Contents

Precalculus

P-0-Sets

A set is a collection of elements.

$x \in A$ means x is an element of the set A. If x is not a member of A, we write $x \notin A$.

$\varnothing$ is the set which contains no element and is called the empty set.

There are finite sets (ex. $\{0, 1, 2\}$) and infinite sets (ex. $\{0, 1, 2, 3, \ldots\}$).

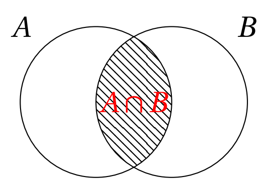
If every element of the set A is an element of the set B, we say that A is subset of B, and write $A \subset B$.

Example 1. *List all the subsets of* $\{0, 1, 2\}$.

For any set A, $A \subset A$ and $\varnothing \subset A$.

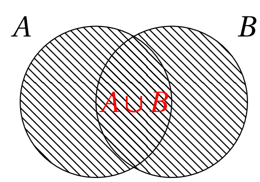
If $A \subset B$ and $B \subset A$, we write $A = B$.

$A \cap B = \{x : x \in A \text{ and } x \in B\}$ is called the intersection of A and B.



If the intersection of two sets is the empty set, those sets are called disjoint.

$A \cup B = \{x : x \in A \text{ or } x \in B\}$ is called the union of A and B.



Example 2. *For example if* $A = \{0, 1, 2, 5, 8\}$ *and* $B = \{1, 3, 5, 6\}$ *then find* $A \cap B$ *and* $A \cup B$.

The set of all elements in A but not in B is denoted $A \setminus B = \{x \in A : x \notin B\}$ and is called the complemet of B in A.

Example 3. $\{0, 2, 3, 5\} \setminus \{2, 5, 7, 8\} = \{0, 3\}$

$A \times B = \{(a, b) : a \in A \text{ and } b \in B\}$ is called the Cartesian product of the sets A and B.

Example 4. *Write the cartesian product of* $A = \{0, 1, 2\}$ *and* $B = \{2, 3, 4\}$.

P-1-Real Numbers

The **integers** are $\mathbb{Z} = \{\ldots,-2,-1,0,1,2,\ldots\}$.

Integers come in a lot varities:

- even integers that are of the form $2k$, for some $k \in \mathbb{Z}$,
- odd integers that are of the form $2k+1$, for some $k \in \mathbb{Z}$
- positive and negative integers,
- primes, etc...

The **rational numbers** are $\mathbb{Q} = \left\{\frac{m}{n} : m, n \in \mathbb{Z} \text{ and } n \neq 0\right\}$.

Pythagoreans preached that all numbers could be expressed as the ratio of integers, and the discovery of irrational numbers is said to have shocked them.

Example 5. *$\sqrt{2}$ is not a rational number.*

Suppose that it is rational. Then $\sqrt{2} = m/n$, where $m, n \in \mathbb{Z}$ and $n \neq 0$. Also assume m and n have no common divisor.

$$m^2/n^2 = 2 \implies m^2 = 2n^2$$

Thus m is even and we can write $m = 2k$, where $k \in \mathbb{Z}$.

$$4k^2 = 2n^2 \implies n^2 = 2k^2$$

Thus n is also even. But m and n cannot both be even. Accordingly, there can be no rational number whose square is 2.

The set of irrational numbers is denoted by $\mathbb{I}$.
The set of real numbers is $\mathbb{R} = \mathbb{Q} \cup \mathbb{I}$.
Note that $\mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$.
The real numbers are ordered such that

1. $a < b \implies a + c < b + c$
2. $a < b$ and $c > 0$ implies $ac < bc$
3. $a < b$ and $c < 0$ implies $ac > bc$
4. $a > 0$ implies $\frac{1}{a} > 0$
5. $0 < a < b$ implies $\frac{1}{b} < \frac{1}{a}$

Intervals

The open interval $(a,b) = \{x \mid a < x < b\}$, closed interval ($[a,b]$), half open intervals $(a,b]$, $[a,b)$. It is possible that $a = -\infty$, $b = \infty$. Draw each interval on the real line.

Example 6. *Solve the following inequalities.*

1. $\frac{2}{x-1} \geq 5$.

Solution. It is not right to multiply both sides by $x-1$ *and say* $5x-5 \leq 2$.

$$\frac{2}{x-1} \geq 5 \iff \frac{2}{x-1} - 5 \geq 0 \iff \frac{7-5x}{x-1} \geq 0.$$

Now make a sign analysis to get interval $(1, 7/5]$

2. $3x - 1 \leq 5x + 3 \leq 2x + 15$.

Solution. $-2 \leq x$ *and* $x \leq 4$.

The absolute value.

$$|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$$

ex. $|3| = |-3| = 3$

Geometrically, $|x|$ is the distance between x and 0 on the real line. And $|x-y|$ is the distance between x and y.

Properties *(can be proved from definition)*:

1. $|-x| = |x|$, (Do not fall into the trap $|-x| = x$, this is not always true!)

2. $|ab| = |a||b|$,

3. $|a+b| \leq |a| + |b|$, (triangle inequality).

From (2), for any x, $x^2 = |x^2| = |x|^2$

If D is a nonnegative number

$$|x| = D \implies x = -D \text{ or } x = D,$$

$$|x| < D \implies -D < x < D$$

$$|x| > D \implies x < -D \text{ or } x > D$$

More generally,

$$|x-a| = D \implies x = a - D \text{ or } x = a + D,$$

$$|x-a| < D \implies a - D < x < a + D$$

$$|x-a| > D \implies x < a - D \text{ or } x > a + D$$

Example 7. *Solve* $|3x-2| \leq 1$.

Solution.

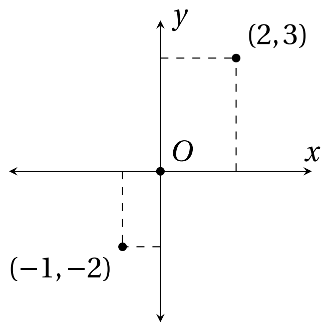
$$-1 \leq 3x - 2 \leq 1 \implies x \geq 1/3 \textit{ and } x \leq 1.$$

Example 8. *Solve the equation* $|x+1| > |x-3|$.

Solution. The distance between x and -1 *is greater than the distance between x and* 3. *So* $x > 1$.

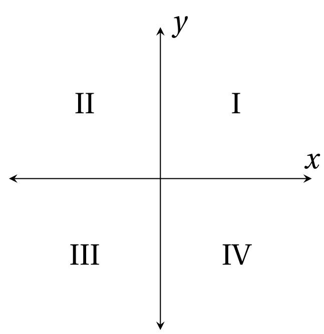
P-2-Cartesian Coordinates

Cartesian plane is $\mathbb{R} \times \mathbb{R} = \{(x, y) \mid x \in \mathbb{R} \text{ and } b \in \mathbb{R}\}$.



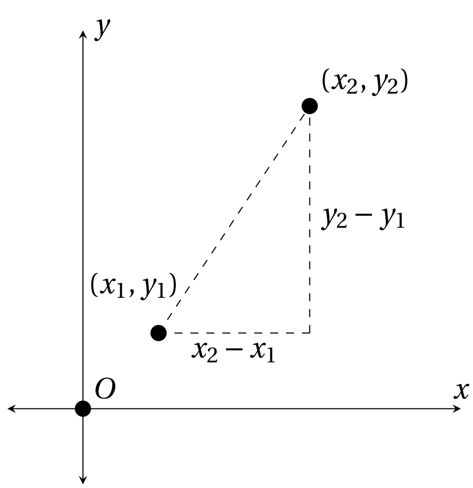
Horizontal axis is usually called the x axis, the vertical axis is called the y axis. Intersection of the axes is called the origin, denoted O.

The coordinate axes divide the Cartesian plane into four quadrants.



Distance of two points (x_1, y_1) and (x_2, y_2) in the plane is

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$



The distance of (x, y) to the origin is $\sqrt{x^2 + y^2}$.

Example 9. *Find the distance between* $(-1, 1)$ *and* $(3, -4)$.

The **graph** of an equation (or inequality) is the set of all (x, y) satisfying that equation (or inequality).

Equations of Lines. For any two points (x_1, y_1) and (x_2, y_2) on a non-vertical line L, the quantity $m = \frac{y_2-y_1}{x_2-x_1}$ is constant and is called the **slope** of the line L.

Let L be a nonvertical line. Let m be the slope of L and (x_1, y_1) be the coordinates of a point on L. If (x, y) is another point on L, then

$$\frac{y-y_1}{x-x_1} = m$$

so that an equation for L is

$$y = m(x-x_1) + y_1$$

All points on a **vertical line** have their x coordinate equal to a constant a. So the equation of a vertical line is $x = a$. **Horizontal lines** have equations of the form $y = a$.

y-intercept of a nonvertical line L is the y-coordinate of the point where L intersects the y-axis. x-intercept of a nonhorizontal is defined similarly.

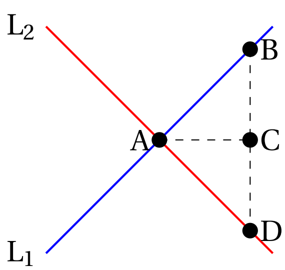
Example 10. *Find an equation of the line through the points* $(1,-1)$ *and* $(3,5)$. *Draw the line. Find the x and y intercepts.*

Example 11. *Find an equation of the line that passes through the point* $(-3,-4)$ *and has slope* 2. *Draw the line.*

Example 12. *Find the slope and the two intercepts of the line with equation* $8x + 5y = 20$. *Draw the line.*

Parallel vs. perpendicular lines;

1. Two nonvertical lines are parallel if and only if their slopes are equal.
2. Two nonvertical lines with slopes m_1 and m_2 are perpendicular if and only if $m_1 m_2 = -1$. This condition is equivalent to their angle of intersection being 90°.



proof. Use the similarity of the triangles ABC and DAC to get

$$\frac{|BC|}{|AC|} = \frac{|AC|}{|CD|} \implies \frac{|BC||CD|}{|AC|^2} = 1$$

Slope of L_1 (m_1) is $|BC|/|AC| = 1$ and slope of L_2 (m_2) is $-|CD|/|AC|$. So $m_1 m_2 = -1$.

Example 13. *Find an equation of the line through* $(1,-2)$ *that is parallel to the line L with equation* $3x - 2y = 1$. *Draw the lines.*

Example 14. *Find an equation of the line through* $(2,-3)$ *that is perpendicular to the line L with equation* $4x + y = 3$. *Draw the lines.*

P-3-Quadratic Equations

Circles and Disks The circle is the set of all points that have the same distance (called radius of the circle) from a given point (called center of the circle).

If (x, y) is a point on a circle with center (a, b) and radius r then

$$\sqrt{(x-a)^2+(y-b)^2}=r \implies (x-a)^2+(y-b)^2=r^2$$

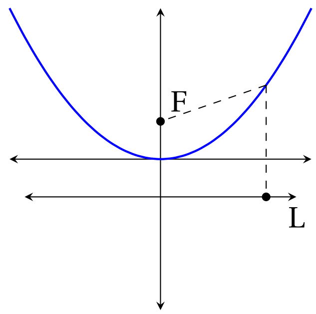
Example 15. *Find the center and radius of the circle* $x^2+y^2-4x+6y=3$.

Solution. Complete to squares to get $(x-2)^2+(y+3)^2=16$.

The equation $(x-a)^2+(y-b)^2<r^2$ represents open disk and the equation $(x-a)^2+(y-b)^2\le r^2$ represents closed disk or simply disk.

Example 16. *Draw* $x^2+2x+y^2\le 8$.

A **parabola** P is the set of all points in the plane that are equidistant from a given line L (called directrix of P) and a point F (called the focus of P).



Example 17. *Find the equation of the parabola having the point* $F(0, p)$*as focus and the line L with equation* $y=-p$ *as directrix.*

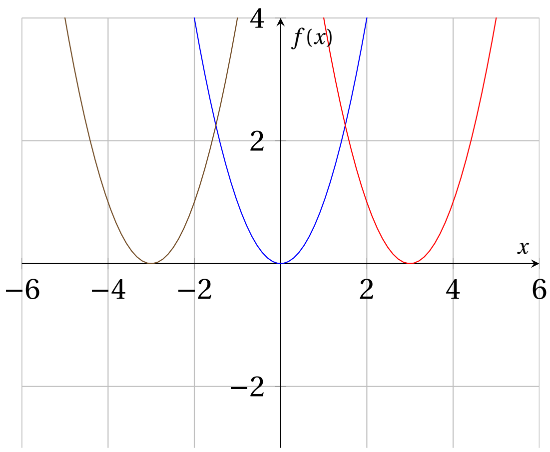
Solution. If $P(x, y)$ *is any point on the parabola then squaring both sides of PF=PQ we get*

$$x^2+(y-p)^2=0^2+(y+p)^2$$

After simplifying, $y=x^2/4p$.

Shifting a Graph Let $c>0$.

- To shift a graph c units to the right, replace x in its equation with $x-c$. To shift to left, replace x by $x+c$.

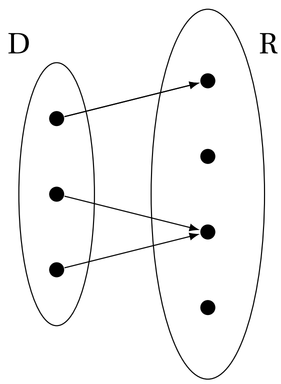


P-4-Functions and Their Graphs

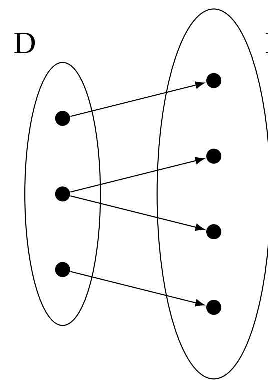
A **function** f on a set D into a set R is a rule that assigns a unique element $f(x)$ in R to each element x in D.

D is called the **domain** of f. The **range** of f is a subset of R containing of all possible values $f(x)$.

This definition is not mathematical as we did not define what a rule is. Formally one defines a function as a relation.



[Not a function]



[A Function]

Example 18. *Define a function on the set of all real numbers by $f(x) = x^2 + 1$. Find $f(0)$, $f(2)$, $f(x+2)$.*

$$f(x) = \frac{1}{x}, \qquad x > 0$$

means that the domain of f is the set $\{x \mid x > 0\}$.

Technically, this function is different from the function

$$f(x) = \frac{1}{x}, \qquad x < 0.$$

If we do not specify the domain of a function f, then the **domain convention** is to assume that the domain of f is the set of all real numbers for which f is defined.

So if we write

$$f(x) = \frac{1}{x},$$

we are assuming f is defined for all real numbers except 0.

Example 19. *Find the domain of $f(x) = \sqrt{2-x}$.*

Solution. *Its domain is all x for which $2 - x \geq 0$, i.e. the interval $(-\infty, 2]$.*

Example 20. *Find the domain of $f(x) = \frac{1}{x^2 - x}$.*

A function $f : D \to R$ is **1-1** if $f(x_1) = f(x_2)$ then $x_1 = x_2$. A function $f : D \to R$ is **onto** if for every $y \in R$, there is an $x \in D$ such that $f(x) = y$.

Example 21. *Draw functions which are 1-1, onto, not 1-1 and not onto, similar to the Figure .*

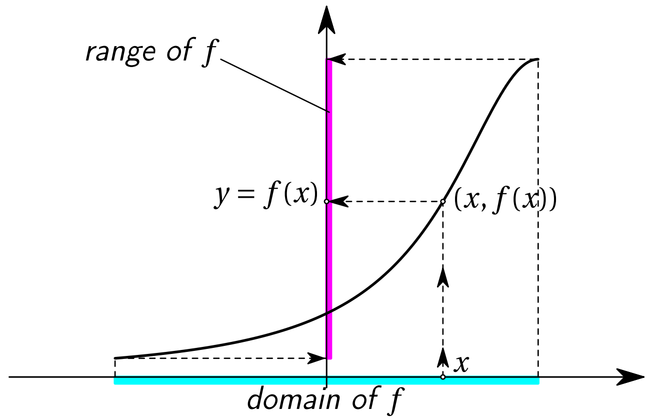


Figure 0.1: The graph of a function f. The domain of f consists of all x values at which the function is defined, and the range consists of all possible values f can have.

Graph of a function

The *graph of a function* f is the set of all points whose coordinates are $(x, f(x))$ where x is in the domain of f.

Example 22. *A function which is given by the formula*

$$f(x) = mx + n$$

where m and n are constants is called a linear function. *Its graph is a straight line. The constants m and n are the* slope *and y-*intercept *of the line.*

Example 23. *The square root function* $f(x) = \sqrt{x}$ *has domain* $[0, \infty)$ *and takes x to its positive square root. Hence it has range* $[0, \infty)$.

Example 24. *The absolute value function* $f(x) = |x| = \sqrt{x^2}$ *has domain* $(-\infty, \infty)$ *and range* $[0, \infty)$.

Example 25. *Draw the graphs of some elementary functions*

$$c, x, x^2, \sqrt{x}, x^3, x^{1/3}, \frac{1}{x}, \frac{1}{x^2}, \sqrt{1-x^2}, |x|.$$

Example 26. *Sketch the graph of* $f(x) = 1 + \sqrt{x-4}$.
Solution: Shift the graph of $y = \sqrt{x}$ *1 unit up and 4 units to the right.*

Example 27. *Sketch the graph of the function* $f(x) = \frac{2-x}{x-1}$.
Solution. $f(x) = \frac{2-x}{x-1} = -1 + \frac{1}{x-1}$. *So shift the graph of* $y = \frac{1}{x}$ *1 unit down and 1 unit to the right.*

Vertical Line Test

The graph of a function cannot intersect a vertical line "$x =$ constant" in more than one point.

The collection of points determined by the equation $x^2 + y^2 = 1$ is a circle. It is not the graph of a function since the vertical line $x = 0$ (the y-axis) intersects the graph in two points $P_1(0, 1)$ and $P_2(0, -1)$. See Figure 0.2.

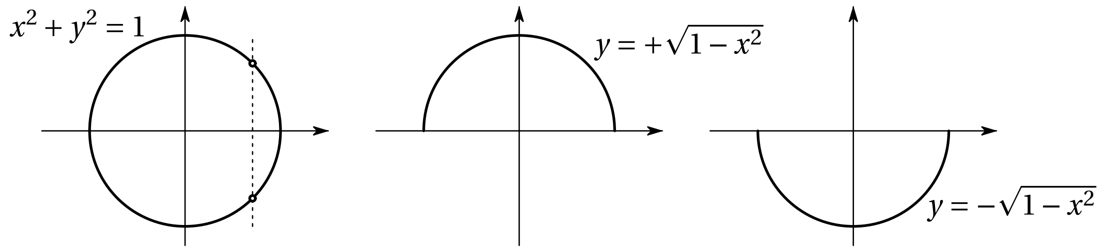


Figure 0.2: The circle determined by $x^2+y^2=1$ is not the graph of a function, but it contains the graphs of the two functions $h_1(x)=\sqrt{1-x^2}$ and $h_2(x)=-\sqrt{1-x^2}$.

Even and Odd Functions

definition 1. *We say that f is an **even function** if $f(-x)=f(x)$ for every $x\in D$. We say that f is an **odd function** if $f(-x)=-f(x)$ for every $x\in D$. See Figure 0.3.*

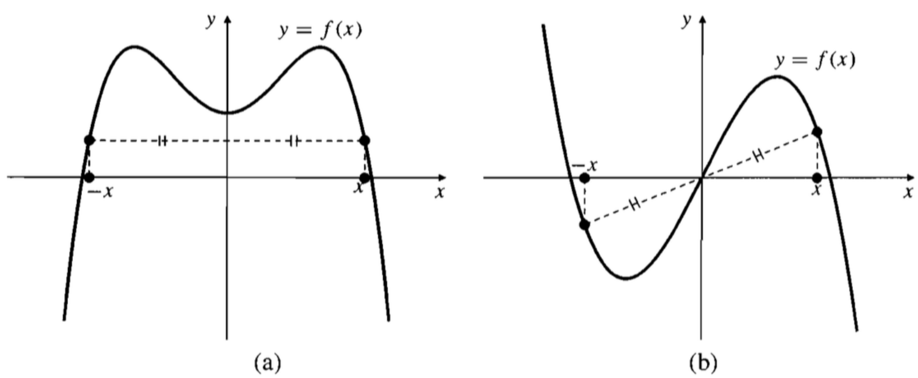


Figure 0.3: a) An even function, b) an odd function.

Example 28. *$f(x)=x$, $f(x)=x^3$ are odd and $f(x)=x^2$ and $f(x)=x^4$ are even and $f(x)=\frac{1}{x+1}$ is neither even or odd.*

Example 29. *$f(x)=x^3+x$ is odd and $f(x)=\frac{1}{x^2-1}$ is even and $f(x)=x^2+x$ is either even or odd.*

P-5-Operations on Functions

If f and g are functions, then for every x that belongs to the domains of both f and g we define functions

$(f+g)(x)=f(x)+g(x)$
$(f-g)(x)=f(x)-g(x)$
$(fg)(x)=f(x)g(x)$
$(f/g)(x)=f(x)/g(x)$ where $g(x)\neq 0$.

Example 30. *Let $f(x)=\frac{1}{x+2}$ and $g(x)=\frac{x}{x-1}$. Find $(f+g)(x)$, $(f-g)(x)$, $(fg)(x)=f(x)g(x)$ and $(f/g)(x)$ where $g(x)\neq 0$.*

Composition of Functions

If f and g are two functions, then

$$f \circ g(x) = f(g(x)).$$

The domain of $f \circ g$ consists of those numbers x in the domain of g for which $g(x)$ is in the domain of f.

Function	Formula	Domain
f	$f(x) = \sqrt{x}$	$[0, \infty)$
g	$g(x) = x + 1$	$\mathbb{R}$
$f \circ g$	$f \circ g(x) = f(g(x)) = f(x+1) = \sqrt{x+1}$	$[-1, \infty)$
$g \circ f$	$g \circ f(x) = g(f(x)) = g(\sqrt{x}) = \sqrt{x} + 1$	$[0, \infty)$
$f \circ f$	$f \circ f(x) = f(f(x)) = f(\sqrt{x}) = \sqrt{\sqrt{x}} = x^{1/4}$	$[0, \infty)$
$g \circ g$	$g \circ g(x) = g(g(x)) = g(x+1) = (x+1) + 1 = x + 2$	$\mathbb{R}$

Example 31.

Piecewise Defined Functions

Functions which are defined by different formulas on different intervals are sometimes called **piecewise defined functions.**

Example 32.

$$g(x) = \begin{cases} 2x & \textit{for } x < 0 \\ x^2 & \textit{for } x \geq 0 \end{cases}$$

Inverse Functions

Remember that a function is **one-to-one** if for every value in the range, there is exactly one value in the domain.

A function is one-to-one if every horizontal line crosses its graph at most once, which is commonly known as the **horizontal line test**.

P-6-Polynomials and Rational Functions

definition 2. *A **polynomial** is a function $P : \mathbb{R} \to \mathbb{R}$ such that*

$$P(x) = a_n x^n + \cdot + a_1 x + a_0.$$

*Here $a_n, \ldots, a_1$ are called the **coefficients** of the polynomial. We assume $a_n \neq 0$. The number n is called the **degree** of the polynomial.*

Example 33. *Write polynomials of degree 0, 1 and 2.*

Just as the quotient of two integers is called a rational number, the quotient of two polynomials is called a **rational function**. Give an example.

Let A_m be a polynomial of degree m, B_n be a polynomial of degree n with $m \geq n$. Then there are polynomial Q_{m-n} of degree $m-n$, R_k of degree $k < n$ such that

$$\frac{A_m}{B_n} = Q_{m-n} + \frac{R_k}{B_n}.$$

The quotient Q_{m-n} and the remainder R_k can be calculated by the "long division".

Example 34. *Using the long division algorithm, show that*

$$\frac{2x^3 - 3x^2 + 3x + 4}{x^2+1} = 2x - 3 + \frac{x+7}{x^2+1}$$

If P is a polynomial and $P(r) = 0$ then r is called a **root** of P.

The Fundamental Theorem of Algebra says every polynomial of degree greater than 0 must have a root. But these roots may be complex.

Example 35. x^2+1 *has no real roots. Its roots are* $i = \sqrt{-1}$ *and* $-i$.

Theorem 1. *If r is a root of the polynomial P then*

$$P(x) = (x-r)Q(x),$$

for some polynomial Q whose degree is 1 less than P.

The polynomial $x(x-7)^3$ has 4 roots: 0 and the other three are each equal to 7. We say that 7 is a root of **multiplicity** 3.

By the Fundamental Theorem of Algebra and the above theorem, every polynomial of degree n has exactly n (not necessarily distinct) roots.

Roots of Quadratic Polynomials

To obtain the solutions of

$$Ax^2 + Bx + C = 0, \qquad A \neq 0$$

Divide by A and complete to square

$$\left(x + \frac{B}{2A}\right)^2 = \frac{B^2}{4A^2} - \frac{C}{A} = \frac{B^2 - 4AC}{4A^2},$$

Taking the square root of both sides gives the quadratic formula

$$x = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}.$$

A form of this formula is known since B.C. 2000 by Babylonians.

The quantity $D = B^2 - 4AC$ is called the **discriminant** of the quadratic equation.

- If $D > 0$ then there are two distinct real roots,
- If $D = 0$ then there is 1 root of multiplicity 2,
- If $D < 0$ then there are two complex conjugate roots.

Example 36. *Find the roots of the polynomials: (a)* $x^2 + x - 1$, *(b)* $9x^2 - 6x + 1$, *(c)* $2x^2 + x + 1$.

Misc Factorings

Limits and Continuity

1.1 Informal definition of limits

Two main problems of calculus are

1. Derivative. Find the rate of change of f.

2. Integral. Find the area under a given curve.

Both are based on the concept of limit.

We say $\lim_{x\to a} f(x) = L$ to mean $f(x)$ is "close enough" to L when x is "close enough" to *but not equal to a*.

Example 37. $\lim_{x\to a} x = a$. *x is close enough to a when x is close enough to a.*

Example 38. $\lim_{x\to a} c = c$ *if c is a constant.*

Example 39.

$$g(x) = \begin{cases} x, & \text{if } x \neq 2 \\ 1, & \text{if } x = 2 \end{cases}$$

$\lim_{x\to 2} g(x) = \lim_{x\to 2} x = 2$ *although* $g(2) = 1$.

One-sided limits

If $f(x)$ is close to L when $x < a$ is close enough to a then we say

$$\lim_{x\to a^-} f(x) = L$$

This is called the *left limit* of f at $x = a$.

Similarly we can define the right limit.

Example 40.

$$f(x) = \begin{cases} -1 & \text{for } x < 0 \\ 0 & \text{for } x = 0 \\ 1 & \text{for } x > 0 \end{cases}$$

$$\lim_{x\to 0} f(x) \text{ does not exist.}$$

In this example the one-sided limits do exist, namely,

$$\lim_{x\searrow 0} f(x) = 1 \text{ and } \lim_{x\nearrow 0} f(x) = -1.$$

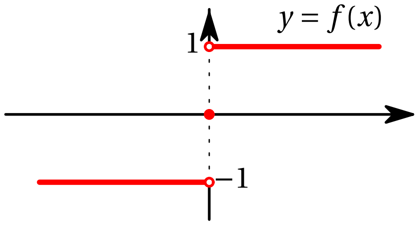


Figure 0.4: The sign function.

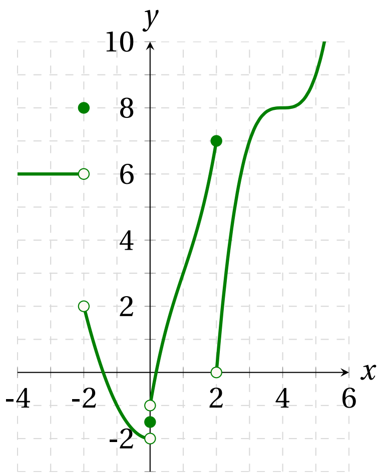


Figure 0.5: A plot of $f(x)$, a piecewise defined function.

Example 41. *Evaluate the expressions by referencing the plot in Figure 0.5.*

1. $\lim_{x\to 4} f(x)$	*5.* $\lim_{x\to 0+} f(x)$	*9.* $\lim_{x\to 0} f(x+1)$
2. $\lim_{x\to -3} f(x)$	*6.* $f(-2)$	*10.* $f(0)$
3. $\lim_{x\to 0} f(x)$	*7.* $\lim_{x\to 2-} f(x)$	*11.* $\lim_{x\to 1-} f(x-4)$
4. $\lim_{x\to 0-} f(x)$	*8.* $\lim_{x\to -2-} f(x)$	*12.* $\lim_{x\to 0+} f(x-2)$

(a) 8, *(b)* 6, *(c) DNE*, *(d)* -2, *(e)* -1, *(f)* 8, *(g)* 7, *(h)* 6, *(i)* 3, *(j)* $-3/2$, *(k)* 6, *(l)* 2

1.2 Formal definition of Limit

The informal description of the limit uses phrases like "close enough" and "really very small". "Fortunately" there is a good definition, i.e. one which is unambiguous and can be used to settle any dispute about the question of whether $\lim_{x\to a} f(x)$ equals some number L or not.

In this section we assume that f is defined in an open interval containing a except possibly at $x = a$.

definition 3. *We say that*

$$\lim_{x\to a} f(x) = L$$

if for every $\epsilon > 0$ *there exists a* $\delta > 0$ *such that*

$$0 < |x-a| < \delta \textit{ implies } |f(x) - L| < \epsilon. \tag{1}$$

Why the absolute values? Recall that the quantity $|x - y|$ is the distance between the points x and y on the number line.

What are ϵ *and* δ*?* The quantity ϵ is how close you would like $f(x)$ to be to its limit L; the quantity δ is how close you have to choose x to a to achieve this. To prove that $\lim_{x\to a} f(x) = L$ you must assume that someone has given you an unknown $\epsilon > 0$, and then find a positive δ for which (1) holds. The δ you find will depend on ϵ.

Example 42. *Show that* $\lim_{x\to 5} 2x+1 = 11$.

We have $f(x) = 2x+1$, $a = 5$ *and* $L = 11$, *and the question we must answer is "how close should* x *be to* 5 *if want to be sure that* $f(x) = 2x+1$ *differs less than* ϵ *from* $L = 11$*?"*

$$|f(x) - L| = |(2x+1) - 11| = |2x - 10| = 2\cdot|x-5| = 2\cdot|x-a|.$$

So choose $\delta = \frac{\epsilon}{2}$. *Then*

$$|f(x) - L| < \epsilon \textit{ whenever } 0 < |x-a| < \frac{\epsilon}{2}.$$

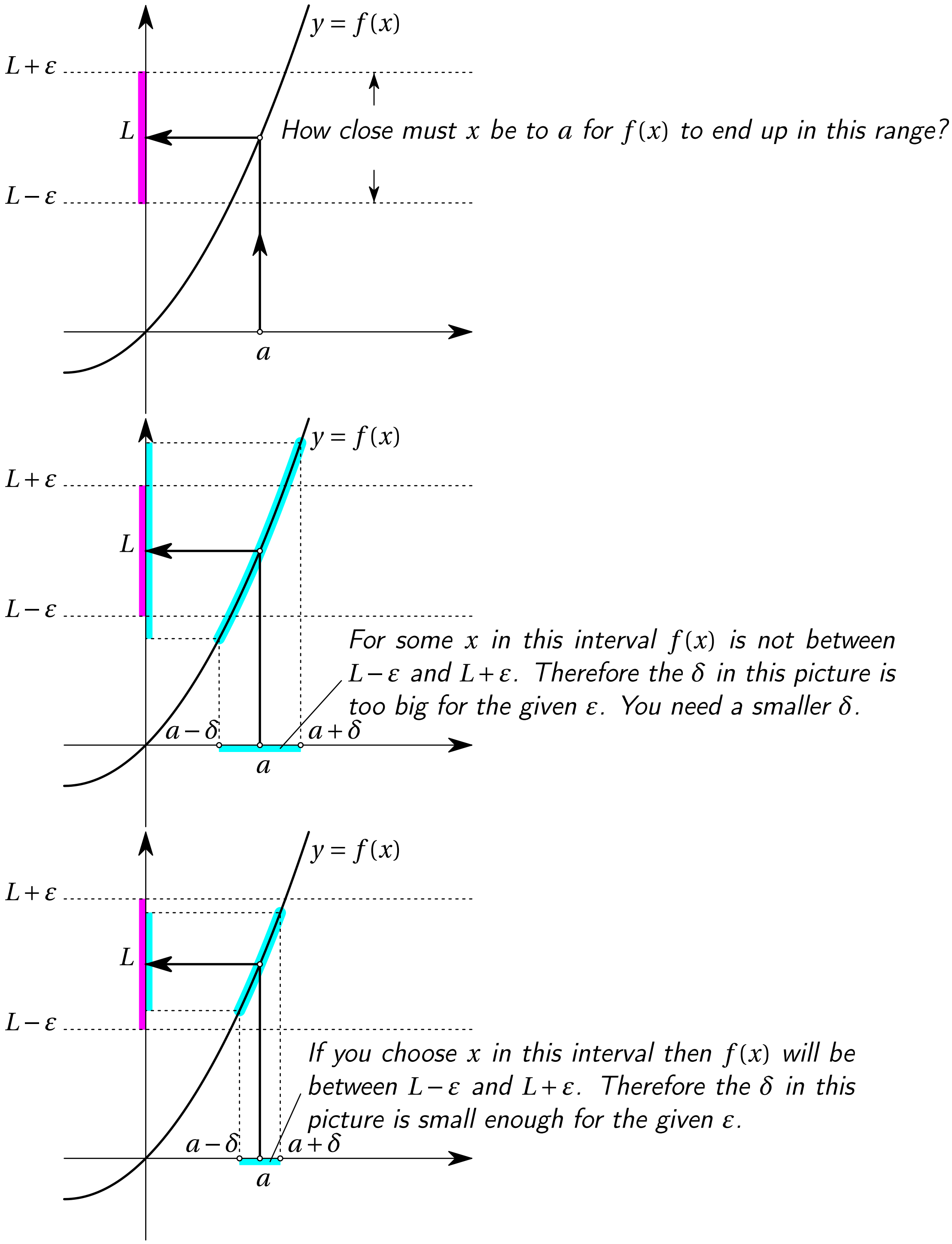
Example 43 ("Don't choose $\delta > 1$" trick)**.** *Show that* $\lim_{x\to 3} x^2 = 9$.

We have $f(x) = x^2$, $a = 3$, $L = 9$, *and again the question is, "how small should* $|x-3|$ *be to guarantee* $|x^2 - 9| < \epsilon$*?"*

$$|x^2 - 9| = |(x-3)(x+3)| = |x+3|\cdot|x-3|.$$

Here is a trick that allows you to replace the factor $|x+3|$ *with a constant. We hereby agree that we always choose our* δ *so that* $\delta \le 1$. *If we do that, then we will always have*

$$|x-3| < \delta \le 1, \textit{i.e. } |x-3| < 1,$$


$y = f(x)$
$L+\varepsilon$
L
$L-\varepsilon$
How close must x be to a for $f(x)$ to end up in this range?
a
$y = f(x)$
$L+\varepsilon$
L
$L-\varepsilon$
For some x in this interval $f(x)$ is not between $L-\varepsilon$ and $L+\varepsilon$. Therefore the δ in this picture is too big for the given ε. You need a smaller δ.
$a-\delta$
$a+\delta$
a
$y = f(x)$
$L+\varepsilon$
L
$L-\varepsilon$
If you choose x in this interval then $f(x)$ will be between $L-\varepsilon$ and $L+\varepsilon$. Therefore the δ in this picture is small enough for the given ε.
$a-\delta$
$a+\delta$
a

or $2 < x < 4$ *or* $|x+1| < 5$. *Therefore*

$$|x^2-1| = |x+1|\cdot|x-1| < 5|x-1|.$$

So choose

$$\delta = \min\{1, \frac{\epsilon}{5}\}.$$

2nd way: Note that $|x+3| = |x-3+6| < |x-3|+6 < \delta+6$

$$|f(x)-9| = |x+3||x-3| < (\delta+6)\delta$$

So choose $(\delta+6)\delta < \epsilon$, *or*

$$(\delta+3)^2 < \epsilon+9 \implies \delta < \sqrt{\epsilon+9}-3$$

Example 44. *Show that* $\lim_{x\to 4} 1/x = 1/4$.

We apply the definition with $a = 4$, $L = 1/4$ *and* $f(x) = 1/x$. *Thus, for any* $\epsilon > 0$ *we try to show that if* $|x-4|$ *is small enough then one has* $|f(x)-1/4| < \epsilon$.

We begin by estimating $|f(x)-\frac{1}{4}|$ *in terms of* $|x-4|$:

$$|f(x)-1/4| = \left|\frac{1}{x}-\frac{1}{4}\right| = \left|\frac{4-x}{4x}\right| = \frac{|x-4|}{|4x|} = \frac{1}{|4x|}|x-4|.$$

As before, things would be easier if $1/|4x|$ *were a constant. To achieve that we again agree not to take* $\delta > 1$. *If we always have* $\delta \le 1$, *then we will always have* $|x-4| < 1$, *and hence* $3 < x < 5$. *How large can* $1/|4x|$ *be in this situation? Answer: the quantity* $1/|4x|$ *increases as you decrease* x, *so if* $3 < x < 5$ *then it will never be larger than* $1/|4\cdot 3| = \frac{1}{12}$.

We see that if we never choose $\delta > 1$, *we will always have*

$$|f(x)-\frac{1}{4}| \le \frac{1}{12}|x-4| \quad \textit{for} \quad |x-4| < \delta.$$

To guarantee that $|f(x)-\frac{1}{4}| < \epsilon$ *we could threfore require*

$$\frac{1}{12}|x-4| < \epsilon, \quad \textit{i.e.} \quad |x-4| < 12\epsilon.$$

Hence if we choose $\delta = 12\epsilon$ *or any smaller number, then* $|x-4| < \delta$ *implies* $|f(x)-4| < \epsilon$. *Of course we have to honor our agreement never to choose* $\delta > 1$, *so our choice of* δ *is*

$$\delta = \textit{the smaller of } 1 \textit{ and } 12\epsilon = \min\big(1, 12\epsilon\big).$$

Example 45. *Verify that* $\lim_{x\to 2} \dfrac{x-2}{1+x^2} = 0$.

Solution: Notice that $\dfrac{|x-2|}{|1+x^2|} < |x-2|$ *since* $1+x^2 > 1$. *Hence choose* $\delta = \epsilon$.

1.3 One Sided Limits and Limits at Infinity

definition 4 (Right Limit)**.** *We say that*

$$\lim_{x\searrow a} f(x) = L \tag{2}$$

if for every $\epsilon > 0$ *one can find a* $\delta > 0$ *such that*

$$a < x < a+\delta \implies |f(x)-L| < \epsilon.$$

This is called right-limit of f *at* $x = a$.

The left-limit, i.e. the one-sided limit in which x approaches a through values less than a is defined in a similar way.

Theorem 2. *If both one-sided limits*

$$\lim_{x\searrow a} f(x) = L_+, \text{ and } \lim_{x\nearrow a} f(x) = L_-$$

exist, then

$$\lim_{x\to a} f(x) \text{ exists} \iff L_+ = L_-.$$

In other words, if a function has both left- and right-limits at some $x = a$, then that function has a limit at $x = a$ if the left- and right-limits are equal.

Example 46. *Show that* $\lim_{x\to 0^+} \sqrt{x} = 0$.
Solution: Choose $\delta = \epsilon^2$.

What happens to $f(x)$ as x becomes "larger and larger" and ask?

definition 5 (Limit at Infinity)**.** *Let f be some function which is defined on some interval $x_0 < x < \infty$. If there is a number L such that for every $\epsilon > 0$ one can find an A such that*

$$x > A \implies |f(x) - L| < \epsilon$$

for all x, then we say that the limit of $f(x)$ for $x \to \infty$ is L.

Example 47. *To* prove *that* $\lim_{x\to\infty} 1/x = 0$ *we apply the definition to* $f(x) = 1/x$, $L = 0$.
For given $\epsilon > 0$ we need to show that

$$\left|\frac{1}{x} - L\right| < \epsilon \text{ for all } x > A \tag{3}$$

provided we choose the right A.
How do we choose A? A is not allowed to depend on x, but it may depend on ϵ.
If we assume for now that we will only consider positive values of x, then (3) simplifies to

$$\frac{1}{x} < \epsilon$$

which is equivalent to

$$x > \frac{1}{\epsilon}.$$

This tells us how to choose A. Given any positive ϵ, we will simply choose

$$A = \frac{1}{\epsilon}$$

Then one has $|\frac{1}{x} - 0| = \frac{1}{x} < \epsilon$ *for all* $x > A$. *Hence we have proved that* $\lim_{x\to\infty} 1/x = 0$.

Infinite Limits

definition 6. $\lim_{x\to a} f(x) = \infty$ *if for every* $B > 0$ *there exists* $\delta > 0$ *such that* $f(x) > B$ *whenever* $0 < |x - a| < \delta$.

Example 48. *Verify that* $\lim_{x\to 0} \frac{1}{x^2} = \infty$.
Solution: Choose $\delta = 1/\sqrt{B}$.

1.4 Properties of limits

The precise definition of the limit is not easy to use. Instead, there are a number of properties that limits have which allow you to compute them without having to resort to the epsilon-delta definition.

The following statements remain true if one replaces each limit by a one-sided limit, or a limit for $x \to \infty$.

Theorem 3 (Limits of constants and x)**.** *If a and c are constants, then*

$$\lim_{x\to a} c = c \tag{P_1}$$

and

$$\lim_{x\to a} x = a. \tag{P_2}$$

Proof. We must find δ. For (P_2) choose $\delta = \epsilon$. For (P_1) any δ will do. □

Theorem 4 (Limits of sums, products, quotients)**.** *Suppose*

$$\lim_{x\to a} f(x) = L, \qquad \lim_{x\to a} g(x) = M.$$

Then

$$\lim_{x\to a}\bigl(f(x)+g(x)\bigr) = L+M, \tag{P_3}$$

$$\lim_{x\to a}\bigl(f(x)-g(x)\bigr) = L-M, \tag{P_4}$$

$$\lim_{x\to a}\bigl(f(x)\cdot g(x)\bigr) = L\cdot M \tag{P_5}$$

If $\lim_{x\to a} g(x) \neq 0$,

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{L}{M}. \tag{P_6}$$

Finally, if m and n are integers such that $L^{m/n}$ is defined

$$\lim_{x\to a}(f(x))^{m/n} = L^{m/n}. \tag{P_7}$$

In other words the limit of the sum is the sum of the limits, etc. One can prove these laws using the definition of limit.

Proof of (P_3). We can find $\delta_1 > 0$ such that $|f(x)-L| < \frac{\epsilon}{2}$ and $\delta_2 > 0$ such that $|g(x)-M| < \frac{\epsilon}{2}$. Now choose $\delta = \min\{\delta_1,\delta_2\}$. If $|x-a| < \delta$, by triangle inequality

$$|f(x)+g(x)-L-M| \le |f(x)-L| + |g(x)-M| < \epsilon.$$

□

Proof of (P_5). Let $\epsilon > 0$ and let

$$\epsilon_1 = \min\{1, \frac{\epsilon}{1+|L|+|M|}\}$$

Then $\epsilon_1 > 0$ and by definition, there exists $\delta_1 > 0$ and $\delta_2 > 0$ such that

$$|f(x)-L| < \epsilon_1 \text{ whenever } 0 < |x-a| < \delta_1$$

$$|g(x) - L| < \epsilon_1 \text{ whenever } 0 < |x-a| < \delta_2$$

Let $\delta = \min\{\delta_1, \delta_2\}$.

Use triangle inequality

$$\begin{aligned}
|f(x)g(x) - LM| &= |(f(x) - L + L) + (g(x) - M + M) - LM| \\
&\leq |f(x) - L||g(x) - M| + |M||f(x) - L| + |L||g(x) - M| \\
&\leq \epsilon_1^2 + |M|\epsilon_1 + |L|\epsilon_1 \\
&\leq \epsilon_1 + |M|\epsilon_1 + |L|\epsilon_1 \\
&\leq \epsilon_1(1 + |M| + |L|) \leq \epsilon
\end{aligned}$$

□

Example 49. *Find* $\lim_{x\to 2} x^2$.

One has

$$\begin{aligned}
\lim_{x\to 2} x^2 &= \lim_{x\to 2} x \cdot x \\
&= \left(\lim_{x\to 2} x\right) \cdot \left(\lim_{x\to 2} x\right) && \textit{by } (P_5) \\
&= 2 \cdot 2 = 4.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\lim_{x\to 2} x^3 &= \lim_{x\to 2} x \cdot x^2 \\
&= \left(\lim_{x\to 2} x\right) \cdot \left(\lim_{x\to 2} x^2\right) && (P_5) \textit{ again} \\
&= 2 \cdot 4 = 8,
\end{aligned}$$

and, by (P_4)

$$\lim_{x\to 2} x^2 - 1 = \lim_{x\to 2} x^2 - \lim_{x\to 2} 1 = 4 - 1 = 3,$$

and, by (P_4) *again,*

$$\lim_{x\to 2} x^3 - 1 = \lim_{x\to 2} x^3 - \lim_{x\to 2} 1 = 8 - 1 = 7,$$

Putting all this together, one gets

$$\lim_{x\to 2} \frac{x^3 - 1}{x^2 - 1} = \frac{2^3 - 1}{2^2 - 1} = \frac{8-1}{4-1} = \frac{7}{3}$$

because of (P_6). *To apply* (P_6) *we must check that the denominator (“L_2”) is not zero. Since the denominator is 3 everything is OK, and we were allowed to use* (P_6).

1.5 Continuity

Let $f(x) = \sqrt{4 - x^2}$. Domain of f is $[-2, 2]$.

- $x = -2$ is the left end point of $\mathrm{Dom}(f)$.
- $x = 2$ is the right end point of $\mathrm{Dom}(f)$.

- Any x with $-2 < x < 2$ is called an interior point of Dom(f).

definition 7. *A function f is **continuous** at an interior point c of its domain if*

$$\lim_{x\to c} f(x) = f(c)$$

Note that f is discontinuous at c if

i) either $\lim_{x\to c} f(x)$ does not exist.

ii) or $\lim_{x\to c} f(x)$ exists but is not equal to $f(c)$.

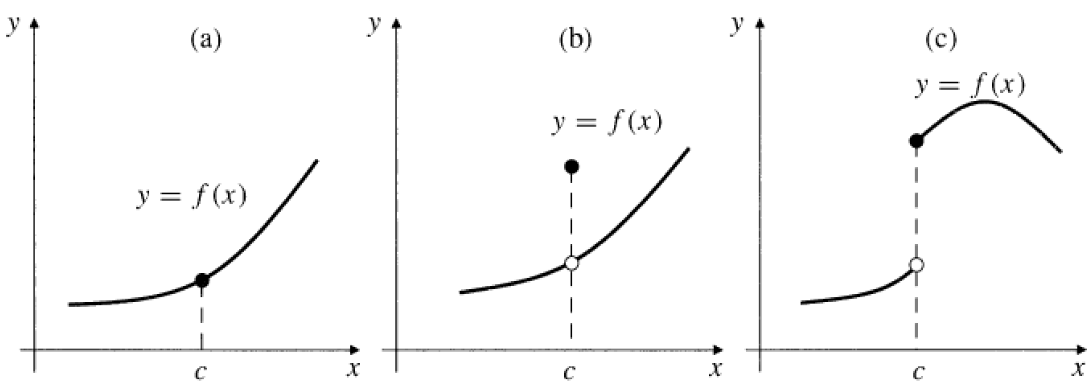


Figure 0.6: a) f is continuous at c, b) f is discontinuous at c because of (ii), c) f is discontinuous at c because of (i)

definition 8. *f is **right continuous** at c if*

$$\lim_{x\to c^+} f(x) = f(c)$$

*and **left continuous** at c if*

$$\lim_{x\to c^-} f(x) = f(c).$$

If Dom(f)= $[a,b]$ then

- f is continuous at a if it is right continuous at a
- f is continuous at b if it is left continuous at b

Example 50. *Show that $f(x) = \sqrt{4-x^2}$ is continuous at every point of its domain.*

definition 9. *f is called a continuous function if f is continuous at every pt of its domain.*

According to this definition $f(x) = \frac{1}{x}$ is continuous!!! 0 is not in domain of f. So we say f is undefined rather than discontinuous at 0.

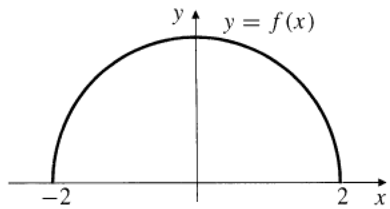


Figure 0.7: $f(x) = \sqrt{4 - x^2}$

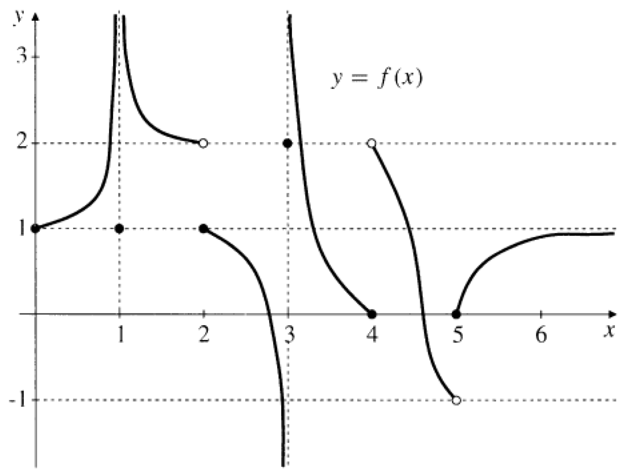


Figure 1.32

4. At what points is the function f, whose graph is shown in Figure 1.32, discontinuous? At which of those points is it left continuous? right continuous?

Example 51.

There are lots of continuous functions:

- polynomials,
- rational functions,
- rational powers $x^{m/n}$
- trig functions
- absolute value function $|x|$

Combining continuous functions: If f and g are continuous at c then

- $f + g$, $f - g$, fg, are continuous at c,
- if k is constant then kf is continuous at c,
- $\frac{f}{g}$ continuous at c provided that $g(c) \neq 0$.
- $f(x)^{1/n}$ continuous at c provided that $f(c) > 0$ if n is even.

Composites of continuous funcs are continuous
If g is continuous at c and f is continuous at $g(c)$ then $f \circ g$ is continuous at c.

Example 52. *Find m so that*

$$g(x) = \begin{cases} x - m, & \text{if } x < 3, \\ 1 - mx, & \text{if } x \geq 3 \end{cases}$$

is continuous for all x.

Continuous extensions and removable discontinuities
If f is not defined at c but $\lim_{x \to c} f(x) = L$ is defined then we can define a new function

$$F(x) = \begin{cases} f(x), & \text{if } x \neq c \\ L, & \text{if } x = c \end{cases}$$

F is continuous at c and is called **continuous extension** of f to $x = c$.

Example 53. *The function* $f(x) = \dfrac{x^2 - x}{x^2 - 1}$ *is not defined at* $x = 1$ *but has a continuous extension* $F(x) = \dfrac{x}{x+1}$ *to* $x = 1$.

If a function is undefined or discontinuous at c but can be redefined at c then we say f has a **removable discontinuity** at c.

Example 54. *The function* $f(x) = \dfrac{1}{x^2}$ *is not defined at 0 but there is no way of redefining f at 0 so that f becomes continuous at 0.*

Continuous Functions on Closed Intervals [a, b] are bounded.

Theorem 5. *If f is continuous on the closed interval $[a, b]$ then there exist numbers p and q s.t.*

$$f(p) \leq f(x) \leq f(q)$$

*for all x in $[a, b]$. $f(p)$ is the **aboslute minimum value** and $f(q)$ is the **absolute maximum value**.*

How to find p and q? This is just an existence theorem.

Example 55. *The conclusions of the theorem may fail if the function f is not continuous or the interval is not closed. Do Figure 1.25 and Figure 1.27.*

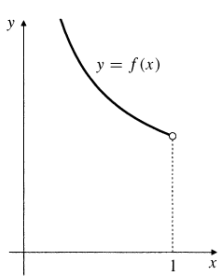


Figure 1.24 $f(x) = 1/x$ is continuous on the open interval $(0, 1)$. It is not bounded and has neither a maximum nor a minimum value

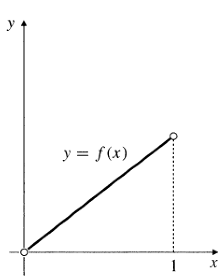


Figure 1.25 $f(x) = x$ is continuous on the open interval $(0, 1)$. It is bounded but has neither a maximum nor a minimum value

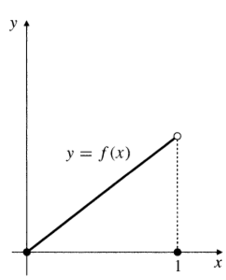


Figure 1.26 This function is defined on the closed interval $[0, 1]$ but is discontinuous at the endpoint $x = 1$. It has a minimum value but no maximum value

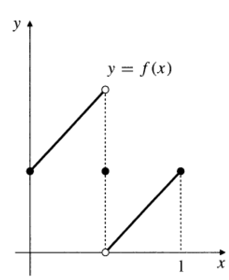


Figure 1.27 This function is discontinuous at an interior point of its domain, the closed interval $[0, 1]$. It is bounded but has neither maximum nor minimum values

Theorem 6 (Intermediate Value Theorem)**.** *If f is continuous on $[a,b]$ and if s is between $f(a)$ and $f(b)$ then there exists c in $[a,b]$ s.t. $f(c)=s$.*

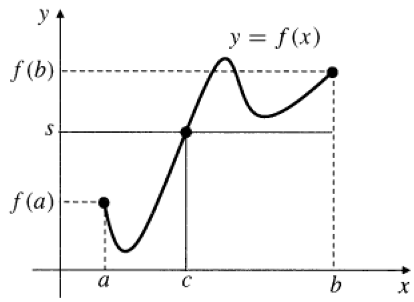


Figure 1.29 The continuous function f takes on the value s at some point c between a and b

Example 56. *Show that the equation $x^3-x-1=0$ has a solution in the interval $[1,2]$.*

Solution 1. *$f(x)=x^3-x-1$ is a polynomial and hence continuous. $f(1)=-1$ and $f(2)=5$. Since 0 lies between -1 and 5, the intermediate value theorem assures us that there must be a number c in $[1,2]$ such that $f(c)=0$.*

Differentiation

2.1 Tangent Lines and Their Slopes

Problem: Find a straight line L that is tangent to a curve C at a point P.

"For simplicity, restrict ourselves to curves which are graphs of functions."

How do we define the tangent line to a curve? The slope of the line PQ is

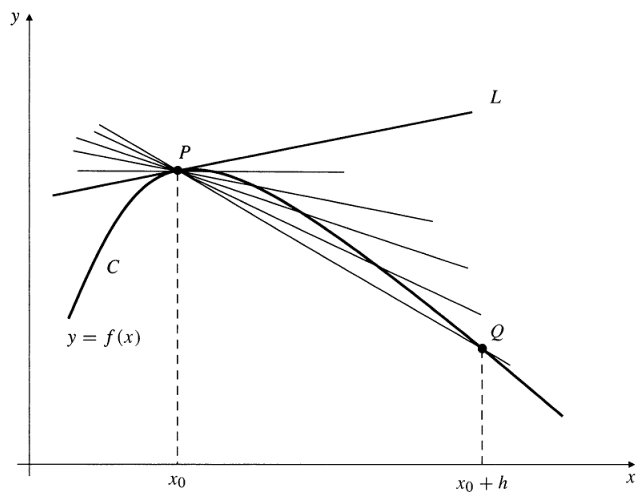


Figure 0.8: caption

$$\frac{f(x_0+h)-f(x_0)}{h}.$$

definition 10. *Suppose f is cts at $x = x_0$ and*

$$\lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h} = m$$

If the limit exists, then the line with equation

$$y = m(x - x_0) + f(x_0)$$

*is called **the tangent line** to the graph of $y = f(x)$ at $P = (x_0, f(x_0))$. If the limit does not exist and $m = \infty$ or $m = -\infty$ then the tangent line is the vertical line $x = x_0$. If the limit does not exist and is not $\pm\infty$ then there is no tangent line at P.*

Example 57. *Find an equation of the tangent line to the curve $y = x^2$ at $(1,1)$.*

Solution 2.

$$m = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = 2.$$

And an equation is $y = 2(x-1) + 1$

Example 58. *Find an equation of the tangent line to the curve* $y = x^{1/3} = \sqrt[3]{x}$ *at the origin.*

Solution 3. *The slope of the tangent line is*

$$m = \lim_{h \to 0} \frac{h^{1/3}}{h} = \infty.$$

So the tangent line is a vertical line $x = 0$ *(in other words the y-axis).*

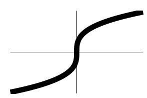


Figure 0.9: $y = x^{1/3}$

Example 59. *Does* $f(x) = x^{2/3}$ *have a tangent line at* $(0,0)$*?*

Solution 4. *The limit of the difference quotient is undefined at* 0*since the right limit is* ∞ *while the left limit is* $-\infty$*. Hence the graph has no tangent line at* $(0,0)$.

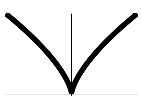


Figure 0.10: $y = x^{2/3}$

"We say that this curve has a cusp at the origin. A cusp is an infinitely sharp point. If you were traveling along the curve, you would have to stop and turn 180° at the origin."

Example 60. *Does* $f(x) = |x|$ *have a tangent line at* $(0,0)$*?*

Solution 5. *The difference quotient is* $\frac{|h|}{h}$ *which has right limit* 1 *and left limit* -1 *at* $h = 0$.

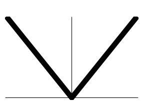


Figure 0.11: $y = |x|$

2.2 Derivative

definition 11. *The* ***derivative*** *of a function* f *at* x *is*

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

whenever the limit exists. If $f'(x)$ *exists,* f *is called* ***differentiable*** *at* x.

f' is a function whose domain is those x at which f is differentiable.
Another way of defining derivative is

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

Two limits are equivalent and can be seen by letting $x = x_0 + h$.

Example 61. *Show that the derivative of the linear function $f(x) = ax + b$ is $f'(x) = a$. In particular the derivative of a constant function is zero.*

Example 62. *Use the definition of the derivative to calculate the derivatives of a) $f(x) = x^2$, b) $f(x) = \frac{1}{x}$, c) $f(x) = \sqrt{x}$.*

The previous three formulas are special cases of the following rule:

$$f(x) = x^r \implies f'(x) = r x^{r-1}$$

whenever x^{r-1} makes sense.

Example 63.

$$f(x) = x^{5/3} \implies f'(x) = x^{2/3},$$

for all x. How about $f'(-1/8)$?

$$f(x) = \frac{1}{\sqrt{x}} \implies f'(x) = -\frac{1}{2} x^{-3/2}$$

for $x > 0$.

Example 64. *Differentiate the absolute value function $f(x) = |x|$ to get*

$$f'(x) = sgn(x) = \begin{cases} -1, & \text{if } x < 0 \\ 1, & \text{if } x > 0 \end{cases}$$

Note that f is differentiable at 0.

Example 65. *How should the function $f(x) = xsgn(x)$ be defined at $x = 0$ so that it is cts there? Is it then differentiable there?*

Let $y = f(x)$. We denote the derivative by

$$y' = f'(x) = \frac{dy}{dx} = \frac{d}{dx} f(x).$$

If we want to evaluate the derivative at point x_0

$$y' |_{x=x_0} = f'(x_0) = \frac{dy}{dx} |_{x=x_0} = \frac{d}{dx} f(x) |_{x=x_0}.$$

The notations dy/dx and $\frac{d}{dx} f(x)$ are called **Leibniz notations** for the derivative. The notation dy/dx is suggested by the definition of the derivative. Let $\Delta y = f(x + h) - f(x)$ be the increment in y and $\Delta x = x + h - x = h$ be the increment in x. Then

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

2.3 Differentiation Rules

The graph of a function can not have a break at a point where it is smooth. Obvious!

Theorem 7. *If f is differentiable at x then f is cts at x.*

Theorem 8.

$$\lim_{h\to 0}(f(x+h)-f(x)) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\lim_{h\to 0} h = f'(x)0 = 0$$

This means

$$\lim_{h\to 0} f(x+h) = f(x)$$

Theorem 9. *If f and g are differentiable at x then*

$$(f+g)'(x) = f'(x)+g'(x),$$

$$(f-g)'(x) = f'(x)-g'(x),$$

and for any constant c

$$(cf)'(x) = cf'(x).$$

The proof of this theorem is easy but let's skip it.
The sum rule extends to any number of functions.

$$(f_1+\cdots+f_n)'(x) = f_1'(x)+\cdots+f_n'(x).$$

Example 66. *Take the derivative of*

$$f(x) = 5\sqrt{x}+\frac{3}{x}-19$$

It is NOT true that derivative of product of functions is a product of their derivatives

Theorem 10. *If f and g are diffentiable at x then*

$$(fg)'(x) = f'(x)g(x)+f(x)g'(x).$$

Example 67. *Find the derivative of $f(x) = (x^2+x+1)(2x+\frac{1}{x})$.*

The product rule can be extended to any number of functions

$$(f_1f_2f_3)' = f_1'f_2f_3+f_1f_2'f_3+f_1f_2f_3'$$

$$(f_1\cdots f_n)' = f_1'f_2\cdots f_3+f_1f_2'f_3\cdots f_n+\cdots+f_1\cdots f_{n-1}f_n'.$$

Theorem 11. *If f is differentiable at x and $f(x)\neq 0$ then $1/f$ is diff at x, and*

$$\left(\frac{1}{f}\right)'(x) = \frac{-f'(x)}{f(x)^2}.$$

Proof.

$$\frac{d}{dx}\frac{1}{f(x)} = \cdots = \lim_{h\to 0}\frac{-1}{f(x+h)f(x)}\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

The result follows by limit rules and continuity of f. □

Example 68. *Differentiate* $y = \dfrac{1}{x^{2/3}+1}$

Use the product rule and reciprocal rule to obtain quotient rule. If f and g are differentiable at x and $g(x) \neq 0$ then

$$\left(\frac{f}{g}\right)'(x) = \left(\frac{1}{g}(x)f(x)\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}$$

Example 69. *Find the derivative of* $f(x) = \dfrac{a+bx}{m+cx}$.

Example 70. *Find an equation of the tangent line to* $y = \dfrac{2}{3-4\sqrt{x}}$ *at the point* $(1,-2)$.

2.4 Chain Rule

If $f(u)$ is differentiable at $u = g(x)$ and $g(x)$ is differentiable at x, then

$$(f \circ g)'(x) = f'(g(x))g'(x)$$

In Leibniz notation, if $y = f(u)$ where $u = g(x)$ then

$$y = f(g(x)) = (f \circ g)(x)$$

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

where $\frac{dy}{du}$ is evaluated at $u = g(x)$.

Example 71. *Find the derivative of* $y = \sqrt{x^2+1}$.

Solution 6. *Here* $y = f(g(x))$ *where* $f(u) = \sqrt{u}$ *and* $u = x^2+1$.

$$\frac{dy}{dx} = f'(g(x))g'(x) = \frac{1}{2\sqrt{g(x)}}g'(x) = \frac{1}{2\sqrt{x^2+1}}2x = \frac{x}{\sqrt{x^2+1}}.$$

Derivative of the composite function = derivative of outer function evaluated at inner function times derivative of inner function.

Example 72. *Differentiate* $y = (x^3-1)^{1000}$. *Let* $u = (x^3-1)$ *then* $y = u^{1000}$. $y' = 1000u^{999}u' = 1000(x^3-1)^{999}3x^2$.

Example 73. *Take the derivative* $f(t) = |t^2-1|$.

$$f'(t) = (sgn(t^2-1))(2t) = \begin{cases} 2t, & \text{if } t < -1, t > 1 \\ -2t, & \text{if } -1 < t < 1 \\ \text{undefined} & \text{if } t \pm 1 \end{cases}$$

Example 74. *In terms of derivative* f' *of* f, *express the derivative of a)* $f(x^2)$, *b)* $[f(\pi - 2f(x))]^4$.

For (a)

$$\frac{d}{dx}f(x^2) = f'(x^2)2x$$

For (b)

$$\frac{d}{dx}[f(\pi-2f(x))]^4 = 4[f(\pi-2f(x))]^3 f'(\pi-2f(x))(-2f'(x)).$$

Example 75. *For*

$$f(x) = \left(1 + \sqrt{x^2 - 1}\right)^{-4/3}$$

evaluate $f'(\sqrt{5})$.

2.5 Derivatives of Trigonometric Functions

Trigonometric functions are useful for investigating many real-world phenomena where quantities fluctuate in a periodic way. Examples: elastic motions, vibrations and waves.

The radian measure of an angle is defined to be the length of the arc of a unit circle corresponding to that angle.

$$\text{angle in degrees} = \text{angle in radians} \cdot \frac{180^\circ}{\pi}.$$

In calculus all angles are measured in radians. When we talk about the angle $\pi/3$ we mean $\pi/3 = 60^\circ$ not $(\pi/3)^\circ \approx 1.04^\circ$.

In the book, it is proved that

$$\frac{d}{dx}\sin x = \cos x, \qquad \frac{d}{dx}\cos x = -\sin x.$$

Notice that both sin and cos are differentiable for all x.

Example 76. *Evaluate the derivative of*

a) $\sin(\pi x) + \cos(3x)$,

b) $x^2\cos(\sqrt{x})$,

c) $\frac{\cos x}{1-\sin x}$,

d) $\sin(\cos(\tan t))$

The derivatives of the other trigonometric functions

$$\tan x = \frac{\sin x}{\cos x}, \qquad \sec x = \frac{1}{\cos x},$$

$$\cot x = \frac{\cos x}{\sin x}, \qquad \csc x = \frac{1}{\sin x}.$$

Since cos and sin are eveywhere differentiable, the above functions are differentiable everywhere except where their denominators are zero. The derivatives of these functions can be derived by using quotient and reciprocal rules.

$$\frac{d}{dx}\tan x = \sec^2 x, \qquad \frac{d}{dx}\sec x \tan x,$$

$$\frac{d}{dx}\cot x = -\csc^2 x, \qquad \frac{d}{dx}\csc = -\csc x \cot x.$$

Example 77. *Verify the formuals for* $\tan x$ *and* $\sec x$.

Example 78. *Find the points on the curve* $y = \tan(2x)$, $-\pi/4 < x < \pi/4$, *where the normal is parallel to the line* $y = -x/8$.

2.6 Higher Order Derivatives

Derivative of derivative is called **second derivative**. If $y = f(x)$ then

$$y'' = f''(x) = \frac{d}{dx}\frac{d}{dx}y = \frac{d^2}{dx^2}y = \frac{d^2}{dx^2}f(x).$$

Similar notations can be used for third, fourth, etc. derivatives. For n-th derivative, we write

$$y^{(n)} = f^{(n)}(x) = \frac{d^n y}{dx^n}$$

Example 79. *Calculate all the derivatives of* $y = x^3$.

Example 80. *Calculate all the derivatives of* $y = x^n$ *where* n *is a positive integer.*

Solution 7.

$$y^{(k)} = \begin{cases} \frac{n!}{(n-k)!}x^{n-k} & if 0 \le k \le n \\ 0 & if k > n \end{cases}$$

Example 81. *Show that if* A, B *and* k *are constants, then the function* $y = A\cos(kt) + B\sin(kt)$ *is a solution of the second order differential equation*

$$\frac{d^2y}{dx^2} + k^2y = 0.$$

Example 82. *If* $y = \tan kx$ *show that* $y'' = 2k^2y(1+y^2)$.

Example 83. *If* f *and* g *are twice differentiable functions, show that*

$$(fg)'' = f''g + 2f'g' + fg''.$$

What do you think about the general formula for $\frac{d^n}{dx^n}(fg)$*?*

2.8 Mean Value Theorem

Suppose you drive from city A to city B which is 200km's in 2 hours. Your average speed is 100km/h. Even if you did not travel constant speed. There is at least one instant where your speed was exactly 100km/h. This is called **mean value theorem**.

Theorem 12 (The Mean-Value Theorem)**.** *Suppose that* f *is cts on the interval* $[a, b]$ *and that it is differentiable on the open inteval* (a, b)*. Then there exists a point* c *in the open interval* (a, b) *s.t.*

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

Let $f(t)$ denote the distance from city A. Then $f(0) = 0$ and $f(2) = 200$. MVT says there is a time $t = c$ s.t. $f'(c) = 100$.

If the assumptions of the Mean Value Theorem are satisfied:

- We don't know how to find c.

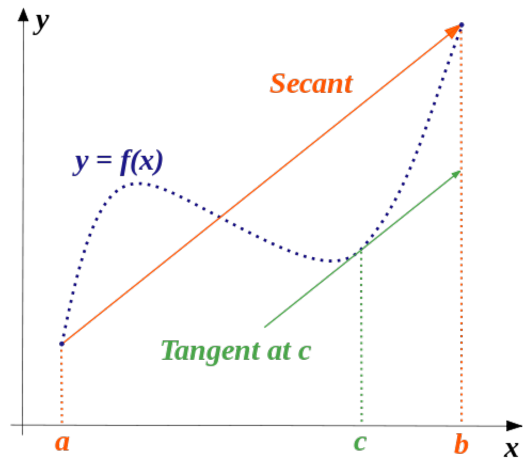


Figure 0.12: Mean Value Theorem says that the slope of the secant line joining two points on the graph of of $f(x)$ is equal to the slope of the tangent line at some point $x = c$ between a and b. from Wikipedia.

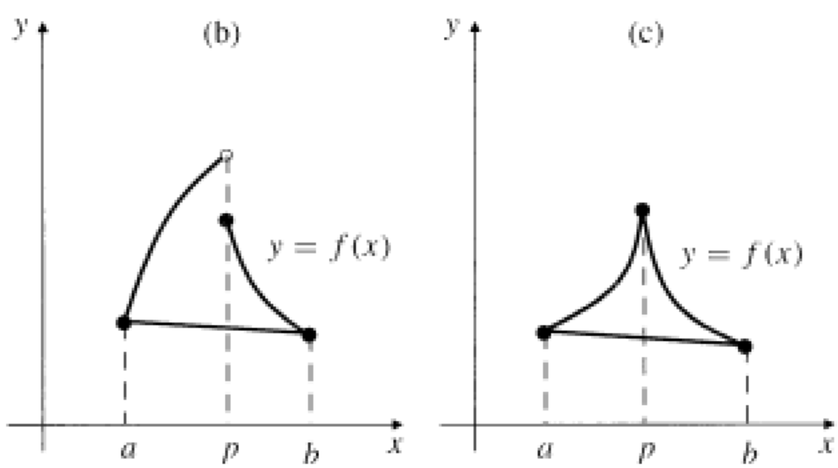


Figure 0.13: Functions that fail to satisfy the Mean Value Theorem. (b) f is discts at p, (c) f is not differentiable at p.

- We don't know how many different c can be found satisfying MVT (there is at least one).

In short, MVT is an *existence theorem* like Intermediate Value Theorem and Min-Max Theorem.

Example 84. *Show that* $\sin x < x$ *for all* $x > 0$.

Solution 8. *For* $x > 2\pi$ *done. For* $0 < x < 2\pi$, *use the MVT and* $\cos c < 1$.

Example 85. *Show that* $\sqrt{1+x} < 1 + \frac{x}{2}$ *for all* $x > 0$.

Solution 9. *Let* $f(x) = \sqrt{1+x}$. *Then* $f'(c) < \frac{1}{2}$ *for* $c > 0$. *Use MVT.*

Suppose f is defined on an interval I. If $f(x_2) > f(x_1)$ for all x_1, x_2 in I s.t. $x_2 > x_1$ then we say f is **increasing** on I. Similarly define **decreasing, non-increasing, non-decreasing** on I.

Theorem 13. *Suppose* f *is differentiable on an open interval* I.

- *If* $f'(x) > 0$ *for all* $x \in I$ *then* f *is increasing on* I.
- *If* $f'(x) \geq 0$ *for all* $x \in I$ *then* f *is nondecreasing on* I.

Similar statements hold for decreasing and nonincreasing functions.

Proof. Prove first statement. Let $x_2 > x_1$ in I. Use MVT, $f(x_2) - f(x_1) = f'(c)(x_2 - x_1) > 0$. □

Example 86. *On what intervals is* $f(x) = x^3 - 12x + 1$ *increasing or decreasing?*

Solution 10. $f'(x) = 3(x-2)(x+2)$. *So* f *is decreasing on* $(-2, 2)$ *and increasing otherwise.*

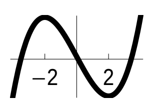


Figure 0.14: Graph of $f(x) = x^3 - 12x + 1$.

We know that if f is a constant function then its derivative is zero. The converse is also true.

Theorem 14. *If* f *is cts on the interval* I *and* $f'(x) = 0$ *at every* x *in* I *then* $f(x) = C$, *a constant on* I.

Proof. Choose x_0 in I. Let $C = f(x_0)$. If x is any other point in I then by MVT, $f(x) - f(x_0) = f'(c)(x - x_0) = 0$. □

2.9 Implicit Differentiation

We learned to find the slope of a curve that is the graph of a function. But not all curves are graphs of functions such as the graph of the equation $x^2 + y^2 = 1$.

Curves are graphs of equations in two variables

$$F(x, y) = 0.$$

For the circle $F(x, y) = x^2 + y^2 - 1$.

Example 87. *Find the slope of the circle* $x^2 + y^2 = 25$ *at the point* $(3, -4)$.

1st method: solve for y explicitly. $y_{1,2} = \pm\sqrt{25 - x^2}$. *The point lies on the graph of* y_2. *Take derivative of* y_2.

Second use ***implicit differentiation****. To diff w.r.t x treat y as a function of x.*

$$2x + 2y\frac{dy}{dx} = 0$$

Plug in the point to find dy/dx.

Example 88. *Find an equation of the tangent line to the curve* $x\sin(xy - y^2) = 0$ *at* $(1, 1)$

Example 89. *Find* y'' *in terms of x and y if* $xy + y^2 = 2x$. *The answer is* $y'' = -\frac{8}{(x+2y)^3}$

2.10 Antiderivatives

An **antiderivative** of a function on an interval I is another function F satisfying

$$F'(x) = f(x).$$

Example 90. *Find an antiderivative of*

- $f(x) = 1$
- $f(x) = \cos x$
- Antiderivatives of continuous functions always exist. We'll see later that $\int_0^x f(x)dx$ is an antiderivative of $f(x)$.
- Antiderivatives are not unique. If C is any constant then $F(x) = x + C$ is an antiderivative of $f(x) = 1$.
- Two antiderivatives of a function differ by a constant. If F and G are two antiderivatives of f then
$$\frac{d}{dx}(F(x) - G(x)) = 0.$$
Hence by the result in Section 2.8, $F(x) = G(x) + C$ for some constant C.

The **indefinite integral** of $f(x)$ is

$$\int f(x)dx = F(x) + C$$

provided $F'(x) = f(x)$.

The symbol $\int$ is called the **integral sign**. The constant C is called a **constant of integration**.

Example 91.

$$\int x\,dx = \frac{1}{2}x^2 + C$$

$$\int (x^3 - 5x^2 + 7)dx = \frac{1}{4}x^4 - \frac{5}{3}x^3 + 7x + C$$

Example 92. *Find the function $f(x)$ whose derivative is $f'(x) = 6x^2 - 1$ for all x and $f(2) = 10$.*

Solution 11.

$$f(x) = \int (6x^2 - 1)\,dx = 2x^3 - x + C$$

Use the condition to find $C = -4$.

Transcendental Functions

3.1 Inverse Functions

definition 12. *f is called **one-to-one** if $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$ or equivalently*

$$f(x_1) = f(x_2) \implies x_1 = x_2$$

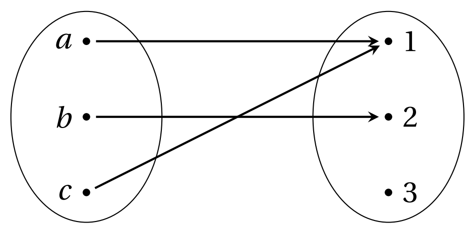


Figure 0.15: A function which is not 1-1.

Horizontal Line Test. Let $f : \mathbb{R} \to \mathbb{R}$. By definition of a function any vertical line intersects the graph at one point. f is 1-1 if its graph is never intersected by any horizontal line more than once.

Increasing or Decreasing Functions are 1-1.

definition 13. *If f is one-to-one then it has an inverse function f^{-1} defined as follows: If x is in the range of f then it is in the domain of f^{-1} and*

$$f^{-1}(x) = y \iff x = f(y).$$

Example 93. *$f(x) = x^2$ is not 1-1 and so does not have an inverse.*

Example 94. *Show that $f(x) = 2x - 1$ is one-to-one and find its inverse $f^{-1}(x)$.*

Solution 12. *Since $f'(x) = 2 > 0$, f is increasing on $\mathbb{R}$ and therefore one-to-one for all x. Let $y = f^{-1}(x)$, solve for y to get*

$$f^{-1}(x) = \frac{x+1}{2}.$$

Properties of inverse functions

1. The domain of f^{-1} is the range of f.
2. The range of f^{-1} is the domain of f.
3. $f(f^{-1}(x)) = x$ for all x in the domain of f^{-1}.

Proof. If $f^{-1}(x) = y$ then $x = f(y)$ and

$$f(f^{-1}(x)) = f(y) = x$$

$\square$

4. $f^{-1}(f(x)) = x$ for all x in the domain of f.
5. $(f^{-1})^{-1}(x) = f(x)$ for all x in the domain of f. (The inverse of inverse of f is f.)

   Proof.
   $$(f^{-1})^{-1}(x) = y \iff f^{-1}(y) = x \iff y = f(x).$$

   $\square$

6. The graph of f^{-1} is the reflection of the graph of f in the line $x = y$. (Because if (a, b) is a point on the graph of $y = f(x)$ then (b, a) is a point on the graph of $y = f^{-1}(x)$).

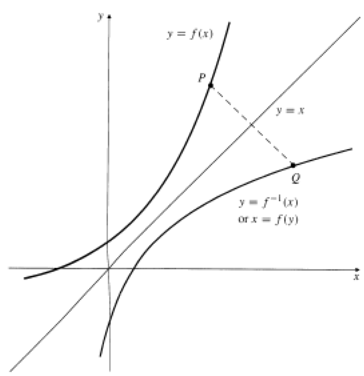


Figure 0.16: The graph of the inverse function is a reflection along $y = x$.

Inverting Non One-to-one Functions

The function $f(x) = x^2$ is not one-to-one and hence not invertible. $f(-a) = f(a)$ for any a. Let us define a new function by restricting the domain of f

$$F(x) = x^2, \quad x \geq 0$$

Then $F^{-1}(x) = \sqrt{x}$.

Derivatives of Inverse Functions

Let $y = f^{-1}(x)$.

$$\frac{d}{dx} f(y) = \frac{d}{dx} x \implies f'(y)\frac{dy}{dx} = 1 \implies \frac{dy}{dx} = \frac{1}{f'(y)}$$

Thus

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$

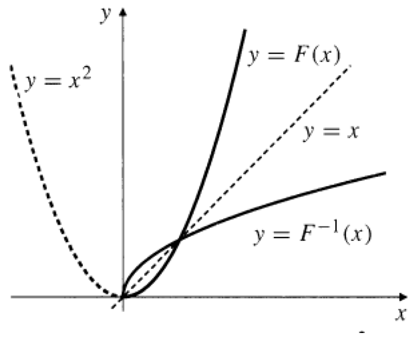


Figure 0.17: The restriction of x^2 to $[0,\infty)$ and its inverse.

Example 95. *Show that* $f(x) = x^3 + x$ *is one-to-one on the whole real line and find* $(f^{-1})'(10)$. *Hint:* $2^3 + 2 = 10$.

Solution 13. *First* $f'(x) = 3x^2 + 1 > 0$. *Hence* f *is 1-1. Let* $y = f^{-1}(x)$.

$$(f^{-1})'(x) = \frac{1}{3y^2+1} = \frac{1}{3(f^{-1}(x))^2+1}.$$

$$(f^{-1})'(10) = \frac{1}{3(f^{-1}(10))^2+1}.$$

$$y = f^{-1}(10) \implies f(y) = 10 \implies y = 2 \implies f^{-1}(10) = 2.$$

Thus

$$(f^{-1})'(10) = \frac{1}{3\times 2^2+1} = \frac{1}{13}.$$

Example 96 (OPTIONAL)**.** *Show that*

$$f(x) = \begin{cases} x^2+1, & \text{if } x \geq 0 \\ x+1, & \text{if } x < 0 \end{cases}$$

is 1-1 and find its inverse.

Solution 14. $f'(x) > 0$ *for* $x < 0$ *and* $x > 0$ *so it is increasing for* $x < 0$ *and* $x < 0$. *Also if* $x < 0$ *then* $f(x) < 1 = f(0)$ *and if* $x > 0$ *then* $1 = f(0) < f(x)$, *hence* f *is increasing everywhere. That proves that* f *is 1-1.*

$$f^{-1}(x) = \begin{cases} \sqrt{x-1}, & \text{if } x \geq 0 \\ x-1, & \text{if } x < 0 \end{cases}$$

3.2 Exponential and Logarithmic Functions

An **exponential function** is a function of the form $f(x) = a^x$ where the **base** a is a positive constant and the **exponent** x is the variable.

- $a^0 = 1$.
- $a^n = a \cdot a \cdots a$ (n -times) if $n = 1, 2, 3, \ldots$.

- $a^{-n} = \frac{1}{a^n}$ if $n = 1, 2, 3, \ldots$.
- $a^{m/n} = \sqrt[n]{a^m}$ if $n = 1, 2, \ldots$ and $m = \pm 1, \pm 2, \ldots$.

How should we define a^x if x is not rational? What does 2^π mean?

If x is irrational, then we define a^x as being the limit values a^r for rational numbers r approaching x

$$a^x = \lim_{r \to x} a^r, \qquad \text{r is rational.}$$

Example 97. *Since the irrational number $\pi = 3.141592\ldots$ is the limit of the sequence of rational numbers*

$$r_1 = 3 \quad r_2 = 3.1 \quad r_3 = 3.14 \quad \ldots$$

we can calculate 2^π as the limit of the sequence

$$2^3 = 8 \quad 2^{3.1} = 8.5741877\ldots \quad 2^{3.14} = 8.8152409\ldots$$

This gives

$$2^\pi = \lim_{n \to \infty} 2^{r_n} = 8.824977\ldots$$

Laws of Exponents

If $a > 0$ and $b > 0$ and x, y are real numbers then

1. $a^0 = 1$,
2. $a^{x+y} = a^x a^y$,
3. $a^{-x} = \dfrac{1}{a^x}$,
4. $a^{x-y} = \dfrac{a^x}{a^y}$,
5. $(a^x)^y = a^{xy}$,
6. $(ab)^x = a^x b^x$.

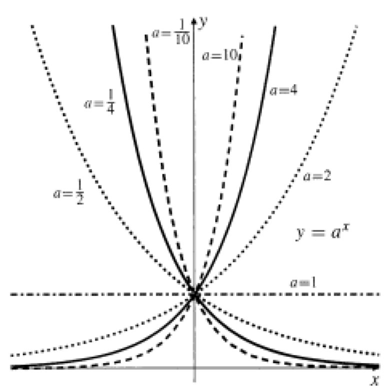
If $a > 1$ then

$$a^x \to \begin{cases} \infty, & \text{if } x \to \infty \\ 0, & \text{if } x \to -\infty \end{cases}$$

If $0 < a < 1$ then

$$a^x \to \begin{cases} 0, & \text{if } x \to \infty \\ \infty, & \text{if } x \to -\infty \end{cases}$$

The domain of a^x is $(-\infty, \infty)$ and its range is $(0, \infty)$.



Logarithm

If $a > 0$ and $a \neq 1$, the function $\log_a x$ called the **logarithm of** x **base** a is the inverse of the 1-1 function a^x:

$$y = \log_a x \iff x = a^y$$

Since a^x has domain $(-\infty, \infty)$, and range $(0, \infty)$, $\log_a x$ has domain $(0, \infty)$ and range $(-\infty, \infty)$. Since a^x and $\log_a x$ are inverse functions

$$\log_a a^x = x \quad \forall x, \qquad a^{\log_a x} = x, \quad x > 0$$

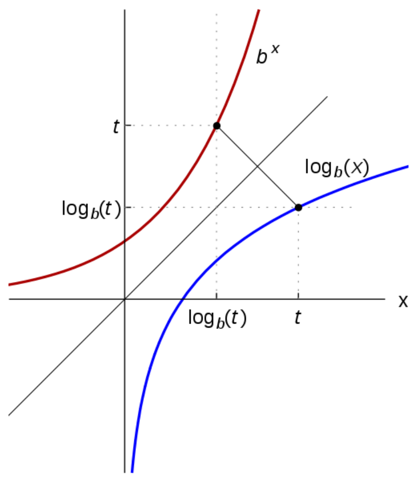


Figure 0.18: The graph of logarithmic function is a reflection of the graph of the exponential function in the line $y = x$ (from Wikipedia).

Laws of Logarithm If $x > 0$, $y > 0$, $a > 0$, $b > 0$, $a \neq 1$, $b \neq 1$, then

1. $\log_a 1 = 0$
2. $\log_a(xy) = \log_a x + \log_a y$
3. $\log_a(\frac{1}{x}) = -\log_a x$
4. $\log_a(\frac{x}{y}) = \log_a x - \log_a y$
5. $\log_a x^y = y \log_a x$
6. $\log_a x = \dfrac{\log_b x}{\log_b a}$

Example 98. *Prove property* $\log_a(xy) = \log_a x + \log_a y$ *using laws of exponent.*

Solution 15. *Take* $u = \log_a x$, $v = \log_a y$ *then* $x = a^u$, $y = a^v$ *and*

$$xy = a^{u+v} \iff u + v = \log(xy)$$

Example 99. *Simplify*

1. $\log_2 10 + \log_2 12 - \log_2 15$

$$= \log_2 \frac{10 \times 12}{15} = \log_2 8 = 3$$

2. $\log_{a^2} a^3$

$$= \frac{\log_a a^3}{\log_a a^2} = \frac{3}{2}$$

3. $3^{\log_9 4}$

$$= 3^{\frac{1}{2}\log_3 4} = 3^{\log_3 2} = 2$$

Example 100. *Solve*

$$3^{x-1} = 2^x.$$

Solution 16.

$$(x-1)\log_{10} 3 = x\log_{10} 2 \iff x = \frac{\log_{10} 3}{\log_{10} 3/2}$$

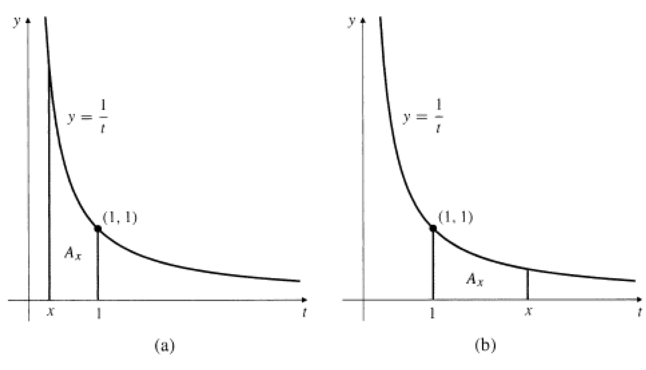
3.3 The Natural Logarithm and Exponential

$f(x)$	$f'(x)$
x^3	$3x^2$
x^2	$2x$
x	1
x^0	0
x^{-1}	$-x^{-2}$
x^{-2}	$-2x^{-3}$

Table 0.1: What is the mysterious function whose derivative is x^{-1}?

definition 14. *For $x > 0$, let A_x be the area bounded by the curve $y = 1/t$, the t-axis and the vertical lines $t = 1$ and $t = x$. The **natural logarithm** function is defined by*

$$\ln x = \begin{cases} A_x & x \geq 1 \\ -A_x & 0 < x < 1 \end{cases}$$

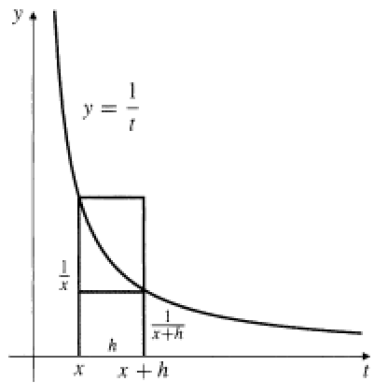


- Domain of $\ln x$ is $(0, \infty)$,
- $\ln 1 = 0$,
- $\ln x > 0$ if $x > 1$,
- $\ln x < 0$ if $0 < x < 1$,

Theorem 15. *If* $x > 0$ *then* $\dfrac{d}{dx}\ln x = \dfrac{1}{x}$

Proof. For $h > 0$, $\ln(x+h) - \ln x$ is the area under $1/x$ between $t = x$ and $t = x+h$. Thus

$$\frac{h}{x+h} < \ln(x+h) - \ln x < \frac{h}{x}$$



Thus

$$\frac{1}{x+h} < \frac{\ln(x+h) - \ln x}{h} < \frac{1}{x}$$

Now use Squeeze Theorem to get

$$\lim_{h \to 0+} \frac{\ln(x+h) - \ln x}{h} = \frac{1}{x}$$

Similar argument holds for $h < 0$.

□

Theorem 16. *If* $x \neq 0$ *then*

$$\frac{d}{dx}\ln|x| = \frac{1}{x},$$

and

$$\int \frac{1}{x}\,dx = \ln|x| + C.$$

Example 101. *Find the derivatives of*

1. $y = \ln|\cos x|$
2. $y = \ln(x + \sqrt{x^2+1})$

Solution 17. *For (1) use the previous theorem to get*

$$y' = \frac{1}{\cos x}(-\sin x) = -\tan x.$$

For (2),

$$y' = \frac{1}{\sqrt{x^2+1}}.$$

The natural logarithm function $\ln x$ satisfies all the rules that the regular logarithms satisfy. Note that

$$\ln 2^n = n\ln 2 \to \infty \text{ as } n \to \infty.$$

$$\ln 2^{-n} = -n\ln 2 \to -\infty \text{ as } n \to \infty.$$

This shows that

$$\lim_{x\to\infty} \ln x = \infty, \qquad \lim_{x\to 0+} \ln x = -\infty.$$

Domain of $\ln x$ is $(0,\infty)$ and the range of $\ln x$ is $(-\infty,\infty)$.

The Exponential Function

Let $f(x) = \ln x$. Since $f'(x) = 1/x > 0 \implies f$ is increasing $\implies f$ is 1-1 $\implies f$ has an inverse. Call its inverse $\exp x$. Thus

$$\exp x = y \iff x = \ln y$$

- $\exp 0 = 1$ (since $\ln 1 = 0$),
- Domain of exp is $(-\infty,\infty)$ (since range of ln is),
- Range of exp = Domain of ln = $(0,\infty)$,
- Cancellation identities

$$\exp\ln x = x, \qquad \ln\exp x = x, \qquad x > 0$$

definition 15. $e = \exp(1) \approx 2.718\ldots.$

It turns out that

$$\exp x = e^x$$

To be rigorous, this holds when x is rational. When x is irrational we use $\exp x$ *to define e^x.*
Since exp is actually an exponential function, its inverse must be a logarithm

$$\ln x = \log_e x$$

The derivative of $y = e^x$ is calculated by implicit differentiation:

$$y = e^x \iff x = \ln y \iff 1 = \frac{y'}{y} \iff y' = y = e^x$$

This is a remarkable property:

$$\frac{d}{dx}e^x = e^x, \qquad \int e^x\,dx = e^x + C$$

Example 102. *Find the derivatives of*

1. e^{x^2-3x},
2. $\sqrt{1+e^{2x}}$

General Exponentials and Logaritms

definition 16. *If $a > 0$ then for all real x, we define*

$$a^x = e^{x \ln a}$$

This coincides with our previous definition that a^x is the limit of a^{r_n} where r_n are rational numbers tending to x.

Example 103. $2^\pi = e^{\pi \ln 2} \approx 8.825$.

Derivative of $y = a^x$.

$$\frac{d}{dx} a^x = \frac{d}{dx} e^{x \ln a} = e^{x \ln a} \ln a = a^x \ln a.$$

Example 104. *Show that the graph of $f(x) = x^\pi - \pi^x$ has negative slope at $x = \pi$.*

Solution 18. $f'(\pi) = \pi^\pi (1 - \ln \pi)$. *Note that* $\ln \pi > \ln e = 1$

definition 17. *Let $y = a^x$. Then $\frac{dy}{dx} = a^x \ln a$ which is negative if $0 < a < 1$ and positive if $a > 1$. Thus a^x is 1-1 and has an inverse function. We define its inverse as $\log_a x$.*

Derivative of $y = \log_a x$.

$$\frac{d}{dx} \log_a x = \frac{d}{dx} \frac{\ln x}{\ln a} = \frac{1}{ax}.$$

Logarithmic Differentiation

Example 105. *Let $y = x^x$, $x > 0$. Find y'.*

Solution 19. *Neither the power rule $d/dx(x^a) = ax^{a-1}$ nor the exponential rule $d/dx(a^x) = \ln a a^x$ works.*

$$\ln y = x \ln x \implies \frac{y'}{y} = 1 \ln x + x \frac{1}{x} \implies y' = x^x (\ln x + 1)$$

This technique is called **logarithmic differentiation** and is used to differentiate functions of the form $y = (f(x))^{g(x)}$ $(f(x) > 0)$.

Example 106. *Find dy/dt if $y = (\sin t)^{\ln t}$ where $0 < t < \pi$.*

Solution 20.

$$y' = (\sin t)^{\ln t} \left(\frac{\ln \sin t}{t} + \ln t \cot t \right).$$

Example 107. *If $y = \dfrac{(x+1)(x+2)(x+3)}{\sqrt{x+4}}$, find y'.*

Solution 21. *Since $(x+1)$ is not necessarily positive, $\ln(x+1)$ may or may not be defined. So we take the absolute value and then logarithm.*

$$\ln|y| = \ln|x+1| + \ln|x+2| + \ln|x+3| - \frac{1}{2} \ln|x+4|$$

$$\frac{y'}{y} = \frac{1}{x+1} + \cdots$$