Calculus I Lecture Notes

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Contents

Chapter 1

Precalculus

1.1 Sets

A **set** is a collection of elements.

$x \in A$ means x is an element of the set A. If x is not a member of A, we write $x \notin A$.

$\varnothing$ is the set which contains no element and is called the **empty set**.

There are finite sets (ex. $\{0,1,2\}$) and infinite sets (ex. $\{0,1,2,3,...\}$).

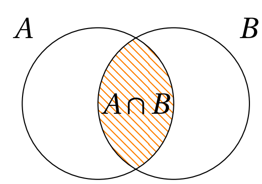
If every element of the set A is an element of the set B, we say that A is **subset** of B, and write $A \subset B$.

Example 1. *List all the subsets of* $\{0,1,2\}$.

For any set A, $A \subset A$ and $\varnothing \subset A$.

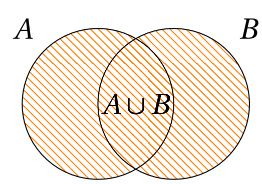
If $A \subset B$ and $B \subset A$, we write $A = B$.

$A \cap B = \{x : x \in A \text{ and } x \in B\}$ is called the **intersection** of A and B.



If the intersection of two sets is the empty set, those sets are called **disjoint**.

$A \cup B = \{x : x \in A \text{ or } x \in B\}$ is called the **union** of A and B.



Example 2. *For example if* $A = \{0,1,2,5,8\}$ *and* $B = \{1,3,5,6\}$ *then find* $A \cap B$ *and* $A \cup B$.

The set of all elements in A but not in B is denoted $A \setminus B = \{x \in A : x \notin B\}$ and is called the **complement** of B in A.

Example 3. $\{0,2,3,5\} \setminus \{2,5,7,8\} = \{0,3\}$

$A \times B = \{(a,b) : a \in A \text{ and } b \in B\}$ is called the **Cartesian** product of the sets A and B.

Example 4. *Write the cartesian product of* $A = \{0,1,2\}$ *and* $B = \{2,3,4\}$.

1.2 Real Numbers

The **integers** are $\mathbb{Z} = \{\ldots,-2,-1,0,1,2,\ldots\}$.

Integers come in a lot varieties:

- even integers that are of the form $2k$, for some $k \in \mathbb{Z}$,
- odd integers that are of the form $2k+1$, for some $k \in \mathbb{Z}$
- positive and negative integers,
- primes, etc...

The **rational numbers** are $\mathbb{Q} = \left\{\frac{m}{n} : m, n \in \mathbb{Z} \text{ and } n \neq 0\right\}$.

Pythagoreans preached that all numbers could be expressed as the ratio of integers, and the discovery of irrational numbers is said to have shocked them.

Example 5. $\sqrt{2}$ *is not a rational number.*

Suppose that it is rational. Then $\sqrt{2} = m/n$, *where* $m, n \in \mathbb{Z}$ *and* $n \neq 0$. *Also assume* m *and* n *have no common divisor.*

$$m^2/n^2 = 2 \implies m^2 = 2n^2$$

Thus m *is even and we can write* $m = 2k$, *where* $k \in \mathbb{Z}$.

$$4k^2 = 2n^2 \implies n^2 = 2k^2$$

Thus n *is also even. But* m *and* n *cannot both be even. Accordingly, there can be no rational number whose square is 2.*

The set of irrational numbers is denoted by $\mathbb{I}$.
The set of real numbers is $\mathbb{R} = \mathbb{Q} \cup \mathbb{I}$.
Note that $\mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$.
The real numbers are ordered such that

1. $a < b \implies a + c < b + c$
2. $a < b$ and $c > 0$ implies $ac < bc$
3. $a < b$ and $c < 0$ implies $ac > bc$
4. $a > 0$ implies $\frac{1}{a} > 0$
5. $0 < a < b$ implies $\frac{1}{b} < \frac{1}{a}$

Intervals

The open interval $(a,b) = \{x \mid a < x < b\}$, closed interval ($[a,b]$), half open intervals $(a,b]$, $[a,b)$. It is possible that $a = -\infty$, $b = \infty$. Draw each interval on the real line.

Example 6. *Solve the following inequalities.*

1. $\frac{2}{x-1} \geq 5$.

   Solution. It is not right to multiply both sides by $x-1$ *and say* $5x-5 \leq 2$.

$$\frac{2}{x-1} \geq 5 \iff \frac{2}{x-1} - 5 \geq 0 \iff \frac{7-5x}{x-1} \geq 0.$$

   Now make a sign analysis to get interval $(1, 7/5]$

2. $3x-1 \leq 5x+3 \leq 2x+15$.

   Solution. $-2 \leq x$ *and* $x \leq 4$.

The absolute value.

$$|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$$

ex. $|3| = |-3| = 3$

Geometrically, $|x|$ is the distance between x and 0 on the real line. And $|x-y|$ is the distance between x and y.

Properties *(can be proved from definition)*:

1. $|-x| = |x|$, (Do not fall into the trap $|-x| = x$, this is not always true!)
2. $|ab| = |a||b|$,
3. $|a+b| \leq |a| + |b|$, (triangle inequality).

From (2), for any x, $x^2 = |x^2| = |x|^2$

If D is a nonnegative number

$$|x| = D \implies x = -D \text{ or } x = D,$$

$$|x| < D \implies -D < x < D$$

$$|x| > D \implies x < -D \text{ or } x > D$$

More generally,

$$|x-a| = D \implies x = a-D \text{ or } x = a+D,$$

$$|x-a| < D \implies a-D < x < a+D$$

$$|x-a| > D \implies x < a-D \text{ or } x > a+D$$

Example 7. *Solve* $|3x-2| \leq 1$.

Solution.

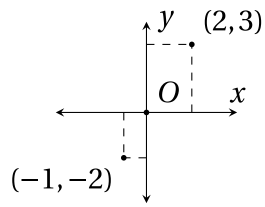
$$-1 \leq 3x-2 \leq 1 \implies x \geq 1/3 \textit{ and } x \leq 1.$$

Example 8. *Solve the equation* $|x+1| > |x-3|$.

Solution. The distance between x and -1 *is greater than the distance between x and* 3. *So* $x > 1$.

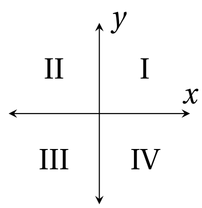
1.3 Cartesian Coordinates

Cartesian plane is $\mathbb{R} \times \mathbb{R} = \{(x, y) \mid x \in \mathbb{R} \text{ and } b \in \mathbb{R}\}$.



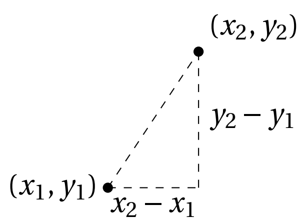
Horizontal axis is usually called the x axis, the vertical axis is called the y axis. Intersection of the axes is called the origin, denoted O.

The coordinate axes divide the Cartesian plane into four quadrants.



By the Pythogerean Theorem, the distance of two points (x_1, y_1) and (x_2, y_2) in the plane is

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$



The distance of (x, y) to the origin is $\sqrt{x^2 + y^2}$.

Example 9. *Find the distance between* $(-1, 1)$ *and* $(3, -4)$.

Equations of Lines

For any two points (x_1, y_1) and (x_2, y_2) on a non-vertical line L, the quantity $m = \frac{y_2 - y_1}{x_2 - x_1}$ is constant and is called the **slope** of the line L.

Let L be a nonvertical line. Let m be the slope of L and (x_1, y_1) be the coordinates of a point on L. If (x, y) is another point on L, then

$$\frac{y - y_1}{x - x_1} = m$$

Hence any (x, y) on L satisfies

$$y = m(x - x_1) + y_1$$

The above is known as an equation for the line L.

All points on a **vertical line** have their x coordinate equal to a constant a. So the equation of a vertical line is $x = a$. **Horizontal lines** have equations of the form $y = a$.

y-intercept of a nonvertical line L is the y-coordinate of the point where L intersects the y-axis. **x-intercept** of a nonhorizontal is defined similarly.

Example 10. *Find an equation of the line through the points* $(1,-1)$ *and* $(3,5)$. *Draw the line. Find the x and y intercepts.*

Example 11. *Find an equation of the line that passes through the point* $(-3,-4)$ *and has slope* 2. *Draw the line.*

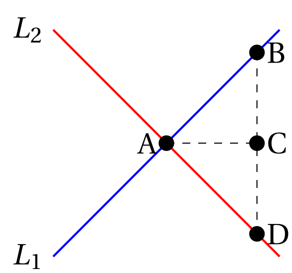
Example 12. *Find the slope and the two intercepts of the line with equation* $8x+5y = 20$. *Draw the line.*

Parallel vs. perpendicular lines

We call two lines **parallel** if their slopes are equal.

We call two lines **perpendicular** if they intersect at right angles (90°).

Theorem 1. *Two nonvertical lines with slopes* m_1 *and* m_2 *are perpendicular if and only if* $m_1 m_2 = -1$.



Proof. Use the similarity of the triangles ABC and DAC to get

$$\frac{|BC|}{|AC|} = \frac{|AC|}{|CD|} \implies \frac{|BC||CD|}{|AC|^2} = 1$$

Slope of L_1 (m_1) is $|BC|/|AC| = 1$ and slope of L_2 (m_2) is $-|CD|/|AC|$. So $m_1 m_2 = -1$. □

Example 13. *Find an equation of the line through* $(1,-2)$ *that is parallel to the line L with equation* $3x-2y = 1$. *Draw the lines.*

Example 14. *Find an equation of the line through* $(2,-3)$ *that is perpendicular to the line L with equation* $4x+y = 3$. *Draw the lines.*

1.4 Quadratic Equations

Circles and Disks

The circle is the set of all points that have the same distance (called radius of the circle) from a given point (called center of the circle).

If (x, y) is a point on a circle with center (a, b) and radius r then

$$\sqrt{(x-a)^2 + (y-b)^2} = r \implies (x-a)^2 + (y-b)^2 = r^2$$

Example 15. *Find the center and radius of the circle $x^2 + y^2 - 4x + 6y = 3$.*

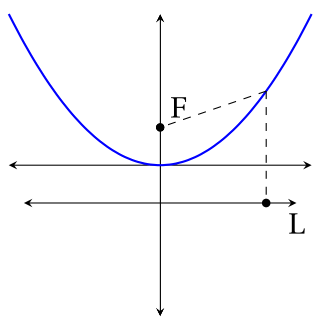
Solution. Complete to squares to get $(x-2)^2 + (y+3)^2 = 16$.

The equation $(x-a)^2 + (y-b)^2 < r^2$ represents open disk and the equation $(x-a)^2 + (y-b)^2 \le r^2$ represents closed disk or simply disk.

Example 16. *Draw $x^2 + 2x + y^2 \le 8$.*

Parabolas

A parabola P is the set of all points in the plane that are equidistant from a given line L (called directrix of P) and a point F (called the focus of P).



Example 17. *Find the equation of the parabola having the point $F(0, p)$as focus and the line L with equation $y = -p$ as directrix.*

Solution. If $P(x, y)$ is any point on the parabola then squaring both sides of PF=PQ we get

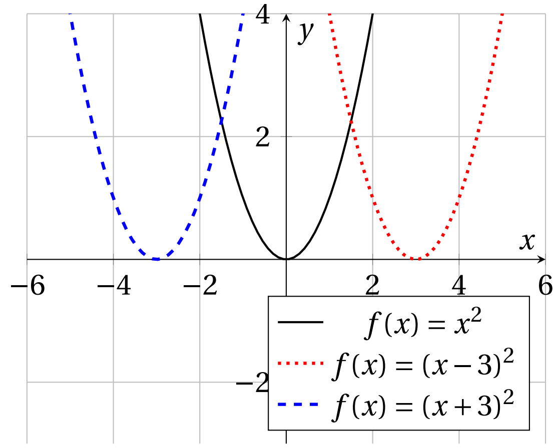
$$x^2 + (y-p)^2 = 0^2 + (y+p)^2$$

After simplifying, $y = x^2/4p$.

Shifting a Graph

Let $c > 0$.

- To shift a graph c units to the right, replace x in its equation with $x - c$. To shift to left, replace x by $x + c$.
- To shift a graph c units up, replace y in its equation with $y - c$. To shift down, replace y by $y + c$.



1.5 Functions and Their Graphs

A **function** f on a set D into a set R is a rule that assigns a unique element $f(x)$ in R to each element x in D.

D is called the **domain** of f. R is called the target or **codomain** of f. The **range** of f is a subset of R containing of all possible values $f(x)$.

This definition is not mathematical as we did not define what a rule is. Formally one defines a function as a relation.

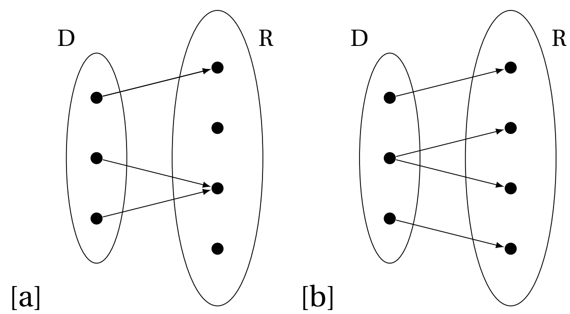


Figure 1.1: a) Not a function. b) A Function

Example 18. *Define a function on the set of all real numbers by* $f(x) = x^2 + 1$. *Find* $f(0)$, $f(2)$, $f(x+2)$.

$$f(x) = \frac{1}{x}, \qquad x > 0$$

means that the domain of f is the set $\{x \mid x > 0\}$.

Technically, this function is different from the function

$$f(x) = \frac{1}{x}, \qquad x < 0.$$

If we do not specify the domain of a function f, then the **domain convention** is to assume that the domain of f is the set of all real numbers for which f is defined.

So if we write

$$f(x) = \frac{1}{x},$$

we are assuming f is defined for all real numbers except 0.

Example 19. *Find the domain of* $f(x) = \sqrt{2-x}$.
Solution. *Its domain is all* x *for which* $2-x \geq 0$, *i.e. the interval* $(-\infty, 2]$.

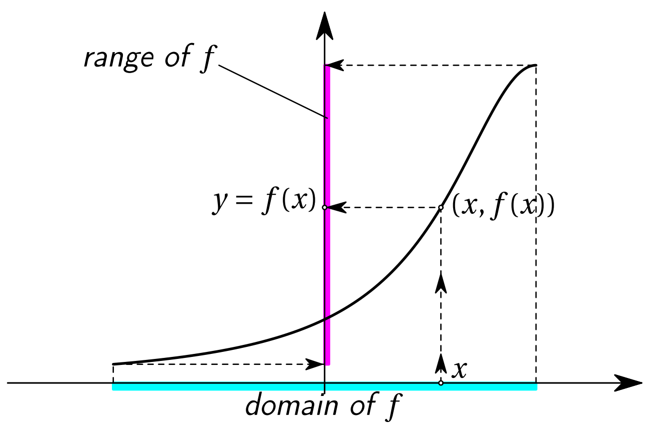
Example 20. *Find the domain of* $f(x) = \frac{1}{x^2-x}$.

A function $f : D \to R$ is **1-1** if $f(x_1) = f(x_2)$ then $x_1 = x_2$. A function $f : D \to R$ is **onto** if for every $y \in R$, there is an $x \in D$ such that $f(x) = y$.

Example 21. *Draw functions which are 1-1, onto, not 1-1 and not onto, similar to the Figure 1.1.*

Graph of a function

The *graph of a function* f is the set of all points whose coordinates are $(x, f(x))$ where x is in the domain of f.



Example 22. *A function which is given by the formula*

$$f(x) = mx + n$$

where m *and* n *are constants is called a* linear function. *Its graph is a straight line. The constants* m *and* n *are the* slope *and* y-intercept *of the line.*

Example 23. *The square root function* $f(x) = \sqrt{x}$ *has domain* $[0, \infty)$ *and takes* x *to its positive square root. Hence it has range* $[0, \infty)$.

Example 24. *The absolute value function* $f(x) = |x| = \sqrt{x^2}$ *has domain* $(-\infty, \infty)$ *and range* $[0, \infty)$.

Example 25. *Draw the graphs of some elementary functions*

$$c, x, x^2, \sqrt{x}, x^3, x^{1/3}, \frac{1}{x}, \frac{1}{x^2}, \sqrt{1-x^2}, |x|.$$

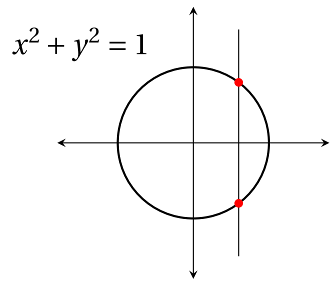
Example 26. *Sketch the graph of* $f(x) = 1 + \sqrt{x-4}$.
Solution: Shift the graph of $y = \sqrt{x}$ *1 unit up and 4 units to the right.*

Example 27. *Sketch the graph of the function* $f(x) = \frac{2-x}{x-1}$.
Solution. $f(x) = \frac{2-x}{x-1} = -1 + \frac{1}{x-1}$. *So shift the graph of* $y = \frac{1}{x}$ *1 unit down and 1 unit to the right.*

Vertical Line Test

The graph of a function cannot intersect a vertical line "$x =$ constant" in more than one point.

For example, the circle $x^2 + y^2 = 1$ is not a graph of a function.



Even and Odd Functions

Definition 1. *We say that f is an* ***even function*** *if $f(-x) = f(x)$ for every $x \in D$. We say that f is an* ***odd function*** *if $f(-x) = -f(x)$ for every $x \in D$.*

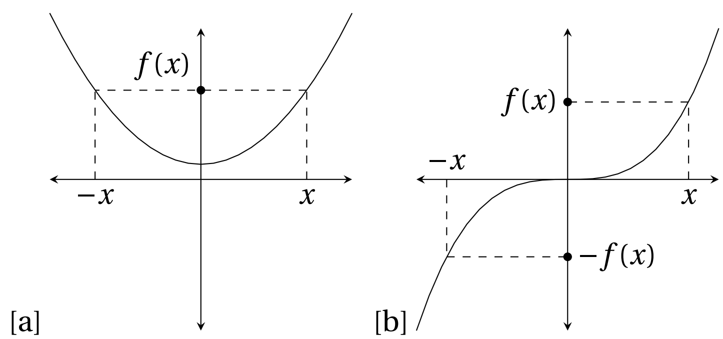


Figure 1.2: a) An even function, b) An odd function.

Odd functions are symmetric with respect to origin and even functions are symmetric with respect to the y-axis.

Example 28. *$f(x) = x$, $f(x) = x^3$ are odd and $f(x) = x^2$ and $f(x) = x^4$ are even and $f(x) = \frac{1}{x+1}$ is neither even or odd.*

Example 29. *$f(x) = x^3 + x$ is odd and $f(x) = \frac{1}{x^2-1}$ is even and $f(x) = x^2 + x$ is either even or odd.*

1.6 Operations on Functions

If f and g are functions, then for every x that belongs to the domains of both f and g we define functions

$(f+g)(x) = f(x) + g(x)$
$(f-g)(x) = f(x) - g(x)$
$(fg)(x) = f(x)g(x)$
$(f/g)(x) = f(x)/g(x)$ where $g(x) \neq 0$.

Example 30. *Let $f(x) = \frac{1}{x+2}$ and $g(x) = \frac{x}{x-1}$. Find $(f+g)(x)$, $(f-g)(x)$, $(fg)(x) = f(x)g(x)$ and $(f/g)(x)$ where $g(x) \neq 0$.*

Composition of Functions

If f and g are two functions, then

$$f \circ g(x) = f(g(x)).$$

The domain of $f \circ g$ consists of those numbers x in the domain of g for which $g(x)$ is in the domain of f.

Example 31. *Let $f(x) = \sqrt{x}$ and $g(x) = x+1$. Find $f \circ g$, $g \circ f$, $f \circ f$ and $g \circ g$. State the domains of each function.*

Function	Formula	Domain
f	$\sqrt{x}$	$[0,\infty)$
g	$x+1$	$\mathbb{R}$
$f \circ g$	$\sqrt{x+1}$	$[-1,\infty)$
$g \circ f$	$\sqrt{x}+1$	$[0,\infty)$
$f \circ f$	$x^{1/4}$	$[0,\infty)$
$g \circ g$	$x+2$	$\mathbb{R}$

Piecewise Defined Functions

Functions such as

$$g(x) = \begin{cases} 2x & \text{for } x < 0 \\ x^2 & \text{for } x \geq 0 \end{cases}$$

which are defined by different formulas on different intervals are sometimes called **piecewise defined functions.**

Inverse Functions

Remember that a function is **one-to-one** if for every value in the range, there is exactly one value in the domain.

A function is one-to-one if every horizontal line crosses its graph at most once, which is commonly known as the **horizontal line test**.

1.7 Polynomials and Rational Functions

Definition 2. *A **polynomial** is a function $P : \mathbb{R} \to \mathbb{R}$ such that*

$$P(x) = a_n x^n + \cdot + a_1 x + a_0.$$

*Here $a_n, \ldots, a_1$ are called the **coefficients** of the polynomial. We assume $a_n \neq 0$. The number n is called the **degree** of the polynomial.*

Example 32. *Write polynomials of degree 0, 1 and 2.*

Just as the quotient of two integers is called a rational number, the quotient of two polynomials is called a **rational function**. Give an example.

Let A_m be a polynomial of degree m, B_n be a polynomial of degree n with $m \geq n$. Then there are polynomial Q_{m-n} of degree $m-n$, R_k of degree $k<n$ such that

$$\frac{A_m}{B_n} = Q_{m-n} + \frac{R_k}{B_n}.$$

The quotient Q_{m-n} and the remainder R_k can be calculated by the "long division".

Example 33. *Using the long division algorithm, show that*

$$\frac{2x^3-3x^2+3x+4}{x^2+1} = 2x-3+\frac{x+7}{x^2+1}$$

If P is a polynomial and $P(r)=0$ then r is called a **root** of P.

The Fundamental Theorem of Algebra says every polynomial of degree greater than 0 must have a root. But these roots may be complex.

Example 34. x^2+1 *has no real roots. Its roots are* $i=\sqrt{-1}$ *and* $-i$.

Theorem 2. *If r is a root of the polynomial P then*

$$P(x) = (x-r)Q(x),$$

for some polynomial Q whose degree is 1 less than P.

The polynomial $x(x-7)^3$ has 4 roots: 0 and the other three are each equal to 7. We say that 7 is a root of **multiplicity** 3.

By the Fundamental Theorem of Algebra and the above theorem, every polynomial of degree n has exactly n (not necessarily distinct) roots.

Roots of Quadratic Polynomials

To obtain the solutions of

$$Ax^2+Bx+C=0, \qquad A \neq 0$$

Divide by A and complete to square

$$\left(x+\frac{B}{2A}\right)^2 = \frac{B^2}{4A^2}-\frac{C}{A} = \frac{B^2-4AC}{4A^2},$$

Taking the square root of both sides gives the quadratic formula

$$x = \frac{-B \pm \sqrt{B^2-4AC}}{2A}.$$

A form of this formula is known since B.C. 2000 by Babylonians.

The quantity $D=B^2-4AC$ is called the **discriminant** of the quadratic equation.

- If $D>0$ then there are two distinct real roots,
- If $D=0$ then there is 1 root of multiplicity 2,
- If $D<0$ then there are two complex conjugate roots.

Example 35. *Find the roots of the polynomials: (a)* x^2+x-1, *(b)* $9x^2-6x+1$, *(c)* $2x^2+x+1$.

Misc Factorings

- Difference of squares:
$$x^2 - a^2 = (x-a)(x+a)$$

- Difference of cubes:
$$x^3 - a^3 = (x-a)(x^2 + ax + a^2)$$

- Difference of nth powers
$$x^n - a^n = (x-a)(x^{n-1} + ax^{n-2} + a^2x^{n-3} + \cdots + a^{n-2}x + a^{n-1})$$

- If n is an odd integer then $x + a$ is a factor of $x^n + a^n$,
$$x^n + a^n = (x+a)(x^{n-1} - ax^{n-2} + a^2x^{n-3} - \cdots - a^{n-2}x + a^{n-1})$$

Chapter 2

Limits and Continuity

2.1 Informal definition of limits

Two main problems of calculus are

1. Derivative. Find the rate of change of f.

2. Integral. Find the area under a given curve.

Both are based on the concept of limit.

We say $\lim_{x\to a} f(x) = L$ to mean that $f(x)$ is “close enough” to L when x is “close enough” to *but not equal to a*. Hence $f(a)$ is unimportant for $\lim_{x\to a} f(x)$.

Example 36. *Which value is x close to when x is close to 2?* $\lim_{x\to 2} x = 2$.

Example 37. *Which value is 3 close to when x is close to 2?* $\lim_{x\to 2} 3 = 3$.

We can generalize these examples.

Theorem 3. *Let a and c be two real numbers. Then*

$$\lim_{x\to a} c = c, \qquad \lim_{x\to a} x = x.$$

The limit $\lim_{x\to a} f(x)$ may be different from $f(a)$ as the next example shows.

Example 38.

$$f(x) = \begin{cases} x, & \text{if } x \neq 2 \\ 1, & \text{if } x = 2 \end{cases}$$

Which value is $f(x)$ close to when x is close to (but not equal to) 2?

$\lim_{x\to 2} f(x) = \lim_{x\to 2} x = 2$ *although* $f(2) = 1$.

Informal definition of left and right limits

If $f(x)$ is close to L when $x < a$ and x is close enough to a then we say

$$\lim_{x\to a^-} f(x) = L$$

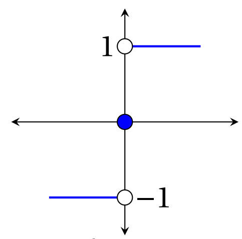
This is called the *left limit* of f at $x = a$.

Similarly we can define the right limit.

Theorem 4. $\lim_{x\to a} f(x) = L$ *if and only if both* $\lim_{x\to a^-} f(x) = L$ *and* $\lim_{x\to a^+} f(x) = L$.

Example 39. *Find the left and right limits of the signum function*

$$f(x) = \begin{cases} -1 & \text{for } x < 0 \\ 0 & \text{for } x = 0 \\ 1 & \text{for } x > 0 \end{cases}$$



In this example the one-sided limits exist, but are not equal

$$\lim_{x\searrow 0} f(x) = 1 \text{ and } \lim_{x\nearrow 0} f(x) = -1.$$

Hence $\lim_{x\to 0} f(x)$ *does not exist.*

Properties of Limits

Theorem 5. *Suppose*

$$\lim_{x\to a} f(x) = L, \qquad \lim_{x\to a} g(x) = M.$$

Then

$$\lim_{x\to a}\big(f(x) + g(x)\big) = L + M, \tag{2.1}$$

$$\lim_{x\to a}\big(f(x) - g(x)\big) = L - M, \tag{2.2}$$

$$\lim_{x\to a}\big(f(x)\cdot g(x)\big) = L \cdot M \tag{2.3}$$

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{L}{M}, \quad \text{if } M \neq 0. \tag{2.4}$$

Finally, if m and n are integers such that $L^{m/n}$ is defined

$$\lim_{x\to a} (f(x))^{m/n} = L^{m/n}. \tag{2.5}$$

Using the above properties we can evaluate the following limits.

Example 40. *Find* $\lim_{x\to 2} x^2 + 1$ *and* $\lim_{x\to 2} \dfrac{x^2+1}{6-x}$.

Solution. Using the product rule of limits and the Theorem 3,

$$\lim_{x\to 2} x^2 = \lim_{x\to 2} x \cdot \lim_{x\to 2} x = 2\cdot 2 = 4$$

Using the sum rule of limits,

$$\lim_{x\to 2} x^2+1 = \lim_{x\to 2} x^2 + \lim_{x\to 2} 1 = 4+1 = 5$$

Using the division rule of limits,

$$\lim_{x\to 2} \frac{x^2+1}{6-x} = \frac{\lim_{x\to 2} x^2+1}{\lim_{x\to 2} 6-x} = \frac{5}{4}.$$

The above example is a special case of the following theorem.

Theorem 6. *If $P(x)$ is a polynomial then,*

$$\lim_{x\to a} P(x) = P(a)$$

If $Q(x)$ is another polynomial with $Q(a) \neq 0$ then

$$\lim_{x\to a} \frac{P(x)}{Q(x)} = \frac{P(a)}{Q(a)}.$$

The Squeeze Theorem

Theorem 7. *Suppose that $f(x) \le g(x) \le h(x)$ and $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$. Then $\lim_{x\to a} g(x) = L$.*

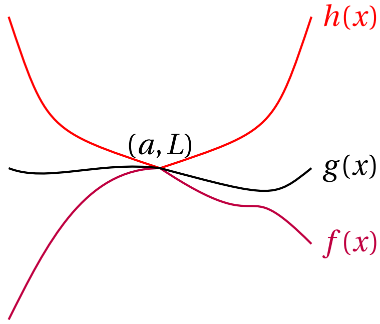


Figure 2.1: The Squeeze Theorem.

Example 41. *If $2-x^2 \le g(x) \le 2\cos x$ for $-1 \le x \le 1$, find $\lim_{x\to 0} f(x)$.*

Example 42. *Show that if $\lim_{x\to a} |f(x)| = 0$ then $\lim_{x\to a} f(x) = 0$.*
Note that $-|f(x)| \le f(x) \le |f(x)|$ and use the Squeeze Theorem.

More examples

Example 43. *Let*

$$f(x) = \frac{|x-2|}{x^2+x-6}.$$

Find $\lim_{x\to 2+} f(x)$, $\lim_{x\to 2-} f(x)$. Does $\lim_{x\to 2} f(x)$ exist?

In these example, we will compute $\lim_{x\to a} f(x)$ even when $f(a)$ does not exist.

Example 44. *Evaluate*

1. $\lim_{x\to-2} \dfrac{x^2+x-2}{x^2+5x+6}$,

   Remember that we consider x values close to but not equal to -2. Hence $x+2\neq 0$ and we can make the simplification

$$\lim_{x\to-2} \frac{x^2+x-2}{x^2+5x+6} = \lim_{x\to-2} \frac{(x+2)(x-1)}{(x+2)(x+3)} = \lim_{x\to-2} \frac{x-1}{x+3} = \frac{-3}{1} = -3.$$

2. $\lim_{x\to5} \dfrac{\frac{1}{x}-\frac{1}{5}}{x-5}$,

3. $\lim_{x\to4} \dfrac{\sqrt{x}-2}{x^2-16}$,

   Trick is to multiply both sides by the conjugate expression.

4. $\lim_{x\to-2} \dfrac{x^2+2x}{x^2-4}$,

5. $\lim_{h\to0} \dfrac{\sqrt{4+h}-2}{h}$,

6. $\lim_{t\to0} \dfrac{t}{\sqrt{4+t}-\sqrt{4-t}}$,

7. $\lim_{x\to-1} \dfrac{x^3+1}{x+1}$,

8. $\lim_{x\to0} \dfrac{|3x-1|-|3x+1|}{x}$,

9. $\lim_{x\to2-} \dfrac{x^2-4}{|x+2|}$.

2.2 Limits at Infinity and Infinite Limits

Limits at Infinity

Definition 3. *We will say that* $\lim_{x\to\infty} f(x) = L$ *if* $f(x)$ *is "close enough" to* L *whenever* $x>0$ *is "large enough".*

Similarly we define $\lim_{x\to-\infty} f(x) = L$ *if* $f(x)$ *is "close enough" to* L *whenever* $x<0$ *is "large enough".*

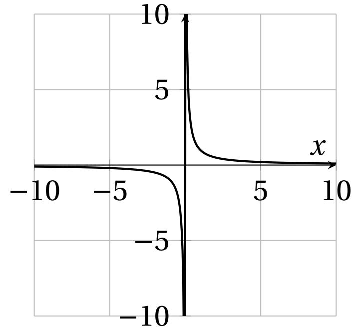
If either $\lim_{x\to\infty} f(x) = L$ *or* $\lim_{x\to-\infty} f(x) = L$*, we say that the line* $y=L$ *is an* ***horizontal asymptote*** *of the graph of* f*.*

Example 45. *Argue that*

$$\lim_{x\to\infty} 1/x = \lim_{x\to\infty} 1/x = 0.$$

by making a table of values of x *and* $1/x$*.*

$$f(x) = \frac{1}{x}$$



Recall that for ordinary limits, limit of product of functions is a product of limits of functions. Same is also true for limits at infinity. Hence

$$\lim_{x\to\infty} \frac{1}{x^2} = \lim_{x\to\infty} \frac{1}{x} \lim_{x\to\infty} \frac{1}{x} = 0 \times 0 = 0.$$

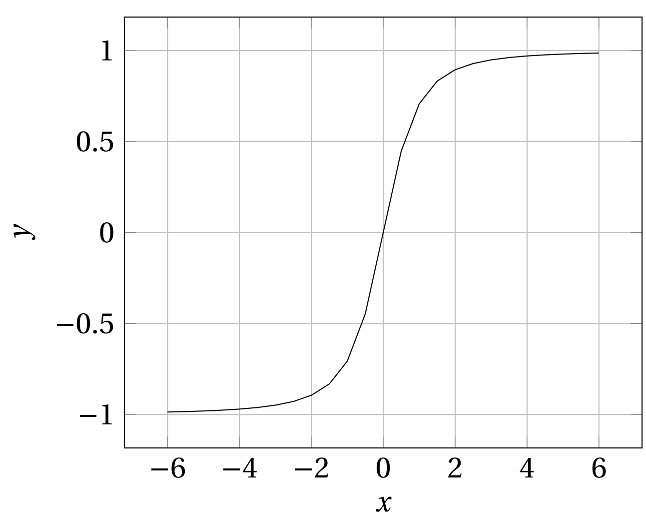
Similarly

$$\lim_{x\to-\infty} \frac{1}{x^2} = 0$$

Finally, for any positive integer n

$$\lim_{x\to\infty} \frac{1}{x^n} = \lim_{x\to-\infty} \frac{1}{x^n} = 0.$$

$$f(x) = \frac{x}{\sqrt{x^2+1}}$$



Example 46. *Let* $f(x) = \dfrac{x}{\sqrt{x^2+1}}$. *Find* $\lim_{x\to\infty} f(x)$, $\lim_{x\to-\infty} f(x)$.

$$\begin{aligned}\lim_{x\to\infty} \frac{x}{\sqrt{x^2+1}} &= \lim_{x\to\infty} \frac{x}{|x|\sqrt{1+1/x^2}} = \lim_{x\to\infty} \frac{x}{x\sqrt{1+1/x^2}} = \lim_{x\to\infty} \frac{1}{\sqrt{1+1/x^2}} = \frac{\lim_{x\to\infty} 1}{\lim_{x\to\infty} \sqrt{1+1/x^2}} \\ &= \frac{1}{\sqrt{\lim_{x\to\infty}(1+1/x^2)}} = \frac{1}{1} = 1.\end{aligned}$$

Similarly,

$$\lim_{x\to-\infty}\frac{x}{\sqrt{x^2+1}}=-1$$

Limits of Rational Functions at Infinity

Recall that a rational function is a ratio of two polynomials.

Strategy. To find limits of rational functions at infinity, divide by the highest power of x appearing in the *denominator.*

Example 47.

$$\lim_{x\to\pm\infty}\frac{2x^2-x+3}{3x^2+5}=\lim_{x\to\pm\infty}\frac{2-\frac{1}{x}+\frac{3}{x^2}}{3+\frac{5}{x}}=\frac{2}{3}.$$

Example 48.

$$\lim_{x\to\pm\infty}\frac{x-5}{2x^2+4x+1}=\lim_{x\to\pm\infty}\frac{\frac{1}{x}-\frac{5}{x^2}}{2+\frac{4}{x}+\frac{1}{x^2}}=\frac{0}{2}=0.$$

We can generalize the above examples.

Theorem 8. *Let $P(x)=a_px^p+a_{p-1}x^{p-1}+\cdots+a_0$ be a polynomial of degree p and $Q(x)=b_qx^q+\cdots+b_0$ be a polynomial of degree q. If $p=q$, then*

$$\lim_{x\to\pm\infty}\frac{P(x)}{Q(x)}=\frac{a_p}{q_p},$$

If $p<q$, then

$$\lim_{x\to\pm\infty}\frac{P(x)}{Q(x)}=0,$$

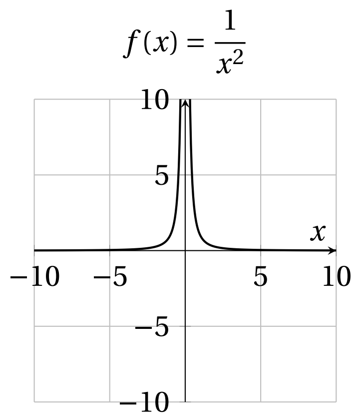
Example 49.

$$\lim_{x\to\infty}\sqrt{x^2+x}-x=\lim_{x\to\infty}\frac{(\sqrt{x^2+x}-x)(\sqrt{x^2+x}+x)}{\sqrt{x^2+x}+x}=\lim_{x\to\infty}\frac{x}{|x|\sqrt{1+\frac{1}{x}}+x}=\lim_{x\to\infty}\frac{1}{\sqrt{1+\frac{1}{x}}+1}=\frac{1}{2}.$$

Infinite Limits

Example 50. *The values of $\frac{1}{x^2}$ gets larger and larger as x approaches to 0. Thus $\lim_{x\to0}\frac{1}{x^2}$ does not exist. Although the limit does not exist, it is useful to state why it does not exist by writing*

$$\lim_{x\to0}\frac{1}{x^2}=\infty.$$



Example 51.

$$\lim_{x\to 0+} \frac{1}{x} = \infty.$$

$$\lim_{x\to 0-} \frac{1}{x} = -\infty.$$

$$\lim_{x\to 0} \frac{1}{x} \textit{ does not exist.}$$

Example 52.

$$\lim_{x\to -\infty} \sqrt{x^2+x} - x = \lim_{x\to -\infty} \frac{(\sqrt{x^2+x}-x)(\sqrt{x^2+x}+x)}{\sqrt{x^2+x}+x} = \lim_{x\to -\infty} \frac{x}{|x|\sqrt{1+\frac{1}{x}}+x} = \lim_{x\to -\infty} \frac{1}{-\sqrt{1+\frac{1}{x}}+1}$$

As $x \to -\infty$, $-\sqrt{1+\frac{1}{x}}$ is close to but less than 1. Hence

$$\lim_{x\to -\infty} \frac{1}{-\sqrt{1+\frac{1}{x}}+1} = \infty.$$

Behaviour of Polynomials at Infinity

Example 53.

$$\lim_{x\to\infty} 4x^3 - 2x + 1 = \lim_{x\to\infty} 4x^3 = \infty.$$

$$\lim_{x\to -\infty} -3x^5 + x^3 + 1 = \lim_{x\to -\infty} -3x^5 = \infty.$$

In general,

Theorem 9. *If $P(x) = a_n x^n + \cdots + a_0$ is a polynomial then*

$$\lim_{x\to\pm\infty} P(x) = \lim_{x\to\pm\infty} a_n x^n.$$

Example 54.

$$\lim_{x\to\infty} \frac{x^3+1}{x^2-2x} = \lim_{x\to\infty} \frac{x+\frac{1}{x^2}}{1-\frac{2}{x}} = \lim_{x\to\infty} \frac{x}{1} = \infty$$

Example 55.

1. $\lim_{x\to 2} \dfrac{(x-2)^2}{x^2-4} = 0$
2. $\lim_{x\to 2+} \dfrac{x-3}{x^2-4} = -\infty$
3. $\lim_{x\to 2-} \dfrac{x-3}{x^2-4} = \infty$
4. $\lim_{x\to 2} \dfrac{x-3}{x^2-4}$ *does not exist.*
5. $\lim_{x\to\infty} \dfrac{2x-1}{\sqrt{3x^2+x+1}}$,
6. $\lim_{x\to 1+} \dfrac{\sqrt{x^2-x}}{x-x^2}$