l'Hopital's rule twice.

The l'Hopital's rule can also be applied to indeterminate forms of type $[\infty/\infty]$.

Example.

$$\lim_{x\to\infty}\frac{x^2}{e^x}.$$

Solution. *The answer is 0.*

To deal with indeterminate forms of types $[0^0]$, $[\infty^0]$ and $[1^\infty]$, we take logarithms.

Example.

$$\lim_{x\to 0+}x^x.$$

Solution. *This is of the form* $[0^0]$. *Let* $y=x^x$. *Then*

$$\lim_{x\to 0+}\ln y=\lim_{x\to 0+}x\ln x=\lim_{x\to 0+}\frac{\ln x}{1/x}=\lim_{x\to 0+}\frac{1/x}{-1/x^2}=0$$

Since ln *is a continuous function*

$$\ln\lim_{x\to 0+}\ln y=\lim_{x\to 0+}\ln y=0,$$

$$\lim_{x\to 0+}x^x=e^0=1.$$

Example. *Evaluate*

$$\lim_{x\to\infty}\left(1+\sin\frac{3}{x}\right)^x$$

Example. *This is of the form* $[1^\infty]$. *Again first evaluate the limit of the logarithm.* $y=\left(1+\sin\frac{3}{x}\right)^x$.

$$\lim_{x\to\infty}\ln y=\lim_{x\to\infty}\frac{\ln\left(1+\sin\frac{3}{x}\right)}{1/x}=3$$

Hence

$$\lim_{x\to\infty}\left(1+\sin\frac{3}{x}\right)^x=e^3.$$

5.3 Extreme Values

A function has an **absolute maximum value** $f(x_0)$ if $f(x) < f(x_0)$ holds for every x in its domain.

Similarly, define **absolute minimum value**.

If it has an absolute min/max, then that value may be achieved at more than one point. For example the function $\cos x$ attains its absolute max at $x = 2n\pi$ for any integer n.

A function may or may not have an absolute min/max value. For example the function $f(x) = x, 0 < x < 1$ does not have an absolute maximum or minimum.

Recall from the section on continuous functions that,

`A continuous function defined on a closed and bounded interval must have an absolute maximum and an absolute minimum.`

Maximum and minimum values of a function are collectively referred to as **extreme values**.

Function f has a **local maximum** value $f(x_0)$ if there exists $h > 0$ such that $f(x) \leq f(x_0)$ whenever x is in the domain of f and $|x - x_0| < h$.

Similarly we define **local minimum**.

We define **critical points** of f where $f'(x) = 0$, **singular points** of f where x is in domain of f and $f'(x)$ does not exist.

Following theorem says where the extreme values are located.

Theorem 33. *If the function f is defined on an interval I and has a local max or local min at $x = x_0$ then x_0 **must** be either a critical point, a singular point or an endpoint of the interval.*

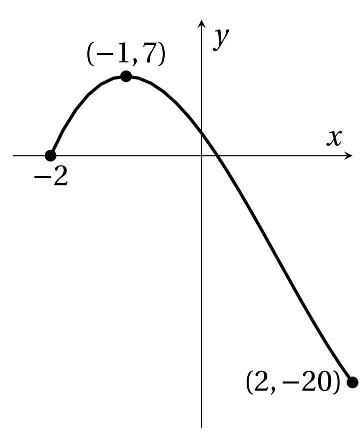
Proof. If $f(x_0)$ is a local extrema and x_0 is not an endpoint or singular point, then $f'(x_0) = 0$. Otherwise, either $f'(x_0) > 0$ which means f is increasing at x_0 or $f'(x_0) < 0$ which means f is decreasing at x_0 so that $f(x_0)$ is neither a local min nor local max. □

This theorem does not say f must have a local min/max at at every singular, critical or endpoint. For example for $f(x) = x^3$, $f'(0) = 0$ but $f(0)$ is not an extremum value.

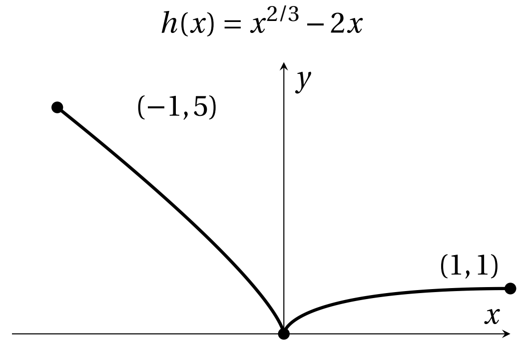
Example. *Find the maximum and minimum values of the function $g(x) = x^3 - 3x^2 - 9x + 2$ on the interval $-2 \leq x \leq 2$.*

Solution. *g is a continuous function defined on a closed and bounded interval so it must have an absolute minimum and absolute maximum.*
Since g is a polynomial, it can't have singular points.
$g'(x) = 3(x^2 - 2x - 3) = 3(x+1)(x-3)$. $g'(x) = 0$ *if* $x = -1$ *or* $x = 3$. $x = 3$ *is not in the domain, so we ignore it.*
We check the values of $g(x)$ at endpoints and critical points, $g(-2) = 0$, $g(-1) = 7$, $g(2) = -20$. *The maximum value is 7, the minimum value is -20.*

$f(x) = x^3 - 3x^2 - 9x + 2$



Example. *Find the maximum and minimum values of $h(x) = 3x^{2/3} - 2x$ on the inteval* $[-1, 1]$.



Solution. $h'(x) = 2(x^{-1/3} - 1)$. $h'(0)$ *is undefined,* 0 *is a singular point of* h*.* h *has a critical point at* $x = 1$ *which is also an endpoint.*
$h(-1) = 5$, $h(0) = 0$, $h(1) = 1$. h *has maximum value* 5 *and minimum value* 0.

The first derivative test

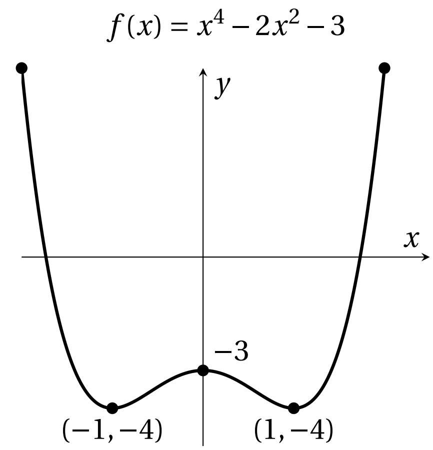
By investigating the sign of the first derivative we can determine whether an extrema is a local minimum or local maximum.

Example. *Find the local and absolute extreme values of* $f(x) = x^4 - 2x^2 - 3$ *on the interval* $[-2,2]$*. Sketch the graph of* f*.*

Solution. $f'(x) = 2x(x^2-1) = 4x(x-1)(x+1)$. *The critical points are* $0, -1, 1$. *There are no singular points.*

x	-2		-1		0		1		2
f'		-		+		-		+	
f	max	↘	min	↗	max	↘	min	↗	max

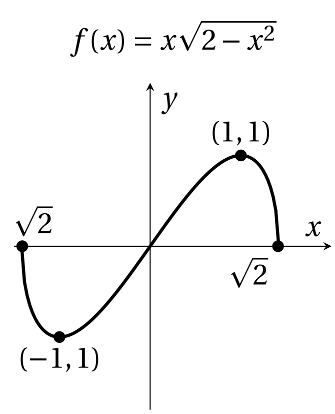
$f(-2) = f(2) = 5$, $f(-1) = f(1) = -4$, $f(0) = -3$.
Since f *is continuous and defined on a closed and bounded interval, it must have an absolute min/max. So* 5 *is the absolute maximum and* -4 *is the absolute minimum.*



Example. *Locate all extreme values of* $f(x) = x\sqrt{2 - x^2}$*. Determine whether any of these extreme values are absolute. Sketch the graph.*

Solution. *Note that* f *has domain* $[-\sqrt{2}, \sqrt{2}]$. $f'(x) = -2\frac{x^2-1}{\sqrt{2-x^2}}$. *Critical points are* ± 1. *Singular points are* $\pm\sqrt{2}$ *and endpoints are also* $\pm\sqrt{2}$.
$f(\pm\sqrt{2}) = 0$, $f(-1) = -1$, $f(1) = 1$. *Since* f *is continuous on a closed bounded interval it must have maximum value* 1 *and minimum value* -1.

x	$-\sqrt{2}$		-1		1		$\sqrt{2}$
f'		-		+		-	
f	max	↘	min	↗	max	↘	min



5.4 Concavity and Inflections

We say f is **concave up** on an interval I if f' is increasing on I and **concave down** on I if f' decreasing on I.

Note that if f is concave up then f lies above its tangents and below its chords while if f is concave down then f lies below its tangents and above its chords.

If f changes its concavity at x_0 then we call x_0 and **inflection point**.

Theorem 34. *Assume f is twice differentiable.*

a) If $f'' > 0$ on an interval I then f is concave up on I,

b) If $f'' < 0$ on an interval I then f is concave down on I,

c) If f has an inflection point at x_0 then $f''(x_0) = 0$.

Note $f''(x_0) = 0$ does not necessarily mean x_0 is an inflection point, for example for $f(x) = x^4$ $f''(0) = 0$ while f does not change concavity at $x = 0$.

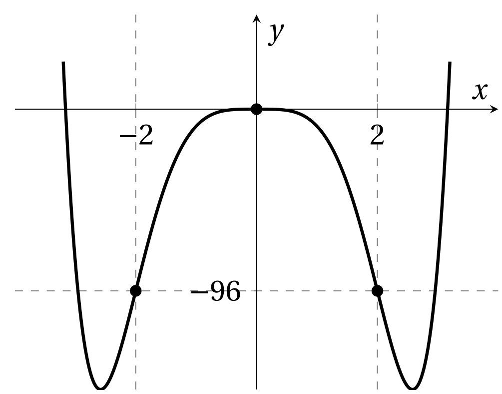
Example. *Determine the intervals of concavity of $f(x) = x^6 - 10x^4$ and the inflection points of its graph.*

Solution. $f'(x) = 2x^3(3x^2 - 20)$, $f''(x) = 30x^2(x-2)(x+2)$. *So possible inflection points are* $0, \pm 2$.

x		-2		0		2	
f''	+	0	-	0	-	0	+
f	c.up	infl.	c.down		c.down	infl	c.up

The inflection points are ± 2.

$f(x) = x^6 - 10x^4$

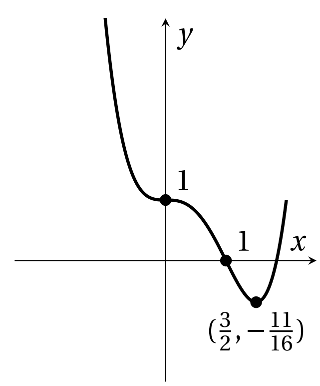


Example. *Determine the intervals of increase, decrease , the local extreme values and the concavity of $f(x) = x^4 - 2x^3 + 1$. Sketch the graph of f.*

Solution. $f'(x) = 4x^3 - 6x^2 = 2x^2(2x-3)$, *critical points are* $x = 0$, $x = 3/2$.
$f''(x) = 12x(x-1)$, *possible inflection points are* $x = 0$, $x = 1$.

x		0		1		3/2	
f'	-	0	-		-	0	+
f''	+	0	-	0	+		+
f	↘		↘		↘	min	↗
	c.up	infl	c.down	infl	c.up		c.up

$f(x) = x^4 - 2x^3 + 1$



The Second Derivative Test

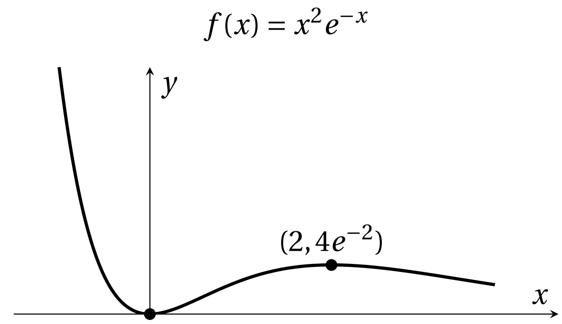
Theorem 35. *a) If $f'(x_0) = 0$ and $f''(x_0) < 0$, then f has a local max at x_0.*

b) If $f'(x_0) = 0$ and $f''(x_0) > 0$, then f has a local min at x_0.

c) *If $f'(x_0) = f''(x_0)$, then no conclusion can be drawn.*

Example. *Find an classify the critical points of $f(x) = x^2 e^{-x}$.*

Solution. $f'(x) = x(2-x)e^{-x} = 0$, *at* $x = 0$, $x = 2$. $f''(x) = (2 - 4x + x^2)e^{-x}$. $f''(0) = 2 > 0$ *and* $f''(2) = -2e^{-2} < 0$. *Thus f has a local min at $x = 0$ and local max at $x = 2$.*



5.5 Graphs of Functions

Definition 16. *The graph of $y = f(x)$ has a **vertical asymptote** at $x = a$ if either* $\lim_{x\to a-} f(x) = \pm\infty$ *or* $\lim_{x\to a+} f(x) = \pm\infty$.

Definition 17. *The graph of $y = f(x)$ has a **horizontal asymptote** at $y = L$ if either* $\lim_{x\to\infty} f(x) = L$ *or* $\lim_{x\to-\infty} f(x) = L$.

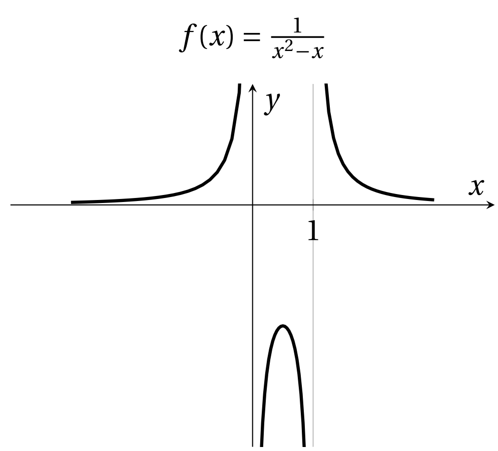
Example. *Find the vertical and the horizontal asymptotes of $f(x) = \frac{1}{x^2-x}$.*

Solution. *The vertical asymptotes are $x = 0$, $x = 1$.*

$$\lim_{x\to 0-} \frac{1}{x^2 - x} = \infty, \qquad \lim_{x\to 0+} \frac{1}{x^2 - x} = -\infty,$$

$$\lim_{x\to 1-} \frac{1}{x^2 - x} = -\infty, \qquad \lim_{x\to 1+} \frac{1}{x^2 - x} = \infty,$$

The function has a horizontal asymptote, $\lim_{x\to\infty} \frac{1}{x^2-x} = \lim_{x\to-\infty} \frac{1}{x^2-x} = 0$. *This is a two-sided horizontal asymptote.*

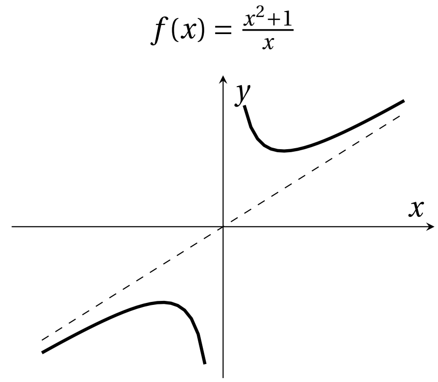


Example. $f(x) = e^x$ *has a left horizontal asymptote* $y = 0$, $\lim_{x\to-\infty} e^x = 0$.

Example. $f(x) = \tan^{-1} x$ *has a two one sided limits,* $\lim_{x\to\infty} \tan^{-1} x = \pi/2$ *and* $\lim_{x\to-\infty} \tan^{-1} x = -\pi/2$.

Definition 18. *The straight line $y = ax + b$ ($a \neq 0$) is an oblique asymptote of the graph $y = f(x)$ if either* $\lim_{x\to\infty}(f(x) - (ax + b)) = 0$ *or* $\lim_{x\to-\infty}(f(x) - (ax + b)) = 0$.

Example. *Let* $f(x) = \frac{x^2+1}{x} = x + \frac{1}{x}$. *Then* $\lim_{x\to\pm\infty}(f(x) - x) = 0$. *Hence* f *has a two-sided oblique asymptote.*



Asymptotes of rational function

Let $f(x) = \frac{P_m(x)}{Q_n(x)}$, where P_m and Q_n are polynomials of degree m and n respectively. Suppose that P_m and Q_n have no common linear factors. The graph of f has

1. a vertical asymptote at every position at every x for which $Q_n(x) = 0$.

2. a two-sided horizontal asymptote $y = 0$ only if $m < n$.

3. a two-sided horizontal asymptote $y = L$ only if $m = n$. L is the ratio of the coefficients of the highest degree terms in P_m and Q_n.

4. a two sided oblique asymptote only if $m = n + 1$.

Example. *Find the oblique asymptote of* $y = \frac{x^3}{x^2+x+1}$.

Solution. *Bu polynomial division, we get* $y = x - 1 + \frac{1}{x^2+x+1}$. $y = x - 1$ *is the oblique asymptote.*

Checklist For Curve Sketching

1. Examine $f(x)$ to find the domain, intercepts, asymptotes and even/odd symmetries.

2. Find points where $f'(x) = 0$ (critical points of f) and where $f'(x)$ is undefined (singular points of f).

3. Find points where $f''(x) = 0$ (critical points of f) and where $f''(x)$ is undefined (singular points of f).

4. Make a table to investigate the signs of $f'(x)$ and $f''(x)$ to find the intervals where f is increasing or decreasing and the intervals where f is concave up and down. Find also the extreme points and inflection points of the graph.

5.6 Extreme Value Problems

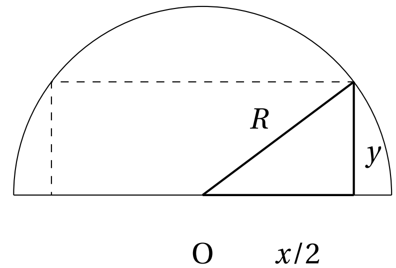
Example. *Find the area of the largest rectangle that can be inscribed in a semicircle of radius R if one side of the rectangle lies along the diameter of the semicircle.*

Solution. $(x/2)^2 + y^2 = R^2$. *So*

$$A = xy = x\sqrt{R^2 - (x/2)^2}.$$

$$\frac{dA}{dx} = \frac{2R^2 - x^2}{\sqrt{4R^2 - x^2}}$$

The derivative is zero when $x = \sqrt{2}R$. *Use the first derivative test to see that this gives max area* $A = R^2$.



Example. *Find the shortest distance from the origin to the curve* $x^2y^4 = 1$.

Example. *A manufacturer has 100 tons of metal that he can sell now with a profit of $5 a ton. For each week that he delays shipment, he can produce another 10 tons of metal. However, for each week he waits, the profit drops 25 cents a ton. If he can sell the metal at any time, when is the best time to sell so that his profit is maximized?*

Solution. *Let x be the number of weeks to wait.*

Ship	Amount of metal	Profit per ton	Total profit
now	100	5	500
in x weeks	$100 + 10x$	$5 - 0.25x$	$500 + 25x - 0.25x^2$

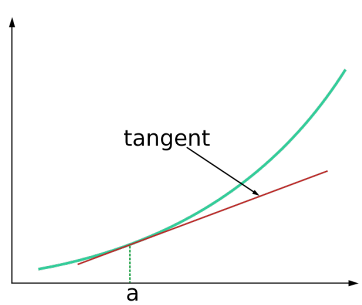
$P(x) = 500 + 25x - 2.5x^2$. *Solve* $P'(x) = 0$ *to find* $x = 5$. *And maximum profit is* $562.50.

Example. *Among all rectangles of perimeter P, show that the square has the greatest area.*

Example. *Among all rectangles of given area A, show that the square has the least perimeter.*

5.7 Linear Approximation

The best line approximating the graph of $y = f(x)$ near $(a, f(a))$ is the tangent line through $(a, f(a))$.



The linearization of the function f about a is the function L defined by

$$L(x) = f(a) + f'(a)(x - a)$$

We say that L approximates f near $x = a$ and write $f(x) \approx L(x)$.

Example. *Using the linearization, approximate* $\sqrt{26}$. *(Hint: use the linearization of* $\sqrt{x}$ *at* $x = 25$.*)*

Solution. $f'(x) = \frac{1}{2\sqrt{x}}$. $f'(25) = \frac{1}{10}$. *So*

$$L(x) = 5 + \frac{1}{10}(x - 25).$$

Hence $f(26) \approx f(26) = 5.1$.

Example. *Approximate* $\cos\pi/5 = \cos 36^{\text{deg}}$ *using the linearization of* $\cos x$ *at* $x = \pi/6$.

Solution. $L(x) = \cos\frac{\pi}{6} - \sin\frac{\pi}{6}(x - \frac{\pi}{6}) = \frac{\sqrt{3}}{2} - \frac{1}{2}(x - \frac{\pi}{6})$.

$$\cos 36^\circ \approx L(\pi/5) = \frac{\sqrt{3}}{2} - \frac{1}{2}\frac{\pi}{30} \approx 0.81367$$

Error Estimation

The error in the linear approximation is

$$\frac{f''(s)}{2}(x-a)^2$$

where s is some number between a and x. (The proof depends on the generalized mean value theorem.)

Since we do not know s, we have to choose $f''(s)$ to be largest (in absolute value) possible value, to get the maximum error.

So for the previous example, $f''(x) = -\sin x$, $a = \pi/6$, $x = \pi/5$, $\pi/6 < s < \pi/5$. Note that $f''(s) \le 1$. So the error is smaller that $\frac{1}{2}(x-a)^2 = \frac{\pi^2}{1800} < 0.00549$. So $0.81367 - 0.00549 < \cos\pi/5 < 0.81367 + 0.00549$.

5.8 Exam 2 Review

Section	Exercises
3.1	27-29
3.3	11-16, 19-48, 55-66
3.5	1-12, 19-32
4.1	1-15
4.3	1-24
4.4	1-17, 18-39
4.5	1-22
4.6	7-39
4.7	1-32
4.9	1-10, 15-22

Table 5.1: Exam 2 Review Problems from Adams & Essex Calculus: A Complete Course 7th Edition

Sample Exam 2

1. Find the dimensions of the right triangle with hypotenuse $h = 2$ and maximum area.

2. Find the min and max values of $f(x) = 2x^3 - 15x^2 + 24x + 19$ on the interval $[0, 5]$

3. Let $f(x) = x^4 - 3x^2 + 2$.

a) Find the intervals on which f is increasing or decreasing. Find all local extrema for f.

b) Find the intervals on which f is concave up or down. Find all inflection points for f.

c) Sketch a graph of $f(x)$ using parts (a) and (b).

4. Find all the horizontal and vertical asymptotes for

$$y = \frac{x^2+3}{1-3x^2}.$$

5. Let $f(x) = 2x^2 + x^3, x > 0$. Show that f is invertible and find $(f^{-1})'(16)$.

6. Find dy/dx by implicit differentiation if

$$y^2 e^x + y \ln x = 2.$$

7. Let $y = (1/x)^{\ln x}$. Find dy/dx.

8. Find $\cos(\sin^{-1} 0.7)$.

Chapter 6

Integration

6.1 The Definite Integral

Our main goal in this section is to find the area between the graph of a function and the x-axis.

Idea is to approximate this region with rectangles.

Let's start with an easy example.

Example. *Find the area of the region lying under the straight line $y = x+1$, above the x-axis and between the lines $x = 0$ and $x = 2$.*

Solution. *Two ways of approximating the area. With "smaller" rectangles and "larger" rectangles.*

With "smaller" rectangles. Divide the interval [0,2] into n equal pieces, call $x_0 = 0$, $x_1 = 2/n$, $x_2 = 4/n$, $\dots$, $x_n = 2$.

$$L_n = f(x_0)(x_1 - x_0) + f(x_1)(x_2 - x_1) + \cdots + f(x_n - 1)(x_n - x_{n-1})$$

For $n = 1$, $L_1 = f(0)(2-0) = 2$ For $n = 2$, $L_2 = f(0)(1-0) + f(1)(2-1) = 1+2 = 3$, For $n = 3$, $L_4 = (f(0) + f(1/2) + f(1) + f(3/2))(1/2) = 7/2$

We can show that

$$L_n = \frac{2(2n-1)}{n}$$

Thus $\lim_{n\to\infty} L_n = 4$. This is in fact the exact area.

We can repeat this with "larger" rectangles. In this case

$$U_n = \frac{2(2n+1)}{n}$$

And again, $\lim_{n\to\infty} U_n = 4$.

This procedure can be used to find the areas under more exotic curves.

In general, if we want to calculate the area between the graph of $y = f(x)$, the x-axis, $a \le x \le b$. We partition the interval $a = x_0 < x_1 < \cdots < x_n = b$. Let $P = \{x_0, x_1, \dots, x_n\}$.

Definition 19. *Let $f(l_i)$ be the smallest value and $f(u_i)$ be the largest value of $f(x)$ on $[x_i, x_{i+1}]$. Then we define the **lower Riemann sum***

$$L(f,P) = f(l_0)(x_1 - x_0) + f(l_1)(x_2 - x_1) + \cdots f(l_{n-1})(x_n - x_{n-1})$$

and the ***upper Riemann sum***

$$U(f,P) = f(u_0)(x_1 - x_0) + f(u_1)(x_2 - x_1) + \cdots f(u_{n-1})(x_n - x_{n-1})$$

Suppose that there exists exactly one number I such that for every partition P of $[a,b]$,

$$L(f,P) \le I \le U(f,P)$$

Then we say f is ***integrable*** *on $[a,b]$ and we call I, the definite integral of f on $[a,b]$ and write*

$$I = \int_a^b f(x)dx.$$

Let R be the region bounded by the graph of $f(x)$, the x-axis and the lines $x = a$ and $x = b$. If $f(x) \ge 0$ on $[a,b]$ then

$$\text{Area(R)} = \int_a^b f(x)dx$$

If $f(x) \le 0$ on $[a,b]$ then

$$\text{Area(R)} = -\int_a^b f(x)dx$$

In general $\int_a^b f(x)$ is the area of the part of R lying above the x-axis minus the area of the part below the x-axis.

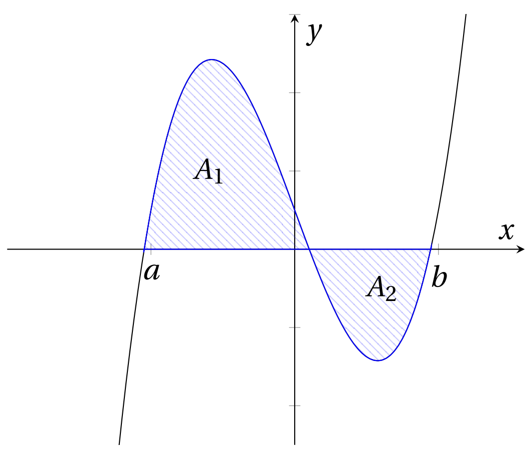


Figure 6.1: $\int_a^b f(x)dx = A_1 - A_2$

Here the variable x is a dummy variable. Replacing x by any other symbol does not change value of the integral. The function f is known as **integrand**. dx is differential of x and if an integrand depends on more than one variable it tells which one is the variable of integration.

In the first example, we showed that

$$\int_0^2 (x+1)dx = 4.$$

Which functions are integrable?

Theorem 36. *If f is continuous on $[a,b]$ then f is integrable.*

Piecewise continuous functions are also integrable.

Properties of Definite Integral

The following properties are easy consequences of the definition of definite integral.

Theorem 37. *Let f and g be integrable on an interval containing the points a, b and c.*

1. $\int_a^a f(x)dx = 0$.

2. *We can define $\int_a^b f(x)dx$ when $a > b$. In this case $x_0 = a > x_1 > \cdots x_n = b$. It is easy to see that*
$$\int_a^b f(x)dx = -\int_b^a f(x)dx.$$

3. *If A and B are constants then*
$$\int_a^b (Af(x) + Bg(x))dx = A\int_a^b f(x)dx + B\int_a^b g(x)dx.$$

4. $$\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx.$$

5. *If $a \le b$ and $f(x) \le g(x)$ then*
$$\int_a^b f(x)dx \le \int_a^b g(x)dx$$

6. *If f is an odd function then*
$$\int_{-a}^a f(x)dx = 0$$

7. *If f is an even function then*
$$\int_{-a}^a f(x)dx = \int_0^a f(x)dx$$

Example. *Show that $\int_a^b c dx = c(b-a)$ and $\int_a^b x dx = \frac{(b^2-a^2)}{2}$ interpreting the integrals as areas.*

Example. *Using the properties of the integral, compute*
$$\int_{-2}^2 (3+5x)dx$$

Example. *Compute $\int_{-3}^3 \sqrt{9-x^2}$.*

Solution. *This is the area of the semicircle with radius 3 and center $(0,0)$. The answer is $\frac{9\pi}{2}$.*

6.2 The Fundamental Theorem of Calculus

In this section we develop the relation between the integral and the derivative.

Antiderivative

We will call $F(x)$ as an antiderivative of $f(x)$ if $F'(x) = f(x)$. For example x is an antiderivative of 1. Note that $x+1$ is also an antiderivative of 1. So antiderivatives are not unique.

If F and G are antiderivatives of f on an interval, so that $F'(x) = G'(x) = f(x)$ then

$$\frac{d}{dx}(F(x) - G(x)) = 0.$$

But Theorem 27 tells that $F(x) - G(x)$ must be a constant. Hence if $F(x)$ is an antiderivative of $f(x)$ then for any C, $F(x) + C$ is also an antiderivative of $f(x)$. Also any antiderivative of $f(x)$ is of the form $F(x) + C$ for some c.

Definition 20. *The **indefinite integral** of $f(x)$ on interval I is*

$$\int f(x)dx = F(x) + C$$

provided $F'(x) = f(x)$ on I.

The Fundamental Theorem of Calculus

Theorem 38.

PART I. *Suppose f is continuous.*

$$\frac{d}{dx}\int_a^x f(t)dt = f(x)$$

PART II. *Suppose f is differentiable.*

$$\int_a^b f'(x)dx = f(b) - f(a)$$

Proof. For the first part, let

$$F(x) = \int_a^x f(x)dx.$$

Then

$$F'(x) = \lim_{h\to 0}\frac{F(x+h) - F(x)}{h} = \lim_{h\to 0}\frac{1}{h}\left(\int_a^{x+h} f(t)dt - \int_a^x f(t)dt\right) = \lim_{h\to 0}\frac{1}{h}\int_x^{x+h} f(t)dt$$

Let $m(h)$ be the minimum, $M(h)$ be the maximum of f on $[x, x+h]$. Then $m(h) \le f(t) \le M(h)$ on $x \le t \le x+h$. Thus

$$m(h)h = \int_x^{x+h} m(h)dx \le \int_x^{x+h} f(t)dt \le \int_x^{x+h} M(h)dx = M(h)h.$$

Or

$$m(h) \le \frac{1}{h}\int_x^{x+h} f(t)dt \le M(h)$$

Since $\lim_{h\to 0} m(h) = \lim_{h\to 0} M(h) = f(x)$, by Sandwich Theorem,

$$F'(x) = \lim_{h\to 0}\frac{1}{h}\int_x^{x+h} f(t)dt = f(x).$$

Proof of the second part. Let

$$F(x) = \int_a^x f'(t)\,dt.$$

Then by part I, $F'(x) = f'(x)$. We have seen that the only function whose derivative is zero on an interval is the constant function. Thus $F'(x) - f'(x) = 0$. Hence $F(x) - f(x) = c$, a constant. Since $0 = F(a)$, $c = -f(a)$. And

$$\int_a^b f'(t)\,dt = F(b) = f(b) + c = f(b) - f(a).$$

□

Second part gives a method to evaluate definite integrals. To compute $\int_a^b f(x)\,dx$, find a function $F(x)$ whose derivative is $f(x)$. Then the value of $\int_a^b f(x)\,dx = F(b) - F(a)$.

We will use the evaluation symbol

$$F(x)\mid_a^b = F(b) - F(a).$$

Example. *Evaluate*

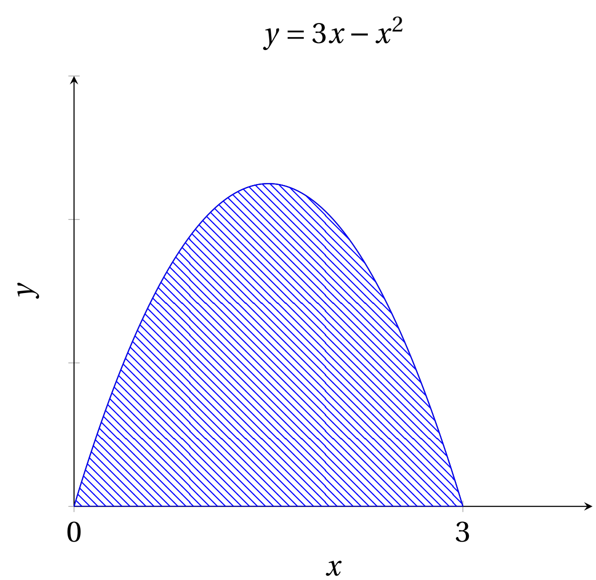
1. $\displaystyle\int_0^a x^2\,dx = \frac{a^3}{3}$

2. $\displaystyle\int_{-1}^2 (x^2 - 3x + 2)\,dx = \frac{9}{2}$

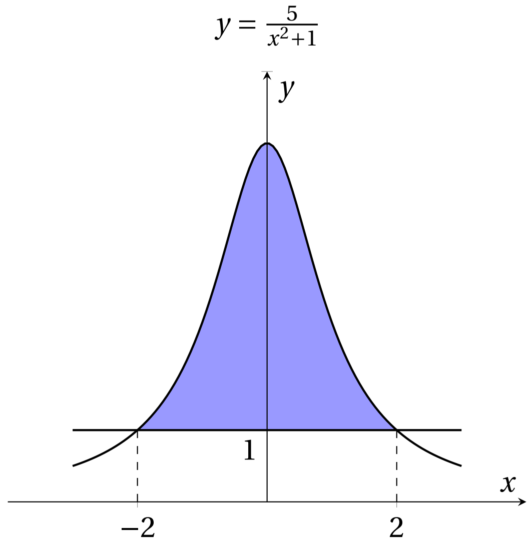
Example. *Find the area of the region lying above the x-axis and under the curve $y = 3x - x^2$.*

Solution. *The points where the graph intersects the x-axis are $y = 0$ which gives $x = 0$, $x =$.*
The area is

$$\int_0^3 (3x - x^2)\,dx = \left(\frac{3}{2}x^2 - \frac{1}{3}x^3\right)\Bigg|_0^3 = \frac{9}{2}.$$



Example. *Find the are under of the region lying above the line $y = 1$ and below the curve $y = \frac{5}{x^2+1}$.*



Solution. *The curves $y = 1$ and $y = \frac{5}{x^2+1}$ intersect at $x = \pm 2$. The area*
The area is

$$\int_{-2}^{2} \frac{5}{x^2+1}\,dx - \int_{-2}^{2} 1\,dx = 5\tan^{-1} x\,|_0^2 - 4 = 10\tan$$

Example. *Find the derivatives of the following functions.*

1. $F(x) = \displaystyle\int_x^3 e^{-t^2}\,dt$

2. $G(x) = \displaystyle\int_{-4}^{5x} e^{-t^2}\,dt$

3. $H(x) = \displaystyle\int_{x^2}^{x^3} e^{-t^2}\,dt$

Solution. *By the Fundamental Theorem of Calculus Part I,*

$$F(x) = -\int_3^x e^{-t^2} \implies F'(x) = -e^{x^2}.$$

Let $g(x) = \int_{-4}^x e^{-t^2}\,dt$. Then $G(x) = g(5x)$ and

$$G'(x) = g'(5x)5 = 5e^{-(5x)^2}$$

$H(x) = \int_{x^2}^a e^{-t^2}\,dt + \int_a^{x^3} e^{-t^2}\,dt$. *Then*

$$H'(x) = e^{-x^6}3x^2 - e^{x^4}2x.$$

In general

$$\frac{d}{dx}\int_{f(x)}^{g(x)} h(t)\,dt = h(f(x))f'(x) - h(g(x))g'(x).$$

6.3 The Method of Substitution

The following should be memorized.

1. $\int x^n dx = \frac{1}{n+1}x^{n+1} + C$, if $n \neq 1$
2. $\int 1 dx = x + C$
3. $\int x dx = \frac{1}{2}x^2 + C$
4. $\int x^2 dx = \frac{1}{3}x^3 + C$
5. $\int \sqrt{x} dx = \frac{2}{3}x^{3/2} + C$
6. $\int \frac{1}{x} dx = \ln|x| + C$
7. $\int \sin x dx = -\cos x + C$
8. $\int \cos x dx = \sin x + C$

1. $\int \sec^2 x dx = \tan x + C$
2. $\int \csc^2 x dx = -\cot x + C$
3. $\int \sec x \tan x dx = \sec x + C$
4. $\int \csc x \cot x dx = -\csc x + C$
5. $\int \frac{1}{\sqrt{1-x^2}} dx = \arcsin x + C$
6. $\int \frac{1}{1+x^2} dx = \arctan x + C$
7. $\int e^x dx = e^x + C$
8. $\int a^x dx = \frac{1}{\ln a} a^x + C$

Example.

1. $\int (x^3 - 3x^2 + 6x - 9) dx = \frac{x^4}{4} - x^3 + 3x^2 - 9x + C$

2. $\int (5x^{3/4} - \frac{1}{\sqrt{x}}) dx$

3. $\int \frac{(x+1)^3}{x} dx$

The Chain Rule says

$$\frac{d}{dx} f(g(x)) = f'(g(x))g'(x).$$

So we have,

$$\int f'(g(x))g'(x) dx = f(g(x)) + C$$

To see this another way, let $u = g(x)$. Then $du/dx = g'(x)$. In differential form $du = g'(x)dx$

$$\int f'(g(x))g'(x) dx = \int f'(u) du = f(u) + C = f(g(x)) + C$$

Example. *Compute the following integrals.*

1. $I = \int x \sin(2x^2) dx.$

Let $2x^2 = u$ *then* $4x dx = du$.

$$I = \frac{1}{4}\int \sin u du = -\frac{\cos u}{4} + C = -\frac{\cos 2x^2}{4} + C$$

2. $I = \int \sec^2(3x+2)dx$

   Let $3x+2 = u$ *then* $3dx = du$.

$$I = \int \sec^2 u \frac{du}{3} = \frac{\tan u}{3} + C = \frac{1}{3}\tan(3x+2) + C$$

3. $I = \int \frac{x}{(x-4)^3}dx$

   Let $x-4 = u$.

$$I = \int \frac{u+4}{u^3}du = \int (u^{-2} + 4u^{-3})du = -u^{-1} - 2u^{-2} = \frac{-1}{x-4} - \frac{2}{(x-4)^2} + C$$

4. $I = \int \tan^2\theta \sec^2\theta d\theta$.

   Let $\tan\theta = u$. *Then* $\sec^2 d\theta = du$.

$$I = \int u^2 du = \frac{u^3}{3} + C = \frac{\tan^3\theta}{3} + C$$

5. $I = \int \sqrt{\frac{x^4}{x^3-1}}dx = \int \frac{x^2}{\sqrt{x^3-1}}dx$.

   Let $x^3 - 1 = u$. *Then* $x^2 dx = \frac{du}{3}$.

$$I = \int \frac{du/3}{\sqrt{u}} = \frac{1}{3}\frac{u^{1/2}}{1/2} + C = \frac{2}{3}\sqrt{x^3-1} + C$$

6. *Let* $y = x\int_2^{x^2} \sin(t^3)dt$. *Find* y'

$$y' = \int_2^{x^2} \sin(t^3)dt + x\sin(x^6)2x$$

7. $I = \int \sec x dx$.

   There is an interesting trick to evaluate this integral!

$$I = \int \sec x \frac{(\sec x + \tan x)}{\sec x + \tan x}dx = \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}dx$$

   Let $u = \sec x + \tan x$, *then*

$$I = \int \frac{du}{u} = \ln|u| + C = \ln|\sec x + \tan x| + C.$$

6.4 Areas of Plane Regions

Suppose $f(x) \le g(x)$ for $a \le x \le b$. Then the area of the region between these two curves and the lines $x = a$ and $x = b$ is

$$\text{Area} = \int_a^b (g(x) - f(x))\,dx.$$

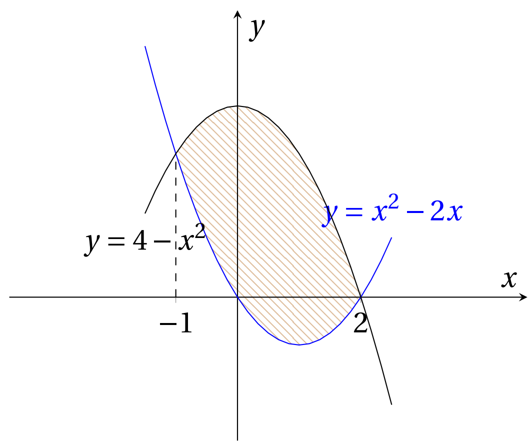
Example. *Find the area of the bounded region lying between the curves $y = x^2 - 2x$ and $y = 4 - x^2$.*

Solution. *The two curves intersect at*

$$x^2 - 2x = 4 - x^2 \implies 2x^2 - 2x - 4 = 0 \implies (x-2)(x+1) = 0$$

So the intersection points are $x = -1$ and $x = 2$. The area of the region is

$$Area = \int_{-1}^{2} \left((4 - x^2) - (x^2 - 2x)\right) dx = 9$$



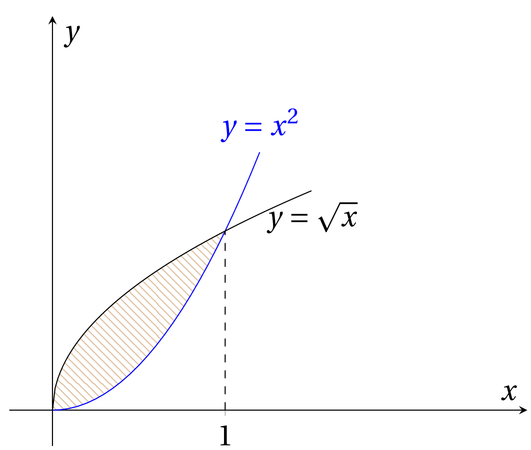
Example. *Find the area of the region bounded by $y = \sqrt{x}$ and $y = x^2$.*

Solution. *The curves intersect at*

$$\sqrt{x} = x^2 \implies x = x^4 \implies x(1 - x^3) = 0.$$

Hence the intersection points are $x = 0$ and $x = 1$.

$$Area = \int_0^1 \left(\sqrt{x} - x^2\right) dx = \frac{2}{3} - \frac{1}{3} = \frac{1}{3}.$$



Example. *Find the area of the region lying to the right of the parabola $x = y^2 - 12$ and to the left of the straight line $y = x$.*

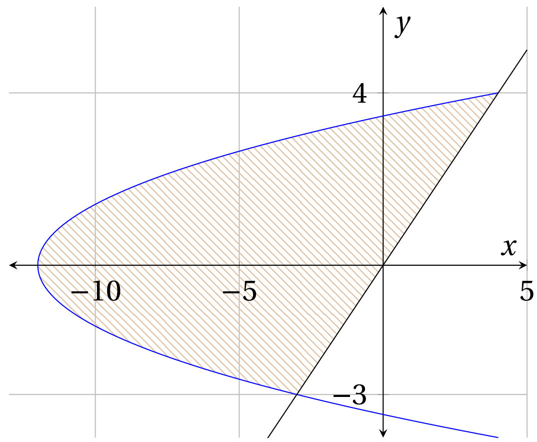
Solution. *The curves intersect at*

$$y^2 - 12 = y \implies (y-4)(y+3) = 0$$

The intersection points are $y = 4$ and $y = -3$.

$$Area = \int_{-3}^{4} \left(y - (y^2 - 12)\right) dy = \frac{343}{6}$$

An alternative way is to make the transformation $x \to y$, $y \to x$. The problem becomes finding the area between $y = x^2 - 12$ and $y = x$.



6.5 Integration by Parts

Integrating both sides of

$$\frac{d}{dx}(u(x)v(x)) = \frac{du}{dx}v + u\frac{dv}{dx}$$

we get

$$\int \frac{d}{dx}(u(x)v(x))dx = \int \frac{du}{dx}v dx + \int u\frac{dv}{dx}dx$$

Since $\int \frac{d}{dx}(u(x)v(x))dx = u(x)v(x)$, in differential notation, we get

$$uv = \int du\, v + \int u\, dv$$

Another way to write this is

$$\int u\, dv = uv - \int v\, du$$

This is one of the most powerful method to integrate, known as the **integration by parts**.

Example. $\displaystyle\int xe^x dx.$

Solution. *Let $u = x$ and $dv = e^x dx$. Then $du = dx$ and $v = e^x$.*

$$\int xe^x dx = xe^x - \int e^x dx = xe^x - e^x + C$$

Example. $\displaystyle\int \ln x dx.$

Solution. *Let $u = \ln x$ and $dv = dx$. Then $du = dx/x$ and $v = x$.*

$$\int \ln x dx = x\ln x - \int x\frac{dx}{x} = x\ln x - x + C$$

Example. $I = \displaystyle\int x^2 \sin x dx$

Solution. *We have to integrate by parts twice. Let* $u = x^2$ *and* $dv = \sin x dx$. *Then* $du = 2x dx$ *and* $v = -\cos x$.

$$I = x^2(-\cos x) - \int (-\cos x) 2x dx = -x^2 \cos x + \int 2x \cos x dx$$

Now let $u = 2x$ *and* $dv = \cos x dx$. *Then* $du = 2dx$ *and* $v = \sin x$. *And*

$$\int 2x \cos x dx = 2x \sin x - \int 2 \sin x dx$$

Hence

$$I = -x^2 \cos x + 2x \sin x + 2\cos x + C$$

Example. $I = \int x \tan^{-1} x dx$

Solution. *Let* $u = \tan^{-1} x, dv = x dx$. *Then* $du = dx/(1+x^2)$ *and* $v = x^2/2$.

$$I = \frac{1}{2}x^2 \tan^{-1} x - \frac{1}{2}\int \frac{x^2}{1+x^2} dx = \frac{1}{2}x^2 \tan^{-1} x - \frac{1}{2}\int \left(1 - \frac{1}{1+x^2}\right) dx$$

And

$$I = \frac{1}{2}x^2 \tan^{-1} x - \frac{1}{2}(x - \tan^{-1} x) + C$$

Example. *Find* $I = \int e^x \sin x dx$.

Solution. *There is a circular argument here. We will integrate by parts twice to return the same integral. Let* $u = \sin x$ *and* $dv = e^x dx$. *Then* $du = \cos x dx$, $v = e^x$.

$$\int e^x \sin x dx = e^x \sin x - \int \cos x e^x dx$$

Now let $u = \cos x$ *and* $dv = e^x dx$.

$$\int \cos x e^x dx = \cos x e^x - \int (-\sin x) e^x = \cos x e^x + I$$

So

$$I = e^x \sin x - e^x \cos x - I$$

Hence

$$2I = e^x(\sin x - \cos x) + C \implies I = \frac{e^x}{2}(\sin x - \cos x) + C.$$

Example. $I = \int \sec^3 x dx$.

Solution. *Let* $u = \sec x$ *and* $dv = \sec^2 x dx$. *Then* $du = \sec x \tan x dx$ *and* $v = \tan x$

$$I = \sec x \tan x - \int \sec x \tan^2 x dx$$

Using $\tan^2 x = \sec^2 x - 1$,

$$I = \sec x \tan x + \int \sec x dx - I$$

Using $\int \sec x dx = \ln|\sec x + \tan x|$, *(see the section on "The Method of Substitution") we get*

$$I = \frac{1}{2}\sec x \tan x + \frac{1}{2}\ln|\sec x + \tan x| + C$$

6.6 Integrals of Rational Function

In this section we are concerned with integrals of the form

$$\int \frac{P(x)}{Q(x)} dx$$

where $P(x)$ and $Q(x)$ are both polynomials.

We will look at methods to deal with such integrals when $\deg(P(x)) < \deg(Q(x))$.

The case deg$(Q(x)) = 1$ and deg$(P(x)) = 0$

Example. $\int \frac{1}{ax+b} dx = \frac{1}{a} \ln ax + b + C.$

Solution. *Let $u = ax + b$ then $du = adx$ and the integral becomes $\frac{1}{a}\int \frac{du}{u}$.*

The case deg$(Q(x)) = 2$ and deg$(P(x)) = 0$

First let's look at two examples where $Q(x)$ does not have real roots.

Example. $\int \frac{dx}{x^2 + a^2} = \frac{1}{a} \tan^{-1} \frac{x}{a} + C.$

Solution. *Let $x = a \tan\theta$ (We will talk about these types of transformations in the next section in detail!), then $dx = a \sec^2\theta d\theta$ and $x^2 + a^2 = a^2(\sec^2\theta + 1) = a^2 \tan^2\theta$.*

If $Q(x) = ax^2 + bx + c$ has no real roots, we have to **complete to squares**.

Example. $\int \frac{dx}{x^2 + 3x + 3}$

Solution. *Notice that $x^2 + 3x + 3$ has no real roots. So we complete to squares*

$$x^2 + 3x + 3 = (x + \frac{3}{2})^2 + \frac{3}{4}$$

Letting $u = (x + 3/2)$ and $du = dx$,

$$\int \frac{dx}{(x + \frac{3}{2})^2 + \frac{3}{4}} = \int \frac{du}{u^2 + \frac{3}{4}} = \frac{2}{\sqrt{3}} \tan^{-1} \frac{2x}{\sqrt{3}} + C$$

The last part follows from the last example.

If $Q(x)$ has real roots then we use **partial fractions**.

Partial Fractions

Let us still assume that deg($P(x)$) <deg($Q(x)$) and that $Q(x)$ has the unique factorization

$$Q(x) = (x-a_1)^{m_1}(x-a_2)^{m_2}\cdots(x-a_j)^{m_j}(x^2+b_1x+c_1)^{n_1}\cdots(x^2+b_kx+c_k)^{n_k}$$

into real linear factors $(x-a_i)$ and real quadratic factors $x^2+b_ix+c_i$ having no real roots.

To each factor of the form $(x-a)^m$, the partial fraction decomposition contains a sum

$$\frac{A_1}{(x-a)}+\frac{A_2}{(x-a)^2}+\cdots+\frac{A_m}{(x-a)^m}$$

To each factor of the form $(x^2+bx+c)^n$, the partial fraction decomposition contains a sum

$$\frac{B_1x+C_1}{(x^2+bx+c)}+\frac{B_2x+C_2}{(x^2+bx+c)^2}+\cdots+\frac{B_nx+C_n}{(x^2+bx+c)^n}$$

Example. $\displaystyle\int\frac{(x+4)}{x^2-5x+6}dx$

Solution.

$$\frac{x+4}{x^2-5x+6}=\frac{A}{x-2}+\frac{B}{x-3}$$
$$x+4=A(x-3)+B(x-2)$$

Plugging $x=2$ gives $A=-6$ and plugging $x=3$ gives $B=7$. So

$$\int\frac{(x+4)}{x^2-5x+6}dx=-6\int\frac{dx}{x-2}+7\int\frac{dx}{x-3}=-6\ln(x-2)+7\ln(x-3)+C$$

Example. $\displaystyle\int\frac{2+3x+x^2}{x(x^2+1)}dx.$

Solution. *The partial fraction decomposition is*

$$\frac{2+3x+x^2}{x(x^2+1)}=\frac{A}{x}+\frac{Bx+C}{x^2+1}\implies A(x^2+1)+x(Bx+C)=2+3x+x^2$$

Since this equation holds for every x, we have $A+B=1$ (coefficient of x^2 term), $C=3$ (coefficient of x term) and $A=2$ (coefficient of constant term). We find $B=-1$.

$$\int\frac{2+3x+x^2}{x(x^2+1)}dx=\int\frac{2}{x}dx+\int\frac{-x+3}{x^2+1}dx=2\ln x-\frac{1}{2}\ln(x^2+1)+3\tan^{-1}x+C.$$

Example. *Evaluate* $\displaystyle\int\frac{1}{x(x-1)^2}dx.$

Solution.

$$\frac{1}{x(x-1)^2}=\frac{A}{x}+\frac{B}{(x-1)}+\frac{C}{(x-1)^2}$$
$$1=A(x-1)^2+Bx(x-1)+Cx$$

Letting $x=0$ gives $A=1$, $x=1$ gives $C=1$. To find B, notice that the coefficient of x^2 is $A+B$ which must be zero. So $B=-1$.

$$\int\frac{1}{x(x-1)^2}dx=\int\frac{1}{x}dx-\int\frac{1}{x-1}dx+\int\frac{1}{(x-1)^2}dx=\ln|x|-\ln|x-1|-\frac{1}{x-1}+C.$$

The last integral can be found by letting $u=x-1$.

The Case $\mathbf{deg}(P(x)) \geq \mathbf{deg}(Q(x))$

If $\deg(P(x)) \geq \deg(Q(x))$ then we divide $P(x)$ to $Q(x)$ so that we get a rational function with the degree of numerator less than the degree of denominator.

Example. *Evaluate* $\displaystyle\int \frac{x^3+3x^2}{x^2+1}dx$

Solution.

$$\frac{x^3+3x^2}{x^2+1} = x+3-\frac{x+3}{x^2+1}$$

$$\int \frac{x+3}{x^2+1}dx = \int \frac{x}{x^2+1}dx + \int \frac{3}{x^2+1}dx$$

$$\frac{x^3+3x^2}{x^2+1} = \frac{x^2}{2}+3x-\frac{1}{2}\ln(x^2+1)-3\tan^{-1}x+C$$

6.7 Inverse Substitutions

The Inverse Sine Substitution

If an integral involves $\sqrt{a^2-x^2}$, try the substitution $x = a\sin\theta$ or $\theta = \sin^{-1}\frac{x}{a}$.

We can assume $a > 0$. Notice that $\sqrt{a^2-x^2}$ makes sense only when $-a \leq x \leq a$ which corresponds to $-\pi/2 \leq \theta \leq \pi/2$ so that $\cos\theta \geq 0$. Hence

$$\sqrt{a^2-x^2} = \sqrt{a^2(1-\sin^2\theta)} = a\sqrt{\cos^2\theta} = a|\cos\theta| = a\cos\theta.$$

Example. *Evaluate* $I = \displaystyle\int \frac{dx}{(5-x^2)^{3/2}}$.

Solution. *Let* $x = \sqrt{5}\sin\theta$, $dx = \sqrt{5}\cos\theta d\theta$.

$$(5-x^2)^{3/2} = (5-5\sin^2\theta)^{3/2} = 5^{3/2}|\cos\theta|^3 = 5^{3/2}\cos^3\theta.$$

since $\cos\theta \geq 0$. *So*

$$I = \int \frac{\sqrt{5}\cos\theta d\theta}{5^{3/2}\cos^3\theta} = \frac{1}{5}\int \sec^2\theta d\theta = \frac{1}{5}\tan\theta + C = \frac{1}{5}\frac{x}{\sqrt{5-x^2}} + C$$

The last equality can be found using $\sin\theta = \frac{x}{\sqrt{5}}$.

The inverse Tangent Substitution

If an integral involves $\sqrt{a^2+x^2}$ or $\dfrac{1}{x^2+a^2}$, try the substitution $x = a\tan\theta$ or $\theta = \tan^{-1}\frac{x}{a}$.

Since x can take any real value, we have $-\pi/2 < \theta < \pi/2$ so that $\sec\theta > 0$. Assuming $a > 0$,

$$\sqrt{a^2+x^2} = \sqrt{a^2(1+\tan^2\theta)} = a\sqrt{\sec^2\theta} = a|\sec\theta| = a\sec\theta.$$

Example. *Evaluate* $I = \displaystyle\int \frac{dx}{\sqrt{4+x^2}}$.

Solution. *Let* $x = 2\tan\theta$, $dx = 2\sec^2\theta d\theta$.

$$\sqrt{4+x^2} = 2\sqrt{\sec^2\theta} = 2|\sec\theta| = 2\sec\theta$$

since $\sec\theta > 0$. *Using* $\tan\theta = x/2$ *we can find* $\sec\theta = \frac{\sqrt{4+x^2}}{2}$ *and*

$$I = \int \sec\theta d\theta = \ln|\sec\theta + \tan\theta| + C = \ln\left|\frac{\sqrt{4+x^2}}{2} + \frac{x}{2}\right| + C$$

You are not responsible for the inverse secant transformation which can be used to solve integrals involving $\sqrt{x^2 - a^2}$.