Calculus I Lecture Notes

Taylan Şengül

November 2, 2015

Contents

Contents **1**

1 Precalculus **3**
- 1.1 Sets . . . 3
- 1.2 Real Numbers . . . 4
- 1.3 Cartesian Coordinates . . . 6
- 1.4 Quadratic Equations . . . 8
- 1.5 Functions and Their Graphs . . . 9
- 1.6 Operations on Functions . . . 12
- 1.7 Polynomials and Rational Functions . . . 13

2 Limits and Continuity **15**
- 2.1 Informal definition of limits . . . 15
- 2.2 Limits at Infinity and Infinite Limits . . . 18
- 2.3 Continuity . . . 22
- 2.4 Formal definition of Limit . . . 27
- 2.5 Review Problems . . . 30

3 Differentiation **31**
- 3.1 Tangent Lines and Their Slopes . . . 31
- 3.2 Derivative . . . 33
- 3.3 Differentiation Rules . . . 34
- 3.4 Chain Rule . . . 36

3.5 Derivatives of Trigonometric Functions 38
3.6 Higher Order Derivatives 40
3.7 Mean Value Theorem 40

Chapter 1

Precalculus

1.1 Sets

A **set** is a collection of elements.

$x \in A$ means x is an element of the set A. If x is not a member of A, we write $x \notin A$.

$\varnothing$ is the set which contains no element and is called the **empty set**.

There are finite sets such as $\{0,1,2\}$ and infinite sets such as $\{0,1,2,3,\ldots\}$.

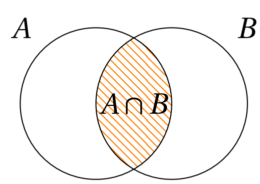
If every element of the set A is an element of the set B, we say that A is **subset** of B, and write $A \subset B$.

Example 1. *List all the subsets of* $\{0,1,2\}$.

For any set A, $A \subset A$ and $\varnothing \subset A$.

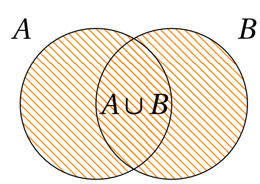
If $A \subset B$ and $B \subset A$, we write $A = B$.

$A \cap B = \{x : x \in A \text{ and } x \in B\}$ is called the **intersection** of A and B.



If the intersection of two sets is the empty set, those sets are called **disjoint**.

$A \cup B = \{x : x \in A \text{ or } x \in B\}$ is called the **union** of A and B.



Example 2. *For example if* $A = \{0,1,2,5,8\}$ *and* $B = \{1,3,5,6\}$ *then find* $A \cap B$ *and* $A \cup B$.

The set of all elements in A but not in B is denoted $A \setminus B = \{x \in A : x \notin B\}$ and is called the **complement** of B in A.

Example 3. $\{0,2,3,5\} \setminus \{2,5,7,8\} = \{0,3\}$

$A \times B = \{(a,b) : a \in A$ and $b \in B\}$ is called the **Cartesian** product of the sets A and B.

Example 4. *Write the cartesian product of* $A = \{0,1,2\}$ *and* $B = \{2,3,4\}$.

1.2 Real Numbers

The **integers** are $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$. Some important subsets of the set of integers are:

- even integers that are of the form $2k$, for some $k \in \mathbb{Z}$,
- odd integers that are of the form $2k+1$, for some $k \in \mathbb{Z}$
- positive and negative integers,
- primes, etc...

The **rational numbers** are $\mathbb{Q} = \left\{\frac{m}{n} : m, n \in \mathbb{Z} \text{ and } n \neq 0\right\}$.
Pythagoreans thought that all numbers are ratios of integers. The discovery of irrational numbers is said to have shocked them.

Example 5. $\sqrt{2}$ *is not a rational number.*
Suppose that it is rational. Then $\sqrt{2} = m/n$, *where* $m, n \in \mathbb{Z}$ *and* $n \neq 0$. *Also assume* m *and* n *have no common divisor.*

$$m^2/n^2 = 2 \implies m^2 = 2n^2$$

Thus m *is even and we can write* $m = 2k$, *where* $k \in \mathbb{Z}$.

$$4k^2 = 2n^2 \implies n^2 = 2k^2$$

Thus n *is also even. But* m *and* n *cannot both be even. Accordingly, there can be no rational number whose square is 2.*

The set of irrational numbers is denoted by $\mathbb{I}$.
The set of real numbers is $\mathbb{R} = \mathbb{Q} \cup \mathbb{I}$.
Note that $\mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$.
The real numbers are ordered such that

1. $a < b \implies a + c < b + c$
2. $a < b$ and $c > 0$ implies $ac < bc$
3. $a < b$ and $c < 0$ implies $ac > bc$
4. $a > 0$ implies $\frac{1}{a} > 0$
5. $0 < a < b$ implies $\frac{1}{b} < \frac{1}{a}$

Intervals

The open interval $(a,b) = \{x \mid a < x < b\}$, closed interval ($[a,b]$), half open intervals $(a,b]$, $[a,b)$. It is possible that $a = -\infty$, $b = \infty$. Draw each interval on the real line.

Example 6. *Solve the following inequalities.*

1. $\frac{2}{x-1} \geq 5$.

   Solution. *It is not right to multiply both sides by $x-1$ and say $5x-5 \leq 2$.*

$$\frac{2}{x-1} \geq 5 \iff \frac{2}{x-1} - 5 \geq 0 \iff \frac{7-5x}{x-1} \geq 0.$$

   Now make a sign analysis to get interval $(1, 7/5]$

2. $3x - 1 \leq 5x + 3 \leq 2x + 15$.

   Solution. $-2 \leq x$ *and* $x \leq 4$.

The absolute value.

$$|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$$

ex. $|3| = |-3| = 3$

Geometrically, $|x|$ is the distance between x and 0 on the real line. And $|x-y|$ is the distance between x and y.

Properties *(can be proved from definition)*:

1. $|-x| = |x|$, (Do not fall into the trap $|-x| = x$, this is not always true!)
2. $|ab| = |a||b|$,
3. $|a+b| \leq |a| + |b|$, (triangle inequality).

From (2), for any x, $x^2 = |x^2| = |x|^2$

If D is a nonnegative number

$$|x| = D \implies x = -D \text{ or } x = D,$$

$$|x| < D \implies -D < x < D$$

$$|x| > D \implies x < -D \text{ or } x > D$$

More generally,

$$|x-a| = D \implies x = a - D \text{ or } x = a + D,$$

$$|x-a| < D \implies a - D < x < a + D$$

$$|x-a| > D \implies x < a - D \text{ or } x > a + D$$

Example 7. *Solve* $|3x-2| \leq 1$.

Solution.

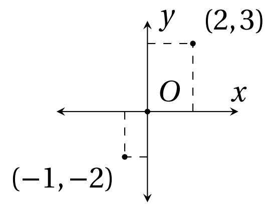
$$-1 \leq 3x - 2 \leq 1 \implies x \geq 1/3 \text{ and } x \leq 1.$$

Example 8. *Solve the equation* $|x+1| > |x-3|$.

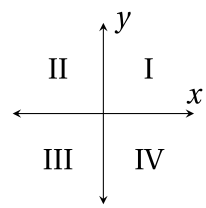
Solution. The distance between x and -1 is greater than the distance between x and 3. So $x > 1$.

1.3 Cartesian Coordinates

Cartesian plane is $\mathbb{R} \times \mathbb{R} = \{(x, y) \mid x \in \mathbb{R} \text{ and } b \in \mathbb{R}\}$. Horizontal axis is usually called the x axis, the vertical axis is called the y axis. Intersection of the axes is called the origin, denoted O.

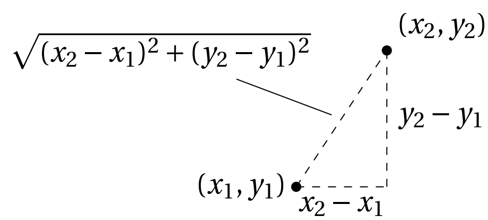


The coordinate axes divide the Cartesian plane into four quadrants.



By the Pythagorean Theorem, the distance between two points (x_1, y_1) and (x_2, y_2) is

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$



The distance of (x, y) to the origin $(0,0)$ is $\sqrt{x^2 + y^2}$.

Example 9. *Find the distance between* $(-1,1)$ *and* $(3,-4)$.

Graphs of Equations

The set of all points (x, y) satisfying an equation in x and y is called the **graph of that equation**.

Example 10. *The graph of* $x^2 + y^2 = 4$ *is the set of all* (x, y) *whose distance to* $(0,0)$ *is 2, i.e. a circle with center at origin and radius 2.*

Equations of Lines

For any two points (x_1, y_1) and (x_2, y_2) on a non-vertical line L, the quantity $m = \dfrac{y_2 - y_1}{x_2 - x_1}$ is constant and is called the **slope** of the line L.

Let L be a nonvertical line. Let m be the slope of L and (x_1, y_1) be the coordinates of a point on L. If (x, y) is another point on L, then

$$\frac{y - y_1}{x - x_1} = m$$

Hence any (x, y) on L satisfies

$$y = m(x - x_1) + y_1$$

The above is known as an equation for the line L.

All points on a **vertical line** have their x coordinate equal to a constant a. So the equation of a vertical line is $x = a$. **Horizontal lines** have equations of the form $y = a$.

y-intercept of a nonvertical line L is the y-coordinate of the point where L intersects the y-axis. **x-intercept** of a nonhorizontal is defined similarly.

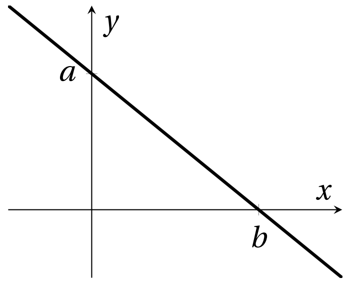


Figure 1.1: a is the x-intercept, b is the y-intercept.

Example 11. *Find an equation of the line through the points* $(1,-1)$ *and* $(3,5)$*. Draw the line. Find the x and y intercepts.*

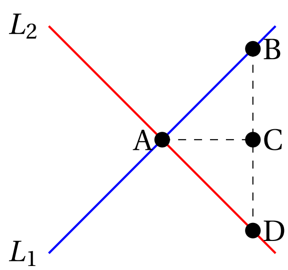
Example 12. *Find an equation of the line that passes through the point* $(-3,-4)$ *and has slope* 2*. Draw the line.*

Example 13. *Find the slope and the two intercepts of the line with equation* $8x+5y=20$*. Draw the line.*

Parallel vs. perpendicular lines

We call two lines **parallel** if their slopes are equal. We call two lines **perpendicular** if they intersect at right angles (90°).

Theorem 1. *Two nonvertical lines with slopes* m_1 *and* m_2 *are perpendicular if and only if* $m_1 m_2 = -1$.



Proof. Use the similarity of the triangles ABC and DAC to get

$$\frac{|BC|}{|AC|} = \frac{|AC|}{|CD|} \implies \frac{|BC||CD|}{|AC|^2} = 1$$

Slope of L_1 (m_1) is $|BC|/|AC| = 1$ and slope of L_2 (m_2) is $-|CD|/|AC|$. So $m_1 m_2 = -1$. □

Example 14. *Find an equation of the line through* $(1,-2)$ *that is parallel to the line L with equation* $3x-2y=1$*. Draw the lines.*

Example 15. *Find an equation of the line through* $(2,-3)$ *that is perpendicular to the line L with equation* $4x+y=3$*. Draw the lines.*

1.4 Quadratic Equations

Circles and Disks

The circle is the set of all points that have the same distance (called radius of the circle) from a given point (called center of the circle).

If (x, y) is a point on a circle with center (a, b) and radius r then

$$\sqrt{(x-a)^2+(y-b)^2} = r \implies (x-a)^2+(y-b)^2 = r^2$$

Example 16. *Find the center and radius of the circle $x^2+y^2-4x+6y=3$.*

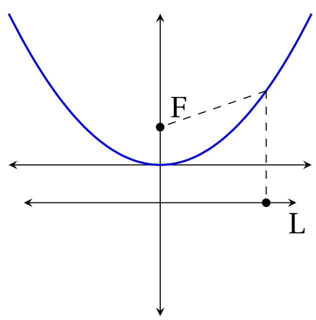
Solution. Complete to squares to get $(x-2)^2+(y+3)^2=16$.

The equation $(x-a)^2+(y-b)^2 < r^2$ represents open disk and the equation $(x-a)^2+(y-b)^2 \le r^2$ represents closed disk or simply disk.

Example 17. *Draw $x^2+2x+y^2 \le 8$.*

Parabolas

A parabola P is the set of all points in the plane that are equidistant from a given line L (called directrix of P) and a point F (called the focus of P).



Example 18. *Find the equation of the parabola having the point $F(0, p)$as focus and the line L with equation $y=-p$ as directrix.*

Solution. *If $P(x, y)$ is any point on the parabola then squaring both sides of PF=PQ we get*

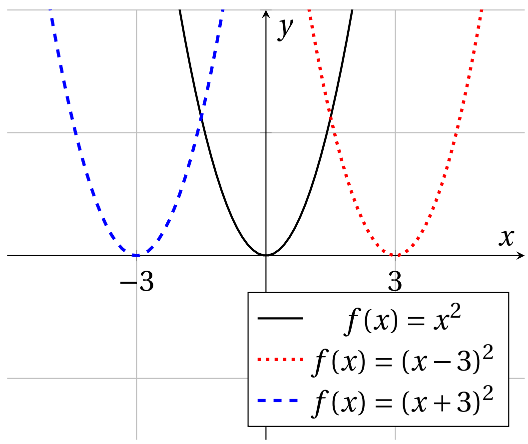
$$x^2+(y-p)^2 = 0^2+(y+p)^2$$

After simplifying, $y = x^2/4p$.

Shifting a Graph

Let $c > 0$.

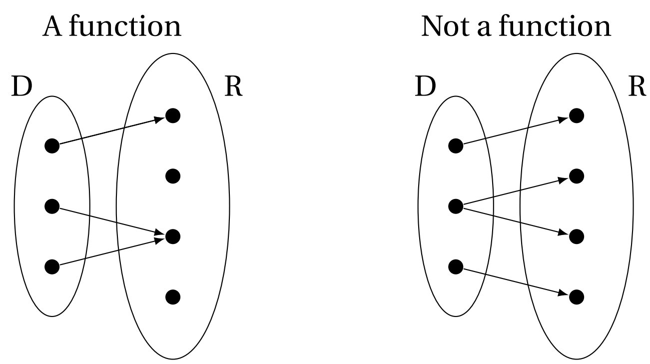
- To shift a graph c units to the right, replace x in its equation with $x-c$. To shift to left, replace x by $x+c$.
- To shift a graph c units up, replace y in its equation with $y-c$. To shift down, replace y by $y+c$.



1.5 Functions and Their Graphs

A **function** f on a set D into a set R is a rule that assigns a unique element $f(x)$ in R to each element x in D.

D is called the **domain** of f. R is called the target or **codomain** of f. The **range** of f is a subset of R containing of all possible values $f(x)$.



Example 19. *Define a function on the set of all real numbers by* $f(x) = x^2 + 1$. *Find* $f(0)$, $f(2)$, $f(x+2)$.

When defining a function, its domain should be defined. For example,

$$f(x) = \frac{1}{x}, \qquad x > 0$$

means that the domain of f is the set $\{x \mid x > 0\}$. This function is different from the function

$$f(x) = \frac{1}{x}, \qquad x < 0.$$

If we do not specify the domain of a function f, then the **domain convention** is to assume that the domain of f is the set of all real numbers for which f is defined.

So if we write

$$f(x) = \frac{1}{x},$$

we are assuming f is defined for all real numbers except 0.

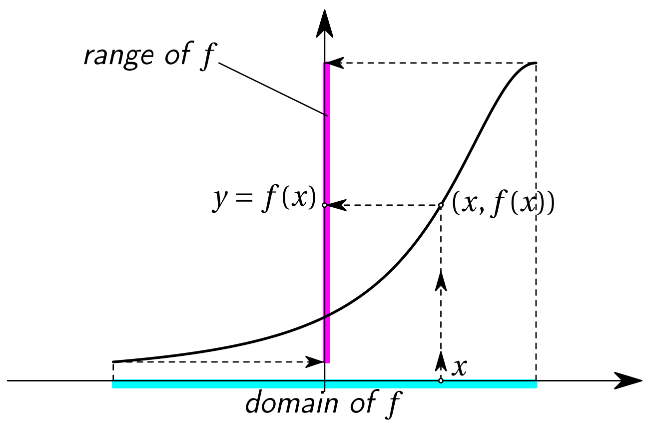
Example 20. *Find the domain of* $f(x) = \sqrt{2-x}$.

Solution. *Its domain is all* x *for which* $2 - x \geq 0$*, i.e. the interval* $(-\infty, 2]$.

Example 21. *Find the domain of* $f(x) = \dfrac{1}{x^2 - x}$.

Graph of a function

The *graph of a function* f is the set of all points whose coordinates are $(x, f(x))$ where x is in the domain of f.



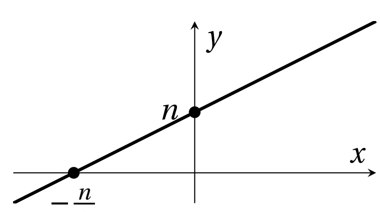
Some Elementary Functions

Linear Function

A function which is given by the formula

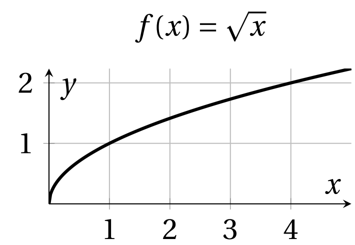
$$f(x) = mx + n$$

where m and n are constants is called a **linear function**. Its graph is a straight line. The constants m and n are the *slope* and y*-intercept* of the line. Its domain is all x and its range is all x.



Square Root Function

The square root function $f(x) = \sqrt{x}$ has domain $[0, \infty)$ and takes x to its positive square root. Hence it has range $[0, \infty)$.

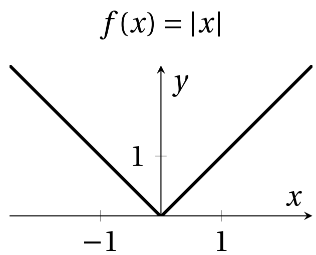


The absolute value function

The absolute value function is

$$|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$$

Its domain $(-\infty, \infty)$ and range $[0, \infty)$. We can only define the absolute value function by $f(x) = |x| = \sqrt{x^2}$.



Example 22. *Draw the graphs of some elementary functions*

$$c, x, x^2, x^3, x^{1/3}, \frac{1}{x}, \frac{1}{x^2}, \sqrt{1-x^2}.$$

Example 23. *Sketch the graph of* $f(x) = 1 + \sqrt{x-4}$.

Solution. *Shift the graph of* $y = \sqrt{x}$ *1 unit up and 4 units to the right.*

Example 24. *Sketch the graph of the function* $f(x) = \frac{2-x}{x-1}$.

Solution. $f(x) = \frac{2-x}{x-1} = -1 + \frac{1}{x-1}$. *So shift the graph of* $y = \frac{1}{x}$ *1 unit down and 1 unit to the right.*

Vertical Line Test

The graph of a function cannot intersect a vertical line "$x = \text{constant}$*" in more than one point.*

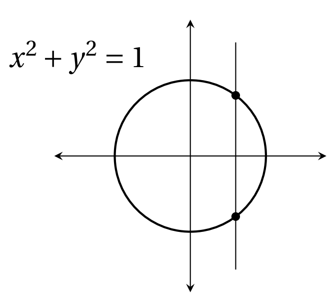
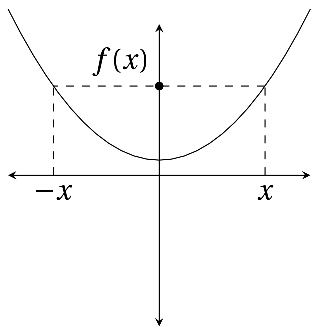


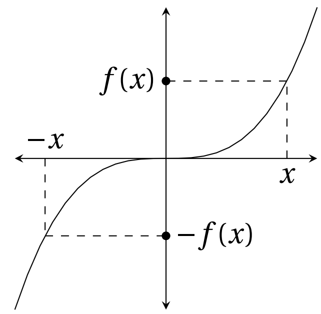
Figure 1.3: The circle $x^2 + y^2 = 1$ is not a graph of a function. It fails the vertical line test.

Even and Odd Functions

Definition 1. *We say that f is an **even function** if $f(-x) = f(x)$ for every $x \in D$. We say that f is an **odd function** if $f(-x) = -f(x)$ for every $x \in D$.*



(a) An even function



(b) An odd function.

Odd functions are symmetric with respect to origin and even functions are symmetric with respect to the y-axis.

Example 25. *$f(x) = x$, $f(x) = x^3$ are odd and $f(x) = x^2$ and $f(x) = x^4$ are even and $f(x) = \frac{1}{x+1}$ is neither even or odd.*

Example 26. *$f(x) = x^3 + x$ is odd and $f(x) = \frac{1}{x^2-1}$ is even and $f(x) = x^2 + x$ is either even or odd.*

1.6 Operations on Functions

If f and g are functions, then for every x that belongs to the domains of both f and g we define functions

- $(f+g)(x) = f(x) + g(x)$
- $(f-g)(x) = f(x) - g(x)$
- $(fg)(x) = f(x)g(x)$
- $(f/g)(x) = f(x)/g(x)$ where $g(x) \neq 0$.

Example 27. *Let $f(x) = \frac{1}{x+2}$ and $g(x) = \frac{x}{x-1}$. Find $(f+g)(x)$, $(f-g)(x)$, $(fg)(x) = f(x)g(x)$ and $(f/g)(x)$ where $g(x) \neq 0$.*

Composition of Functions

If f and g are two functions, then

$$f \circ g(x) = f(g(x)).$$

The domain of $f \circ g$ consists of those numbers x in the domain of g for which $g(x)$ is in the domain of f.

Example 28. *Let $f(x) = \sqrt{x}$ and $g(x) = x+1$. Find $f \circ g$, $g \circ f$, $f \circ f$ and $g \circ g$. State the domains of each function.*

Function	Formula	Domain
f	$\sqrt{x}$	$[0,\infty)$
g	$x+1$	$\mathbb{R}$
$f \circ g$	$\sqrt{x+1}$	$[-1,\infty)$
$g \circ f$	$\sqrt{x}+1$	$[0,\infty)$
$f \circ f$	$x^{1/4}$	$[0,\infty)$
$g \circ g$	$x+2$	$\mathbb{R}$

Piecewise Defined Functions

Functions such as

$$g(x) = \begin{cases} 2x & \text{for } x < 0 \\ x^2 & \text{for } x \geq 0 \end{cases}$$

which are defined by different formulas on different intervals are sometimes called **piecewise defined functions.**

1.7 Polynomials and Rational Functions

Definition 2. *A **polynomial** is a function $P : \mathbb{R} \to \mathbb{R}$ such that*

$$P(x) = a_n x^n + \cdot + a_1 x + a_0.$$

*Here $a_n, \ldots, a_1$ are called the **coefficients** of the polynomial. We assume $a_n \neq 0$. The number n is called the **degree** of the polynomial.*

Example 29. *Write polynomials of degree 0, 1 and 2.*

Just as the quotient of two integers is called a rational number, the quotient of two polynomials is called a **rational function**. Give an example.

Let A_m be a polynomial of degree m, B_n be a polynomial of degree n with $m \geq n$. Then there are polynomial Q_{m-n} of degree $m-n$, R_k of degree $k < n$ such that

$$\frac{A_m}{B_n} = Q_{m-n} + \frac{R_k}{B_n}.$$

The quotient Q_{m-n} and the remainder R_k can be calculated by the "long division".

Example 30. *Using the long division algorithm, show that*

$$\frac{2x^3 - 3x^2 + 3x + 4}{x^2 + 1} = 2x - 3 + \frac{x+7}{x^2+1}$$

If P is a polynomial and $P(r) = 0$ then r is called a **root** of P.

The Fundamental Theorem of Algebra says every polynomial of degree greater than 0 must have a root. But these roots may be complex.

Example 31. *x^2+1 has no real roots. Its roots are $i=\sqrt{-1}$ and $-i$.*

Theorem 2. *If r is a root of the polynomial P then*

$$P(x)=(x-r)Q(x),$$

for some polynomial Q whose degree is 1 less than P.

The polynomial $x(x-7)^3$ has 4 roots: 0 and the other three are each equal to 7. We say that 7 is a root of **multiplicity** 3.

By the Fundamental Theorem of Algebra and the above theorem, every polynomial of degree n has exactly n (not necessarily distinct) roots.

Roots of Quadratic Polynomials

To obtain the solutions of

$$Ax^2+Bx+C=0, \qquad A\neq 0$$

Divide by A and complete to square

$$\left(x+\frac{B}{2A}\right)^2=\frac{B^2}{4A^2}-\frac{C}{A}=\frac{B^2-4AC}{4A^2},$$

Taking the square root of both sides gives the quadratic formula

$$x=\frac{-B\pm\sqrt{B^2-4AC}}{2A}.$$

A form of this formula is known since B.C. 2000 by Babylonians.

The quantity $D=B^2-4AC$ is called the **discriminant** of the quadratic equation.

- If $D>0$ then there are two distinct real roots,
- If $D=0$ then there is 1 root of multiplicity 2,
- If $D<0$ then there are two complex conjugate roots.

Example 32. *Find the roots of the polynomials: (a) x^2+x-1, (b) $9x^2-6x+1$, (c) $2x^2+x+1$.*

Misc Factorings

- Difference of squares:
$$x^2-a^2=(x-a)(x+a)$$
- Difference of cubes:
$$x^3-a^3=(x-a)(x^2+ax+a^2)$$
- Difference of nth powers
$$x^n-a^n=(x-a)(x^{n-1}+ax^{n-2}+a^2x^{n-3}+\cdots+a^{n-2}x+a^{n-1})$$
- If n is an odd integer then $x+a$ is a factor of x^n+a^n,
$$x^n+a^n=(x+a)(x^{n-1}-ax^{n-2}+a^2x^{n-3}-\cdots-a^{n-2}x+a^{n-1})$$

Chapter 2

Limits and Continuity

2.1 Informal definition of limits

Two main problems of calculus are

1. Derivative. Find the rate of change of f.

2. Integral. Find the area under a given curve.

Both are based on the concept of limit.

We say $\lim_{x\to a} f(x) = L$ to mean that $f(x)$ is "close enough" to L when x is "close enough" to *but not equal to* a. Hence $f(a)$ is unimportant for $\lim_{x\to a} f(x)$.

Example 33. *Which value is x close to when x is close to 2?*

$$\lim_{x\to 2} x = 2$$

Example 34. *Which value is 3 close to when x is close to 2?*

$$\lim_{x\to 2} 3 = 3$$

We can generalize these examples.

Theorem 3. *Let a and c be two real numbers. Then*

$$\lim_{x\to a} c = c, \qquad \lim_{x\to a} x = a.$$

The limit $\lim_{x\to a} f(x)$ may be different from $f(a)$ as the next example shows.

Example 35.

$$f(x) = \begin{cases} x, & \text{if } x \neq 2 \\ 1, & \text{if } x = 2 \end{cases}$$

Which value is $f(x)$ close to when x is close to (but not equal to) 2?

$\lim_{x\to 2} f(x) = \lim_{x\to 2} x = 2$ *although* $f(2) = 1$.

Informal definition of left and right limits

If $f(x)$ is close to L when $x<a$ and x is close enough to a then we say

$$\lim_{x\to a^-} f(x) = L$$

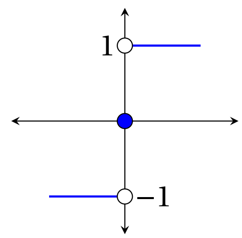
This is called the *left limit* of f at $x=a$.

Similarly we can define the right limit.

Theorem 4. $\lim_{x\to a} f(x) = L$ *if and only if both* $\lim_{x\to a^-} f(x) = L$ *and* $\lim_{x\to a^+} f(x) = L$.

Example 36. *Find the left and right limits of the signum function*

$$f(x) = \begin{cases} -1 & \textit{for } x<0 \\ 0 & \textit{for } x=0 \\ 1 & \textit{for } x>0 \end{cases}$$



Solution. *The one-sided limits exist, but are not equal*

$$\lim_{x\searrow 0} f(x) = 1 \textit{ and } \lim_{x\nearrow 0} f(x) = -1.$$

Hence $\lim_{x\to 0} f(x)$ *does not exist.*

Properties of Limits

Theorem 5. *Suppose*

$$\lim_{x\to a} f(x) = L, \qquad \lim_{x\to a} g(x) = M.$$

Then

$$\lim_{x\to a}\big(f(x)+g(x)\big) = L+M, \tag{2.1}$$
$$\lim_{x\to a}\big(f(x)-g(x)\big) = L-M, \tag{2.2}$$
$$\lim_{x\to a}\big(f(x)\cdot g(x)\big) = L\cdot M \tag{2.3}$$
$$\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{L}{M}, \quad \textit{if } M\neq 0. \tag{2.4}$$

Finally, if m and n are integers such that $L^{m/n}$ is defined

$$\lim_{x\to a}(f(x))^{m/n} = L^{m/n}. \tag{2.5}$$

Using the above properties we can evaluate the following limits.

Example 37. *Find* $\lim_{x\to 2} x^2+1$ *and* $\lim_{x\to 2}\dfrac{x^2+1}{6-x}$.

Solution. *Using the product rule of limits and the Theorem 3,*

$$\lim_{x\to 2} x^2 = \lim_{x\to 2} x \cdot \lim_{x\to 2} x = 2\cdot 2 = 4$$

Using the sum rule of limits,

$$\lim_{x\to 2} x^2 + 1 = \lim_{x\to 2} x^2 + \lim_{x\to 2} 1 = 4+1 = 5$$

Using the division rule of limits,

$$\lim_{x\to 2} \frac{x^2+1}{6-x} = \frac{\lim_{x\to 2} x^2+1}{\lim_{x\to 2} 6-x} = \frac{5}{4}.$$

The above example is a special case of the following theorem.

Theorem 6. *If $P(x)$ is a polynomial then,*

$$\lim_{x\to a} P(x) = P(a)$$

If $Q(x)$ is another polynomial with $Q(a) \neq 0$ then

$$\lim_{x\to a} \frac{P(x)}{Q(x)} = \frac{P(a)}{Q(a)}.$$

The Squeeze Theorem

Theorem 7. *Suppose that $f(x) \le g(x) \le h(x)$ and $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$. Then $\lim_{x\to a} g(x) = L$.*

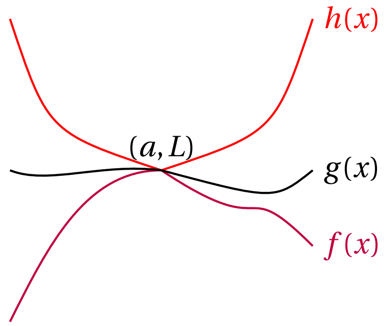


Figure 2.1: The Squeeze Theorem.

Example 38. *If $2 - x^2 \le g(x) \le 2\cos x$ for $-1 \le x \le 1$, find $\lim_{x\to 0} f(x)$.*

Example 39. *Show that if $\lim_{x\to a} |f(x)| = 0$ then $\lim_{x\to a} f(x) = 0$.*

Solution. *Note that $-|f(x)| \le f(x) \le |f(x)|$ and use the Squeeze Theorem.*

More examples

Example 40. *Let*

$$f(x) = \frac{|x-2|}{x^2+x-6}.$$

Find $\lim_{x\to 2+} f(x)$, $\lim_{x\to 2-} f(x)$. *Does* $\lim_{x\to 2} f(x)$ *exist?*

In these example, we will compute $\lim_{x\to a} f(x)$ even when $f(a)$ does not exist.

Example 41. *Evaluate*

1. $\lim_{x\to -2} \frac{x^2+x-2}{x^2+5x+6}$,

   Solution. *Remember that we consider x values close to but not equal to* -2. *Hence* $x+2 \neq 0$ *and we can make the simplification*

   $$\lim_{x\to -2} \frac{x^2+x-2}{x^2+5x+6} = \lim_{x\to -2} \frac{(x+2)(x-1)}{(x+2)(x+3)} = \lim_{x\to -2} \frac{x-1}{x+3} = \frac{-3}{1} = -3.$$

2. $\lim_{x\to 5} \frac{\frac{1}{x}-\frac{1}{5}}{x-5}$,
3. $\lim_{x\to 4} \frac{\sqrt{x}-2}{x^2-16}$, *Hint: multiply both sides by the conjugate expression.*
4. $\lim_{x\to -2} \frac{x^2+2x}{x^2-4}$,
5. $\lim_{h\to 0} \frac{\sqrt{4+h}-2}{h}$,
6. $\lim_{t\to 0} \frac{t}{\sqrt{4+t}-\sqrt{4-t}}$,
7. $\lim_{x\to -1} \frac{x^3+1}{x+1}$,
8. $\lim_{x\to 0} \frac{|3x-1|-|3x+1|}{x}$,
9. $\lim_{x\to 2-} \frac{x^2-4}{|x+2|}$.

2.2 Limits at Infinity and Infinite Limits

Limits at Infinity

Definition 3. *We will say that* $\lim_{x\to\infty} f(x) = L$ *if* $f(x)$ *is "close enough" to* L *whenever* $x > 0$ *is "large enough".*

Similarly we define $\lim_{x\to -\infty} f(x) = L$ *if* $f(x)$ *is "close enough" to* L *whenever* $x < 0$ *is "large enough".*

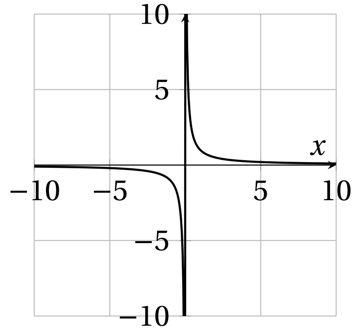
If either $\lim_{x\to\infty} f(x) = L$ *or* $\lim_{x\to -\infty} f(x) = L$, *we say that the line* $y = L$ *is an* ***horizontal asymptote*** *of the graph of* f.

Example 42. *Argue that*

$$\lim_{x\to\infty} 1/x = \lim_{x\to\infty} 1/x = 0.$$

by making a table of values of x and $1/x$.

$$f(x) = \frac{1}{x}$$



Recall that for ordinary limits, limit of product of functions is a product of limits of functions. Same is also true for limits at infinity. Hence

$$\lim_{x\to\infty} \frac{1}{x^2} = \lim_{x\to\infty} \frac{1}{x} \cdot \lim_{x\to\infty} \frac{1}{x} = 0 \times 0 = 0.$$

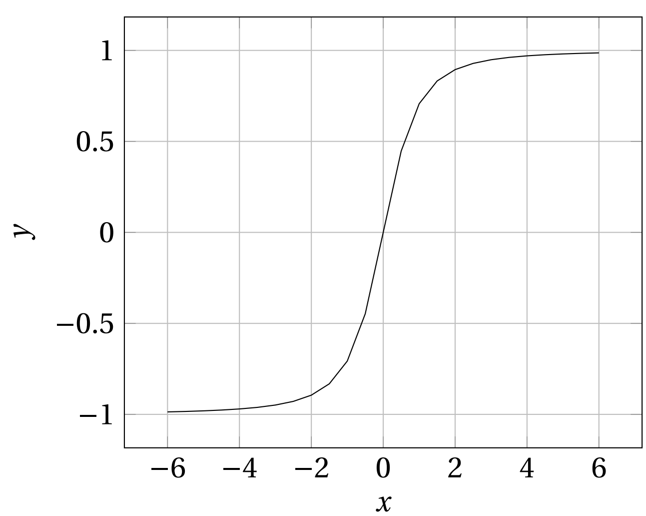
Similarly

$$\lim_{x\to-\infty} \frac{1}{x^2} = 0$$

Finally, for any positive integer n

$$\lim_{x\to\infty} \frac{1}{x^n} = \lim_{x\to-\infty} \frac{1}{x^n} = 0.$$

$$f(x) = \frac{x}{\sqrt{x^2+1}}$$



Example 43. *Let* $f(x) = \dfrac{x}{\sqrt{x^2+1}}$. *Find* $\lim_{x\to\infty} f(x)$, $\lim_{x\to-\infty} f(x)$.

Solution.

$$\begin{aligned}\lim_{x\to\infty}\frac{x}{\sqrt{x^2+1}} &= \lim_{x\to\infty}\frac{x}{|x|\sqrt{1+1/x^2}} = \lim_{x\to\infty}\frac{x}{x\sqrt{1+1/x^2}} = \lim_{x\to\infty}\frac{1}{\sqrt{1+1/x^2}} = \frac{\lim_{x\to\infty}1}{\lim_{x\to\infty}\sqrt{1+1/x^2}}\\ &= \frac{1}{\sqrt{\lim_{x\to\infty}(1+1/x^2)}} = \frac{1}{1} = 1.\end{aligned}$$

Similarly,

$$\lim_{x\to-\infty}\frac{x}{\sqrt{x^2+1}} = -1$$

Limits of Rational Functions at Infinity

Recall that a rational function is a ratio of two polynomials.

Strategy. To find limits of rational functions at infinity, divide by the highest power of x appearing in the *denominator*.

Example 44.

$$\lim_{x\to\pm\infty}\frac{2x^2-x+3}{3x^2+5} = \lim_{x\to\pm\infty}\frac{2-\frac{1}{x}+\frac{3}{x^2}}{3+\frac{5}{x}} = \frac{2}{3}.$$

Example 45.

$$\lim_{x\to\pm\infty}\frac{x-5}{2x^2+4x+1} = \lim_{x\to\pm\infty}\frac{\frac{1}{x}-\frac{5}{x^2}}{2+\frac{4}{x}+\frac{1}{x^2}} = \frac{0}{2} = 0.$$

We can generalize the above examples.

Theorem 8. *Let* $P(x) = a_p x^p + a_{p-1}x^{p-1} + \cdots + a_0$ *be a polynomial of degree* p *and* $Q(x) = b_q x^q + \cdots + b_0$ *be a polynomial of degree* q. *If* $p = q$, *then*

$$\lim_{x\to\pm\infty}\frac{P(x)}{Q(x)} = \frac{a_p}{q_p},$$

If $p < q$, *then*

$$\lim_{x\to\pm\infty}\frac{P(x)}{Q(x)} = 0,$$

Example 46.

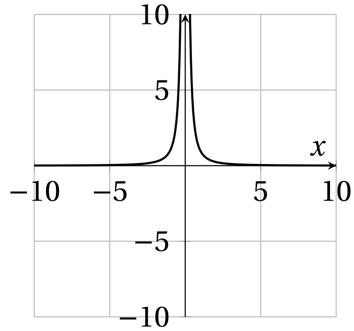
$$\lim_{x\to\infty}\sqrt{x^2+x}-x = \lim_{x\to\infty}\frac{(\sqrt{x^2+x}-x)(\sqrt{x^2+x}+x)}{\sqrt{x^2+x}+x} = \lim_{x\to\infty}\frac{x}{|x|\sqrt{1+\frac{1}{x}}+x} = \lim_{x\to\infty}\frac{1}{\sqrt{1+\frac{1}{x}}+1} = \frac{1}{2}.$$

Infinite Limits

Example 47. *The values of* $\frac{1}{x^2}$ *gets larger and larger as* x *approaches to 0. Thus* $\lim_{x\to 0}\frac{1}{x^2}$ *does not exist. Although the limit does not exist, it is useful to state why it does not exist by writing*

$$\lim_{x\to 0}\frac{1}{x^2}=\infty.$$

$$f(x)=\frac{1}{x^2}$$



Example 48.

$$\lim_{x\to 0+}\frac{1}{x}=\infty.$$

$$\lim_{x\to 0-}\frac{1}{x}=-\infty.$$

$$\lim_{x\to 0}\frac{1}{x}\ \textit{does not exist.}$$

Example 49.

$$\lim_{x\to-\infty}\sqrt{x^2+x}-x=\lim_{x\to-\infty}\frac{(\sqrt{x^2+x}-x)(\sqrt{x^2+x}+x)}{\sqrt{x^2+x}+x}=\lim_{x\to-\infty}\frac{x}{|x|\sqrt{1+\frac{1}{x}}+x}=\lim_{x\to-\infty}\frac{1}{-\sqrt{1+\frac{1}{x}}+1}$$

If $x<0$ *then* $\sqrt{1+\frac{1}{x}}<1$ *and* $-\sqrt{1+\frac{1}{x}}+1>0$*. Hence the denominator is positive and approaches to zero. So*

$$\lim_{x\to-\infty}\frac{1}{-\sqrt{1+\frac{1}{x}}+1}=\infty.$$

Behaviour of Polynomials at Infinity

Example 50.

$$\lim_{x\to\infty}4x^3-2x+1=\lim_{x\to\infty}4x^3=\infty.$$

$$\lim_{x\to-\infty}-3x^5+x^3+1=\lim_{x\to-\infty}-3x^5=\infty.$$

In general,

Theorem 9. *If $P(x) = a_n x^n + \cdots + a_0$ is a polynomial then*

$$\lim_{x\to\pm\infty} P(x) = \lim_{x\to\pm\infty} a_n x^n.$$

Example 51.

$$\lim_{x\to\infty} \frac{x^3+1}{x^2-2x} = \lim_{x\to\infty} \frac{x+\frac{1}{x^2}}{1-\frac{2}{x}} = \lim_{x\to\infty} \frac{x}{1} = \infty$$

Example 52. 1. $\lim_{x\to 2} \dfrac{(x-2)^2}{x^2-4} = 0$

2. $\lim_{x\to 2+} \dfrac{x-3}{x^2-4} = -\infty$

3. $\lim_{x\to 2-} \dfrac{x-3}{x^2-4} = \infty$

4. $\lim_{x\to 2} \dfrac{x-3}{x^2-4}$ *does not exist.*

5. $\lim_{x\to\infty} \dfrac{2x-1}{\sqrt{3x^2+x+1}}$,

6. $\lim_{x\to 1+} \dfrac{\sqrt{x^2-x}}{x-x^2}$

   Solution. *If $x > 1$ then $x - x^2 = x(1-x) < 0$. So*

$$\lim_{x\to 1+} \frac{\sqrt{x^2-x}}{x-x^2} = \lim_{x\to 1+} \frac{-\sqrt{x^2-x}}{x^2-x} = \lim_{x\to 1+} \frac{-\sqrt{x^2-x}}{\sqrt{x^2-x}\sqrt{x^2-x}} = \lim_{x\to 1+} \frac{-1}{\sqrt{x^2-x}} = -\infty$$

2.3 Continuity

Let $f(x) = \sqrt{4-x^2}$. Domain of f is $[-2,2]$.

- $x = -2$ is the left end point of $\mathrm{Dom}(f)$.
- $x = 2$ is the right end point of $\mathrm{Dom}(f)$.
- Any x with $-2 < x < 2$ is called an interior point of $\mathrm{Dom}(f)$.

Definition 4. *A function f is **continuous** at an interior point c of its domain if*

$$\lim_{x\to c} f(x) = f(c)$$

f is continuous at its left endpoint c if

$$\lim_{x\to c+} f(x) = f(c)$$

f is continuous at its right endpoint c if

$$\lim_{x\to c-} f(x) = f(c)$$

The following theorem gives an alternative definition of continuity which is sometimes useful.

Theorem 10. *A function f is **continuous** at an interior point c of its domain if and only if*

$$\lim_{h\to 0} f(c+h) = f(c)$$

f is continuous at its left endpoint c if

$$\lim_{h\to 0+} f(c+h) = f(c)$$

f is continuous at its right endpoint c if

$$\lim_{h\to 0-} f(c+h) = f(c)$$

Proof. Let $h = x - c$. Then $x \to c$ if and only if $h \to 0$. So $\lim_{h\to 0} f(c+h) = f(c)$ is the same as $\lim_{h\to 0} f(c+h) = f(c)$. □

Note that f is discontinuous at c if

i) either $\lim_{x\to c} f(x)$ does not exist.

ii) or $\lim_{x\to c} f(x)$ exists but is not equal to $f(c)$.

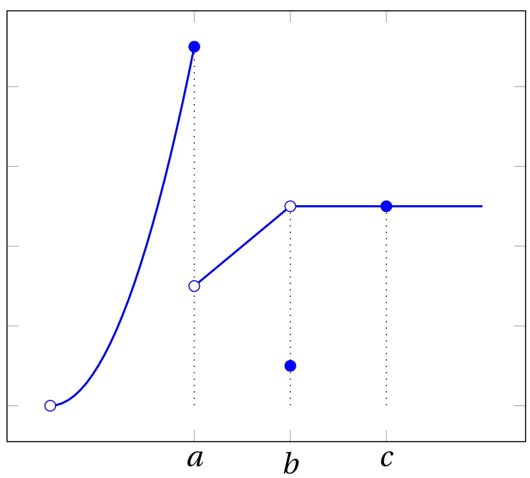
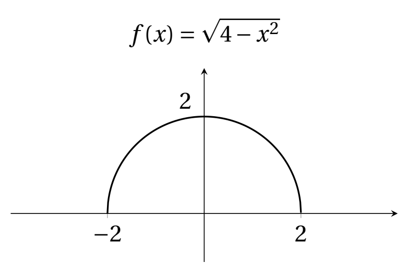


Figure 2.2: f is discontinuous at a because of (ii) and discontinuous at b because of (i). f is continuous at c.

Example 53. $f(x) = \sqrt{4-x^2}$ *is continuous at every point of its domain.*



Definition 5. *f is called a continuous function if f is continuous at every pt of its domain.*

According to this definition $f(x) = \frac{1}{x}$ is continuous!!! 0 is not in domain of f. So we say f is undefined rather than discontinuous at 0.

There are lots of continuous functions:

- polynomials,
- rational functions,
- rational powers $x^{m/n}$
- trigonometric functions
- absolute value function $|x|$

Theorem 11. *If f and g are continuous at c then*

- *$f+g$, $f-g$, fg, are continuous at c,*
- *if k is constant then kf is continuous at c,*
- *$\dfrac{f}{g}$ continuous at c provided that $g(c) \neq 0$.*
- *$f(x)^{1/n}$ continuous at c provided that $f(c) > 0$ if n is even.*

Proof. Let's prove that if f and g are continuous at c then so is $f+g$. If f and g are continuous at c then

$$\lim_{x\to c} f(x) = f(c), \qquad \lim_{x\to c} g(x) = g(c),$$

By the limit rule,

$$\lim_{x\to c}(f(x)+g(x)) = \lim_{x\to c} f(x) + \lim_{x\to c} g(x) = f(c) + g(c).$$

The other proofs are similar. □

Composites of continuous functions are continuous

If g is continuous at c and f is continuous at $g(c)$ then $f \circ g$ is continuous at c. In other words,

$$\lim_{x\to c} f(g(x)) = f(\lim_{x\to c} g(x)) = f(g(c)).$$

Example 54. *Find m so that*

$$g(x) = \begin{cases} x - m, & \text{if } x < 3, \\ 1 - mx, & \text{if } x \geq 3 \end{cases}$$

is continuous for all x.

Continuity of Trigonometric Functions

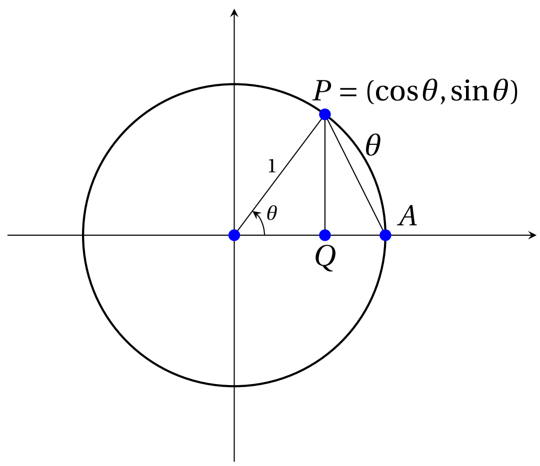
Theorem 12. $\sin x$ *and* $\cos x$ *are continuous at* $x = 0$, *i.e.*

$$\lim_{x\to 0} \sin x = \sin 0 = 0, \qquad \lim_{x\to 0} \cos x = \cos 0 = 1.$$

Proof.

$$|1 - \cos\theta| = |AQ| \le |AP| \le \theta,$$
$$|\sin\theta| = |PQ| \le |AP| \le \theta$$

In other words, $-\theta \le \sin\theta \le \theta$ and using the squeeze theorem we get $\lim_{\theta\to 0} \sin\theta = 0$. Similarly, we get $\lim_{\theta\to 0} 1 - \cos\theta = 0$ or $\lim_{\theta\to 0} \cos\theta = 1$.



□

Theorem 13. $\sin x$ *and* $\cos x$ *are continuous for all* x.

Proof. By Theorem 10, we need to prove $\lim_{h\to 0} \sin(x+h) = \sin x$ for any x.

$$\lim_{h\to 0} \sin(x+h) = \lim_{h\to 0} \sin x \cos h + \cos x \sin h = \sin x \lim_{h\to 0} \cos h + \cos x \lim_{h\to 0} \sin h = \sin x.$$

Prove the continuity of $\cos x$ as an exercise. □

Continuous Functions on Closed Intervals [a, b] are bounded

We say a function f is **bounded** if there exists M and N such that $M \le f(x) \le N$ for all x in the domain of f.

Theorem 14. *If* f *is continuous on the closed interval* $[a, b]$ *then there exist numbers* p *and* q *in the interval* $[a, b]$ *s.t.*

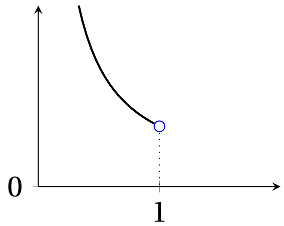
$$f(p) \le f(x) \le f(q)$$

for all x *in* $[a, b]$. $f(p)$ *is called the* ***absolute minimum value*** *and* $f(q)$ *is called the* ***absolute maximum value***.

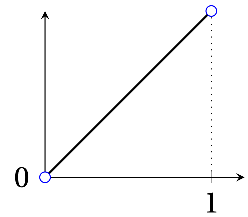
This theorem is an existence theorem. It only guarantees the existence of p and q but does not tell how to actually find them.

Also the theorem says that continuous functions on closed intervals must be bounded.

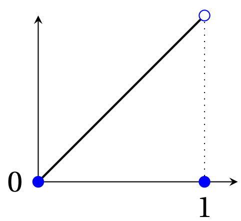
Example 55. *The conclusions of the theorem may fail if the function* f *is not continuous or the interval is not closed.*



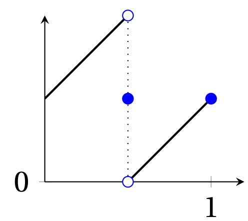
(a) The function $f(x) = 1/x$ on the open interval $(0,1)$ is continuous but unbounded and has no minimum and no maximum.



(b) The function $f(x) = x$ on $(0,1)$ is discontinuous, bounded and has no minimum and no maximum.



(a) This function is defined on the closed interval $[0,1]$, discontinuous, has a minimum but no maximum.



(b) This function is defined on the closed interval $[0,1]$, discontinuous, bounded, has no minimum but no maximum.

Theorem 15 (Intermediate Value Theorem)**.** *If f is continuous on $[a,b]$ and if s is between $f(a)$ and $f(b)$ then there exists c in $[a,b]$ s.t. $f(c) = s$.*

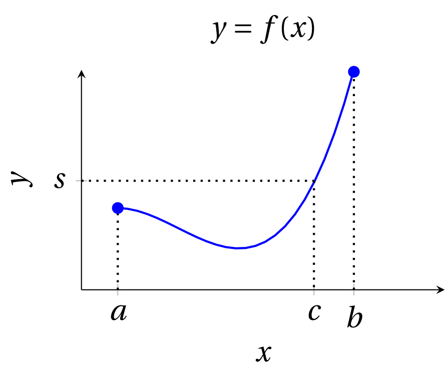


Figure 2.5: Illustration of the intermediate value theorem.

In particular, a continuous function on a closed interval takes every value between its minimum m and maximum M. Hence its range is a closed interval $[m,M]$.

Example 56. *Show that the equation $x^3 - x - 1 = 0$ has a solution in the interval $[1,2]$.*

Solution. *$f(x) = x^3 - x - 1$ is a polynomial and hence continuous. $f(1) = -1$ and $f(2) = 5$. Since 0 lies between -1 and 5, the intermediate value theorem assures us that there must be a number c in $[1,2]$ such that $f(c) = 0$.*

Bisection Algorithm

Intermediate Value Theorem is also an existence theorem. It does not say how to find c in its statement. Let's try to better estimate the root of previous example. Write $f(x) = x^3 - x - 1$ and try to find a smaller interval where a root lies of

$$f(x) = 0.$$

We know that a root lies in $[1,2]$, if say that the root is 1.5 the maximum error will be 0.5.

Now $f(1.5) = 0.875 > 0$. So a root lies in $[1,1.5]$, and if we say the root is 1.25 then the maximum error will be 0.25.

If this is not sufficient then compute $f(1.25) = -0.2969$, now if we say the root is 1.375 then the error is less than 0.125.

Next step is $f(1.1375) = 0.2246$. So a root must lie in $[1.25, 1375]$. The error is less than 0.0625 if we say the root is 1.315.

Going this way, we find the approximations, 1.3438, 1.3282, 1.3204. Hence the root must lie in $[1.3204, 1.3282]$. So the first two decimal digits of the root are 1.32.

In engineering, you almost never get exact results. All you can do is lower your error below an acceptable threshold.

2.4 Formal definition of Limit

The informal description of the limit uses phrases like "close enough" and "really very small". "Fortunately" there is a good definition, i.e. one which is unambiguous and can be used to settle any dispute about the question of whether $\lim_{x\to a} f(x)$ equals some number L or not.

In this section we assume that f is defined in an open interval containing a except possibly at $x = a$.

Definition 6. *We say that*

$$\lim_{x\to a} f(x) = L$$

if for every $\epsilon > 0$ *there exists a* $\delta > 0$ *such that*

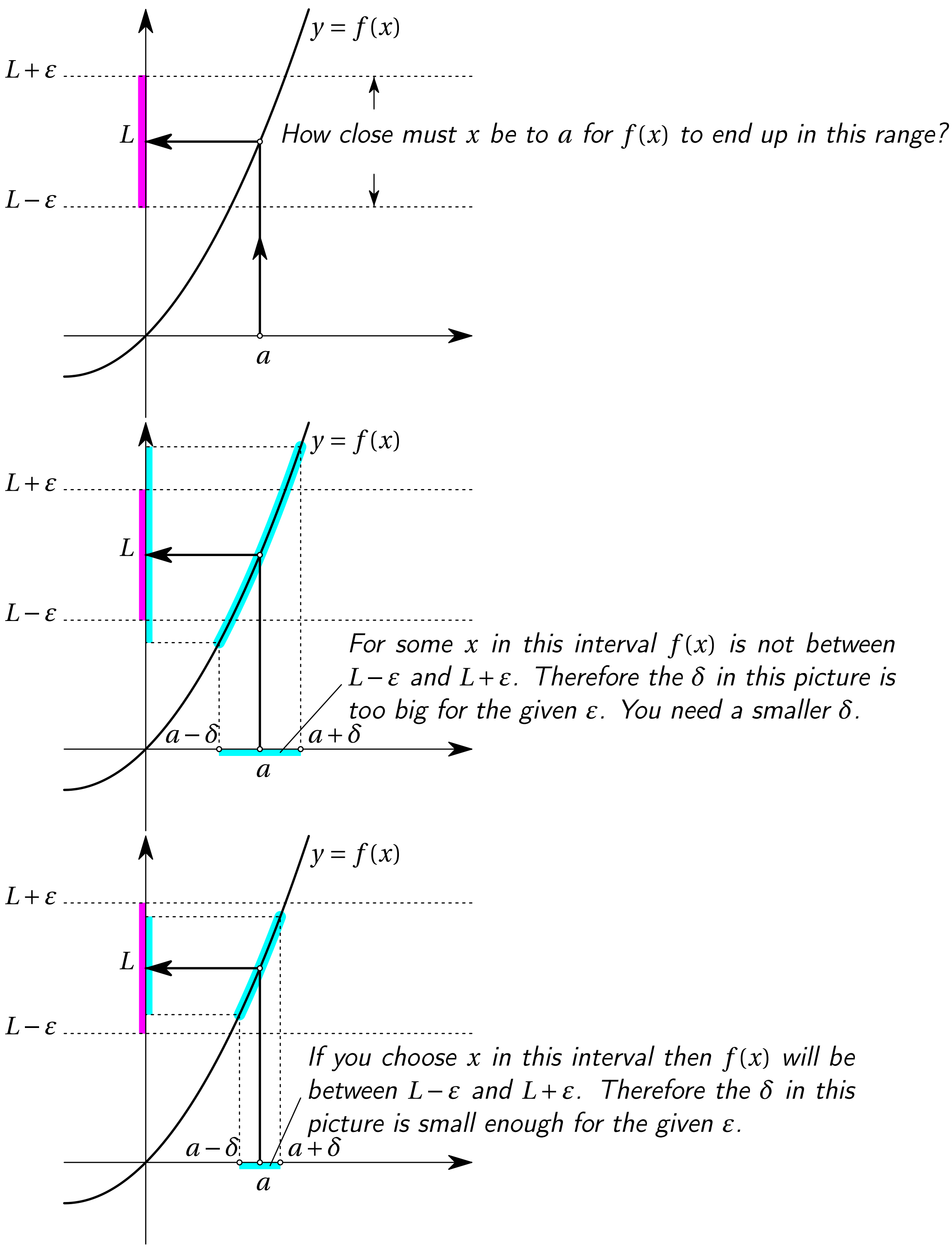
$$0 < |x - a| < \delta \text{ implies } |f(x) - L| < \epsilon. \tag{2.6}$$

Why the absolute values? Recall that the quantity $|x - y|$ is the distance between the points x and y on the number line.

What are ϵ *and* δ*?* The quantity ϵ is how close you would like $f(x)$ to be to its limit L; the quantity δ is how close you have to choose x to a to achieve this. To prove that $\lim_{x\to a} f(x) = L$ you must assume that someone has given you an unknown $\epsilon > 0$, and then find a positive δ for which (2.6) holds. The δ you find will depend on ϵ.

When we first discussed the limit, say $\lim_{x\to 5} 2x + 1$, we made a table,

x	$f(x) = 2x + 1$
5.1	11.2
5.01	11.02
5.001	11.002
4.9	10.8
4.99	10.98
4.999	10.998



This table can be written also in this form.

$\lvert x-5\rvert$	$\lvert f(x)-11\rvert$
0.1	0.2
0.01	0.02
0.001	0.002

It looks like for any $\epsilon > 0$, if $|x-5| < \frac{\epsilon}{2}$ then $|f(x)-11| < \epsilon$. Now let's prove this!

Example 57. *Show that* $\lim_{x\to 5} 2x+1 = 11$.

Solution. *We have* $f(x) = 2x+1$, $a = 5$ *and* $L = 11$, *and the question we must answer is "how close should* x *be to* 5 *if you want to be sure that* $f(x) = 2x+1$ *differs less than* ϵ *from* $L = 11$*?"*

$$|f(x) - L| = |(2x+1) - 11| = |2x - 10| = 2\cdot|x-5| = 2\cdot|x-a|.$$

So choose $\delta = \frac{\epsilon}{2}$. *Then*

$$|f(x) - L| < \epsilon \text{ whenever } 0 < |x-a| < \frac{\epsilon}{2}.$$

Example 58 ("Don't choose $\delta > 1$" trick)**.** *Show that* $\lim_{x\to 3} x^2 = 9$.

Solution. *We have* $f(x) = x^2$, $a = 3$, $L = 9$, *and again the question is, "how small should* $|x-3|$ *be to guarantee* $|x^2 - 9| < \epsilon$*?"*

$$|x^2 - 9| = |(x-3)(x+3)| = |x+3|\cdot|x-3|.$$

Here is a trick that allows you to replace the factor $|x+3|$ *with a constant. We hereby agree that we always choose our* δ *so that* $\delta \le 1$. *If we do that, then we will always have*

$$|x-3| < \delta \le 1, \text{ i.e. } |x-3| < 1,$$

or $2 < x < 4$ *or* $|x+1| < 5$. *Therefore*

$$|x^2 - 1| = |x+1|\cdot|x-1| < 5|x-1|.$$

So choose

$$\delta = \min\{1, \frac{\epsilon}{5}\}.$$

2nd way: Note that $|x+3| = |x-3+6| < |x-3| + 6 < \delta + 6$

$$|f(x) - 9| = |x+3||x-3| < (\delta+6)\delta$$

So choose $(\delta+6)\delta < \epsilon$, *or*

$$(\delta+3)^2 < \epsilon + 9 \implies \delta < \sqrt{\epsilon+9} - 3$$

Example 59. *Show that* $\lim_{x\to 4} 1/x = 1/4$.

Solution. *We apply the definition with* $a = 4$, $L = 1/4$ *and* $f(x) = 1/x$. *Thus, for any* $\epsilon > 0$ *we try to show that if* $|x-4|$ *is small enough then one has* $|f(x) - 1/4| < \epsilon$.

We begin by estimating $|f(x) - \frac{1}{4}|$ *in terms of* $|x-4|$:

$$|f(x) - 1/4| = \left|\frac{1}{x} - \frac{1}{4}\right| = \left|\frac{4-x}{4x}\right| = \frac{|x-4|}{|4x|} = \frac{1}{|4x|}|x-4|.$$

As before, things would be easier if $1/|4x|$ *were a constant. To achieve that we again agree not to take* $\delta > 1$. *If we always have* $\delta \le 1$, *then we will always have* $|x-4| < 1$, *and hence* $3 < x < 5$. *How large can* $1/|4x|$ *be in this situation? Answer: the quantity* $1/|4x|$ *increases as you decrease* x, *so if* $3 < x < 5$ *then it will never be larger than* $1/|4\cdot 3| = \frac{1}{12}$.

We see that if we never choose $\delta > 1$, *we will always have*

$$|f(x) - \frac{1}{4}| \le \frac{1}{12}|x-4| \quad \text{for} \quad |x-4| < \delta.$$

To guarantee that $|f(x)-\frac{1}{4}|<\epsilon$ *we could threfore require*

$$\frac{1}{12}|x-4|<\epsilon, \quad \textit{i.e.} \quad |x-4|<12\epsilon.$$

Hence if we choose $\delta=12\epsilon$ *or any smaller number, then* $|x-4|<\delta$ *implies* $|f(x)-4|<\epsilon$. *Of course we have to honor our agreement never to choose* $\delta>1$, *so our choice of* δ *is*

$$\delta = \textit{the smaller of } 1 \textit{ and } 12\epsilon = \min\big(1,12\epsilon\big).$$

Example 60. *Verify that* $\lim_{x\to 2}\dfrac{x-2}{1+x^2}=0$.

Solution. *Notice that* $\dfrac{|x-2|}{|1+x^2|}<|x-2|$ *since* $1+x^2>1$. *Hence choose* $\delta=\epsilon$.

2.5 Review Problems

Example 61. *Evaluate the limits if they exist. If they do not exist, state wheter they are* ∞, $-\infty$ *or just does not exist.*

1. $\lim_{x\to 2}\dfrac{x^2+1}{1-x^2}$,
2. $\lim_{x\to 1}\dfrac{x^2}{1-x^2}$,
3. $\lim_{x\to \infty}\dfrac{\cos x}{x}$, *(Hint: Use Sandwich Theorem)*
4. $\lim_{x\to -\infty}\dfrac{2x^3+2x-1}{-3x^3+x^2}$,
5. $\lim_{x\to -\infty} x+\sqrt{x^2-4x+1}$,

   Solution.

$$\begin{aligned}\lim_{x\to-\infty} x+\sqrt{x^2-4x+1} &= \lim_{x\to-\infty} x+|x|\sqrt{1-\frac{4}{x}+\frac{1}{x^2}} = \lim_{x\to-\infty} x\left(1-\sqrt{1-\frac{4}{x}+\frac{1}{x^2}}\right)\\ &= \lim_{x\to-\infty} x\left(1-\sqrt{1-\frac{4}{x}+\frac{1}{x^2}}\right)\frac{\left(1+\sqrt{1-\frac{4}{x}+\frac{1}{x^2}}\right)}{\left(1+\sqrt{1-\frac{4}{x}+\frac{1}{x^2}}\right)}\\ &= \lim_{x\to-\infty} x\left(1-\left(1-\frac{4}{x}+\frac{1}{x^2}\right)\right)\lim_{x\to-\infty}\frac{1}{\left(1+\sqrt{1-\frac{4}{x}+\frac{1}{x^2}}\right)}\\ &= \lim_{x\to-\infty} x\left(\frac{4}{x}-\frac{1}{x^2}\right)\frac{1}{2} = \lim_{x\to-\infty}\left(4-\frac{1}{x}\right)\frac{1}{2}=2.\end{aligned}$$

6. $\lim_{x\to 0}\dfrac{x}{|x-1|-|x+1|}$.

Chapter 3

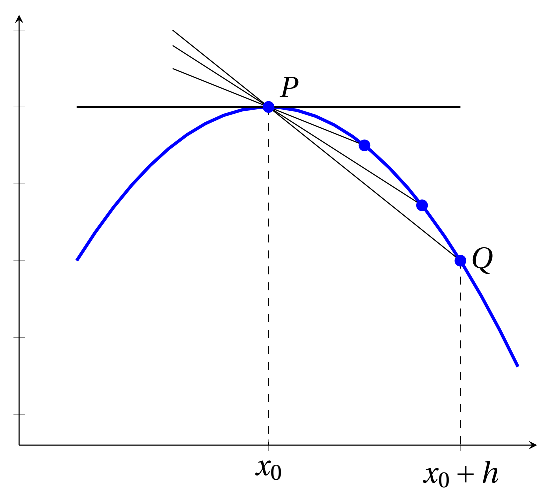
Differentiation

3.1 Tangent Lines and Their Slopes

Problem: Find a straight line L that is tangent to a curve C at a point P.

"For simplicity, restrict ourselves to curves which are graphs of functions."

How do we define the tangent line to a curve?



The slope of the line PQ is

$$\frac{f(x_0+h)-f(x_0)}{h}.$$

Definition 7. *Suppose f is cts at $x = x_0$ and*

$$\lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h} = m$$

If the limit exists, then the line with equation

$$y = m(x - x_0) + f(x_0)$$

is called ***the tangent line*** *to the graph of $y = f(x)$ at $P = (x_0, f(x_0))$. If the limit does not exist and $m = \infty$ or $m = -\infty$ then the tangent line is the vertical line $x = x_0$. If the limit does not exist and is not $\pm\infty$ then there is no tangent line at P.*

Example 62. *Find an equation of the tangent line to the curve $y = x^2$ at $(1,1)$.*

Solution. *The slope is*

$$m = \lim_{h\to 0} \frac{f(1+h) - f(1)}{h} = 2.$$

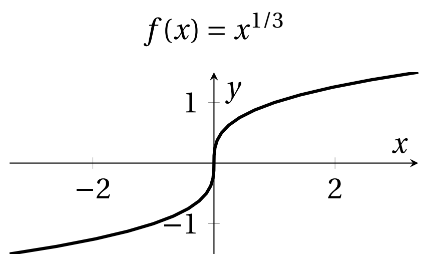
And an equation is $y = 2(x-1)+1$.

Example 63. *Find an equation of the tangent line to the curve $y = x^{1/3} = \sqrt[3]{x}$ at the origin.*

Solution. *The slope of the tangent line is*

$$m = \lim_{h\to 0} \frac{h^{1/3}}{h} = \infty.$$

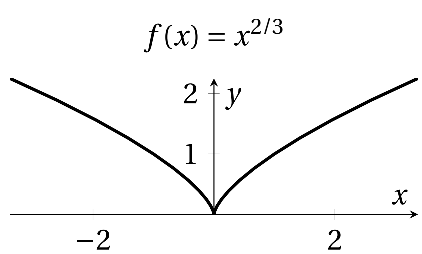
So the tangent line is a vertical line $x = 0$ (in other words the y-axis).



Remark. *Tangent lines to curves such as circles and parabolas do not cross these curves, they just touch at a single point. However, for graphs of functions tangent lines may cross the curve such as above. In fact at inflection points (which we will define later) they always do! For example the tangent line to the graph of $f(x) = x^3$ at $x = 0$ is the y-axis.*

Example 64. *Does $f(x) = x^{2/3}$ have a tangent line at $(0,0)$?*

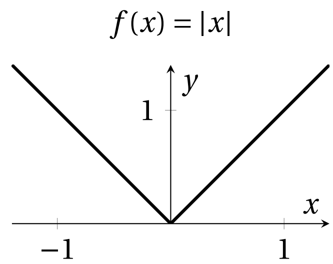
Solution. *The limit of the difference quotient is undefined at 0since the right limit is ∞ while the left limit is $-\infty$. Hence the graph has no tangent line at $(0,0)$.*



"We say that this curve has a cusp at the origin. A cusp is an infinitely sharp point. If you were traveling along the curve, you would have to stop and turn 180° at the origin."

Example 65. *Does $f(x) = |x|$ have a tangent line at $(0,0)$?*

Solution. *The difference quotient is $\frac{|h|}{h}$ which has right limit 1 and left limit -1 at $h = 0$.*



3.2 Derivative

Definition 8. *The **derivative** of a function f at x is*

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

*whenever the limit exists. If $f'(x)$ exists, f is called **differentiable** at x.*

`f'(x) is the slope of the tangent line to the graph of f at (x, f(x)).`

We will regard f' as a function whose domain is those x at which f is differentiable.
Another way of defining derivative is

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

Two limits are equivalent. This can be seen by letting $x = x_0 + h$.

Example 66. *Show that the derivative of the linear function $f(x) = ax + b$ is $f'(x) = a$. In particular the derivative of a constant function is zero.*

Example 67. *Use the definition of the derivative to calculate the derivatives of a) $f(x) = x^2$, b) $f(x) = \frac{1}{x}$, c) $f(x) = \sqrt{x}$.*

The previous three formulas are special cases of the following **Power Rule for Derivative**:

$$f(x) = x^r \implies f'(x) = r x^{r-1}$$

whenever x^{r-1} makes sense.

Example 68.

$$f(x) = x^{5/3} \implies f'(x) = x^{2/3},$$

for all x. How about $f'(-1/8)$?

$$f(x) = \frac{1}{\sqrt{x}} \implies f'(x) = -\frac{1}{2} x^{-3/2}$$

for $x > 0$.

Example 69. *Differentiate the absolute value function $f(x) = |x|$ to get*

$$f'(x) = sgn(x) = \begin{cases} -1, & \text{if } x < 0 \\ 1, & \text{if } x > 0 \end{cases}$$

Note that f is not differentiable at 0.

Example 70. *How should the function $f(x) = x sgn(x)$ be defined at $x = 0$ so that it is continuous there? Is it then differentiable there?*

Notations for Derivative

Let $y = f(x)$. We denote the derivative by

$$y' = f'(x) = \frac{dy}{dx} = \frac{d}{dx} f(x).$$

If we want to evaluate the derivative at point x_0

$$y' \mid_{x=x_0} = f'(x_0) = \frac{dy}{dx} \mid_{x=x_0} = \frac{d}{dx} f(x) \mid_{x=x_0} .$$

The notations y' and $f'(x)$ are *Lagrange notations* for the derivative. The notations $\frac{dy}{dx}$ and $\frac{d}{dx} f(x)$ are called *Leibniz notations* for the derivative.

The Leibniz notation is suggested by the definition of the derivative. Let $\Delta y = f(x+h) - f(x)$ be the increment in y and $\Delta x = x + h - x = h$ be the increment in x. Then

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

3.3 Differentiation Rules

Differentiability is stronger than continuity.

Theorem 16. *If f is differentiable at x then f is cts at x.*

Proof.

$$\lim_{h\to 0} (f(x+h) - f(x)) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \lim_{h\to 0} h = f'(x)0 = 0$$

This means

$$0 = \lim_{h\to 0} f(x+h) - \lim_{h\to 0} f(x) = \lim_{h\to 0} f(x+h) - f(x)$$

Hence

$$\lim_{h\to 0} f(x+h) = f(x)$$

□

Theorem 17. *If f and g are differentiable at x then*

$$(f+g)'(x) = f'(x) + g'(x),$$

$$(f-g)'(x) = f'(x) - g'(x),$$

and for any constant c

$$(cf)'(x) = cf'(x).$$

Proof. Let's prove the derivative of sums is sum of derivatives. The others are similar.

$$\begin{aligned}(f+g)'(x) &= \lim_{h\to 0} \frac{(f+g)(x+h) - (f+g)(x)}{h} = \lim_{h\to 0} \frac{f(x+h) + g(x+h) - f(x) + g(x)}{h} \\ &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} + \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} = f'(x) + g'(x),\end{aligned}$$

□

The sum rule extends to any number of functions.

$$(f_1 + \cdots + f_n)'(x) = f_1'(x) + \cdots + f_n'(x).$$

Example 71. *Take the derivative of*

$$f(x) = 5\sqrt{x} + \frac{3}{x} - 19$$

It is NOT true that derivative of product of functions is a product of their derivatives. Usually $(fg)'(x) \neq f(x)g(x)$.

Theorem 18. *If f and g are differentiable at x then*

$$(fg)'(x) = f'(x)g(x) + f(x)g'(x).$$

Example 72. *Find the derivative of $f(x) = (x^2 + x + 1)(2x + \frac{1}{x})$.*

The product rule can be extended to any number of functions

$$(f_1 f_2 f_3)' = f_1' f_2 f_3 + f_1 f_2' f_3 + f_1 f_2 f_3'$$

$$(f_1 \cdots f_n)' = f_1' f_2 \cdots f_3 + f_1 f_2' f_3 \cdots f_n + \cdots + f_1 \cdots f_{n-1} f_n'.$$

Theorem 19. *If f is differentiable at x and $f(x) \neq 0$ then $1/f$ is diff at x, and*

$$\left(\frac{1}{f}\right)'(x) = \frac{-f'(x)}{f(x)^2}.$$

Proof.

$$\frac{d}{dx}\frac{1}{f(x)} = \lim_{h\to 0} \frac{\dfrac{1}{f(x+h)} - \dfrac{1}{f(x)}}{h} = \lim_{h\to 0} \frac{-1}{f(x+h)f(x)} \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

The result follows by limit rules and continuity of f. □

Example 73. *Differentiate $y = \dfrac{x^5}{x^{2/3} + 1}$.*

Theorem 20. *If f and g are differentiable at x and $g(x) \neq 0$ then*

$$\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}$$

Proof. Using the product rule and reciprocal rule,

$$\left(\frac{f}{g}\right)'(x) = \left(\frac{1}{g}(x)f(x)\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}$$

□

Example 74. *Find the derivative of $f(x) = \dfrac{a + bx}{m + cx}$.*

Example 75. *Find an equation of the tangent line to $y = \dfrac{2}{3 - 4\sqrt{x}}$ at the point $(1, -2)$.*

Solution. *Let us define* $g(x) = 3 - 4\sqrt{x}$*. Then* $g'(x) = -4\frac{1}{2\sqrt{x}} = -\frac{2}{\sqrt{x}}$ *and*

$$y' = 2\frac{-g'(x)}{g(x)^2} = 2\frac{\frac{2}{\sqrt{x}}}{(3-4\sqrt{x})^2} = \frac{4}{\sqrt{x}(3-4\sqrt{x})^2}$$

Hence $y'(1) = 4$*. And the equation of the tangent line is* $y = 4(x-1) - 2$*.*

Example 76. *Find the x-coordinates of points on the curve* $y = \frac{x+1}{x+2}$ *where the tangent line is parallel to the line* $y = 4x$*.*

Solution. *Solving* $y' = 4$*, we find* $x = -3/2$ *and* $x = -5/2$*.*

Example 77. *If* $f(2) = 2$ *and* $f'(2) = 3$*, calculate*

$$\frac{d}{dx}\left(\frac{x^2}{f(x)}\right)\bigg|_{x=2}$$

Solution. *Answer is*

$$\frac{2\cdot 2f(2) - 2^2 f'(2)}{f(2)^2} = \frac{8-12}{4} = -1.$$

3.4 Chain Rule

The following theorem is known as the chain rule.

Theorem 21. *If* $f(u)$ *is differentiable at* $u = g(x)$ *and* $g(x)$ *is differentiable at* x*, then*

$$(f \circ g)'(x) = f'(g(x))g'(x)$$

Derivative of the composite function = derivative of outer function evaluated at inner function times derivative of inner function.

In Leibniz notation, if $y = f(u)$ where $u = g(x)$ then

$$y = f(g(x)) = (f \circ g)(x)$$

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

where $\frac{dy}{du}$ is evaluated at $u = g(x)$.

Example 78. *Find the derivative of* $y = \sqrt{x^2+1}$*.*

Solution. *Here* $y = f(g(x))$ *where* $f(u) = \sqrt{u}$ *and* $u = x^2 + 1$*.*

$$\frac{dy}{dx} = f'(g(x))g'(x) = \frac{1}{2\sqrt{g(x)}}g'(x) = \frac{1}{2\sqrt{x^2+1}}2x = \frac{x}{\sqrt{x^2+1}}.$$

Example 79. *Differentiate* $y = (x^3 - 1)^{1000}$*.*

Solution. *Let* $u = (x^3 - 1)$ *then* $y = u^{1000}$*.* $y' = 1000u^{999}u' = 1000(x^3-1)^{999}3x^2$*.*

Example 80. *Take the derivative* $f(t) = |t^2 - 1|$*.*

Solution.

$$f'(t) = (sgn(t^2-1))(2t) = \begin{cases} 2t, & if\, t < -1, t > 1 \\ -2t, & if -1 < t < 1 \\ undefined & if\, t \pm 1 \end{cases}$$

Example 81. *Express in terms of f and f'.*

a) $\frac{d}{dx} f(x^2)$,

b) $\frac{d}{dx}(f(\pi - 2f(x)))^4$.

Solution. *For (a)*

$$\frac{d}{dx} f(x^2) = f'(x^2)2x$$

For (b)

$$\frac{d}{dx}[f(\pi - 2f(x))]^4 = 4[f(\pi - 2f(x))]^3 f'(\pi - 2f(x))(-2f'(x)).$$

Example 82. *For*

$$f(x) = \left(1 + \sqrt{2x+1}\right)^{-4/3}$$

evaluate $f'(0)$.

Solution.

$$\begin{aligned} f'(x) &= \frac{-4}{3}(1+\sqrt{2x+1})^{-7/3} \frac{d}{dx}\sqrt{2x+1} = \frac{-4}{3}(1+\sqrt{2x+1})^{-7/3} \frac{1}{2\sqrt{2x+1}} \frac{d}{dx}(2x+1) \\ &= \frac{-4}{3}(1+\sqrt{2x+1})^{-7/3} \frac{1}{2\sqrt{2x+1}} 2 \end{aligned}$$

Hence

$$f'(0) = -\frac{1}{2^{1/3}3}.$$

Example 83. *Find an equation of the tangent line to the graph of*

$$y = (1 + x^{2/3})^{3/2}$$

at $x = -1$.

Solution.

$$y' = \frac{3}{2}(1 + x^{2/3})^{1/2} \frac{2}{3} x^{-1/3}.$$

$$y'(-1) = \frac{3}{2}(1+1)^{1/2} \frac{2}{3}(-1) = -\sqrt{2}.$$

3.5 Derivatives of Trigonometric Functions

The radian measure of an angle is defined to be the length of the arc of a unit circle corresponding to that angle.

$$\text{angle in degrees} = \text{angle in radians} \cdot \frac{180^\circ}{\pi}.$$

In calculus all angles are measured in radians. By an angle of $\pi/3$ we mean $\pi/3$ radians or 60° not $(\pi/3)^\circ \approx 1.04^\circ$.

Theorem 22. $\lim_{\theta\to 0} \dfrac{\sin\theta}{\theta} = 1.$

Proof.

Suppose $0 < \theta < \frac{\pi}{2}$.
Area of OQP triangle is $\frac{1}{2}\sin\theta\cos\theta$.
Area of OAP arc is $\frac{\theta}{2\pi}\pi 1^2$.
Area of OAT triangle is $\frac{1}{2}\tan\theta = \frac{\sin\theta}{2\cos\theta}$.

$$\frac{1}{2}\sin\theta\cos\theta \le \frac{\theta}{2} \le \frac{\sin\theta}{2\cos\theta}$$

Multiply by $\frac{2}{\sin\theta} > 0$

$$\cos\theta \le \frac{\theta}{\sin\theta} \le \frac{1}{\cos\theta}$$

Take reciprocal to get

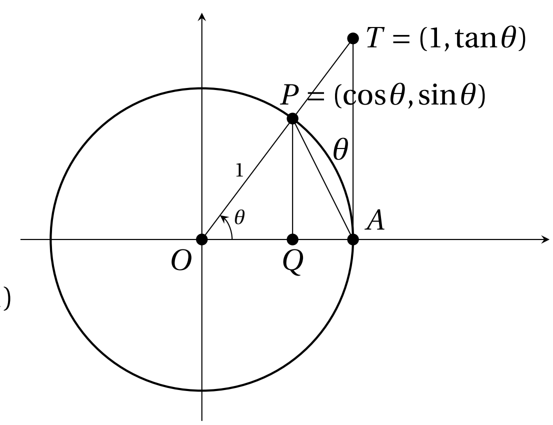
$$\cos\theta \le \frac{\sin\theta}{\theta} \le \frac{1}{\cos\theta}, \tag{3.1}$$

for $0 < \theta < \frac{\pi}{2}$.
Use the squeeze theorem to show that

$$\lim_{\theta\to 0+} \frac{\sin\theta}{\theta} = 1$$

Similarly, we can show that (3.1) holds for $-\frac{\pi}{2} < \theta < 0$ and hence

$$\lim_{\theta\to 0-} \frac{\sin\theta}{\theta} = 1$$



□

Example 84. *Show that* $\lim_{h\to 0} \frac{\cos h - 1}{h} = 0$.

Solution.

$$\begin{aligned}\lim_{h\to 0}\frac{\cos h - 1}{h} &= \lim_{h\to 0}\frac{(\cos h - 1)(\cos h + 1)}{h(\cos h + 1)} = \lim_{h\to 0}\frac{\cos^2 h - 1}{h(\cos h + 1)}\\ &= \lim_{h\to 0}\frac{-\sin^2 h}{h(\cos h + 1)} = -\lim_{h\to 0}\frac{\sin h}{h}\frac{\sin h}{\cos h + 1} = -1\cdot 0 = 0\end{aligned}$$

Theorem 23. $\sin x$ *is differentiable for every* x *and*

$$\frac{d}{dx}\sin x = \cos x$$

Proof.

$$\begin{aligned}\frac{d}{dx}\sin x &= \lim_{h\to 0}\frac{\sin(x+h)+\sin x}{h} = \lim_{h\to 0}\frac{\sin x\cos h+\cos x\sin h-\sin x}{h}\\ &= \lim_{h\to 0}\frac{\sin x(\cos h-1)}{h}+\lim_{h\to 0}\frac{\cos x\sin h}{h} = \sin x\lim_{h\to 0}\frac{(\cos h-1)}{h}+\cos x\lim_{h\to 0}\frac{\sin h}{h} = \cos x\end{aligned}$$

□

Theorem 24. $\cos x$ *is differentiable for every* x *and*

$$\frac{d}{dx}\cos x = -\sin x.$$

Proof.

$$\frac{d}{dx}\cos x = \frac{d}{dx}\sin\left(\frac{\pi}{2}-x\right) = -\cos\left(\frac{\pi}{2}-x\right) = -\sin x.$$

□

Example 85. *Evaluate the derivative of*

a) $\sin(\pi x)+\cos(3x)$,

b) $x^2\cos(\sqrt{x})$,

c) $\dfrac{\cos x}{1-\sin x}$,

d) $\sin(\cos(\tan t))$

The derivatives of the other trigonometric functions

$$\tan x = \frac{\sin x}{\cos x},\quad \sec x = \frac{1}{\cos x},\quad \cot x = \frac{\cos x}{\sin x},\quad \csc x = \frac{1}{\sin x}.$$

Since cos and sin are eveywhere differentiable, the above functions are differentiable everywhere except where their denominators are zero. The derivatives of these functions can be derived by using quotient and reciprocal rules.

$$\frac{d}{dx}\tan x = \sec^2 x,\quad \frac{d}{dx}\sec x\tan x,\quad \frac{d}{dx}\cot x = -\csc^2 x,\quad \frac{d}{dx}\csc = -\csc x\cot x.$$

Example 86. *Verify the derivative formulas for* $\tan x$ *and* $\sec x$.

Example 87. *Find the points on the curve* $y = \tan(2x)$, $-\pi/4 < x < \pi/4$, *where the normal is parallel to the line* $y = -x/8$.

3.6 Higher Order Derivatives

Derivative of derivative is called **second derivative**. If $y = f(x)$ then

$$y'' = f''(x) = \frac{d}{dx}\frac{d}{dx}y = \frac{d^2}{dx^2}y = \frac{d^2}{dx^2}f(x).$$

Similar notations can be used for third, fourth, etc. derivatives. For n-th derivative, we write

$$y^{(n)} = f^{(n)}(x) = \frac{d^n y}{dx^n}$$

Example 88. *Calculate all the derivatives of $y = x^3$.*

Example 89. *Calculate all the derivatives of $y = x^n$ where n is a positive integer.*

Solution.

$$y^{(k)} = \begin{cases} \frac{n!}{(n-k)!}x^{n-k} & if 0 \le k \le n \\ 0 & if k > n \end{cases}$$

Example 90. *Show that if A, B and k are constants, then the function $y = A\cos(kt) + B\sin(kt)$ is a solution of the second order differential equation*

$$\frac{d^2y}{dx^2} + k^2 y = 0.$$

Example 91. *If $y = \tan kx$ show that $y'' = 2k^2 y(1 + y^2)$.*

Example 92. *If f and g are twice differentiable functions, show that*

$$(fg)'' = f''g + 2f'g' + fg''.$$

What do you think about the general formula for $\frac{d^n}{dx^n}(fg)$?

3.7 Mean Value Theorem

Suppose you drive in 2 hours from city A to city B which are 200km apart. That means your average speed was 100km/h. Even if you did not travel constant speed, there was at least one instant where your speed was exactly 100km/h. This is called **mean value theorem**.

Theorem 25 (The Mean-Value Theorem)**.** *Suppose that f is continuous on the interval $[a, b]$ and that it is differentiable on the open interval (a, b). Then there exists a point c in the open interval (a, b) s.t.*

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

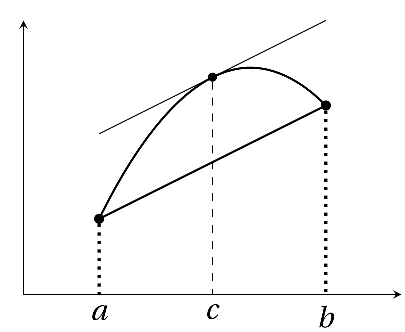


Figure 3.1: Mean Value Theorem says that the slope of the secant line joining two points on the graph of of $f(x)$ is equal to the slope of the tangent line at some point $x = c$ between a and b.

Let $f(t)$ denote the distance from city A. Then $f(0) = 0$ and $f(2) = 200$. Mean Value Theorem says there is a time $t = c$ s.t. $f'(c) = 100$.

Example 93. *Let* $f(x) = |x|$ *on* $[-1,1]$. *Show that there is no* $c \in [-1,1]$ *satisfying the conclusion of the Mean Value Theorem. Why?*

The Mean Value Theorem is an *existence theorem* like Intermediate Value Theorem. In particular

- We don't know how to find c.
- We don't know how many different c can be found satisfying Mean Value Theorem (there is at least one).

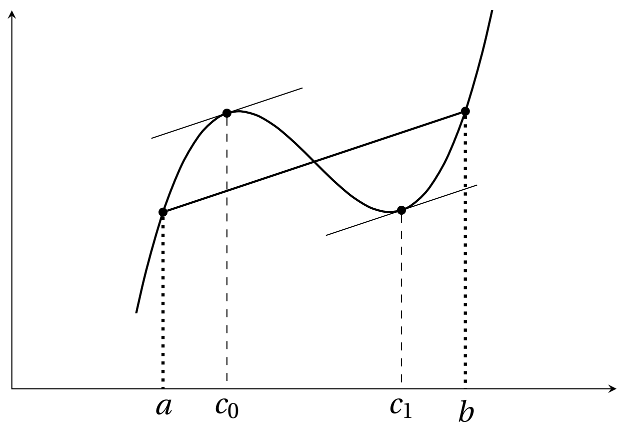


Figure 3.2: There may be more than one c satisfying the conclusion of the Mean Value Theorem.

Example 94. *Show that* $\sin x < x$ *for all* $x > 0$.

Solution. *For* $x > 2\pi$, *we have* $\sin x < 1 < 2\pi < x$. *Now assume* $0 < x < 2\pi$. *By the Mean Value Theorem there exists* c, $0 < c < x$ *such that*

$$\frac{\sin x - \sin 0}{x - 0} = \cos c.$$

Hence $\sin x = x \cos c$. *Since* $0 < c < 2\pi$, $\cos c < 1$. *Since also* $x > 0$, *we have* $x \cos c < x$. *So* $\sin x = x \cos c < x$.

Example 95. *Show that* $\sqrt{1+x} < 1 + \frac{x}{2}$ *for all* $x > 0$.

Solution. *Let* $f(x) = \sqrt{1+x}$. *Then* $f'(c) < \frac{1}{2}$ *for* $c > 0$. *Use Mean Value Theorem.*

Example 96. *Determine all the numbers* c *which satisfy the conclusions of the Mean Value Theorem for*

$$f(x) = x^3 + 2x^2 - x, \qquad x \in [-1,2]$$

Solution. *Solve*

$$3c^2 + 4c - 1 = f'(c) = \frac{f(2) - f(-1)}{2 - (-1)} = \frac{14 - 2}{3} = 4$$

Solutions of $3c^2 + 4c - 5 = 0$ *are*

$$c_{\pm} = \frac{-4 \pm \sqrt{76}}{6}.$$

Notice that only $\frac{-4+\sqrt{76}}{6}$ *lies in* $[-1,2]$.

Example 97. *Suppose f is continuous and differentiable on $[3,9]$. Suppose $f(3) = -4$, and $f'(x) \le 10$ for all x. What is the largest value possible for $f(9)$?*

Solution. *By Mean Value Theorem, there exists $c \in (3,9)$ such that*

$$f(9) - f(3) = f'(c)(9-3) \le 10 \times 6 = 60.$$

So $f(9) \le 60 + f(3) = 56$.

Definition 9. *Suppose f is defined on an interval I. If for all x_1, x_2 in I s.t. $x_2 > x_1$,*

If	Then on I, f is
$f(x_2) > f(x_1)$	increasing
$f(x_2) < f(x_1)$	decreasing
$f(x_2) \ge f(x_1)$	non-decreasing
$f(x_2) \le f(x_1)$	non-increasing

Theorem 26. *Suppose f is differentiable on an open interval I. If for all $x \in I$,*

If	Then on I, f is
$f'(x) > 0$	increasing
$f'(x) < 0$	decreasing
$f'(x) \ge 0$	non-decreasing
$f'(x) \le 0$	non-increasing

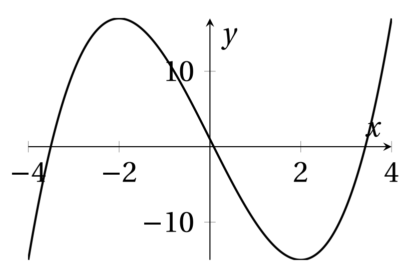
Proof. Let's prove the first statement. Let $x_2 > x_1$ in I. By the Mean Value Theorem, there exists c, $x_1 < c < x_2$, such that $f(x_2) - f(x_1) = f'(c)(x_2 - x_1)$. Since $f'(c) > 0$ and $x_2 - x_1 > 0$, we have $f(x_2) > f(x_1)$. So f is increasing. ☐

Example 98. *On what intervals is $f(x) = x^3 - 12x + 1$ increasing or decreasing?*

Solution.

$f'(x) = 3(x-2)(x+2)$. So f is decreasing on $(-2,2)$ and increasing otherwise.

$f(x) = x^3 - 12x + 1$



We know that if f is a constant function then its derivative is zero. The converse is also true.

Theorem 27. *If f is continuous on the interval I and $f'(x) = 0$ at every interior point x of I then $f(x) = C$, a constant on I.*

Proof. Choose x_0 in I. Let $C = f(x_0)$. If x is any other point in I then by Mean Value Theorem, $f(x) - f(x_0) = f'(c)(x - x_0) = 0$. ☐