According to Hilbert's well-known theorem (see, for example, [4]), the countable set of polynomials Ω_r has a finite basis and therefore the countable set of relations (12) is a consequence of a finite number of them for $0 \leqq r \leqq M_2(F, \sigma, m_1, m_2, n)$. The same thing holds for the relations (7), q.e.d.

4. We now consider Theorem 1. Observe that since the independence of the statistics $Q_1(\xi)$ and $Q_2(\xi)$ is equivalent to the fact that the statistics

$$t_1 Q_1(\xi)+t_2 Q_2(\xi) \cong t_1 Q_1(\xi)+t_2 Q_2(\xi')$$

are identically distributed for any real t_1 and t_2, where ξ' is a random vector independent of and distributed identically as ξ, then by the conditions of Theorem 1, the equivalence of $Q_1(\xi)$ and $Q_2(\xi)$ is equivalent to (6) holding for all non-negative integers r and s.

Arguing just as in Theorem 2 in the proof of the equivalence of (b) and (c), we can establish that the fulfillment of (6) for all non-negative integers r and s implies the fulfillment of (6) for a finite collection of values r and s, which is required to complete the proof of Theorem 1.

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REFERENCES

[1] Yu. V. Linnik, *Polynomial statistics and polynomial ideals*, Golden Jubilee Commemoration Volume, 1958–1959, Calcutta Math. Soc., I, pp. 95–98.
[2] A. A. Zinger, *On the distribution of polynomial statistics in samples from a normal population*, Dokl. Akad. Nauk SSSR, 149, 1, 1963, pp. 20–21. (In Russian.)
[3] J. A. Shohat and J. D. Tamarkin, *The Problem of Moments*, New York, 1947.
[4] B. L. Van der Waerden, *Modern Algebra*, Ungar, New York, 1949.

ON POLYNOMIAL STATISTICS FOR THE NORMAL AND RELATED LAWS

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(*Summary*)

Polynomial statistics are studied for a normal vector and a random vector related to the normal one (this concept is defined by means of differential equations). It is proved that the independence property of polynomial statistics is, roughly speaking, equivalent to the absence of correlation between a finite number of special functions of them. A similar statement holds for the property of equidistribution of polynomial statistics.

SOME NEW ESTIMATES FOR DISTRIBUTION FUNCTIONS

E. A. NADARAYA

(*Translated by B. Seckler*)

1. Let $X_n = (x_1, x_2, \cdots, x_n)$ be a sample of independent observations of a random variable X with distribution function $F(x)$. Suppose $F(x)$ has a continuous density $f(x)$ over the whole x-axis.

As an approximation to $F(x)$ based on the empirical data we take the statistic

$$F_n(x) = \int_{-\infty}^{x} f_n(x')dx',$$

where

$$f_n(x) = \frac{1}{nh} \sum_{i=1}^{n} K\left(\frac{x-x_i}{h}\right),$$

and $K(x)$ is some density function such that $K(x) < c < \infty$, $\lim_{x \to \pm\infty} |xK(x)| = 0$, and $h \to 0$ with increasing n [1].

Our aim is to establish certain properties of $F(x)$ as $n \to \infty$.

2. Asymptotic unbiasedness and consistency of the estimate. Uniform convergence with probability one.

Lemma. *As n increases,*

$$\int_{-\infty}^{\infty} |\mathbf{E}f_n(x) - f(x)| dx \to 0.$$

This lemma is a consequence of a theorem of Scheffé.

Theorem 1. *$F_n(x)$ is an asymptotically unbiased and consistent estimate for $F(x)$. Moreover, for large n,*

$$\mathbf{D}F_n(x) \sim \frac{F(x)(1-F(x))}{n}.$$

Proof. The first part of the theorem follows immediately from the lemma and the inequality

$$|\mathbf{E}F_n(x) - F(x)| \leqq \int_{-\infty}^{\infty} |\mathbf{E}f_n(x') - f(x')| dx'.$$

To prove the consistency of the estimate $F_n(x)$, we compute its variance:

$$\begin{aligned} \mathbf{D}F_n(x) &= \frac{1}{n} \mathbf{D} \frac{1}{h} \int_{-\infty}^{x} K\left(\frac{x'-X}{h}\right) dx' \\ &= \frac{1}{n} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{(x-u)/h} K(t)dt \right)^2 f(u)du - \frac{1}{n} \left[\int_{-\infty}^{\infty} \left(\int_{-\infty}^{(x-u)/h} K(t)dt \right) f(u)du \right]^2. \end{aligned}$$

The application of Lebesgue's theorem to the latter integrals yields

$$\text{(1)} \quad \mathbf{D}F_n(x) \sim \frac{1}{n} \int_{-\infty}^{\infty} \varepsilon(x-u) f(u)du - \frac{1}{n} \left(\int_{-\infty}^{\infty} \varepsilon(x-u) f(u)du \right)^2 = \frac{F(x)(1-F(x))}{n},$$

with

$$\varepsilon(x-u) = \begin{cases} 0 & \text{for } u > x, \\ 1 & \text{for } u \leqq x. \end{cases}$$

From (1) and the first part of the theorem, it follows that

$$\mathbf{E}[F_n(x) - F(x)]^2 = \mathbf{D}F_n(x) + [\mathbf{E}F_n(x) - F(x)]^2 \to 0 \quad \text{as} \quad n \to \infty.$$

The theorem is proved.

Theorem 2. *With probability one,*

$$v_n = \sup_{-\infty < x < \infty} |F_n(x) - F(x)| \to 0 \qquad \textit{as } n \to \infty \textit{ and } h \to 0.$$

The proof follows from the lemma and a theorem of V.I. Glivenko [2].

3. Estimation of the distribution quantiles. Let ζ_p denote the p-th order quantile of the distribution $F(x)$, i.e., a root of the equation $F(\zeta) = p$, with $0 < p < 1$. We assume it to be unique.

[1] In what follows, h should be properly written as $h(n)$. However, for simplicity of writing, we shall make use of the first notation.

As an approximation to the quantile ζ_p, we take the root of

$$F_n(\bar{\zeta}_p) = p.$$

We call $\bar{\zeta}_p$ the sample quantile. Let us show that $\bar{\zeta}_p$ is a consistent estimate for the quantile ζ_p. Let

$$\delta(\varepsilon) = \min\{F(\zeta_p+\varepsilon)-F(\zeta_p), F(\zeta_p)-F(\zeta_p-\varepsilon)\}.$$

The assumption that ζ_p is unique implies that $\delta(\varepsilon)$ is positive. The following inequality holds:

$$\mathbf{P}\{|\bar{\zeta}_p-\zeta_p| > \varepsilon\} \leqq \mathbf{P}\{|F(\bar{\zeta}_p)-F(\zeta_p)| > \delta(\varepsilon)\}. \tag{2}$$

Now

$$|F(\bar{\zeta}_p)-F(\zeta_p)| = |F(\bar{\zeta}_p)-F_n(\bar{\zeta}_p)| \leqq \sup_{-\infty<x<\infty} |F_n(x)-F(x)|,$$

where $\sup_{-\infty<x<\infty} |F_n(x)-F(x)| \to 0$ in probability by Theorem 2. From (2), it then follows that

$$\mathbf{P}\{|\bar{\zeta}_p-\zeta_p| > \varepsilon\} \to 0 \quad \text{as} \quad n \to \infty.$$

Thus, the consistency of the estimate $\bar{\zeta}_p$ has been proved.

In order to study the behavior of the distribution of $\bar{\zeta}_p$ for large n, we assume:

1) the characteristic function corresponding to $K(x)$ is absolutely integrable;
2) the density function $f(x)$ is bounded and uniformly continuous;
3) $\sqrt{n}(\mathbf{E}F_n(x)-F(x)) \to 0$ as $n \to \infty$; [2]
4) h is chosen so that $nh^2 \to \infty$.

Consider the Taylor expansion

$$p = F_n(\bar{\zeta}_p) = F_n(\zeta_p)+(\bar{\zeta}_p-\zeta_p)f_n(\xi), \tag{3}$$

where ξ is some random point between $\bar{\zeta}_p$ and ζ_p. From (3) we obtain

$$\frac{\sqrt{n}f(\zeta_p)(\bar{\zeta}_p-\zeta_p)}{\sqrt{F(\zeta_p)(1-F(\zeta_p))}} = -\sqrt{n}\left\{\frac{F_n(\zeta_p)-\mathbf{E}F_n(\zeta_p)}{\sqrt{F(\zeta_p)(1-F(\zeta_p))}}\right\}\frac{f(\zeta_p)}{f_n(\xi)} - \sqrt{n}\left\{\frac{\mathbf{E}F_n(\zeta_p)-F(\zeta_p)}{\sqrt{F(\zeta_p)(1-F(\zeta_p))}}\right\}\frac{f(\zeta_p)}{f_n(\xi)}.$$

Let us show that $f_n(\xi)$ tends to $f(\zeta_p)$ in probability. Indeed,

$$|f_n(\xi)-f(\zeta_p)| \leqq |f_n(\xi)-f(\xi)|+|f(\xi)-f(\zeta_p)| \leqq \sup_{-\infty<x<\infty} |f_n(x)-f(x)|+|f(\xi)-f(\zeta_p)|.$$

Since under conditions 1), 2), and 4), $\sup_{-\infty<x<\infty} |f_n(x)-f(x)| \to 0$ in probability [1], and also $|f(\xi)-f(\zeta_p)| \to 0$ in probability because $f(x)$ is continuous and $\xi \to \zeta_p$ in probability, our assertion follows. Hence, the variables

$$\frac{\sqrt{n}f(\zeta_p)(\bar{\zeta}_p-\zeta_p)}{\sqrt{F(\zeta_p)(1-F(\zeta_p))}} \quad \text{and} \quad \frac{\sqrt{n}[F_n(\zeta_p)-\mathbf{E}F_n(\zeta_p)]}{\sqrt{F(\zeta_p)(1-F(\zeta_p))}}$$

have the same limit distribution. From (1), it then follows that

$$\lim_{n\to\infty} \mathbf{P}\left\{\sqrt{n}\,\frac{F_n(\zeta_p)-\mathbf{E}F_n(\zeta_p)}{\sqrt{F(\zeta_p)(1-F(\zeta_p))}} < \lambda\right\} = \lim_{n\to\infty} \mathbf{P}\left\{\frac{F_n(\zeta_p)-\mathbf{E}F_n(\zeta_p)}{\sqrt{\mathbf{D}F_n(\zeta_p)}} < \lambda\right\}.$$

Next,

$$\lim_{n\to\infty} \mathbf{P}\left\{\frac{F_n(\zeta_p)-\mathbf{E}F_n(\zeta_p)}{\sqrt{\mathbf{D}F_n(\zeta_p)}} < \lambda\right\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\lambda} e^{-x^2/2}\,dx. \tag{4}$$

[2] This condition is fulfilled if, for instance, $K(x)$ and $F(x)$ satisfy the following conditions: $K(x) = K(-x)$, $\int_{-\infty}^{\infty} x^2K(x)dx < \infty$, $F(x)$ has a bounded second derivative, and finally, $\sqrt{n}\,h^2 \to 0$.

In fact, for (4), it is necessary and sufficient that

$$(5) \qquad n\mathbf{P}\left\{\left|\frac{Z_n-\mathbf{E}Z_n}{\sqrt{\mathbf{D}Z_n}}\right| > \varepsilon n^{1/2}\right\} \to 0$$

(see [3], Chapt. 5), where

$$Z_n = \frac{1}{h}\int_{-\infty}^{\zeta_p} K\left(\frac{t-X}{h}\right) dt.$$

Condition (5) is fulfilled. For, suppose δ is a positive quantity. Then

$$n\mathbf{P}\left\{\left|\frac{Z_n-\mathbf{E}Z_n}{\sqrt{\mathbf{D}Z_n}}\right| \geqq \varepsilon n^{1/2}\right\} \leqq \frac{\mathbf{E}|Z_n-\mathbf{E}Z_n|^{2+\delta}}{n^{\delta/2}(\sqrt{\mathbf{D}Z_n})^{2+\delta}} \leqq \frac{1}{n^{\delta/2}(\sqrt{\mathbf{D}Z_n})^{2+\delta}} = O\left(\frac{1}{n^{\delta/2}}\right).$$

Hence, it follows that under conditions 1)—4), the sampe quantile $\bar{\zeta}_p$ is asymptotically normal with parameters $(\zeta_p, \sqrt{pq/n}(f(\zeta_p))^{-1})$, where $p = F(\zeta_p)$ and $1-p = q$.

It is of interest to note that an estimate for ζ_p using order statistics is asymptotically normal just as is $\bar{\zeta}_p$ ([4], [5]).

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REFERENCES

[1] E. Parzen, *On estimation of a probability density function and mode*, Ann. Math. Statist., 33, 3, 1962, pp. 1065–1076.

[2] V. Glivenko, *Sulla determinazione empirica delle leggi di probabilita*, Giornale dell' Istituto degli Attuari, 4, 1933, pp. 92–99.

[3] B. V. Gnedenko and A. N. Kolmogorov, *Limit Distributions for Sums of Independent Random Variables*, Addison-Wesley, Mass., 1954.

[4] N. V. Smirnov, *Approximation of distribution laws of random variables from empirical data*, Uspekhi Matem. Nauk, X, 1944, pp. 179–206. (In Russian.)

[5] H. Cramér, *Mathematical Methods of Statistics*, Princeton University Press, 1946.

[6] H. Scheffé, *A useful convergence theorem for probability distributions*, Ann. Math. Statist., 18, 3, 1947, pp. 434–438.

SOME NEW ESTIMATES FOR DISTRIBUTION FUNCTIONS

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(Summary)

In the paper an approximation $F_n(x)$ to the distribution function $F(x)$ which makes use of a sample of size n is constructed on the basis of Parzen's work [1]. Some of its properties are studied when $n \to \infty$.

MINIMAX THEOREMS FOR GAMES ON THE UNIT SQUARE

E. B. YANOVSKAYA

(Translated by B. Seckler)

Let $K(x, y)$ be the core of an antagonistic game in the unit square. One of the basic problems of antagonistic game theory for the unit square is the determination of conditions on K such that the relation