1 The Truncated Power Basis

1.1 Piecewise Polynomial Functions

Let $\xi = \{\xi_1 < \xi_2 < \cdots < \xi_{l+1}\}$ be a set of strictly increasing series of points, and let k be a positive integer. Further, let $P_1, \ldots, P_l$ denote a sequence of l polynomials of order k. Then the correponding piecewise polynomial (pp) function of order k is defined as follows:

$$f\left(x\right) = P_i\left(x\right) \quad \text{if } \xi_i < x < \xi_{i+1}$$

for $i = 1, \ldots, l$. $\{\xi\}$ are known as the breakpoints of f. At the interior breakpoints, $\xi_2, \ldots, \xi_l$, the function value is defined by specifying f to be right continuous; that is,

$$f\left(\xi_i\right) = f\left(\xi_i^+\right), \quad i = 2, \ldots, l$$

However, in a sense, without this specification, the function has two values at any interior breakpoint: the value it gets from the polynomial piece to the left of the breakpoint, $f\left(\xi_i^-\right) = P_{i-1}\left(\xi_i\right)$, in addition to the value it gets from the polynomial piece to the right of the breakpoint, $f\left(\xi_i^+\right) = P_i\left(\xi_i\right)$. To properly define the function, one can specify f to be right-continuous:

$$f\left(\xi_i\right) \equiv f\left(\xi_i^+\right) \tag{1}$$

Denote the set of pp functions of order k with breakpoints $\xi = \{\xi_1, \ldots, \xi_{l+1}\}$ by

$$\mathcal{P}_{k,\xi}.$$

$\mathcal{P}_{k,\xi}$ is a linear space having dimension kl, as it consists of l polynomials, each having k polynomial coefficients. The j^{th} derivative of a pp f,

$$D^j f$$

is a pp function of order $k - j$ having the same breakpoint sequence and constructed from the same j^{th} derivatives of the polynomial pieces from which f was constructed. This "definition" dodges much of the complicated discussion of the derivatives of a pp function at its breakpoints and thus must be treated with considerable care in context of the fundamental theorem of calculus.

Proposition pp function, f satisfies

$$f\left(x\right) - f\left(a\right) = \int_a^x \left(Df\right)\left(t\right) dt \quad \text{for all} \quad x$$

if and only if f is a continuous function.

Consider a piecewise constant function f: by the previous definition, its first derivative is identically zero, and is therefore equal to the usual derivative of f if and only if f is constant.

This prerequisite information is merely for the ability to responsibly refer to the set of piecewise polynomial functions and have a shorthand way of doing so. These means enable us to introduce two sets of basis functions: first, the truncated power basis, followed by B-spline basis functions. We will see that both are closely related, with the former having some properties which leave them unattractive for function approximation and thus present the construction of B-splines and how to use them to construct a representation of $\mathcal{P}_k$. In practice, one typically is given some information about an unknown function, g, and the task is to construct a function $f \in \mathcal{P}_{k,\xi}$ which satisfies conditions that g also satisfies, and in addition, has a certain number of continuous derivatives. These conditions define a subspace of $\mathcal{P}_{k,\xi}$, $\mathcal{P}_{k,\xi,\nu}$ for which we will need a corresponding basis.

For illustrative purposes, consider the task of smoothing a histogram using parabolic splines. Suppose we are given points

$$\tau_1 < \tau_2 < \cdots < \tau_{n+1}$$

and non-negative numbers $h_1, h_2, \ldots, h_n$, with h_i denoting the height of the histogram over the interval (τ_i, τ_{i+1}). The histogram is an approximate representation of some underlying density function, g. Letting $\Delta\tau_i = \tau_{i+1} - \tau_i$, one may interpret $h_i \Delta\tau_i$ as (approximately) equal to the integral of g over $[\tau_i, \tau_{i+1}]$. One may impose the following interpolation conditions on our smooth function, f:

$$\int_{\tau_i}^{\tau_{i+1}} f(x)\, dx = h_i \Delta\tau_i$$

for $i = 1, \ldots, n$. Let f be a piecewise polynomial of order 3 having continuous first derivative:

$$f \in \mathcal{P}_{3,\xi} \cap \mathcal{C}^{(1)}$$

Choose the breakpoint sequence ξ to coincide with $\tau = \{\tau_1, \ldots, \tau_{n+1}\}$. If g is smooth and vanishes outside its support, $[\tau_1, \tau_{n+1}]$, then

$$g^{(j)}(\tau_1) = g^{(j)}(\tau_{n+1}) = 0,$$

for $j = 0, 1, \ldots, d$, where d characterizes the extent of the smoothness of g, we may also wish to require f to obey two additional interpolation constraints:

$$f(\tau_1) = f(\tau_{n+1}) = 0,$$

giving a total of $n + 2$ interpolation conditions. These, along with the $2(n-1)$ continuity conditions yield a total $3n$ constraints on the $3n$ polynomial coefficients,

$$c_{ji} \equiv D^{j-1} f\left(\xi_i^+\right).$$

These conditions lead to the system of equations:

$$
\begin{aligned}
c_{11} &= 0 \\
c_{11} + \frac{c_{21}}{2!}\Delta\tau_1 + \frac{c_{31}}{3!}\left(\Delta\tau_1\right)^2 &= h_1 \\
c_{11} + c_{21}\Delta\tau_1 + \frac{c_{31}}{2!}\left(\Delta\tau_1\right)^2 - c_{12} &= 0
\end{aligned}
$$