ASYMPTOTE

The graph of $y = f(x)$ has a **vertical asymptote** at $x = a$ if

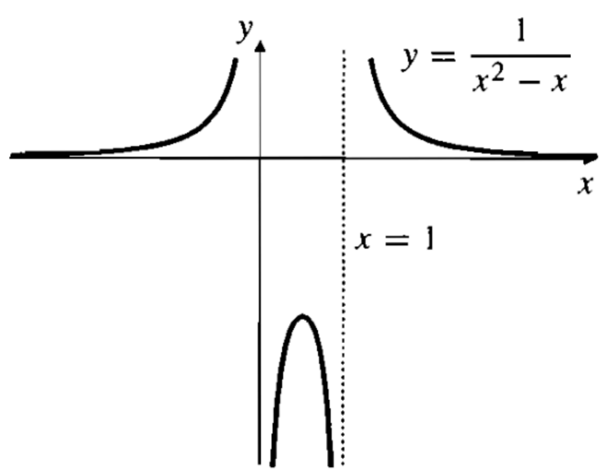
$$\text{either} \quad \lim_{x \to a-} f(x) = \pm\infty \qquad \text{or} \quad \lim_{x \to a+} f(x) = \pm\infty, \qquad \text{or both.}$$

EXAMPLE 1 Find the vertical asymptotes of $f(x) = \dfrac{1}{x^2 - x}$. How does the graph approach these asymptotes?

Solution The denominator $x^2 - x = x(x - 1)$ approaches 0 as x approaches 0 or 1, so f has vertical asymptotes at $x = 0$ and $x = 1$ (Figure 4.35). Since $x(x - 1)$ is positive on $(-\infty, 0)$ and on $(1, \infty)$ and is negative on $(0, 1)$, we have

$$\lim_{x \to 0-} \frac{1}{x^2 - x} = \infty, \qquad \lim_{x \to 1-} \frac{1}{x^2 - x} = -\infty,$$

$$\lim_{x \to 0+} \frac{1}{x^2 - x} = -\infty, \qquad \lim_{x \to 1+} \frac{1}{x^2 - x} = \infty.$$



The graph of $y = f(x)$ has a **horizontal asymptote** $y = L$ if

$$\text{either} \quad \lim_{x \to \infty} f(x) = L \qquad \text{or} \quad \lim_{x \to -\infty} f(x) = L, \qquad \text{or both.}$$

EXAMPLE 2 Find the horizontal asymptotes of

(a) $f(x) = \dfrac{1}{x^2 - x}$ and (b) $g(x) = \dfrac{x^4 + x^2}{x^4 + 1}$.

Solution

(a) The function f has horizontal asymptote $y = 0$ (Figure 4.35) since

$$\lim_{x\to\pm\infty} \frac{1}{x^2 - x} = \lim_{x\to\pm\infty} \frac{1/x^2}{1 - (1/x)} = \frac{0}{1} = 0.$$

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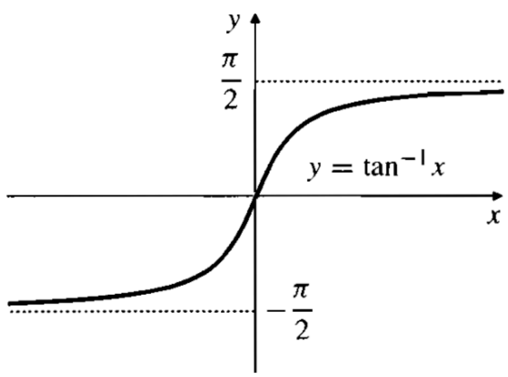
(b) The function g has horizontal asymptote $y = 1$ (Figure 4.36) since

$$\lim_{x\to\pm\infty} \frac{x^4+x^2}{x^4+1} = \lim_{x\to\pm\infty} \frac{1+(1/x^2)}{1+(1/x^4)} = \frac{1}{1} = 1.$$

Observe that the graph of g crosses its asymptote twice. (There is a popular misconception among students that curves cannot cross their asymptotes. Exercise 41 below gives an example of a curve that crosses its asymptote infinitely often.)

The horizontal asymptotes of both functions f and g in Example 2 are **two-sided**, which means that the graphs approach the asymptotes as x approaches both infinity and negative infinity. The function $\tan^{-1} x$ has two **one-sided** asymptotes, $y = \pi/2$ (as $x \to \infty$) and $y = -(\pi/2)$ (as $x \to -\infty$). See Figure 4.37.



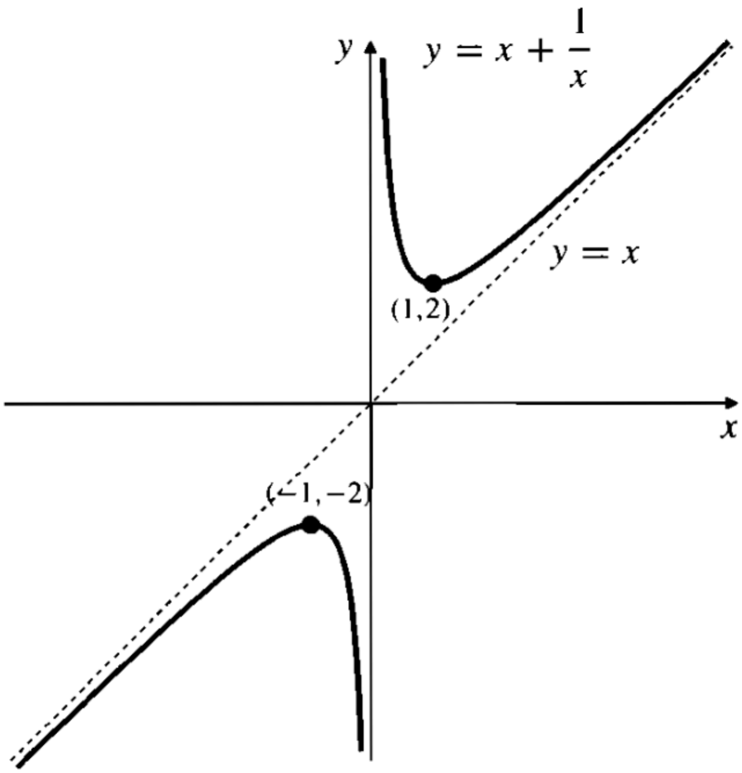
The straight line $y = ax + b$ (where $a \neq 0$) is an **oblique asymptote** of the graph of $y = f(x)$ if

$$\text{either} \quad \lim_{x\to-\infty} \big(f(x) - (ax+b)\big) = 0 \quad \text{or} \quad \lim_{x\to\infty} \big(f(x) - (ax+b)\big) = 0,$$

or both.

EXAMPLE 3 Consider the function $f(x) = \dfrac{x^2+1}{x} = x + \dfrac{1}{x}$, whose graph is shown in Figure 4.38(a). The straight line $y = x$ is a *two-sided* oblique asymptote of the graph of f because

$$\lim_{x\to\pm\infty} \big(f(x) - x\big) = \lim_{x\to\pm\infty} \frac{1}{x} = 0.$$

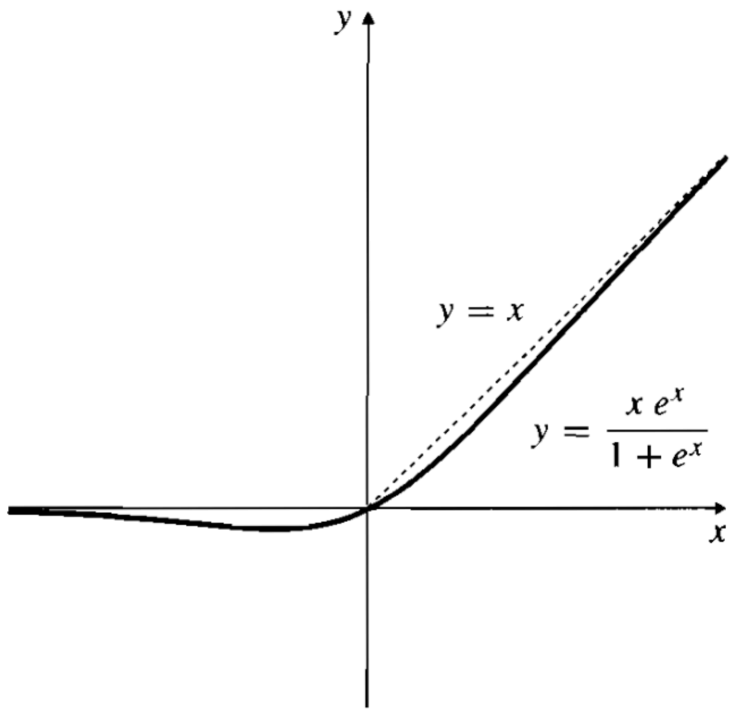


(a)

EXAMPLE 4 The graph of $y = \dfrac{x\,e^x}{1+e^x}$, shown in Figure 4.38(b), has a horizontal asymptote $y = 0$ at the left and an oblique asymptote $y = x$ at the right:

$$\lim_{x\to-\infty} \frac{x\,e^x}{1+e^x} = \frac{0}{1} = 0 \qquad \text{and}$$

$$\lim_{x\to\infty} \left(\frac{x\,e^x}{1+e^x} - x \right) = \lim_{x\to\infty} \frac{x(e^x - 1 - e^x)}{1+e^x} = \lim_{x\to\infty} \frac{-x}{1+e^x} = 0.$$


y
$y = x$
$y = \frac{x\,e^x}{1+e^x}$
x

Asymptotes of a rational function

Suppose that $f(x) = \dfrac{P_m(x)}{Q_n(x)}$, where P_m and Q_n are polynomials of degree m and n, respectively. Suppose also that P_m and Q_n have no common linear factors. Then

(a) The graph of f has a vertical asymptote at every position x such that $Q_n(x) = 0$.

(b) The graph of f has a two-sided horizontal asymptote $y = 0$ if $m < n$.

(c) The graph of f has a two-sided horizontal asymptote $y = L$, ($L \neq 0$) if $m = n$. L is the quotient of the coefficients of the highest degree terms in P_m and Q_n.

(d) The graph of f has a two-sided oblique asymptote if $m = n + 1$. This asymptote can be found by dividing Q_n into P_m to obtain a linear quotient, $ax + b$, and remainder, R, a polynomial of degree at most $n - 1$. That is,

$$f(x) = ax + b + \frac{R(x)}{Q_n(x)}.$$

The oblique asymptote is $y = ax + b$.

(e) The graph of f has no horizontal or oblique asymptotes if $m > n + 1$.

Checklist for curve sketching

1. Calculate $f'(x)$ and $f''(x)$, and express the results in factored form.
2. Examine $f(x)$ to determine its domain and the following items:
   (a) Any vertical asymptotes. (Look for zeros of denominators.)
   (b) Any horizontal or oblique asymptotes. (Consider $\lim_{x\to\pm\infty} f(x)$.)
   (c) Any obvious symmetry. (Is f even or odd?)
   (d) Any easily calculated intercepts (points with coordinates $(x, 0)$ or $(0, y)$) or endpoints or other "obvious" points. You will add to this list when you know any critical points, singular points, and inflection points. Eventually you should make sure you know the coordinates of at least one point on every component of the graph.
3. Examine $f'(x)$ for the following:
   (a) Any critical points.
   (b) Any points where f' is not defined. (These will include singular points, endpoints of the domain of f, and vertical asymptotes.)
   (c) Intervals on which f' is positive or negative. It's a good idea to convey this information in the form of a chart such as those used in the examples. Conclusions about where f is increasing and decreasing and classification of some critical and singular points as local maxima and minima can also be indicated on the chart.
4. Examine $f''(x)$ for the following:
   (a) Points where $f''(x) = 0$.
   (b) Points where $f''(x)$ is undefined. (These will include singular points, endpoints, vertical asymptotes, and possibly other points as well, where f' is defined but f'' isn't.)
   (c) Intervals where f'' is positive or negative and where f is therefore concave up or down. Use a chart.
   (d) Any inflection points.

EXAMPLE 6 Sketch the graph of $y = \dfrac{x^2 + 2x + 4}{2x}$.

Solution It is useful to rewrite the function y in the form

$$y = \frac{x}{2} + 1 + \frac{2}{x},$$

since this form not only shows clearly that $y = (x/2) + 1$ is an oblique asymptote, but also makes it easier to calculate the derivatives

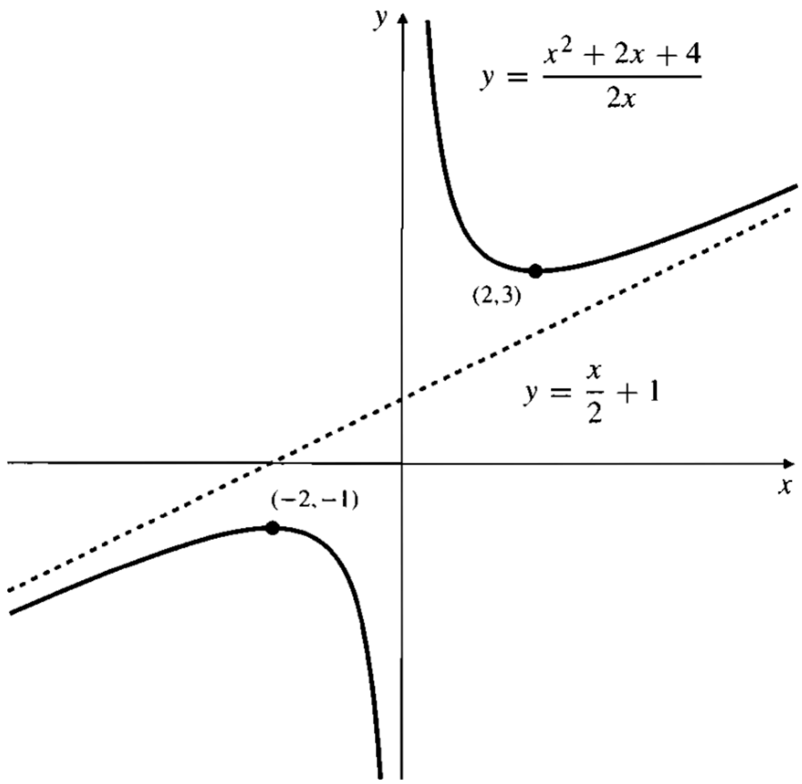
$$y' = \frac{1}{2} - \frac{2}{x^2} = \frac{x^2 - 4}{2x^2}, \qquad y'' = \frac{4}{x^3}.$$

From y: Domain: all x except 0. Vertical asymptote: $x = 0$,
Oblique asymptote: $y = \dfrac{x}{2} + 1$, $\quad y - \left(\dfrac{x}{2} + 1\right) = \dfrac{2}{x} \to 0$ as $x \to \pm\infty$.
Symmetry: none obvious (y is neither odd nor even).
Intercepts: none. $x^2 + 2x + 4 = (x + 1)^2 + 3 \geq 3$ for all x, and y is not defined at $x = 0$.

From y': Critical points: $x = \pm 2$; points $(-2, -1)$ and $(2, 3)$.
y' not defined at $x = 0$ (vertical asymptote).

From y'': $y'' = 0$ nowhere; y'' undefined at $x = 0$.

		CP		ASY		CP	
x		-2		0		2	
y'	$+$	0	$-$	undef	$-$	0	$+$
y''	$-$		$-$	undef	$+$		$+$
y	$\nearrow$	max	$\searrow$	undef	$\searrow$	min	$\nearrow$
	$\frown$		$\frown$		$\smile$		$\smile$



EXAMPLE 8 Sketch the graph of $y = xe^{-x^2/2}$.

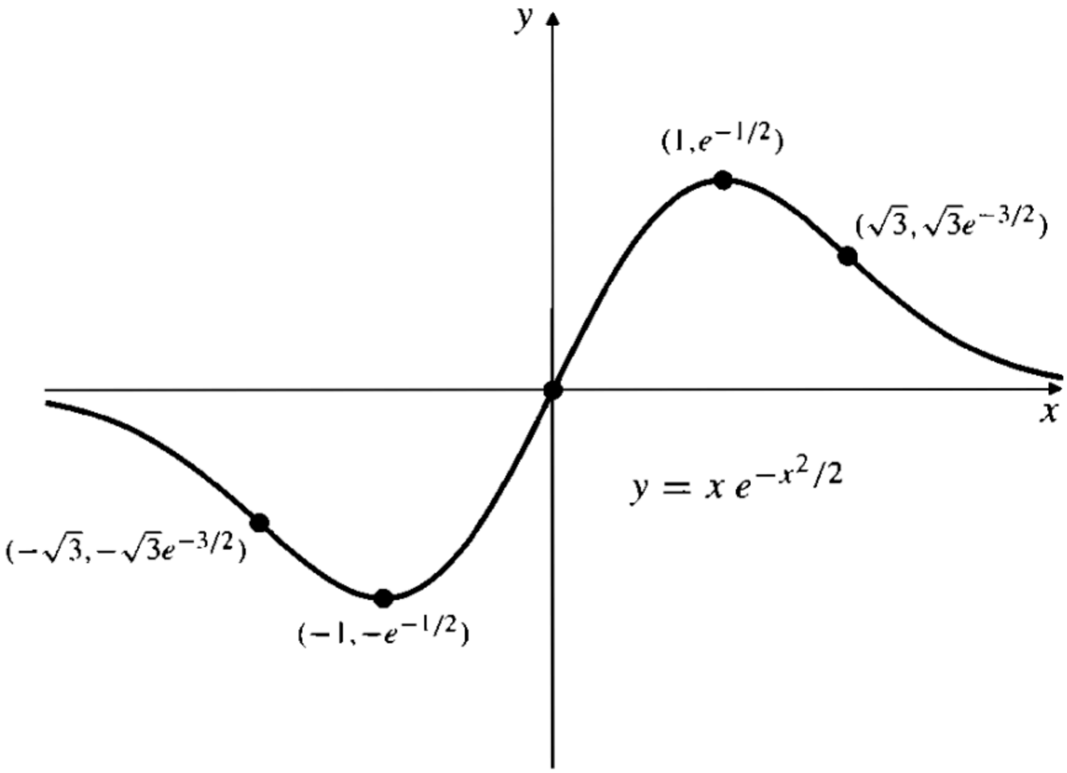
Solution We have $y' = (1 - x^2)e^{-x^2/2}$, $\quad y'' = x(x^2 - 3)e^{-x^2/2}$.

From y: Domain: all x.
Horizontal asymptote: $y = 0$. Note that if $t = x^2/2$, then $|xe^{-x^2/2}| = \sqrt{2t}\, e^{-t} \to 0$ as $t \to \infty$ (hence as $x \to \pm\infty$).
Symmetry: about the origin (y is odd). Intercepts: $(0, 0)$.

From y': Critical points: $x = \pm 1$; points $(\pm 1, \pm 1/\sqrt{e}) \approx (\pm 1, \pm 0.61)$.

From y'': $y'' = 0$ at $x = 0$ and $x = \pm\sqrt{3}$;
points $(0, 0)$, $(\pm\sqrt{3}, \pm\sqrt{3}e^{-3/2}) \approx (\pm 1.73, \pm 0.39)$.

				CP				CP			
x		$-\sqrt{3}$		-1		0		1		$\sqrt{3}$	
y'	$-$		$-$	0	$+$		$+$	0	$-$		$-$
y''	$-$	0	$+$		$+$	0	$-$		$-$	0	$+$
y	$\searrow$		$\searrow$	min	$\nearrow$		$\nearrow$	max	$\searrow$		$\searrow$
	$\frown$	infl	$\smile$		$\smile$	infl	$\frown$		$\frown$	infl	$\smile$



EXAMPLE 9 Sketch the graph of $f(x) = (x^2 - 1)^{2/3}$. (See Figure 4.42.)

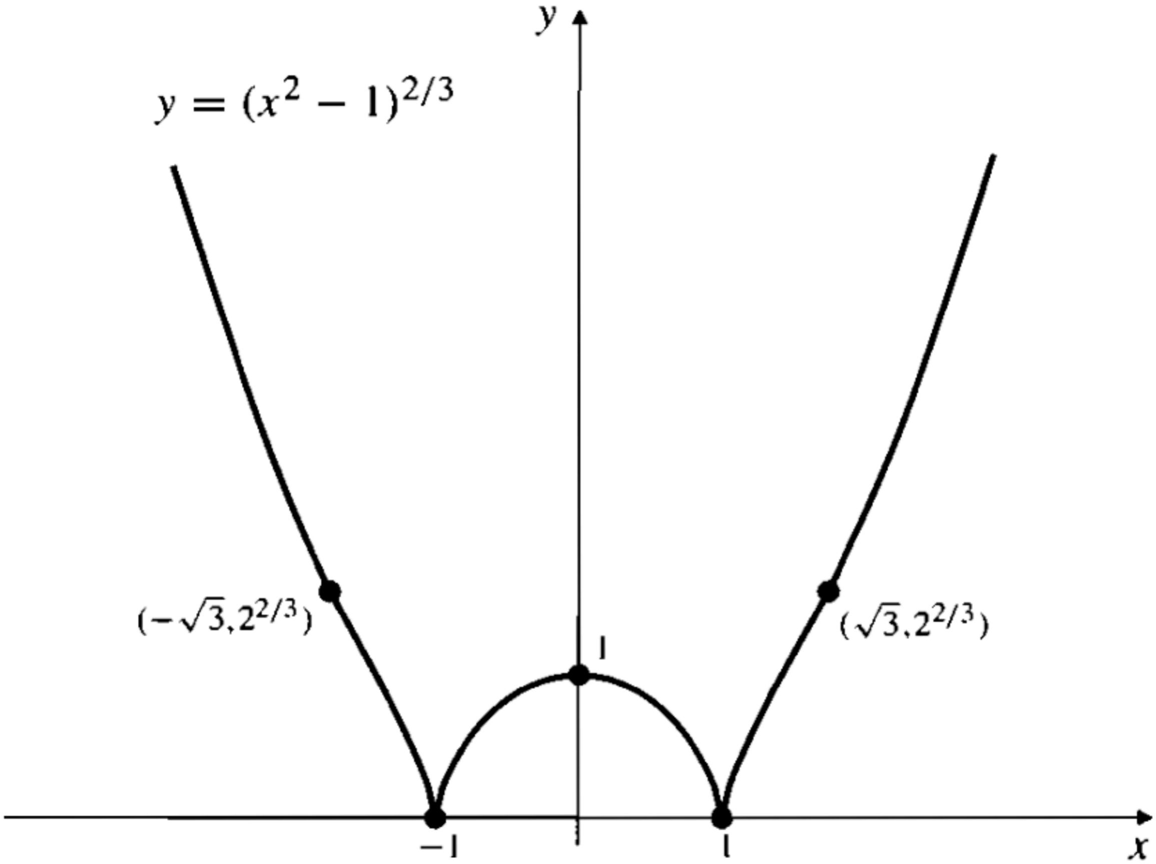
Solution $f'(x) = \dfrac{4}{3}\dfrac{x}{(x^2-1)^{1/3}}, \qquad f''(x) = \dfrac{4}{9}\dfrac{x^2-3}{(x^2-1)^{4/3}}.$

From f: Domain: all x.
Asymptotes: none. ($f(x)$ grows like $x^{4/3}$ as $x \to \pm\infty$.)
Symmetry: about the y-axis (f is an even function).
Intercepts: $(\pm 1, 0)$, $(0, 1)$.

From f': Critical points: $x = 0$; singular points: $x = \pm 1$.

From f'': $f''(x) = 0$ at $x = \pm\sqrt{3}$; points $(\pm\sqrt{3}, 2^{2/3}) \approx (\pm 1.73, 1.59)$;
$f''(x)$ not defined at $x = \pm 1$.

				SP		CP		SP			
x		$-\sqrt{3}$		-1		0		1		$\sqrt{3}$	
f'	$-$		$-$	undef	$+$	0	$-$	undef	$+$		$+$
f''	$+$	0	$-$	undef	$-$		$-$	undef	$-$	0	$+$
f	$\searrow$		$\searrow$	min	$\nearrow$	max	$\searrow$	min	$\nearrow$		$\nearrow$
	$\smile$	infl	$\frown$		$\frown$		$\frown$		$\frown$	infl	$\smile$



4.8 Extreme-Value Problems

EXAMPLE 1 A rectangular animal enclosure is to be constructed having one side along an existing long wall and the other three sides fenced. If 100 m of fence are available, what is the largest possible area for the enclosure?

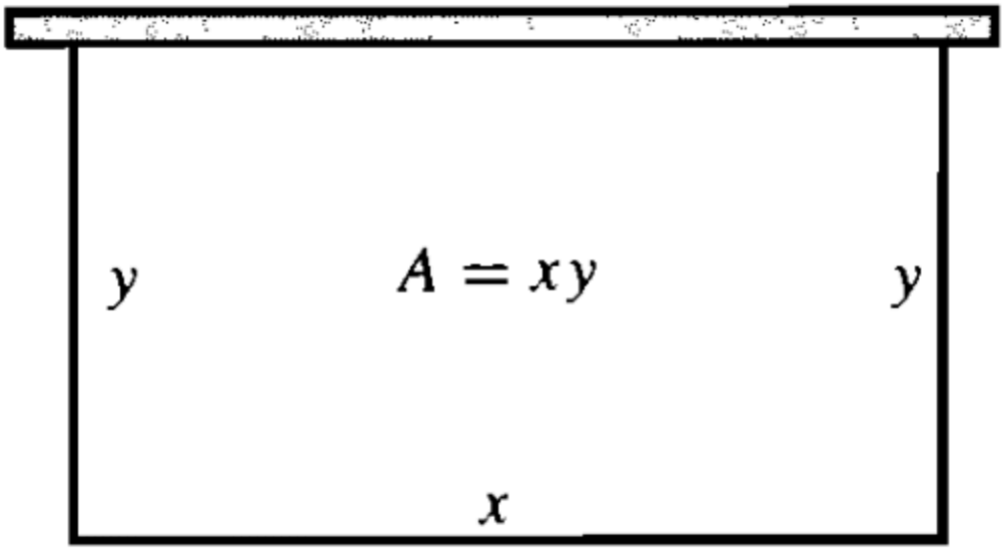
Solution This problem, like many others, is essentially a geometric one. A sketch should be made at the outset, as we have done in Figure 4.50. Let the length and width of the enclosure be x and y m, respectively, and let its area be A m^2. Thus $A = xy$. Since the total length of the fence is 100 m, we must have $x + 2y = 100$. A appears to be a function of two variables, x and y, but these variables are not independent; they are related by the *constraint* $x + 2y = 100$. This constraint equation can be solved for one variable in terms of the other, and A can therefore be written as a function of only one variable:

$$x = 100 - 2y,$$
$$A = A(y) = (100 - 2y)y = 100y - 2y^2.$$

Evidently, we require $y \geq 0$ and $y \leq 50$ (i.e., $x \geq 0$), in order that the area make sense. (It would otherwise be negative.) Thus, we must maximize the function $A(y)$ on the interval $[0, 50]$. Being continuous on this closed, finite interval, A must have a maximum value, by Theorem 5. Clearly, $A(0) = A(50) = 0$ and $A(y) > 0$ for $0 < y < 50$. Hence, the maximum cannot occur at an endpoint. Since A has no singular points, the maximum must occur at a critical point. To find any critical points, we set

$$0 = A'(y) = 100 - 4y.$$

Therefore, $y = 25$. Since A must have a maximum value and there is only one possible point where it can be, the maximum must occur at $y = 25$. The greatest possible area for the enclosure is therefore $A(25) = 1{,}250$ m^2.


y
A = xy
y
x