Marmara University
Faculty of Engineering
Math 101.1/Math 1001.1
First Exam

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Question:	1	2	3	4	5	6	7	Total
Points:	10	10	15	30	15	10	10	100
Score:								

STUDENT NO: ______________________

NAME: ______________________

SIGNATURE: ______________________

Give detailed work. State the names of the theorems you use.

1. (10 points) Find $F'(\pi/4)$ if

$$F(x) = \left(\tan^2 x + x - \frac{\pi}{4}\right)^3$$

Solution:

$$F'(x) = 3\left(\tan^2 x + x - \frac{\pi}{4}\right)^2 \left(2\tan x(1+\tan^2 x)+1\right)$$

Since $\tan\pi/4 = 1$,

$$F'(\pi/4) = 3(1^2+0)^2(2\cdot 1\cdot(1+1^2)+1) = 3\cdot 1\cdot(4+1) = 15$$

2. (10 points) Find c such that the function

$$f(x) = \begin{cases} x^2\sin(1/x), & \text{if } x \neq 0 \\ c, & \text{if } x = 0 \end{cases}$$

is continuous at $x = 0$. (Hint: Use the Squeeze Theorem.)

Solution: To be continuous at $x = 0$, we need

$$\lim_{x\to 0} x^2 \sin(1/x) = f(0) = c$$

To find this limit, $-1 \le \sin(1/x) \le 1$, if $x \neq 0$.
Hence $-x^2 \le x^2 \sin(1/x) \le x^2$ if $x \neq 0$.
Since $\lim_{x\to 0} -x^2 = \lim_{x\to 0} x^2 = 0$, by squeeze theorem,

$$\lim_{x\to 0} f(x) = 0$$

Thus $c = 0$.

3. (15 points) Use **the definition of derivative** to find $f'(0)$ if

$$f(x) = \frac{1}{x+2}$$

Solution:

$$\begin{aligned} f'(0) &= \lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0} \frac{\frac{1}{h+2} - \frac{1}{2}}{h} = \lim_{h\to 0} \frac{2-(2+h)}{(h+2)2h} \\ &= \lim_{h\to 0} \frac{-h}{(h+2)2h} = \lim_{h\to 0} \frac{-1}{(h+2)2} = -\frac{1}{4} \end{aligned}$$

4. Find the following limits if they exist. If a limit does not exist, state whether it is $+\infty$, $-\infty$ or neither.

(a) (10 points)

$$\lim_{x\to 0} \frac{|x|}{2x}$$

(b) (10 points)

$$\lim_{x\to\infty} x^2 - \sqrt{x^4 + 2x^2}$$

(c) (10 points)

$$\lim_{x\to 2+} \frac{2x - x^2}{(x-2)^2}$$

Solution: Part (a)

$$\lim_{x\to 0+}\frac{|x|}{2x}=\lim_{x\to 0+}\frac{x}{2x}=\frac{1}{2}$$

$$\lim_{x\to 0-}\frac{|x|}{2x}=\lim_{x\to 0-}\frac{-x}{2x}=\frac{-1}{2}$$

Since the left limit and right limits are not equal, the limit does not exist.

Part (b)

$$\begin{aligned}\lim_{x\to\infty} x^2-\sqrt{x^4+2x^2} &= \lim_{x\to\infty}\frac{(x^2-\sqrt{x^4+2x^2})(x^2+\sqrt{x^4+2x^2})}{x^2+\sqrt{x^4+2x^2}}\\ &= \lim_{x\to\infty}\frac{(x^4-x^4-2x^2)}{x^2+\sqrt{x^4+2x^2}}=\lim_{x\to\infty}\frac{-2x^2}{x^2+x^2\sqrt{1+2\frac{1}{x^2}}}\\ &= \lim_{x\to\infty}\frac{-2}{1+\sqrt{1+2\frac{1}{x^2}}}=\frac{-2}{1+\sqrt{1+2\cdot 0}}=\frac{-2}{2}=-1\end{aligned}$$

Part (c), the denominator goes to zero by taking positive values.

$$\lim_{x\to 2+}\frac{2x-x^2}{(x-2)^2}=\lim_{x\to 2+}\frac{-x(x-2)}{(x-2)^2}=\lim_{x\to 2+}\frac{-x}{(x-2)}$$

The expression $\frac{-x}{(x-2)}$ is negative for $x>2$, its denominator goes to zero while its numerator goes to -2 as $x\to 2+$. This means

$$\lim_{x\to 2+}\frac{-x}{(x-2)}=-\infty$$

5. (15 points) Find the equation of the tangent line to the graph of the curve $x^3y+2xy^3=12$ at the point $(2,1)$.

Solution: Let $y'=\frac{dy}{dx}$.

$$3x^2y+x^3y'+2y^3+6xy^2y'=0$$

For $x=2$, $y=1$,

$$3\cdot 4+8y'+2+12y'=0 \implies y'=-\frac{7}{10}$$

The equation of the tangent line is

$$y-1=-\frac{7}{10}(x-2) \implies y+\frac{7}{10}x=\frac{24}{10}$$

6. (10 points) If

$$(c-2)^4(c-4)^3+c=3$$

find two numbers x_1 and x_2 such that $x_1 < c < x_2$. Explain your answer.

Solution: Let $f(x)=(x-2)^4(x-4)^3+x$. Then $f(2)=2$ and $f(4)=4$. Since f is continuous, by intermediate value theorem, there exists a number c between 2 and 4 such that $f(c)=3$. Thus $x_1=2<c<x_2=4$.

7. (10 points) If $f(4)=1$, $f'(4)=4$, $f''(4)=-4$, find

$$\frac{d}{dx}\left(\frac{f(x)f'(x)}{\sqrt{x}}\right)\bigg|_{x=4}$$

Solution:

$$\begin{aligned}\frac{d}{dx}\left(\frac{f(x)f'(x)}{\sqrt{x}}\right)\bigg|_{x=4} &= \left(\frac{(f'(x)f'(x)+f(x)f''(x))\sqrt{x}-f(x)f'(x)\frac{1}{2\sqrt{x}}}{x}\right)\Bigg|_{x=4} \\ &= \left(\frac{(16-4)\cdot 2-4\cdot\frac{1}{4}}{4}\right)=\frac{24-1}{4}=\frac{23}{4}\end{aligned}$$