

MATH2056 LINEAR ALGEBRA - WEEK 6

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3.2 PROPERTIES OF DETERMINANT (CONTINUED)

Let A be an upper (or lower) triangular matrix. Then $\det(A)$ is equal to the product of the diagonal entries of A .

Proof depends on the determinant formula which is a sum of products of elements of the matrix. Each product contains a single term from every column from each row and each column.

EXAMPLE

For a 3×3 upper triangular matrix

$$\begin{vmatrix} a & * & * \\ 0 & b & * \\ 0 & 0 & c \end{vmatrix} = abc$$

For a 4×4 lower triangular matrix

$$\begin{vmatrix} a & 0 & 0 & 0 \\ * & b & 0 & 0 \\ * & * & c & 0 \\ * & * & * & d \end{vmatrix} = abcd$$

Since a diagonal matrix is both lower and upper triangular, the determinant of a diagonal matrix is the product of elements in its diagonal as well.

$$\begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = abc$$

EXAMPLE

$\det(I_n) = 1$ where I_n is the $n \times n$ identity matrix.

EXAMPLE

Compute

$$\begin{vmatrix} 3 & 4 & 5 \\ 5 & 2 & 0 \\ -1 & 0 & 0 \end{vmatrix}$$

After $r_1 \leftrightarrow r_3$ determinant changes by -1 and the matrix is in triangular form. The answer is $-1 \times -1 \times 2 \times 5 = 10$.

Since $\det(A) = \det(A^T)$, column operations can be used too instead of row operations. That is

- ① $\det(A) = -\det(A_{c_i \leftrightarrow c_j}), i \neq j.$
- ② $\det(A_{k c_i \rightarrow c_i}) = k \det(A).$
- ③ $\det(A_{c_i + k c_j \rightarrow c_i}) = \det(A), i \neq j.$

EXAMPLE

Compute

$$\begin{vmatrix} 3 & 6 & 5 \\ 4 & 8 & 0 \\ -1 & -2 & 1 \end{vmatrix}$$

Second column can be made zero by $c_2 - 2c_1 \rightarrow c_2$. So the determinant is zero.

EXAMPLE

Compute $\det(A)$ for

$$A = \begin{pmatrix} 2 & 4 & 6 \\ 0 & 1 & 2 \\ 2 & -3 & 1 \end{pmatrix}$$

After the operations

- ① $1/2r_1 \rightarrow r_1$
- ② $-2r_1 + r_3 \rightarrow r_3$,
- ③ $7r_2 + r_3 \rightarrow r_3$

$$\det(A) = 2 \begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 9 \end{vmatrix} = 2(1)(1)9 = 18$$

This method is called to as computation of determinant via reduction to triangular form.

Determinant of three types of elementary matrices:

- $\det(I_{r_i \leftrightarrow r_j}) = -1,$
- $\det(I_{kr_i \rightarrow r_i}) = k,$
- $\det(I_{kr_j + r_i \rightarrow r_i}) = 1.$

Lemma 3.1 If E is an elementary matrix then $\det(EA) = \det(E) \det(A)$.

proof. Take $E = I_{r_i \leftrightarrow r_j}$ then EA is the matrix A with row i and j swapped. Thus $\det(EA) = -\det(A)$. On the other hand $\det(E) \det(A) = -\det(A)$. Thus $\det(EA) = \det(E) \det(A)$. The same is true for other types of elementary matrices.

If B is row equivalent to A then $\det(A)$ is non-zero if and only if $\det(B)$ is non-zero.

proof. Suppose B is row equivalent to A . We know $B = E_k E_{k-1} \cdots E_1 A$.

$$\begin{aligned}\det(B) &= \det(E_k E_{k-1} \cdots E_1 A) = \det(E_k) \det(E_{k-1} \cdots E_1 A) \\ &= \det(E_k) \det(E_{k-1}) \cdots \det(E_1) \det(A)\end{aligned}$$

$A_{n \times n}$ is nonsingular if and only if $\det(A) \neq 0$.

proof. If A is nonsingular then A is a product of elementary matrices and hence its determinant is non zero. If A is singular, then A is row equivalent to a matrix B that has a row of zeros. Hence its determinant is zero.

Corollary 3.1 If A is $n \times n$ matrix, then $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution if and only if $\det(A) = 0$.

EXAMPLE

Let A be a 4×4 matrix with $\det(A) = -2$

- ❶ Describe the set of all solutions to the homogeneous system $A\mathbf{x} = \mathbf{0}$. answer. since $\det(A) \neq 0$, the homogeneous system has only the trivial solution.
- ❷ If A is transformed to reduced row echelon form B , what is B ? answer. I_n .
- ❸ Can the linear system $A\mathbf{x} = \mathbf{b}$ have more than one solution? Explain. answer. no the system has unique solution $\mathbf{x} = A^{-1}\mathbf{b}$.
- ❹ Does A^{-1} exist? answer. Yes.

Theorem 3.9 If A and B are $n \times n$ matrices then $\det(AB) = \det(A) \det(B)$.

proof. If A is nonsingular then $A = E_k \cdots E_1$, product of elementary matrices.

$$\det(AB) = \det(E_k \cdots E_1 B) = \det(E_k) \cdots \det(E_1) \det(B) = \det(A) \det(B)$$

If A is singular then A is row equivalent to C which has a row of zeros. Thus

$$\det(AB) = \det(E_k \cdots E_1 CB) = \det(E_k) \cdots \det(E_1) \det(CB)$$

Notice that CB has a row of zeros and $\det(CB) = 0$.

EXAMPLE

If $\det(A) = 5$, $\det(B) = -3$ then $\det(AB) = -15$ and $\det(A^2) = 25$.

Corollary 3.2 If A is nonsingular then $\det(A^{-1}) = \frac{1}{\det(A)}$.

proof. $1 = \det(I) = \det(AA^{-1}) = \det(A)\det(A^{-1})$.

In general (most usually) $\det(A + B) \neq \det(A) + \det(B)$.

Exercises 3.2 1-5, 8, 9, 10, 13, 14, 15, 17, 22, 26, 30, 31.