Solving the Vehicle Routing Problem using Classical and Quantum Optimisation Techniques

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ABSTRACT

This literature review investigates the vehicle routing problem (VRP). The development of the problem over time, its $\mathcal{NP}$-Hard complexity status, and its industry-related implications are investigated. Solutions to the VRP are compared and categorised through the lens of classical approaches like exact, heuristic and metaheuristic algorithms, as well as through quantum techniques. The current state of the art is assessed for both classical solutions to combinatorial optimisation problems (COPs), quantum algorithms, and today's current quantum hardware. The review sets the context for a discussion about a gap in the body of research representing the benchmarking of classical to quantum techniques in solving the VRP.

1 INTRODUCTION

The Vehicle routing problem (VRP) is a $\mathcal{NP}$-Hard combinatorial optimisation problem (COP) in the realm of logistics [107]. It involves a set of vehicles, often with capacity constraints, to be optimally routed to serve a individual customers to minimise the overall route length. The VRP is significant because of its relevance and implications for several industries, as well as for research into $\mathcal{NP}$-Hard COPs. This research considers the application of Simulated Annealing (SA) and the Variational quantum eigensolver (VQE) to the VRP problem. It touches on classical and quantum approaches to the problem in general. The promise that quantum computing holds for optimization in the current era is investigated, as well as possible future directions for research and algorithmic development.

2 BACKGROUND

In the body of this paper, an overview of the solution approaches and various algorithms will be presented. A more in-depth look at both classical and quantum approaches to the VRP is then expanded upon.

2.1 Overview

The VRP is an $\mathcal{NP}$-Hard COP, initially proposed by Dantzig and Ramser in 1959 as "the truck dispatching problem"[31]. It is one of the most widely researched combinatorial problems, both for its complexity and relevance in industry-related problems and applicability [6]. Solutions for the VRP have been a rich and diverse research ground for many decades, with the field evolving from fairly basic and limited approaches to today's solutions that are quite capable. In this paper, solutions to the VRP will be investigated from a classical perspective, namely exact approaches, heuristics and metaheuristics. Quantum computing will be investigated as an alternative solution paradigm.

2.1.1 A timeline of classical approaches. Classical approaches contain exact methods and heuristics. Heuristics are further split between classical heuristics and metaheuristics. Exact approaches involve finding optimal solutions through rigorous mathematical techniques and can only really be used to solve small problems to optimality [29]. Early examples of exact approaches include the set partitioning formulation and algorithms [61] proposed by Balinksi and Quandt in 1964, with the VRP being turned into an Integer Programming problem [10]. One of the first significant exact approaches was proposed in 1969 by Christofides and Eilon [23]. Their proposal was a branch and bound algorithm, in which they addressed a problem that was based on but slightly deviated from the original problem proposed by Dantzig and Ramser[31]. One of the first formulations of a solution to the VRP as a dynamic programming problem was done by Eilon, Watson-Gandy and Christofides in 1971 [18]. In 1976, Christofides proposed a branch and bound algorithm in which instead of branches being made on arcs, they were made on routes [22], and was one of the first papers that specifically proposed an exact solution to "The vehicle routing problem". [61]. According to [65], two papers published by Christofides, Mingozzi and Toth in 1981 were influential in starting an influx of work on exact algorithms. The first paper was about a dynamic programming solution, and the second involved two mathematical formulations of the VRP that use q-paths and k-shortest spanning trees [24]. In 1984, the first cutting plane algorithm for VRP was introduced by [62]. More deterministic approaches based on mathematical programming have been introduced following these developments. In 1985, an effective branch-and-cut algorithm was produced [64]. The first branch-and-cut solution for VRP was introduced in 1995 by [8]. In the late 1990s, Lagrangian relaxations and the additive approach were incorporated into tree-based methods, improving performance [107]. The first branch and price algorithm was introduced in 2003 by [43]. Successful set partitioning formulations were done by Fukasawa in 2006 and Baldacci in 2008[65]. Other improvements and exact approaches were put forward throughout the early 2000s until a branch-cut-and-price algorithm that combined and improved upon multiple elements from algorithms before it was proposed in 2014 by [90],[107].

The original Dantzig and Ramser paper [31] proposed the first heuristic solution, involving matching vertices to partial routes to form vehicle routes [61]. From this point and until the '90s, numerous constructive and improvement heuristics were proposed [61] based on several things, including geographical proximity, savings, customer matchings and intra and inter-route improvement [65]. Clarke and Wright's savings algorithm [26] is widely considered to be the first significant heuristic for the VRP. Several papers were published over the following decades to improve upon this important algorithm [61], [44], [114], [101]. The first set partitioning heuristic, the "Sweep" algorithm, was proposed in 1974 by [46]. In 1981, [41] proposed the first "cluster first route second" approach [61]. From the '80s to the early '90s, several other savings-based algorithms were proposed, including a matching-based savings algorithm proposed by [73] in 1989 and a parallel savings-based heuristic proposed by [5] in 1991 [107]. A minimum K trees approach was proposed by [40] in 1994 [107].

From the 1990s, research kicked off on Metaheuristic algorithms [65], which is considered to be the beginning of modern heuristics for the VRP. This coincided with accelerated research in the 90s on VRP and increased complexity of algorithmic solutions due to increased microcomputer capability and availability [35]. Metaheuristic approaches include population-based, local search-based, and learning mechanisms [27]. Local search/neighbourhood-based algorithms were some of the earliest proposed methods. Tabu search was applied by [88], [104], and simulated annealing by [88] in 1993. Adaptive memory procedures [27] were applied by [97] in 1995. The greedy randomised adaptive search procedure (GRASP) was initially proposed in 1995 by [53]. In 1999, [57] applied a guided local search algorithm and [17] applied the Ant Colony Optimization algorithm (ACO) [109]. Some population-based approaches, like the use of genetic algorithms, were applied in 2003 by [13] and in 2004 by [94]. Adaptive large neighbourhood search (ALNS) was applied to VRP by [93] in 2007. Swarm-based methods like the bee colony algorithm were applied by [75] in 2011, and particle swarms were applied by [74] in 2010. Hybrid metaheuristics, which combine metaheuristics to gain the various advantages of different methods [59], have been a focus in terms of VRP solutions, particularly through the late 2000s and to now [109]. According to [59], there has also been a trend towards researchers favouring genetic algorithmic approaches to the VRP from the 2010s to now. Local search heuristics also remain popular for their strong results as well as their place in hybridising metaheuristics algorithms [59].

2.1.2 Simulated annealing. Simulated Annealing was first developed in 1983 by [58]. Simulated Annealing is a randomised local search heuristic whose analogy is based on actual Annealing, being the process whereby metals are cooled to form optimal crystalline structures [45]. It starts at a random initial solution. Then, the local solution space is explored. Modifications are made to parts of the solution, and new values of the objective function are evaluated, with energy in the system mapping to a particular VRP solution cost, and the algorithm accepts better solutions as the new current solution, as well as accepts worse solutions to be the current solution with a probability [45] that depends on the current value of the temperature parameter as well as the Boltzmann distribution [45]. Higher temperatures increase the chance of accepting worse solutions. This leads to early exploration of the search space and allows the algorithm to escape from local optima [45]. Multiple iterations are performed whereby modifications are made, and new solutions are accepted until some stopping criterion is reached. The temperature starts high and is lowered according to some predefined cooling schedule [45].

2.1.3 Quantum Computing for COP. The first major contributions to the quantum computing realm are attributed to Richard Feynman [39], where he talked about classical computers being insufficient for simulating physics [98], and Paul Benioff [11]. The first quantum computer was described by [32], [32] where he suggested that quantum superposition may offer parallel computation speedups over classical computers [98]. In 1994, Peter Shor [99] shook the world with his integer factoring algorithm that had major implications for current cryptography technologies [7]. The first paper linking vehicle routing and quantum computing was published in 2003 by [81], [86]. The first ever proposed solution for the Travelling Salesman Problem (TSP) was introduced in 2004 by [77]. The VRP is an extension of the TSP [31]. In 2006, this algorithm was improved with an alternative formulation by [20],[86]. A contribution to formulating routing problems for quantum computing was made in 2007 by [33]. The first-ever experiments to solve TSP on simulated quantum hardware using quantum Annealing were demonstrated by [19]. In 2013, the TSP was formulated and adapted for an adiabatic computer by [112]. Also, in 2013, the first paper addressing the VRP using quantum Annealing (although under the name "vehicle scheduling") was put forward by [1], [86]. The first practical tests of solutions to TSP using actual quantum hardware were performed by [102], in which instances with a maximum vertex degree of 3 were used for testing. Also in 2017, quadratic speedup of the TSP using a quantum backtracking algorithm was demonstrated by [82]. In 2019, a hybrid quantum-classical algorithm was used on instances of the capacitated vehicle routing problem with between 22 and 101 clients by [38], [86]. The first major publication directly addressing the VRP by name was by [54] in 2019. This was the first usage of quantum hardware on the VRP, using a DWAVE quantum annealer [86]. This was followed in 2020 by [50] using quantum Annealing for the dynamic multi-depot, capacitated vehicle routing problem (CVRP), and [9] using the Quantum Approximate Optimization Algorithm (QAOA) to address the VRP in 2022 [86].

2.1.4 Variational Quantum Eigensolver. The Variational Quantum Eigensolver (VQE) was introduced in 2014 by [92]. It uses both quantum and classical computation, referred to as a hybrid quantum-classical model, and it determines solutions by solving for the eigenvalue of a Hamiltonian matrix [78]. It was originally proposed for quantum chemistry applications, and it is used to approximate the ground state of a Hamiltonian [2]. The two parts of the algorithm are a quantum subroutine and a classical loop [56]. First, a parametrised quantum state, or "ansatz" (best guess), is initialised. The algorithm uses a quantum computer to evaluate the value of the energy of the ansatz [56] and uses a classical optimiser to minimise the expected value by optimising the parameters [2]. It is through this quantum-classical iterative loop that the algorithm converges toward an answer.

2.1.5 Comparison of quantum and classical approaches for the VRP. Quantum computing has been applied to COPs substantially less than classical methods like metaheuristics. Quantum computing in optimisation is newer and more limited in comparison to classical methods like heuristics and metaheuristics, which have been proven to obtain satisfactory answers for decades in several practical scenarios. In a benchmark study comparing VQE and QAOA to Simulated Annealing (SA) and Branch and Bound (BNB) on instances of the TSP and the Quadratic assignment problem (QAP), it was found that current classical algorithms significantly dominated the variational quantum algorithms in solving these instances [56]. The study looked at success rate, feasibility and computational time to find solutions, and classical methods beat the quantum methods in all regards [56]. The limitations of current noisy intermediate scale quantum (NISQ) devices meant that only smaller instances were able to be tested with the quantum algorithms [56].

2.2 Vehicle routing problem overview and complexity

The vehicle routing problem, as mentioned earlier, is a widely researched combinatorial optimisation problem (COP) and can be considered as an extension to the TSP [31], [60]. It has many practical applications, and efficient solutions to the problem have major implications for industry as well as for society as a whole [63]. The annual distribution industry in the US and UK combined adds up to several hundreds of billions of dollars, and hence, any increased efficiency in the operations within this industry holds promises to drastically decrease costs [63]. Minimising distances also has a relation to optimising for environmental concerns, which is a growing trend within society and industry as a whole. In terms of other economic effects, distribution cost has a significant contribution in deciding the final cost of a good of sale, and hence, bringing down distribution costs makes things available at cheaper costs [59].

2.2.1 Brief timeline. The "truck dispatching problem" was introduced by Dantzig and Ramser in 1959, and it became known as the first instance of the VRP [31]. Versions of the VRP started emerging in the early 70s [35]. These include but are not limited to routing of public services in 1970 [76], fleet routing in 1971 [69], and distribution management in 1974 [34]. In 1978, stochastic considerations were added by [48]. 1983 saw the addition of time windows by [100], forming the vehicle routing problem with time windows (VRPTW). Lack of computational resources in the 80s prevented much work from being done on very complex (i.e. stochastic and dynamically changing) forms of the VRP [35]. In recent years, dynamic VRP and time-dependent VRP have been more widely studied, enabled by real-time access to large amounts of data [59]. Researchers generally pay more attention now to problem-specific versions of the VRP that incorporate realistic constraints (often referred to as "rich VRPs"), as the industry meets various forms of the challenge [14].

2.2.2 Problem definition. The VRP, as a general problem, can be adapted to many situations. In real life, there are particular use cases and constraints that create variants of the VRP, and as such, there is no one individual satisfactory formulation [60]. Most research that is done on the VRP as a problem can be translated to specific instances [60]. The CVRP is the most widely studied variant, as it involves the practical implication that a vehicle has a finite carrying capacity [59]. Other forms of the problem involve adding time windows, adding 2D and 3D loading constraints, dynamic VRP, and time-dependent VRP, multiple depots, among others [59]. The fundamental goal of the problem is to efficiently deliver items from a central depot to several customers, minimising distance [60]. A fleet of size n (number is variable) vehicles start at the central depot. The depot and the customers are represented on an undirected graph as vertices, and the paths between them as arcs on the graph [60]. Customers have a demand value associated with them, and the arcs between vertices represent some metric of cost (be it time, distance, etc.) [60]. The output of an algorithm to solve such a problem is a set of m routes that start and end at the depot, such that each customer is satisfied by only one vehicle, the capacity is respected, and the total distance is minimised [60].

2.2.3 Complexity of the problem in classical approaches. Vehicle routing problems, in general, are usually $\mathcal{NP}$-Hard, and the VRP is no exception [67]. This implies that the solution cannot be solved to optimality in polynomial time. Another feature of $\mathcal{NP}$-Hard problems is that they tend to scale exponentially in complexity (and hence, time and resources required to solve them) relative to an increase in the problem size. This makes solving many practical versions of the VRP (involving many hundreds or potentially even thousands of customers, for e.g. waste collection) to optimality, intractable [6]. This explains why most attempts at solving the VRP in research usually use a heuristic or metaheuristic approach to attain good enough solutions in reasonable time [59].

2.2.4 Complexity of the problem in quantum approaches. Quantum computing operates off of a different paradigm than classical computers and hence holds promise to beat solutions generated by classical computers or even generate entirely new solutions. The problem of factoring a number into its prime constituents is not believed to be efficiently solvable by a classical computer, whereas Shors' algorithm [99] demonstrates an efficient solution using a quantum computer [2]. Another reason to believe that quantum computing may provide a speedup for solving optimisation problems is the speedup provided in going from a basic search algorithm to a quantum search algorithm for an $\mathcal{NP}$- complete problem [56]. Whilst it is not widely believed that quantum computers will be able to provide a significant speedup for $\mathcal{NP}$-Hard COPs in general, there are certain subsets of problems that hold promise for the quantum advantage [2]. There are inapproximability bounds for many $\mathcal{NP}$-Hard problems which represent bounds on current approximation ratios [2]. The potential for quantum computers to provide a large increase in performance over classical computers lies in potentially being able to improve upon inapproximability bounds and decrease the difference between these bounds and the provable approximation factors [2]. Another potential area for the quantum advantage is in "$\mathcal{NP}$-intermediate" problems, which lie in $\mathcal{NP}$ but are neither in P nor in $\mathcal{NP}$-Complete [2]. Shor's algorithm is considered to be one of these problems [99]. According to [2], an exponential speedup via quantum algorithms is potentially possible on these sorts of problems, as they may have some sort of structure that can be taken advantage of by quantum algorithms.

2.3 Classical approaches to the problem

Classical approaches are divided into deterministic/exact approaches and non-deterministic approaches, which are further split up into classical heuristics and metaheuristics.

2.3.1 Deterministic approaches. Exact methods can only find the optimal solution in cases where the problem instance is small, partially due to the difficulty in estimating accurate lower bounds on the objective function of a problem causing slow convergence rates[29], making them inefficient for real-world applications [72].

Exact methods can be defined under three main categories: Integer linear programming (ILP), dynamic programming, and direct tree-based methods [63]. ILP can further be broken down into set partitioning formulations, vehicle flow formulations and commodity flow formulations [63]. Set partitioning involves representing routes in a matrix and selecting the optimal subset of routes [63]. These algorithms tend to contain huge numbers of variables and, hence, do not perform or scale well [63]. Vehicle flow formulations use binary variables to indicate which vehicles travel between which cities, and commodity flow formulations use flow variables to associate the flow of goods, or demand, to particular cities [63]. Tree-based methods categorise some of the best-known exact algorithms for the VRP, such as branch and bound and its extensions (branch and cut, branch and price) [60]. Tree-based methods sequentially build routes [60] by exploring a tree-based structure of optimal and suboptimal routes [56]. Nodes represent objective function values for each route, and the best options are further broken down and explored [56]. In general, ILP-related solutions require complex mathematical programming and do not scale well to realistic instances [60].

2.3.2 Non-deterministic approaches: Classical heuristics. The main non-deterministic approaches to VRP are classical heuristics and metaheuristics. Classical heuristics are named classical, as they do not allow worse solutions in decision steps, which prevents exploration of the search space, whereas metaheuristics do allow for this and, hence, can avoid local minima [60]. Classic heuristics can further be broken down into constructive heuristics and improvement heuristics. Constructive heuristics are necessary to generate the starting feasible solution; generally, one that is far from optimal [72]. The main four frameworks that encompass constructive heuristics are the nearest neighbour method, the sweep method, the savings method, and the insert method [72]. The nearest neighbour involves iteratively adding the nearest unrouted customer at the end of each route [72]. The insert method adds customers into various places in the route (not just the end), based on whichever incurs the lowest cost [72]. The Savings method, initially proposed by [26], works by combining an initial solution of short, per-customer routes into a large and complete route, combining based on the maximum distance saved [72]. Lastly, Sweep algorithms, first proposed by [46], work by identifying a central depot, ordering customers based on their polar coordinates (creating a "sweep" around the depot), and grouping nearby customers into routes until capacity is reached for a particular trip/route [72].

Improvement heuristics iteratively improve a given solution (usually given via a constructive heuristic) by using local search techniques on the surrounding search space of the solution [72]. They are quite efficient at obtaining acceptable solutions but cannot escape from local minima as they do not allow for much exploration of the search space [72]. They converge quite quickly, which makes them efficient at solving relatively large-scale problems, and can be broken down into inter-route improvement and intra-route improvement [72]. COPs generally work well with local improvement algorithms, with the exploration being done by modifications to parameters of solutions [109]. In the context of the realm of non-deterministic solutions, the combination of constructive and improvement heuristics provides "good" but not "excellent" solutions [60]. They should not necessarily be directly compared with metaheuristics, considering constructive heuristics are often used to form inputs for metaheuristic algorithms and improvement heuristics are often used as parts of the algorithms themselves[59].

2.3.3 Non-deterministic approaches: Metaheuristics. Metaheuristics have been the dominant force in terms of solutions for the VRP in the past few decades, partially owing to increased computational ability and, hence, increasingly complex and capable algorithms [35]. Metaheuristic algorithms aim to balance exploration and exploitation, allowing for the best solutions to be used, as well as allowing for worse and even infeasible intermediate solutions in the name of search space exploration [60]. Metaheuristics are often nature-inspired and are less problem-dependent than classical heuristics [72]. Metaheuristics provide good solutions, are efficient concerning time, and can scale well [36], which makes it understandable that they are the most applied solution technique to the VRP according to a survey done by [59]. All metaheuristic algorithms allow the escaping of local minima and usually have incorporated aspects of classical constructive and improvement heuristics [45]. The main categories of metaheuristics are as follows: Local/neighbourhood-based search/ single solution centred, population-based search, and learning mechanisms [45]. Neighbourhood-based search metaheuristics start at an initial solution and iteratively modify some aspect of their solution to acquire and test new candidates, stopping upon reaching some pre-specified condition [45]. These include Simulated annealing, Tabu search, deterministic annealing, and neighbourhood searches (e.g. Very large neighbourhood search, variable neighbourhood search) [107], [61]. Population-based methods like genetic algorithms encode populations as chromosomes (in practice, bitstrings) and evolve these populations through mixing genes of parents (using crossover techniques) and mutating individual genes (mutation techniques) [45]. Memetic algorithms are borne from replacing mutations with simple improvement heuristics [45]. Also under population-based methods (considered to be learning mechanisms as well in some respect) are things like ant colony optimisation (ACO) [45] and particle swarm optimisation (PSO) [59], which use populations of solutions to explore the search space, and support global learning through incorporation of the best global solutions into the local consideration of each individual entity [59]. Learning mechanisms refer to ACO and PSO, as was mentioned before, but also refer to neural networks [45]. In terms of performance, metaheuristics are the state of the art in terms of solutions to almost any form of the VRP. A large benchmarking of various metaheuristics and hybrid metaheuristics can be seen in [109]. There are two main datasets used to benchmark capacitated VRP (CVRP, the most researched form of VRP, also referred to as the "canonical" VRP) solutions, namely the datasets of [25] and [47]. The first dataset includes 14 benchmark instances of between 50 and 199 customers [25], and many metaheuristics can reach the best-known solution (BKS) on all of the instances [109]. The second has 20 large-scale instances of between 200 and 483 customers [47]. A benchmark of metaheuristic algorithms was carried out on these instances, with the best-performing methods being hybridised metaheuristic algorithms [109]. Fourteen algorithms produced solutions within one % of the BKS on these large-scale instances. For a more comprehensive breakdown, I refer the reader to [109]. The top seven best performing (considering both computational efficiency and solution quality) metaheuristic algorithms are listed: VCGLR11s (a hybrid genetic algorithm), VCGLR11f (A hybrid genetic algorithm), MB07s (a hybrid EA(evolutionary algorithm) and ELS (Evolutionary local search) algorithm, P09 (GRASP (greedy randomised adaptive search procedure) and ELS algorithm), T05 (an adaptive memory and tabu search algorithm), GGW10 (A hybrid record-to-record travel, evolutionary computation algorithm), MB07f (An evolutionary algorithm and evolutionary local search hybrid)[109]. Some of the best solutions tested on instances in [28] for the CVRP include those created by adaptive large neighbourhood search (ALNS) of [93], the hybrid GA of [66], the hybrid genetic search algorithm of [108] and the Parallel iterated tabu search of [30], [109]. For the heterogeneous fleet VRP (HVRP), the HVRP instances of [103], [70] were used. The best algorithms in this case were Tabu search [15], hybrid GA [95] or ILS and VNS [91]. For multiple trips, VRP (MTVRP), instances of [105] were used [109]. The best results were produced by the original tabu search and adaptive memory procedure [104] and adaptive memory-based search [85], [109]. For time windows (VRPTW), the best results were produced by guided EA [96] and Hybrid GA of [83]. For other breakdowns by VRP variant, refer to [109]. Our discussion on metaheuristics will conclude with some notable applications of metaheuristic simulated annealing.

2.3.4 Simulated Annealing, notable applications. The application of simulated Annealing to the CVRP with 2-dimensional loading constraints by [113] showed that SA outperforms (or matches best of) all other existing algorithms, with SA matching and improving the BKS in most instances. A parallel simulated annealing method was applied to VRP with simultaneous pickup-and-delivery with time windows [110]. It was able to outperform a GA on several instances, ranging from small to very large [113],[21]. Simulated Annealing has also been applied to the following, with reasonable success: to the VRPTW by [21], to VRP with independent route length [106], to the truck and trailer routing problem by [71] and the open location-routing problem by [115],[72].

2.4 Quantum approaches to the problem

Quantum computers operate through the use of qubits [86], a data representation that is fundamentally different to that of classical computing (namely, a bit), and hence, quantum computing is a different paradigm from that of classical computing. A qubit, representing a quantum particle, is able to contain more information than a regular bit by virtue of its ability to be in superposition [87]. A qubit can be 0, 1, or in a state of superposition anywhere in between, with a certain probability of being measured at either 0 or 1 [87]. Quantum computers can also represent the correlation between qubits through entanglement [87]. It is the utilisation of both superposition and entanglement that will potentially allow quantum computers to overcome classical limitations and have potential success in optimisation [87], [86]. There have been some examples in the literature of quantum algorithms that have already beat their classical counterparts. Shor's factorisation and Grover's search have shown a quantum advantage over classical algorithms [89], but they are limited by current computational hardware. These algorithms require fault-tolerant quantum computers, which in turn require millions of physical qubits, and hence such hardware is currently technologically out of reach [89], [80]. Therefore, the use of current quantum machinery can be investigated in the meantime. In the current era of quantum computing hardware, we have what are labelled Noisy intermediate scale quantum (NISQ) devices [12]. "Intermediate scale" refers to the limited size of qubits, and "noisy" refers to the high chances of error; both factors mean that the current capabilities of quantum computers are quite limited [12]. Variational quantum algorithms are a promising avenue in making use of NISQ devices [89]. They are hybrid quantum-classical algorithms that make use of shallow quantum circuits and classical optimisers to find a quantum state that minimises a cost function [89], [80]. They are capable of operating with the noise that is present in NISQ devices and only use as many quantum resources as is necessary, leaving optimisation of parameters to the classical optimisers [89].

2.4.1 Algorithms. The main categories of quantum algorithms fall into either annealing-based or quantum gate-based [111], [52]. Quantum Annealing is Simulated Annealing's quantum alternative, using quantum fluctuations and quantum tunnelling to speed up convergence to the optimal state [4], [55]. Quantum Annealing is specifically designed for certain domains of problem-solving, such as optimisation [55]. Gate-based models evolve qubit states towards a solution through the use of quantum circuits, which are comprised of quantum gates [86] (which apply operations to qubits) [3], [49]. Variational quantum algorithms (VQAs) leverage the gate-based model and utilise quantum resources efficiently by only using quantum hardware on the most difficult problem parts, and therefore may be able to provide a quantum advantage using NISQ devices [12]. The encoded problem is passed to the quantum circuit, which in turn estimates a global minimum for the problem, and this is passed to a classical optimiser to optimise parameters [7], [42]. To use quantum algorithms on optimisation problems, it is necessary to mathematically formulate the problems in a form that is conducive to being solved by a quantum algorithm. Commonly, problems are formulated by turning them into a quadratic unconstrained binary optimisation problem (QUBO), where the solution is then found by minimising a quadratic polynomial of binary variables [12]. Many optimisation problems, including the VRP, can be formulated in QUBO form [68]. QUBOs can be mapped easily to an Ising formulation [89], and both of these can be mapped to a representation whereby the objective is to solve for the ground state of a Hamiltonian matrix[2], [84]. It should be noted that other models may be better suited to problem formulation for certain specific problems [12]. There are multiple VQAs, differing mainly in ansatz, cost function and the classical optimiser used [2]. VQE and QAOA are some of the current best methods to utilise NISQ devices effectively [84]. The objective of the VQE (explanation of functionality lies in the overview of this review) is to find the optimum solution to a problem, which is analogous to finding the lowest eigenvalue or ground state of a Hamiltonian matrix [84]. VQE depends on the Rayleigh-Ritz variational principle, which is a quantum mechanics principle that states that the ground state for a given Hamiltonian is the lowest possible expectation value of the Hamiltonian [37].

2.4.2 Applications. VQE and QAOA were applied to the TSP (significant because the VRP is an extension of the TSP [16]), and the QAP by [56]. It was found that VQE and QAOA produced similar results in terms of success and feasibility rate, but VQE was more computationally efficient and more capable of solving larger instances of the problems [56]. QAOA and VQE were used to find optimal solutions for three types of COPs: frequency allocation, register allocation and flight gate allocation [84]. The VQE has been used on several COPs. It was applied to the Max cut problem (An $\mathcal{NP}$-Complete binary optimisation problem) using an ideal quantum simulator on problem instances with up to 4 nodes, finding the solution 95 % of the time [80]. The VQE has also been used in hybridised quantum approaches, such as using QAOA and VQE in quantum machine learning [79]. The VRP was converted into a quantum support vector machine (QSVM) and solved using VQE, solving 3 and 4 city instances successfully [79]. Small instances of the VRPTW were solved using IBM's quantum simulators and their hardware devices through Qiskit [51]. It was found that VQE outperformed QAOA when the number of samples was limited [51]. The VRP was decomposed into a clustering phase solved as a multiple knapsack problem and a routing phase modelled by TSP's [89]. On instances of 4, 5 and 6 cities, VQE performed well with little hyper-parameter tuning being necessary [89].

3 DISCUSSION

There are limited publications that directly compare and benchmark quantum computing methods and classical methods. This leaves open a space for this sort of research to be undertaken. Using a classical algorithm that has a long history of applications to the VRP with consistent success, like SA, provides a solid context and reference against which to test emerging quantum technologies. Simulated Annealing represents a solid example of a metaheuristic algorithm, with metaheuristics being shown in the literature to be the state of the art in terms of VRP solutions. It is important to investigate the efficacy of emerging quantum technologies as NISQ technology evolves using a consistent standard of solution quality. The VRP is a good candidate for investigation as it is a useful representation of the $\mathcal{NP}$-Hard complexity class. We know how classical algorithms behave in the face of $\mathcal{NP}$-Hard complexity, and thus, it is incredibly useful to be able to gain insights into quantum algorithms' behaviour in comparison. There is potential to illuminate any sort of quantum advantage that may be possible to obtain when using algorithms that make the best use of current NISQ technology, as well as advantages that may be possible with future generations of quantum machinery like the idealised fault-tolerant quantum computers. Faster solution times for large instances of the VRP or better solutions to instances that have historically been hard for classical methods to solve will have significant implications for the logistics industry, as well as the $\mathcal{NP}$-Hard complexity field of research. The VQE has been shown in the literature to be an efficient and versatile VQA, with VQAs, in turn, being one of the most effective approaches to utilising the limited quantum hardware of today. A benchmark of the VQE and SA on the VRP will provide key insights to researchers in the field and potentially advance the development of algorithms for such COPs. The limitation of today's NISQ devices in terms of error rates, as well as a low number of qubits, presents a limitation to the size of the problems that will be able to be tackled by algorithms like VQE. Even so, a measure of the VQE's performance against a classical method like SA will contribute to knowledge of quantum capability as the hardware's capability continues to advance. Insight into the VQE's performance on the VRP has the potential to illuminate key advantages of quantum algorithms, which could inspire the development of further algorithms like hybridised quantum algorithms, for example. Metrics to be considered for benchmarking include computational efficiency (time-wise), solution quality, algorithm flexibility, scaling ability, and simplicity of implementation.

4 CONCLUSIONS

The history of solutions for the VRP is long and diverse, ranging from early exact methods that optimally tackle small-scale instances to the development of powerful metaheuristic and hybrid metaheuristic algorithms that are championed as state-of-the-art in terms of VRP solutions today. Metaheuristic algorithms are generally the best current approach, used in practical settings to get good solutions in a reasonable computation time. The formidable nature of $\mathcal{NP}$-Hard problems, as well as the potential financial implications for many industries, motivates the study of any promising avenues of solution approaches. Thus, investigating the potential of quantum algorithms on current NISQ devices to solve COPs provides the potential to unlock powerful insights into potential quantum advantages over classical computation. Whilst quantum computing is still limited, benchmarking the performance of the VQE against SA will provide valuable insights into the potential future directions of research into quantum approaches to COP's. Also, algorithms like Shor's and Grover's show us that there is potential for significant gains to be made in terms of solutions to these problems as the hardware continues to evolve. A comparison between SA and the VQE has the potential to unlock information about a wide range of $\mathcal{NP}$-Hard problems. Benchmarking these algorithms will be useful, as it can set up a comparison framework for these two paradigms, namely classical versus quantum approaches to COP's. There is a clear call for more research into quantum optimisation, as well as into the development of quantum hardware. Algorithmic development should also be encouraged by the power that hybrid metaheuristic algorithms display, as well as the recent work on combining quantum algorithmic approaches successfully.

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