MATH2056 Linear Algebra - Week 5

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Determinant of a 2×2 matrix

For

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

we have

$$\det(A) = a_{11}a_{22} - a_{12}a_{21}$$

Example

For

$$A = \begin{bmatrix} 2 & -3 \\ 4 & 5 \end{bmatrix}$$

$\det(A) = (2)(5) - (-3)(4) = 22.$

DETERMINANT OF 3×3 MATRIX

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \Longrightarrow \begin{aligned} \det(A) =& a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} \\ & - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31} \end{aligned}$$

$$\begin{array}{ccc:cc} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{array}$$

FIGURE: Sarrus rule to compute 3×3 determinants. This method does not work for 4×4 matrices.

EXAMPLE

Check that $\begin{vmatrix} 0 & 3 & 7 \\ 5 & 6 & 5 \\ -1 & 5 & 5 \end{vmatrix} = 127$

PERMUTATIONS

How do we define the determinant of $n \times n$ matrix? Let's do it.

Define $S_n = \{1, 2, \dots, n\}$. A rearrangement (with no repetation of elements) $(j_1, j_2 \dots, j_n)$ of the elements of S_n is called a **permutation** of S_n. The set S_n has a total of $n!$ permutations.

A permutation is said to have an **inversion** if a larger integer comes before than a smaller integer. If the total number of inversions is even, then the permutation is called **even** otherwise it is called **odd**.

EXAMPLE

The permutations of the set $S_2 = \{1, 2\}$ are $(1, 2)$ and $(2, 1)$. The permutation $(1, 2)$ is even (zero inversions), and the permutation $(2, 1)$ is odd (1 inversion).

EXAMPLE

The permutations of the set $S_3 = \{1, 2, 3\}$ are $(1, 2, 3)$, $(2, 3, 1)$, $(3, 1, 2)$ which are even and $(1, 3, 2)$, $(3, 2, 1)$, $(2, 1, 3)$ which are odd.

EXAMPLE

The permutation $(4, 1, 3, 2)$ of S_4 has 4 inversions: $4 > 1$, $4 > 3$, $4 > 2$, $3 > 2$ so it is even.

Definition of Determinant

Let $A = [a_{ij}]$ be an $n \times n$ matrix. The **determinant** of A is

$$\det(A) = \sum (\pm) a_{1j_1} a_{2j_2} \cdots a_{nj_n}$$

where the summation is over all permutations $j_1 j_2 \cdots j_n$ of the set $S = \{1, 2, \ldots, n\}$. The sign is taken as $+$ or $-$ according to whether the permutation $j_1 j_2 \cdots j_n$ is even or odd.

Example

Check that $\begin{vmatrix} 0 & 3 & 7 \\ 5 & 6 & 5 \\ -1 & 5 & 5 \end{vmatrix} = 127$

Positive permutations are $(1, 2, 3)$, $(2, 3, 1)$, $(3, 1, 2)$.
Negative permutations are $(1, 3, 2)$, $(2, 1, 3)$, $(3, 2, 1)$.

$$\begin{aligned}\det(A) &= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31} \\ &= 0 \cdot 6 \cdot 6 + 3 \cdot 5 \cdot (-1) + 7 \cdot 5 \cdot 5 - 0 \cdot 5 \cdot 5 - 3 \cdot 5 \cdot 5 - 7 \cdot 6 \cdot (-1) \\ &= -15 + 175 - 75 + 42 = 127\end{aligned}$$

Solve the following from the textbook. Exercises 3.1: 8, 11, 13

3.2 Properties of Determinant

$$\det(A) = \det\left(A^T\right)$$

Example

$$|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & c \\ b & d \end{vmatrix} = \left|A^T\right| = ad - bc$$

$$\det(A_{r_i \leftrightarrow r_j}) = -\det(A) \qquad \text{if } i \neq j$$

Example

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = -\begin{vmatrix} c & d \\ a & b \end{vmatrix}$$

If two rows (or columns) of A are equal then $\det(A) = 0$.

proof Suppose $row_i = row_j$, $i \neq j$ then $A = A_{r_i \leftrightarrow r_j}$ so that $\det A = \det A_{r_i \leftrightarrow r_j} = -\det A$.

Example

$$\begin{vmatrix} a & b \\ a & b \end{vmatrix} = 0 \text{ and } \begin{vmatrix} 1 & 2 & 3 \\ -1 & 0 & 7 \\ 1 & 2 & 3 \end{vmatrix} = 0$$

If a row (or column) of A consists entirely of zeros then $\det(A) = 0$.

proof. This follows from the fact that each summand in the determinant sum formula contains exactly one element from each row (or column).

Example

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 0 & 0 & 0 \end{vmatrix} = 0$$

$\det(A_{kr_i \to r_i}) = k \det(A)$.

Example

$$\begin{vmatrix} ka & kb \\ c & d \end{vmatrix} = k(ad - bc) = k \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

$$\det(A_{r_i+kr_j\to r_i}) = \det(A),\ i \neq j.$$

EXAMPLE

$$\begin{vmatrix} a & b \\ ka+c & kb+d \end{vmatrix} = (a(kb+d) - b(ka+c) = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

EXAMPLE

$\begin{vmatrix} 1 & 2 & 3 \\ 2 & -1 & 3 \\ 1 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 5 & 0 & 9 \\ 2 & -1 & 3 \\ 1 & 0 & 1 \end{vmatrix} = 4$, obtained by adding twice the second row to the first row.

EXAMPLE

If $\det(A) = 4$ and $B = A_{r_1+2r_2\to r_1}$ then $\det(B) = 4$.