Calculus I Lecture Notes

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Chapter 1

Precalculus

1.1 Sets

A **set** is a collection of elements.

$x \in A$ means x is an element of the set A. If x is not a member of A, we write $x \notin A$.

$\varnothing$ is the set which contains no element and is called the **empty set**.

There are finite sets (ex. $\{0,1,2\}$) and infinite sets (ex. $\{0,1,2,3,\ldots\}$).

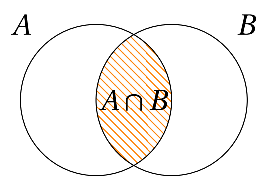
If every element of the set A is an element of the set B, we say that A is **subset** of B, and write $A \subset B$.

Example 1. *List all the subsets of* $\{0,1,2\}$.

For any set A, $A \subset A$ and $\varnothing \subset A$.

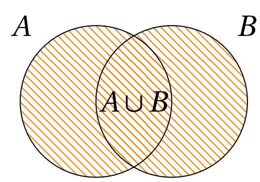
If $A \subset B$ and $B \subset A$, we write $A = B$.

$A \cap B = \{x : x \in A \text{ and } x \in B\}$ is called the **intersection** of A and B.



If the intersection of two sets is the empty set, those sets are called **disjoint**.

$A \cup B = \{x : x \in A \text{ or } x \in B\}$ is called the **union** of A and B.



Example 2. *For example if* $A = \{0,1,2,5,8\}$ *and* $B = \{1,3,5,6\}$ *then find* $A \cap B$ *and* $A \cup B$.

The set of all elements in A but not in B is denoted $A \setminus B = \{x \in A : x \notin B\}$ and is called the **complement** of B in A.

Example 3. $\{0,2,3,5\} \setminus \{2,5,7,8\} = \{0,3\}$

$A \times B = \{(a,b) : a \in A$ and $b \in B\}$ is called the **Cartesian** product of the sets A and B.

Example 4. *Write the cartesian product of* $A = \{0,1,2\}$ *and* $B = \{2,3,4\}$.

1.2 Real Numbers

The **integers** are $\mathbb{Z} = \{\dots,-2,-1,0,1,2,\dots\}$.

Integers come in a lot varieties:

- even integers that are of the form $2k$, for some $k \in \mathbb{Z}$,

- odd integers that are of the form $2k+1$, for some $k \in \mathbb{Z}$

- positive and negative integers,

- primes, etc...

The **rational numbers** are $\mathbb{Q} = \left\{\frac{m}{n} : m,n \in \mathbb{Z} \text{ and } n \neq 0\right\}$.

Pythagoreans preached that all numbers could be expressed as the ratio of integers, and the discovery of irrational numbers is said to have shocked them.

Example 5. $\sqrt{2}$ *is not a rational number.*

Suppose that it is rational. Then $\sqrt{2} = m/n$, *where* $m,n \in \mathbb{Z}$ *and* $n \neq 0$. *Also assume* m *and* n *have no common divisor.*

$$m^2/n^2 = 2 \implies m^2 = 2n^2$$

Thus m *is even and we can write* $m = 2k$, *where* $k \in \mathbb{Z}$.

$$4k^2 = 2n^2 \implies n^2 = 2k^2$$

Thus n *is also even. But* m *and* n *cannot both be even. Accordingly, there can be no rational number whose square is 2.*

The set of irrational numbers is denoted by $\mathbb{I}$.
The set of real numbers is $\mathbb{R} = \mathbb{Q} \cup \mathbb{I}$.
Note that $\mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$.
The real numbers are ordered such that

1. $a < b \implies a + c < b + c$

2. $a < b$ and $c > 0$ implies $ac < bc$

3. $a < b$ and $c < 0$ implies $ac > bc$

4. $a > 0$ implies $\frac{1}{a} > 0$

5. $0 < a < b$ implies $\frac{1}{b} < \frac{1}{a}$

Intervals

The open interval $(a,b) = \{x \mid a < x < b\}$, closed interval ($[a,b]$), half open intervals $(a,b]$, $[a,b)$. It is possible that $a = -\infty$, $b = \infty$. Draw each interval on the real line.

Example 6. *Solve the following inequalities.*

1. $\frac{2}{x-1} \geq 5$.

   Solution. It is not right to multiply both sides by $x-1$ and say $5x-5 \leq 2$.

   $$\frac{2}{x-1} \geq 5 \iff \frac{2}{x-1} - 5 \geq 0 \iff \frac{7-5x}{x-1} \geq 0.$$

   Now make a sign analysis to get interval $(1, 7/5]$

2. $3x - 1 \leq 5x + 3 \leq 2x + 15$.

   Solution. $-2 \leq x$ *and* $x \leq 4$.

The absolute value.

$$|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$$

ex. $|3| = |-3| = 3$

Geometrically, $|x|$ is the distance between x and 0 on the real line. And $|x-y|$ is the distance between x and y.

Properties *(can be proved from definition)*:

1. $|-x| = |x|$, (Do not fall into the trap $|-x| = x$, this is not always true!)

2. $|ab| = |a||b|$,

3. $|a+b| \leq |a| + |b|$, (triangle inequality).

From (2), for any x, $x^2 = |x^2| = |x|^2$

If D is a nonnegative number

$$|x| = D \implies x = -D \text{ or } x = D,$$

$$|x| < D \implies -D < x < D$$

$$|x| > D \implies x < -D \text{ or } x > D$$

More generally,

$$|x-a| = D \implies x = a - D \text{ or } x = a + D,$$

$$|x-a| < D \implies a - D < x < a + D$$

$$|x-a| > D \implies x < a - D \text{ or } x > a + D$$

Example 7. *Solve* $|3x-2| \leq 1$.

Solution.

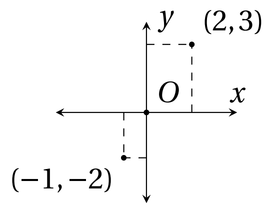
$$-1 \leq 3x - 2 \leq 1 \implies x \geq 1/3 \textit{ and } x \leq 1.$$

Example 8. *Solve the equation* $|x+1| > |x-3|$.

Solution. The distance between x and -1 is greater than the distance between x and 3. So $x > 1$.

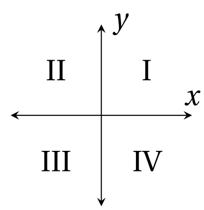
1.3 Cartesian Coordinates

Cartesian plane is $\mathbb{R} \times \mathbb{R} = \{(x, y) \mid x \in \mathbb{R} \text{ and } b \in \mathbb{R}\}$.



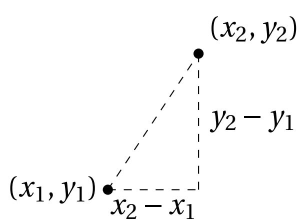
Horizontal axis is usually called the x axis, the vertical axis is called the y axis. Intersection of the axes is called the origin, denoted O.

The coordinate axes divide the Cartesian plane into four quadrants.



By the Pythogerean Theorem, the distance of two points (x_1, y_1) and (x_2, y_2) in the plane is

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$



The distance of (x, y) to the origin is $\sqrt{x^2 + y^2}$.

Example 9. *Find the distance between* $(-1, 1)$ *and* $(3, -4)$.

Equations of Lines

For any two points (x_1, y_1) and (x_2, y_2) on a non-vertical line L, the quantity $m = \frac{y_2 - y_1}{x_2 - x_1}$ is constant and is called the **slope** of the line L.

Let L be a nonvertical line. Let m be the slope of L and (x_1, y_1) be the coordinates of a point on L. If (x, y) is another point on L, then

$$\frac{y - y_1}{x - x_1} = m$$

Hence any (x, y) on L satisfies

$$y = m(x - x_1) + y_1$$

The above is known as an equation for the line L.

All points on a **vertical line** have their x coordinate equal to a constant a. So the equation of a vertical line is $x = a$. **Horizontal lines** have equations of the form $y = a$.

y-intercept of a nonvertical line L is the y-coordinate of the point where L intersects the y-axis. **x-intercept** of a nonhorizontal is defined similarly.

Example 10. *Find an equation of the line through the points* $(1,-1)$ *and* $(3,5)$. *Draw the line. Find the x and y intercepts.*

Example 11. *Find an equation of the line that passes through the point* $(-3,-4)$ *and has slope* 2. *Draw the line.*

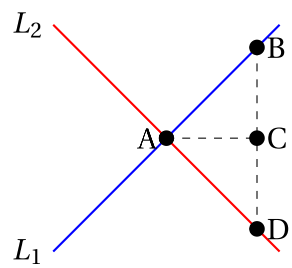
Example 12. *Find the slope and the two intercepts of the line with equation* $8x+5y=20$. *Draw the line.*

Parallel vs. perpendicular lines

We call two lines **parallel** if their slopes are equal.

We call two lines **perpendicular** if they intersect at right angles (90°).

Theorem 1. *Two nonvertical lines with slopes* m_1 *and* m_2 *are perpendicular if and only if* $m_1 m_2 = -1$.



Proof. Use the similarity of the triangles ABC and DAC to get

$$\frac{|BC|}{|AC|} = \frac{|AC|}{|CD|} \implies \frac{|BC||CD|}{|AC|^2} = 1$$

Slope of L_1 (m_1) is $|BC|/|AC| = 1$ and slope of L_2 (m_2) is $-|CD|/|AC|$. So $m_1 m_2 = -1$. □

Example 13. *Find an equation of the line through* $(1,-2)$ *that is parallel to the line L with equation* $3x-2y=1$. *Draw the lines.*

Example 14. *Find an equation of the line through* $(2,-3)$ *that is perpendicular to the line L with equation* $4x+y=3$. *Draw the lines.*

1.4 Quadratic Equations

Circles and Disks

The circle is the set of all points that have the same distance (called radius of the circle) from a given point (called center of the circle).

If (x, y) is a point on a circle with center (a, b) and radius r then

$$\sqrt{(x-a)^2+(y-b)^2} = r \implies (x-a)^2+(y-b)^2 = r^2$$

Example 15. *Find the center and radius of the circle $x^2 + y^2 - 4x + 6y = 3$.*

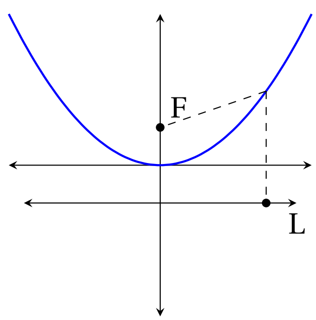
Solution. Complete to squares to get $(x-2)^2 + (y+3)^2 = 16$.

The equation $(x-a)^2+(y-b)^2 < r^2$ represents open disk and the equation $(x-a)^2+(y-b)^2 \le r^2$ represents closed disk or simply disk.

Example 16. *Draw* $x^2 + 2x + y^2 \le 8$.

Parabolas

A parabola P is the set of all points in the plane that are equidistant from a given line L (called directrix of P) and a point F (called the focus of P).



Example 17. *Find the equation of the parabola having the point $F(0, p)$as focus and the line L with equation $y = -p$ as directrix.*

Solution. If $P(x, y)$ is any point on the parabola then squaring both sides of PF=PQ we get

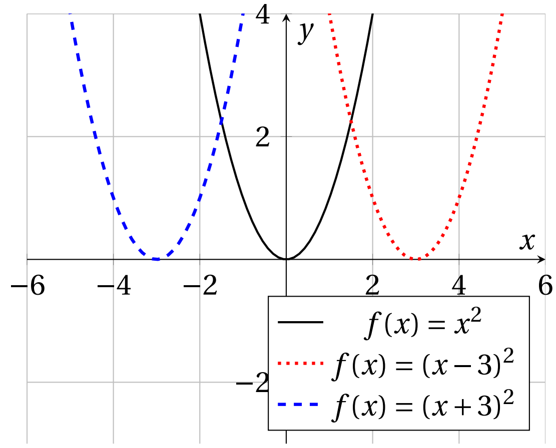
$$x^2 + (y-p)^2 = 0^2 + (y+p)^2$$

After simplifying, $y = x^2/4p$.

Shifting a Graph

Let $c > 0$.

- To shift a graph c units to the right, replace x in its equation with $x - c$. To shift to left, replace x by $x + c$.
- To shift a graph c units up, replace y in its equation with $y - c$. To shift down, replace y by $y + c$.



1.5 Functions and Their Graphs

A **function** f on a set D into a set R is a rule that assigns a unique element $f(x)$ in R to each element x in D.

D is called the **domain** of f. R is called the target or **codomain** of f. The **range** of f is a subset of R containing of all possible values $f(x)$.

This definition is not mathematical as we did not define what a rule is. Formally one defines a function as a relation.

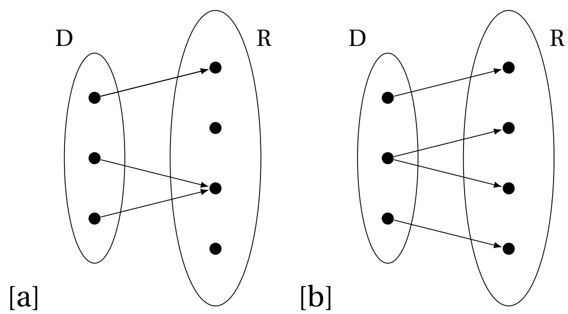


Figure 1.1: a) Not a function. b) A Function

Example 18. *Define a function on the set of all real numbers by* $f(x) = x^2 + 1$. *Find* $f(0)$, $f(2)$, $f(x+2)$.

$$f(x) = \frac{1}{x}, \qquad x > 0$$

means that the domain of f is the set $\{x \mid x > 0\}$.

Technically, this function is different from the function

$$f(x) = \frac{1}{x}, \qquad x < 0.$$

If we do not specify the domain of a function f, then the **domain convention** is to assume that the domain of f is the set of all real numbers for which f is defined.

So if we write

$$f(x) = \frac{1}{x},$$

we are assuming f is defined for all real numbers except 0.

Example 19. *Find the domain of $f(x) = \sqrt{2-x}$.*
Solution. *Its domain is all x for which $2-x \geq 0$, i.e. the interval $(-\infty, 2]$.*

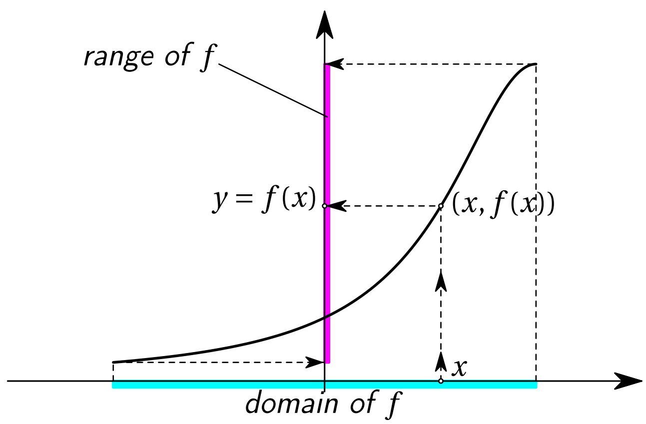
Example 20. *Find the domain of $f(x) = \frac{1}{x^2-x}$.*

A function $f: D \to R$ is **1-1** if $f(x_1) = f(x_2)$ then $x_1 = x_2$. A function $f: D \to R$ is **onto** if for every $y \in R$, there is an $x \in D$ such that $f(x) = y$.

Example 21. *Draw functions which are 1-1, onto, not 1-1 and not onto, similar to the Figure 1.1.*

Graph of a function

The *graph of a function* f is the set of all points whose coordinates are $(x, f(x))$ where x is in the domain of f.



Example 22. *A function which is given by the formula*

$$f(x) = mx + n$$

where m and n are constants is called a linear function. *Its graph is a straight line. The constants m and n are the* slope *and y*-intercept *of the line.*

Example 23. *The square root function $f(x) = \sqrt{x}$ has domain $[0, \infty)$ and takes x to its positive square root. Hence it has range $[0, \infty)$.*

Example 24. *The absolute value function $f(x) = |x| = \sqrt{x^2}$ has domain $(-\infty, \infty)$ and range $[0, \infty)$.*

Example 25. *Draw the graphs of some elementary functions*

$$c, x, x^2, \sqrt{x}, x^3, x^{1/3}, \frac{1}{x}, \frac{1}{x^2}, \sqrt{1-x^2}, |x|.$$

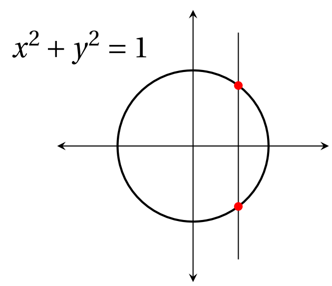
Example 26. *Sketch the graph of $f(x) = 1 + \sqrt{x-4}$.*
Solution: Shift the graph of $y = \sqrt{x}$ 1 unit up and 4 units to the right.

Example 27. *Sketch the graph of the function $f(x) = \frac{2-x}{x-1}$.*
Solution. $f(x) = \frac{2-x}{x-1} = -1 + \frac{1}{x-1}$. So shift the graph of $y = \frac{1}{x}$ 1 unit down and 1 unit to the right.

Vertical Line Test

The graph of a function cannot intersect a vertical line "$x =$ constant" in more than one point.

For example, the circle $x^2 + y^2 = 1$ is not a graph of a function.



Even and Odd Functions

Definition 1. *We say that f is an **even function** if $f(-x) = f(x)$ for every $x \in D$. We say that f is an **odd function** if $f(-x) = -f(x)$ for every $x \in D$.*

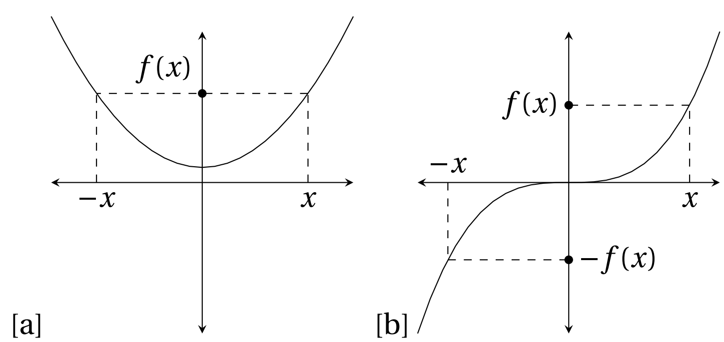


Figure 1.2: a) An even function, b) An odd function.

Odd functions are symmetric with respect to origin and even functions are symmetric with respect to the y-axis.

Example 28. *$f(x) = x$, $f(x) = x^3$ are odd and $f(x) = x^2$ and $f(x) = x^4$ are even and $f(x) = \frac{1}{x+1}$ is neither even or odd.*

Example 29. *$f(x) = x^3 + x$ is odd and $f(x) = \frac{1}{x^2-1}$ is even and $f(x) = x^2 + x$ is either even or odd.*

1.6 Operations on Functions

If f and g are functions, then for every x that belongs to the domains of both f and g we define functions

$(f+g)(x) = f(x) + g(x)$
$(f-g)(x) = f(x) - g(x)$
$(fg)(x) = f(x)g(x)$
$(f/g)(x) = f(x)/g(x)$ where $g(x) \neq 0$.

Example 30. *Let* $f(x) = \frac{1}{x+2}$ *and* $g(x) = \frac{x}{x-1}$. *Find* $(f+g)(x)$, $(f-g)(x)$, $(fg)(x) = f(x)g(x)$ *and* $(f/g)(x)$ *where* $g(x) \neq 0$.

Composition of Functions

If f and g are two functions, then

$$f \circ g(x) = f(g(x)).$$

The domain of $f \circ g$ consists of those numbers x in the domain of g for which $g(x)$ is in the domain of f.

Example 31. *Let* $f(x) = \sqrt{x}$ *and* $g(x) = x+1$. *Find* $f \circ g$, $g \circ f$, $f \circ f$ *and* $g \circ g$. *State the domains of each function.*

Function	Formula	Domain
f	$\sqrt{x}$	$[0,\infty)$
g	$x+1$	$\mathbb{R}$
$f \circ g$	$\sqrt{x+1}$	$[-1,\infty)$
$g \circ f$	$\sqrt{x}+1$	$[0,\infty)$
$f \circ f$	$x^{1/4}$	$[0,\infty)$
$g \circ g$	$x+2$	$\mathbb{R}$

Piecewise Defined Functions

Functions such as

$$g(x) = \begin{cases} 2x & \text{for } x < 0 \\ x^2 & \text{for } x \geq 0 \end{cases}$$

which are defined by different formulas on different intervals are sometimes called **piecewise defined functions.**

Inverse Functions

Remember that a function is **one-to-one** if for every value in the range, there is exactly one value in the domain.

A function is one-to-one if every horizontal line crosses its graph at most once, which is commonly known as the **horizontal line test**.

1.7 Polynomials and Rational Functions

Definition 2. *A* ***polynomial*** *is a function* $P : \mathbb{R} \to \mathbb{R}$ *such that*

$$P(x) = a_n x^n + \cdot + a_1 x + a_0.$$

Here $a_n, \ldots, a_1$ *are called the* ***coefficients*** *of the polynomial. We assume* $a_n \neq 0$. *The number* n *is called the* ***degree*** *of the polynomial.*

Example 32. *Write polynomials of degree 0, 1 and 2.*

Just as the quotient of two integers is called a rational number, the quotient of two polynomials is called a **rational function**. Give an example.

Let A_m be a polynomial of degree m, B_n be a polynomial of degree n with $m \geq n$. Then there are polynomial Q_{m-n} of degree $m-n$, R_k of degree $k < n$ such that

$$\frac{A_m}{B_n} = Q_{m-n} + \frac{R_k}{B_n}.$$

The quotient Q_{m-n} and the remainder R_k can be calculated by the "long division".

Example 33. *Using the long division algorithm, show that*

$$\frac{2x^3 - 3x^2 + 3x + 4}{x^2 + 1} = 2x - 3 + \frac{x + 7}{x^2 + 1}$$

If P is a polynomial and $P(r) = 0$ then r is called a **root** of P.

The Fundamental Theorem of Algebra says every polynomial of degree greater than 0 must have a root. But these roots may be complex.

Example 34. $x^2 + 1$ *has no real roots. Its roots are* $i = \sqrt{-1}$ *and* $-i$.

Theorem 2. *If r is a root of the polynomial P then*

$$P(x) = (x - r)Q(x),$$

for some polynomial Q whose degree is 1 less than P.

The polynomial $x(x-7)^3$ has 4 roots: 0 and the other three are each equal to 7. We say that 7 is a root of **multiplicity** 3.

By the Fundamental Theorem of Algebra and the above theorem, every polynomial of degree n has exactly n (not necessarily distinct) roots.

Roots of Quadratic Polynomials

To obtain the solutions of

$$Ax^2 + Bx + C = 0, \qquad A \neq 0$$

Divide by A and complete to square

$$\left(x + \frac{B}{2A}\right)^2 = \frac{B^2}{4A^2} - \frac{C}{A} = \frac{B^2 - 4AC}{4A^2},$$

Taking the square root of both sides gives the quadratic formula

$$x = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}.$$

A form of this formula is known since B.C. 2000 by Babylonians.

The quantity $D = B^2 - 4AC$ is called the **discriminant** of the quadratic equation.

- If $D > 0$ then there are two distinct real roots,
- If $D = 0$ then there is 1 root of multiplicity 2,
- If $D < 0$ then there are two complex conjugate roots.

Example 35. *Find the roots of the polynomials: (a)* $x^2 + x - 1$*, (b)* $9x^2 - 6x + 1$*, (c)* $2x^2 + x + 1$.

Misc Factorings

- Difference of squares:
$$x^2 - a^2 = (x-a)(x+a)$$
- Difference of cubes:
$$x^3 - a^3 = (x-a)(x^2 + ax + a^2)$$
- Difference of nth powers
$$x^n - a^n = (x-a)(x^{n-1} + ax^{n-2} + a^2x^{n-3} + \cdots + a^{n-2}x + a^{n-1})$$
- If n is an odd integer then $x + a$ is a factor of $x^n + a^n$,
$$x^n + a^n = (x+a)(x^{n-1} - ax^{n-2} + a^2x^{n-3} - \cdots - a^{n-2}x + a^{n-1})$$