Multivariate B-Splines on Triangulated Rectangles

Article *in* Journal of Mathematical Analysis and Applications · April 1983

DOI: 10.1016/0022-247X(83)90265-2

2 authors:

Charles K. Chui
Stanford University

Ren-Hong Wang
Dalian University of Techno…

JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **92**, 533–551 (1983)

Multivariate *B*-Splines on Triangulated Rectangles

CHARLES K. CHUI*

Department of Mathematics, Texas A&M University, College Station, Texas 77843

AND

REN-HONG WANG†

Institute of Mathematics and Department of Mathematics, Jilin University, Changchun, Jilin, People's Republic of China

Submitted by K. Fan

1. INTRODUCTION

Although many results in univariate spline theory have been extended to the higher-dimensional settings by taking tensor products, very little is known on the general theory of multivariate spline functions. Since a univariate spline function is a "smooth" piecewise polynomial separated by a set of points which are called knots, a bivariate spline function is a "smooth" piecewise polynomial in two variables separated by a grid of curves, and so on. In the two-dimensional setting, for example, if a domain $\mathscr{D}$ in $\mathbb{R}^2$ is divided into a finite or countable number of cells by a grid partition Δ, then the space $S_k^\mu(\Delta)$ of multivariate (or, more precisely, bivariate) spline functions is the collection of all functions in $C^\mu(\mathscr{D})$ such that the restriction of every $s \in S_k^\mu(\Delta)$ to each cell of the grid partition is a polynomial $p(x, y)$ of total degree k, namely,

$$p(x, y) = \sum_{0 \leqslant i+k \leqslant k} a_{ij} x^i y^j.$$

If Δ is a simple crosscut partition of a simply connected domain $\mathscr{D}$, the dimension of $S_k^\mu(\Delta)$ is determined in [3] and a basis of $S_k^\mu(\Delta)$ is also given explicitly. This result allows us to study the approximability properties of

* The work of this author was supported by the U. S. Army Research Office under Contract DAAG 29-81-K-0133.

† This author was a visiting scholar at Texas A&M University during 1981–82.

0022-247X/83 $3.00

bivariate spline functions in $S_k^\mu(\Delta)$ and to investigate the existence of locally supported functions in $S_k^\mu(\Delta)$. A detailed study for the rectangular grid partitions is also included in [3]. Similar but less explicit results for an arbitrary crosscut partition are given in [6]. In addition, Schumaker [11] obtained a lower bound for the dimensions of bivariate spline spaces on triangulated polygons and proved that this lower bound is attained for certain special cases.

A locally supported bivariate spline function which is also positive inside the supporting Jordan curve that consists of certain grid-segments of the partition is called a bivariate B-spline. In [1] de Boor introduced a notion of multivariate B-splines which are locally supported nonnegative C^{k-1} piecewise polynomials of total degree k, and Micchelli [9] studied these B-splines in detail. Dahman [7] also provided the truncated power representations of these B-splines. The B-splines they studied are determined by a given set of knots instead of a grid partition. In fact, the knots determine certain simplices which in turn give the grid lines that separate the polynomial pieces. Since the grid partition cannot be assigned in advance, the wide range of applications by univariate B-splines cannot be easily extended to the multivariate setting. In an attempt to give explicit expressions of bivariate B-splines on a preassigned grid partition, we obtained bivariate B-splines in $S_3^1(\Delta)$, where Δ is a crosscut triangulation of the first kind and we also proved that $S_3^2(\Delta)$ has no nontrivial locally supported functions. These results along with the results on the approximation properties of the corresponding variation-diminishing spline operators V_δ are contained in [4]. It should be pointed out that these operators V_δ preserve all linear polynomials in two variables, and hence provide very efficient and good bivariate C^1 cubic spline approximants. In fact, the approximation is of optimal (or Jackson) order for functions in C and C^1 and of order $O(\delta^2)$ for functions in C^2. There also are bivariate B-splines obtained by Zwart [14], Powell [10], and Fredrikson [8] on two types of triangulations. These triangulations, however, are rigid, and in fact, the above-mentioned bivariate B-splines cannot be transformed linearly to nonuniform triangulated rectangles without losing the smoothness joining conditions.

In this paper we shall first study bivariate B-splines on nonuniform triangulated rectangles. Although this result can be applied to an arbitrary polygon, we only consider the square

$$R = \{(x, y) : 0 \leqslant x, y \leqslant 1\}.$$

Let $0 = x_0 < \cdots < x_m = 1$ and $0 = y_0 < \cdots < y_n = 1$. Then the grid lines $x - x_1 = 0, \ldots, x - x_{m-1} = 0$, $y - y_1 = 0, \ldots, y - y_{n-1} = 0$ give a rectangular grid partition Δ_{mn} of R. It was observed in [3] that $S_k^\mu(\Delta_{mn})$ contains locally supported spline functions if and only if $\mu \leqslant (k-2)/2$. Hence, the important space $S_3^1(\Delta_{mn})$ does not contain any B-spline function. To obtain nontrivial locally supported C^1 cubic spline functions, further grid partition is necessary. A very useful practice in finite element methods is to triangulate each of the mn rectangular cells. We divide each cell into four tiangular cells by adding the two diagonals of the rectangle. This refined grid partition which will be denoted by $\bar{\Delta}_{mn}$ divides R into $4mn$ triangular cells. We shall obtain explicit expressions of bivariate B-spline functions $B_{ij}(x, y) = B_{ij;m}(x, y)$ in $S_3^1(\bar{\Delta}_{mn})$. The bivariate B-splines we obtain form a partition of unity, and therefore, the corresponding B-spline series serve as efficient approximants and interpolants.

If all the rectangles in Δ_{mn} are of the same size; that is, $x_i - x_{i-1} = 1/m$ and $y_j - y_{j-1} = 1/n$ for $i = 1,..., m$ and $j = 1,..., n$, then the triangulation $\bar{\Delta}_{mn}$ is a crosscut grid partition. We shall list several important bivariate B-spline functions on this partition. We note, however, that with the exception of $B_{ij}(x, y)$ above, none of these bivariate B-splines can be transformed linearly to spline functions on nonuniform grid partitions $\bar{\Delta}_{mn}$. Bivariate B-splines have a wide range of applications in approximation, interpolation, numerical analysis, and finite element methods. We shall only discuss some integration quadratures that arise from the variation-diminishing spline series, and compare them with the classical product trapezoidal and product Simpson's formulas.

For nonuniform triangulated rectangles, since the grid lines determined by $0 = x_0 < \cdots < x_m = 1$ and $0 = y_0 < \cdots < y_n = 1$ are arbitrary, they can be moved appropriately to fit the (usually discrete) given data. In fact, adaptive schemes can be developed and the problems of approximation by bivariate C^1 cubic splines with variable grid partitions can be investigated by using the bivariate B-splines $B_{ij}(x, y)$ in $S_3^1(\bar{\Delta}_{mn})$ given in Section 3. The study of these problems will be delayed to a later date.

2. Preliminary

We first introduce the necessary notation and discuss the basic properties of bivariate spline functions. It will be clear that the contents in this paper can be generalized to the multivariate setting.

Let $\mathscr{D}$ be a domain in $\mathbb{R}^2$ and Δ a grid partition of $\mathscr{D}$ consisting of algebraic curves (or segments of algebraic curves). Then Δ divides $\mathscr{D}$ into a finite or countable number of cells. The points of intersection of the grid curves are called grid-points (or vertices) and the segments of the curves separated by the grid-points are called grid-segments (or edges) of the partition Δ. Let $\mathbb{P}_k$ denote the collection of all polynomials in two real variables with total degree k over the real field. A function $s(x, y)$ in $C^\mu(\mathscr{D})$ is called a bivariate spline function with smoothness condition C^μ and total degree k on the grid partition Δ if the restriction of $s(x,y)$ to each cell of this partition is in $\mathbb{P}_k$. The space of all of these bivariate spline functions will be denoted by $S_k^\mu(\Delta)=S_k^\mu(\Delta, D)$. Clearly, if $\mu \geqslant k$, then $S_k^\mu(\Delta)=\mathbb{P}_k$. Hence, we always assume that $0 \leqslant \mu \leqslant k-1$.

Let D_i and D_j be two adjacent cells of Δ sharing a common grid-segment Γ_{ij} which lies on an algebraic curve $l_{ij}\ (x,y)=0$, where $l_{ij}(x,y)$ is an irreducible algebraic polynomial. If $p_i(x,y)$ and $p_j(x,y)$ in $\mathbb{P}_k$ are the restrictions of an $s(x,y)\in S_k^\mu(\Delta)$ on the cells D_i and D_j respectively, then by using an old result of Bezout, it can be proved [3, 13] that

$$p_j(x,y)-p_i(x,y)=Q_{ij}(x,y)[l_{ij}(x,y)]^{\mu+1} \tag{2.1}$$

for all (x,y), where $Q_{ij}(x,y)$ is a polynomial of total degree $k-\mu-1$. We shall call $Q_{ij}(x,y)$ the smoothing cofactor of $s(x,y)$ from D_i to D_j across Γ_{ij}. Note that since $l_{ij}(x,y)=l_{ji}(x,y)$, we always have $Q_{ij}(x,y)=-Q_{ij}(x,y)$.

Let A be a grid-point of Δ in $\mathscr{D}$ and $\Gamma_1,\ldots,\Gamma_N$ be the grid-segments with A as the common endpoint ordered in the counterclockwise direction, such that Γ_1 separates a cell D_2 from a cell D_1, Γ_2 separates a cell D_3 from $D_2,\ldots,$ and Γ_N separates the first cell D_1 from a cell D_N. Also, let $l_1(x,y),\ldots,l_N(x,y)$ be irreducible algebraic polynomials such that $\Gamma_1,\ldots,\Gamma_N$ lie on $l_1(x,y)=0,\ldots,$ $l_N(x,y)=0$, respectively. If $Q_{i,i+1}(x,y)\in\mathbb{P}_{k-\mu-1}$ is the smoothing cofactor of a bivariate spline function $s(x,y)\in S_k^\mu(\Delta)$ from D_i to D_{i+1} across Γ_i, where $Q_{N,N+1}(x,y):=Q_{N,1}(x,y)$, then we have

$$\sum_{i=1}^{N} Q_{i,i+1}(x,y)[l_i(x,y)]^{\mu+1}=0 \tag{2.2}$$

for all (x,y) by using (2.1). This identity is called the conformality condition of $s(x,y)$ at the grid-point A (cf. [3, 13]). Hence, every bivariate spline function in $S_k^\mu(\Delta)$ must satisfy the conformality conditions at all grid-points of Δ. The conformality conditions of bivariate spline functions are also useful in studying the general properties of $S_k^\mu(\Delta)$ and in constructing functions in $S_k^\mu(\Delta)$ satisfying certain conditions. One important condition is the local support property. We shall therefore utilize the conformality conditions to construct bivariate B-splines.

3. Triangulation with Two Diagonals

Let $R=\{(x,y): 0\leqslant x,y\leqslant 1\}$, $0=x_0<\cdots<x_m=1$ and $0=y_0<\cdots<y_n=1$. Hence, the lines $x-x_i=0$ and $y-y_j=0$, $1\leqslant i\leqslant m-1$ and $0\leqslant j\leqslant n-1$, divide R into mn rectangles which will be denoted by

$$R_{ij}=\{(x,y): x_{i-1}\leqslant x\leqslant x_i, y_{j-1}\leqslant y\leqslant y_j\}.$$

In this section, each of these rectangles is divided into four triangles by adding its two diagonals. The four triangular subregions of R_{ij} which we denote by $D_1(i,j)$, $D_2(i,j)$, $D_3(i,j)$, and $D_4(i,j)$ are ordered in the counterclockwise direction such that the vertices of $D_1(i,j)$ are (x_{i-1}, y_{j-1}), (x_i, y_{j-1}), and $((x_{i-1}+x_i)/2, (y_{j-1}+y_j)/2)$. Each grid-point (x_i, y_j), $0 \leqslant i \leqslant m$ and $0 \leqslant j \leqslant n$, is the common vertex of four rectangles R_{ij}, $R_{i+1,j}$, $R_{i+1,j+1}$, $R_{i,j+1}$ where additional rectangles are arbitrarily attached to R by introducing x_{-1}, x_{m+1}, y_{-1} and y_{n+1}, where x_{-1}, $y_{-1} < 0$ and x_{m+1}, $y_{n+1} > 1$. Let

$$T_{ij} = R_{ij} \cup R_{i+1,j} \cup R_{i+1,j+1} \cup R_{i,j+1}.$$

We shall obtain a bivariate C^1 cubic B-spline $B_{ij}(x,y)$ supported on T_{ij}. That is, $B_{ij}(x,y)$ vanishes outside T_{ij} and is made up of sixteen pieces of cubic polynomials which are positive on the corresponding triangular cells $D_1(i,j)$,..., $D_4(i,j)$, $D_1(i+1,j)$,..., $D_4(i+1,j)$,..., $D_1(i,j+1)$,..., $D_4(i,j+1)$. These polynomials will be denoted by $p_{1;i,j}(x,y)$,..., $p_{4;i,j}(x,y)$, $p_{1;i+1,j}(x,y)$,..., $p_{4;i+1,j}(x,y)$,..., $p_{1;i,j+1}(x,y)$,..., $p_{4;i,j+1}(x,y)$.

We first start with $p_{1;i+1,j+1}(x,y) := P_1(x,y)$, which is determined by the following ten interpolation conditions: $P_1(x_i, y_j) = 1$, $P_1(x_{i+1}, y_j) = 0$, $P_1((x_i + x_{i+1})/2, (y_j + y_{j+1})/2) = h$, $P((x_i + x_{i+1})/2, (5y_j + y_{j+1})/6) = t$, $\partial P_1/x = 0$ at (x_i, y_j) and (x_{i+1}, y_j), the directional derivative of P_1 along the line $(x_{i+1} - x_i)(y - y_j) - (y_{j+1} - y_j)(x - x_i) = 0$ at the point $((x_i + x_{i+1})/2, (y_j + y_{j+1})/2)$ toward (x_i, y_j) is v and that along the line $(x_{i+1} - x_i)(y - y_j) + (y_{j+1} - y_j)(x - x_i) = 0$ at the point $((x_i + x_{i+1})/2, (y_j + y_{j+1})/2)$ toward (x_{i+1}, y_j) is u. Here, h, t, u, and v are parameters to be determined by the conditions on smoothness and "symmetry" of $B_{ij}(x,y)$. The polynomial $p_{3;i+1,j}(x,y)$ which is the restriction of $B_{ij}(x,y)$ on $D_3(i+1,j)$ that shares a common grid-segment $y - y_j = 0$ with $D_1(i+1,j+1)$ is obtained by "symmetry," namely,

$$p_{3;i+1,j}(x, (y_j - y_{j-1})\, y) = p_{1;i+1,j+1}(x, -(y_{j+1} - y_j)\, y).$$

By (2.1), we see that $p_{3;i+1,j}(x,y) - p_{1;i+1,j+1}(x,y)$ is divisible by $(y - y_j)^2$. This condition gives one restriction on the parameters. Similarly, we define $p_{4;i+1,j+1}(x,y)$ from $p_{1;i+1,j+1}(x,y)$ by "symmetry" with respect to the line $(x_{i+1} - x_i)(y - y_j) - (y_{j+1} - y_j)(x - x_i) = 0$. The C^1 condition now gives two restrictions on the parameters. Next, we use conformality condition (2.2) of $B_{ij}(x,y)$ at the points $((x_i + x_{i+1})/2, (y_j + y_{j+1})/2)$ and (x_{i+1}, y_{j+1}) simultaneously, using the fact that $B_{ij}(x,y) = 0$ outside T_{ij} to obtain the other restrictions on the parameters that define $p_{1;i+1,j+1}(x,y)$. Hence, $p_{1;i+1,j+1}(x,y)$ is uniquely determined. It turns out that if we define the other polynomial pieces by the same type of "symmetry" as above, we obtain a (unique) C^1 bivariate cubic spline function $B_{ij}(x, y)$. In fact, by writing down the relationships (2.1) across each grid-segment, our B-spline $B_{ij}(x, y)$ is in $C^2(T_{ij})$ if and only if $y_{j+1} - y_j = y_j - y_{j-1}$ and $x_{i+1} - x_i = x_i - x_{i-1}$. We have

THEOREM 3.1. *There exists a bivariate B-spline $B_{ij}(x, y)$ in $S_3^1(\bar{\Delta}_{mn})$ supported on T_{ij} such that its restrictions on the cells $D_1(i+1, j+1), \ldots, D_4(i+1, j+1), D_1(i, j+1), \ldots, D_4(i, j+1), \ldots, D_1(i+1, j), \ldots, D_4(i+1, j)$ are given by the following corresponding polynomial pieces:*

$$\begin{aligned} p_{1;i+1,j+1}(x, y) &= \left[1 - 3\left(\frac{x - x_i}{x_{i+1} - x_i}\right)^2 + 2\left(\frac{x - x_i}{x_{i+1} - x_i}\right)^3\right] \\ &\quad + \left[-3 + 3\left(\frac{x - x_i}{x_{i+1} - x_i}\right)\right]\left(\frac{y - y_j}{y_{j+1} - y_j}\right)^2 + \left(\frac{y - y_j}{y_{j+1} - y_j}\right)^3, \end{aligned}$$

$$\begin{aligned} p_{2;i+1,j+1}(x, y) &= \left[2 - 3\left(\frac{x - x_i}{x_{i+1} - x_i}\right) + \left(\frac{x - x_i}{x_{i+1} - x_i}\right)^3\right] \\ &\quad + \left[-3 + 6\left(\frac{x - x_i}{x_{i+1} - x_i}\right) - 3\left(\frac{x - x_i}{x_{i+1} - x_i}\right)^2\right]\left(\frac{y - y_j}{y_{j+1} - y_j}\right), \end{aligned}$$

$$\begin{aligned} p_{3;i+1,j+1}(x, y) &= \left[2 - 3\left(\frac{x - x_i}{x_{i+1} - x_i}\right)\right] + \left[-3 + 6\left(\frac{x - x_i}{x_{i+1} - x_i}\right)\right]\left(\frac{y - y_j}{y_{j+1} - y_j}\right) \\ &\quad - 3\left(\frac{x - x_i}{x_{i+1} - x_i}\right)\left(\frac{y - y_j}{y_{j+1} - y_j}\right)^2 + \left(\frac{y - y_j}{y_{j+1} - y_j}\right)^3, \end{aligned}$$

$$\begin{aligned} p_{4;i+1,j+1}(x, y) &= \left[1 - 3\left(\frac{x - x_i}{x_{i+1} - x_i}\right)^2 + \left(\frac{x - x_i}{x_{i+1} - x_i}\right)^3\right] \\ &\quad + 3\left(\frac{x - x_i}{x_{i+1} - x_i}\right)^2\left(\frac{y - y_j}{y_{j+1} - y_j}\right) - 3\left(\frac{y - y_j}{y_{j+1} - y_j}\right)^2 + 2\left(\frac{y - y_j}{y_{j+1} - y_j}\right)^3, \end{aligned}$$

$$p_{1;i,j+1}(x, y) = \left[1 - 3\left(\frac{x - x_i}{x_i - x_{i-1}}\right)^2 - 2\left(\frac{x - x_i}{x_i - x_{i-1}}\right)^3\right] + \left[-3 - 3\left(\frac{x - x_i}{x_i - x_{i-1}}\right)\right]\left(\frac{y - y_j}{y_{j+1} - y_j}\right)^2 + \left(\frac{y - y_j}{y_{j+1} - y_j}\right)^3,$$

$$p_{2;i,j+1}(x, y) = \left[1 - 3\left(\frac{x - x_i}{x_i - x_{i-1}}\right)^2 - \left(\frac{x - x_i}{x_i - x_{i-1}}\right)^3\right] + 3\left(\frac{x - x_i}{x_i - x_{i-1}}\right)^2\left(\frac{y - y_j}{y_{j+1} - y_j}\right) - 3\left(\frac{y - y_j}{y_{j+1} - y_j}\right)^2 + 2\left(\frac{y - y_j}{y_{j+1} - y_j}\right)^3,$$

$$p_{3;i,j+1}(x, y) = \left[2 + 3\left(\frac{x - x_i}{x_i - x_{i-1}}\right)\right] + \left[-3 - 6\left(\frac{x - x_i}{x_i - x_{i-1}}\right)\right]\left(\frac{y - y_j}{y_{j+1} - y_j}\right) + 3\left(\frac{x - x_i}{x_i - x_{i-1}}\right)\left(\frac{y - y_j}{y_{j+1} - y_j}\right)^2 + \left(\frac{y - y_j}{y_{j+1} - y_j}\right)^3,$$

$$p_{4;i,j+1}(x, y) = \left[2 + 3\left(\frac{x - x_i}{x_i - x_{i-1}}\right) - \left(\frac{x - x_i}{x_i - x_{i-1}}\right)^3\right] + \left[-3 - 6\left(\frac{x - x_i}{x_i - x_{i-1}}\right) - 3\left(\frac{x - x_i}{x_i - x_{i-1}}\right)^2\right]\left(\frac{y - y_j}{y_{j+1} - y_j}\right),$$

$$p_{1;i,j}(x, y) = \left[2 + 3\left(\frac{x - x_i}{x_i - x_{i-1}}\right)\right] + \left[3 + 6\left(\frac{x - x_i}{x_i - x_{i-1}}\right)\right]\left(\frac{y - y_j}{y_j - y_{j-1}}\right) + 3\left(\frac{x - x_i}{x_i - x_{i-1}}\right)\left(\frac{y - y_j}{y_j - y_{j-1}}\right)^2 - \left(\frac{y - y_j}{y_j - y_{j-1}}\right)^3,$$

$$p_{2;i,j}(x, y) = \left[1 - 3\left(\frac{x - x_i}{x_i - x_{i-1}}\right)^2 - \left(\frac{x - x_i}{x_i - x_{i-1}}\right)^3\right] - 3\left(\frac{x - x_i}{x_i - x_{i-1}}\right)^2\left(\frac{y - y_j}{y_j - y_{j-1}}\right) - 3\left(\frac{y - y_j}{y_j - y_{j-1}}\right)^2 - 2\left(\frac{y - y_j}{y_j - y_{j-1}}\right)^3,$$

$$
\begin{aligned}
p_{3;i,j}&(x, y)\\
&= \left[1 - 3\left(\frac{x-x_i}{x_i-x_{i-1}}\right)^2 - 2\left(\frac{x-x_i}{x_i-x_{i-1}}\right)^3\right]\\
&\quad + \left[-3 - 3\left(\frac{x-x_i}{x_i-x_{i-1}}\right)\right]\left(\frac{y-y_j}{y_j-y_{j-1}}\right)^2 - \left(\frac{y-y_j}{y_j-y_{j-1}}\right)^3,
\end{aligned}
$$

$$
\begin{aligned}
p_{4;i,j}&(x, y)\\
&= \left[2 + 3\left(\frac{x-x_i}{x_i-x_{i-1}}\right) - \left(\frac{x-x_i}{x_i-x_{i-1}}\right)^3\right]\\
&\quad + \left[3 + 6\left(\frac{x-x_i}{x_i-x_{i-1}}\right) + 3\left(\frac{x-x_i}{x_i-x_{i-1}}\right)^2\right]\left(\frac{y-y_j}{y_j-y_{j-1}}\right),
\end{aligned}
$$

$$
\begin{aligned}
p_{1;i+1,j}&(x, y)\\
&= \left[2 - 3\left(\frac{x-x_i}{x_{i+1}-x_i}\right)\right] + \left[3 - 6\left(\frac{x-x_i}{x_{i+1}-x_i}\right)\right]\left(\frac{y-y_j}{y_j-y_{j-1}}\right)\\
&\quad - 3\left(\frac{x-x_i}{x_{i+1}-x_i}\right)\left(\frac{y-y_j}{y_j-y_{j-1}}\right)^2 - \left(\frac{y-y_j}{y_j-y_{j-1}}\right)^3,
\end{aligned}
$$

$$
\begin{aligned}
p_{2;i+1,j}&(x, y)\\
&= \left[2 - 3\left(\frac{x-x_i}{x_{i+1}-x_i}\right) + \left(\frac{x-x_i}{x_{i+1}-x_i}\right)^3\right]\\
&\quad + \left[3 - 6\left(\frac{x-x_i}{x_{i+1}-x_i}\right) + 3\left(\frac{x-x_i}{x_{i+1}-x_i}\right)^2\right]\left(\frac{y-y_j}{y_j-y_{j-1}}\right),
\end{aligned}
$$

$$
\begin{aligned}
p_{3;i+1,j}&(x, y)\\
&= \left[1 - 3\left(\frac{x-x_i}{x_{i+1}-x_i}\right)^2 + 2\left(\frac{x-x_i}{x_{i+1}-x_i}\right)^3\right]\\
&\quad + \left[-3 + 3\left(\frac{x-x_i}{x_{i+1}-x_i}\right)\right]\left(\frac{y-y_j}{y_j-y_{j-1}}\right)^2 - \left(\frac{y-y_j}{y_j-y_{j-1}}\right)^3,
\end{aligned}
$$

$$
\begin{aligned}
p_{4;i+1,j}&(x, y)\\
&= \left[1 - 3\left(\frac{x-x_i}{x_{i+1}-x_i}\right)^2 + \left(\frac{x-x_i}{x_{i+1}-x_i}\right)^3\right] - 3\left(\frac{x-x_i}{x_{i+1}-x_i}\right)^2\left(\frac{y-y_j}{y_j-y_{j-1}}\right)\\
&\quad - 3\left(\frac{y-y_j}{y_j-y_{j-1}}\right)^2 - 2\left(\frac{y-y_j}{y_j-y_{j-1}}\right)^3.
\end{aligned}
$$

To verify the above result, we can use the smoothing cofactors. We only list some of them since the rest can be obtained from these by "symmetry:"

$$Q_{(1;i+1,j+1),(2;i+1,j+1)}(x,y)$$
$$= 1 - \left(\frac{x-x_i}{x_{i+1}-x_i}\right) - \left(\frac{y-y_j}{y_{j+1}-y_j}\right),$$

$$Q_{(2;i+1,j+1),(3;i+1,j+1)}(x,y)$$
$$= -\left(\frac{x-x_i}{x_{i+1}-x_i}\right) + \left(\frac{y-y_j}{y_{j+1}-y_j}\right),$$

$$Q_{(3;i+1,j+1),(4;i+1,j+1)}(x,y)$$
$$= -Q_{(1;i+1,j+1),(2;i+1,j+1)}(x,y)$$

$$Q_{(1;i+1,j+1),(4;i+1,j+1)}(x,y)$$
$$= Q_{(2;i+1,j+1),(3;i+1,j+1)}(x,y),$$

$$Q_{(3;i+1,j),(1;i+1,j+1)}(x,y)$$
$$= \frac{3(y_{j+1}-y_{j-1})(y_{j+1}-2y_j+y_{j-1})}{(y_j-y_{j-1})^2}\left[1 - \left(\frac{x-x_i}{x_{i+1}-x_i}\right)\right]$$
$$+ \left[1 + \left(\frac{y_{j+1}-y_j}{y_j-y_{j-1}}\right)^3\right]\left(\frac{y-y_j}{y_{j+1}-y_j}\right),$$

$$Q_{(4;i+1,j+1),(2;i,j+1)}(x,y)$$
$$= \frac{3(x_{i+1}-x_{i-1})(x_{i+1}-2x_i+x_{i-1})}{(x_i-x_{i-1})^2}\left[-1 + \left(\frac{y-y_j}{y_{j+1}-y_j}\right)\right]$$
$$- \left[1 + \left(\frac{x_{i+1}-x_i}{x_i-x_{i-1}}\right)^3\right]\left(\frac{x-x_i}{x_{i+1}-x_i}\right),$$

and the smoothing cofactors from $D_2(i+1, j+1)$ and $D_3(i+1, j+1)$ to $\mathbb{R}^2 \backslash T_{ij}$ across the corresponding grid-segments are, respectively,

$$-2 - \left(\frac{x-x_i}{x_{i+1}-x_i}\right) + 3\left(\frac{y-y_j}{y_{j+1}-y_j}\right), \quad -2 + 3\left(\frac{x-x_i}{x_{i+1}-x_i}\right) - \left(\frac{y-y_j}{y_{j+1}-y_j}\right).$$

From these smoothing cofactors, it is clear that $B_{ij}(x,y)$ is in $C^1(\mathbb{R}^2)$ and is in $C^2(T_{ij})$ if and only if $x_{i+1}-x_i = x_i - x_{i-1}$ and $y_{j+1}-y_j = y_j - y_{j-1}$.

We also have a partition of unity:

THEOREM 3.2. *For all* $(x, y) \in R$,

$$\sum_{j=0}^{n} \sum_{i=0}^{m} B_{ij}(x, y) = 1.$$

Proof. To prove this result, we first observe that the values of $B_{ij}(x, y)$ at the vertices of the triangular cells are 0, $\frac{1}{4}$, and 1, and the values at the midpoints of the "interior" and "exterior" triangular cells in T_{ij} are $\frac{25}{54}$ and $\frac{2}{54}$ respectively. Also, the directional derivatives at $((x_i + x_{i+1})/2, (y_j + y_{j+1})/2)$, $((x_{i-1} + x_i)/2, (y_j + y_{j+1})/2)$, $((x_{i-1} + x_i)/2, (y_{j-1} + y_j)/2)$, $((x_i + x_{i+1})/2, (y_{j-1} + y_j)/2)$ toward (x_i, y_j) are, respectively,

$$\begin{aligned}
&\tfrac{3}{2}((x_{i+1} - x_i)^2 + (y_{j+1} - y_j)^2)^{-1/2},\\
&\tfrac{3}{2}((x_i - x_{i-1})^2 + (y_{j+1} - y_j)^2)^{-1/2},\\
&\tfrac{3}{2}((x_i - x_{i-1})^2 + (y_j - y_{j-1})^2)^{-1/2},\\
&\tfrac{3}{2}((x_{i+1} - x_i)^2 + (y_j - y_{j-1})^2)^{-1/2},
\end{aligned}$$

and the directional derivaties at these points toward the points (x_{i+1}, y_j), (x_i, y_{j+1}), (x_{i-1}, y_j), and (x_i, y_{j-1}), respectively, are all zero. Hence, the piecewise polynomial $\sum_{i,j} B_{ij}(x, y)$ is equal to 1 in each of the cells $D_l(i, j)$, $l = 1, \ldots, 4$, $1 \leqslant i \leqslant m$, and $1 \leqslant j \leqslant n$. This completes the proof of the theorem.

As an immediate consequence, we note that the Lagrange interpolants of continuous functions at the points (x_i, y_j) from $S_3^1(\bar{\Delta}_{mn})$ by using the B-splines $B_{ij}(x, y)$ also approximate. Let δ_{ij} be the maximum of the four numbers

$$\sqrt{(x_{i+1} - x_i)^2 + (y_{j+1} - y_j)^2}, \qquad \sqrt{(x_{i+1} - x_i)^2 + (y_j - y_{j-1})^2},$$
$$\sqrt{(x_i - x_{i-1})^2 + (y_{j+1} - y_j)^2}, \qquad \text{and} \qquad \sqrt{(x_i - x_{i-1})^2 + (y_j - y_{j-1})^2},$$

and let $\delta = \delta(m, n) = \max\{\delta_{ij} : 1 \leqslant i \leqslant m-1 \text{ and } 1 \leqslant j \leqslant n-1\}$. For each $f \in C(R)$, let

$$(Lf)(x, y) = \sum_{j=0}^{n} \sum_{i=0}^{m} f(x_i, y_j)\, B_{ij}(x, y).$$

We have

COROLLARY 3.1. *For every* $f \in C(R)$, $(Lf)(x_i, y_j) = f(x_i, y_j)$ *where* $0 \leqslant i \leqslant m$, $0 \leqslant j \leqslant n$, *and*

$$\| f - Lf \|_R \leqslant \omega(f, \delta).$$

4. B-SPLINES ON UNIFORM TRIANGULATED RECTANGLES

We now consider the special case where the horizontal and vertical grid lines are equally spaced; that is, $x_i - x_{i-1} = 1/m$ and $y_j - y_{j-1} = 1/n$, $i = 1,\ldots, m$ and $j = 1,\ldots, n$. Hence, the partitions $\bar{\Delta}_{mn}$ become crosscut partitions of R. By a simple linear transformation, the grid partitions (of the supports) of the bivariate B-splines will be assumed to be as in Figs. 1, 2, or 3 below. We shall give three bivariate C^1 cubic splines with different supports. The first one is a special case of $B_{ij}(x, y)$ in the previous section, the second will be on the same grid partition but has different support, while the third one was obtained in [4].

Let Δ_1 be the grid partition given in Fig. 1, where the vertices of the square Q_1 are $(1, 1)$, $(-1, 1)$, $(-1, -1)$, and $(1, -1)$, and the sixteen cells inside Q_1 are denoted by $1,\ldots, 16$. Our B-spline function $B(x, y)$ will be in $C^1(\mathbb{R}^2)$ and vanishes outside Q_1. Let $p_i(x, y)$ denote the restriction of $B(x, y)$ on the cell i. Then we have the following expressions for $p_i(x, y)$:

$$
\begin{aligned}
&p_1(x, y) = (1 - 3x^2 + 2x^3) + (-3 + 3x)y^2 + y^3,\\
&p_2(x, y) = (1 - 3x^2 + x^3) + 3x^2y - 3y^2 + 2y^3,\\
&p_3(x, y) = p_2(-x, y), \qquad p_4(x, y) = p_1(-x, y),\\
&p_5(x, y) = p_1(-x, -y), \qquad p_6(x, y) = p_2(-x, -y),\\
&p_7(x, y) = p_2(x, -y), \qquad p_8(x, y) = p_1(x, -y),\\
&p_9(x, y) = (2 - 3x + x^3) + (-3 + 6x - 3x^2)y\\
&p_{10}(x, y) = (2 - 3x) + (-3 + 6x)y - 3xy^2 + y^3,\\
&p_{11}(x, y) = p_{10}(-x, y), \qquad p_{12}(x, y) = p_9(-x, y),\\
&p_{13}(x, y) = p_9(-x, -y), \qquad p_{14}(x, y) = p_{10}(-x, -y),\\
&p_{15}(x, y) = p_{10}(x, -y), \qquad p_{16}(x, y) = p_9(x, -y).
\end{aligned}
$$

The second B-spline function $C(x, y)$ will be on the grid partition Δ_2 given in Fig. 2, where the vertices of the square Q_2 are $(3, 3)$, $(-3, 3)$, $(-3, -3)$, and $(3, -3)$, with center at the origin. The thirty-six cells inside Q_2 will again be denoted by $1,\ldots, 36$. Our B-spline function $C(x, y)$ is in $C^1(\mathbb{R}^2)$ and vanishes outside Q_2, and its restriction on cell i will be given by $q_i(x, y)$ below. This B-spline function is constructed by using the techniques given in Section 2.

$$
\begin{aligned}
&q_1(x, y) = (\tfrac{5}{12} - \tfrac{1}{8}x^2 + \tfrac{1}{48}x^3) + (-\tfrac{1}{8} + \tfrac{1}{16}x)y^2,\\
&q_2(x, y) = q_1(y, x), \qquad q_3(x, y) = q_1(-x, y), \qquad q_4(x, y) = q_1(-y, x),
\end{aligned}
$$

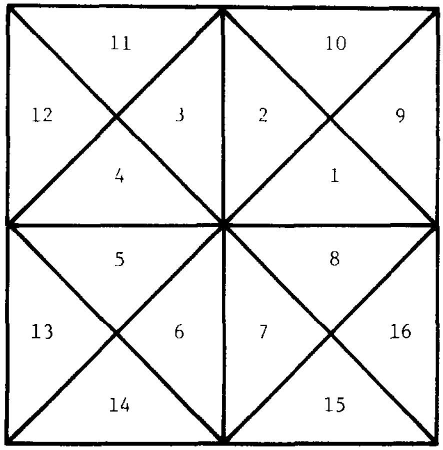

FIG. 1. The square Q_1.

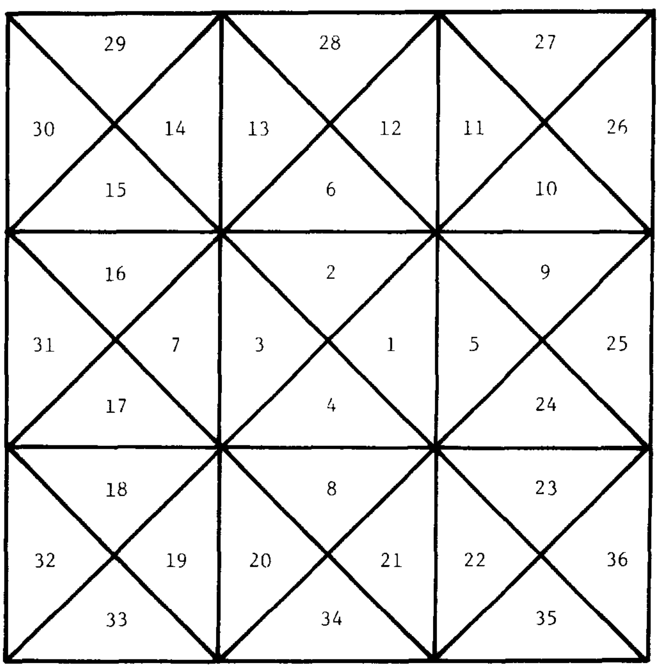

FIG. 2. The square Q_2.

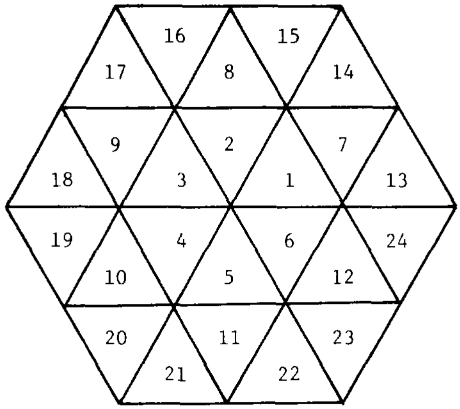

FIG. 3. The hexagon Q_3.

$$q_5(x,y) = (\tfrac{19}{48} + \tfrac{1}{16}x - \tfrac{3}{16}x^2 + \tfrac{1}{24}x^3) + (-\tfrac{1}{8} + \tfrac{1}{16}x)\,y^2,$$

$$q_6(x,y) = q_5(y,x), \qquad q_7(x,y) = q_5(-x,y), \qquad q_8(x,y) = q_5(-y,x),$$

$$q_9(x,y) = (\tfrac{23}{48} - \tfrac{1}{16}x - \tfrac{1}{8}x^2 + \tfrac{1}{32}x^3) + (-\tfrac{1}{8} + \tfrac{1}{8}x - \tfrac{1}{32}x^2)\,y$$
$$+ (-\tfrac{1}{16} + \tfrac{1}{32}x)\,y^2 + \tfrac{1}{96}\,y^3,$$

$$q_{10}(x,y) = (\tfrac{11}{24} - \tfrac{1}{16}x - \tfrac{1}{8}x^2 + \tfrac{1}{32}x^3) + (-\tfrac{1}{16} + \tfrac{1}{8}x - \tfrac{1}{32}x^2)\,y$$
$$+ (-\tfrac{1}{8} + \tfrac{1}{32}x)\,y^2 + \tfrac{1}{96}\,y^3,$$

$$q_{11}(x,y) = q_{10}(y,x), \qquad q_{12}(x,y) = q_9(y,x),$$

$$q_{13}(x,y) = q_9(y,-x), \qquad q_{14}(x,y) = q_{10}(y,-x),$$

$$q_{15}(x,y) = q_{10}(-x,y), \qquad q_{16}(x,y) = q_9(-x,y),$$

$$q_{17}(x,y) = q_9(-x,-y), \qquad q_{18}(x,y) = q_{10}(-x,-y)$$

$$q_{19}(x,y) = q_{10}(-y,-x), \qquad q_{20}(x,y) = q_9(-y,-x),$$

$$q_{21}(x,y) = q_9(-y,x), \qquad q_{22}(x,y) = q_{10}(-y,x),$$

$$q_{23}(x,y) = q_{10}(x,-y), \qquad q_{24}(x,y) = q_9(x,-y),$$

$$q_{25}(x,y) = \tfrac{9}{16} - \tfrac{3}{16}x - \tfrac{1}{16}x^2 + \tfrac{1}{48}x^3,$$

$$q_{26}(x,y) = (\tfrac{9}{8} - \tfrac{9}{16}x + \tfrac{1}{48}x^3) + (-\tfrac{9}{16} + \tfrac{3}{8}x - \tfrac{1}{16}x^2)\,y,$$

$$q_{27}(x,y) = q_{26}(y,x), \qquad q_{28}(x,y) = q_{25}(y,x),$$

$$q_{29}(x,y) = q_{26}(y,-x), \qquad q_{30}(x,y) = q_{26}(-x,y),$$

$$q_{31}(x,y)=q_{25}(-x,y), \qquad q_{32}(x,y)=q_{26}(-x,-y),$$
$$q_{33}(x,y)=q_{26}(-y,-x), \qquad q_{34}(x,y)=q_{25}(-y,x),$$
$$q_{35}(x,y)=q_{26}(-y,x), \qquad q_{36}(x,y)=q_{26}(x,-y).$$

To describe the third bivariate B-spline, it is more convenient to consider a different grid partition Δ_3 as in Fig. 3, where the regular hexagon Q_3 is centered at the origin and has vertices at $(4,0)$, $(2,2\sqrt{3})$, $(-2, 2\sqrt{3})$, $(-4,0)$, $(-2,-2\sqrt{3})$, and $(2,-2\sqrt{3})$. The twenty-four regular triangular cells inside Q_3 are denoted by 1,..., 24. Our bivariate cubic B-spline function $D(x,y)$ is in $C^1(\mathbb{R}^2)$, vanishes outside Q_3, and its restriction on cell i is $r_i(x,y)$. This B-spline is contained in [4], where we also discuss its approximation properties.

$$r_1(x,y)=(\tfrac{1}{3}-\tfrac{1}{12}x^2+\tfrac{1}{72}x^3)+(-\tfrac{1}{12}+\tfrac{1}{72}x)y^2+\tfrac{1}{162}\sqrt{3}\,y^3,$$
$$r_2(x,y)=(\tfrac{1}{3}-\tfrac{1}{12}x^2)+\tfrac{1}{72}\sqrt{3}\,x^2y-\tfrac{1}{12}y^2+\tfrac{5}{648}\sqrt{3}\,y^3,$$
$$r_3(x,y)=r_1(-x,y), \qquad r_4(x,y)=r_1(-x,-y),$$
$$r_5(x,y)=r_2(x,-y), \qquad r_6(x,y)=r_1(x,-y)$$
$$r_7(x,y)=(\tfrac{5}{9}-\tfrac{1}{4}x+\tfrac{1}{144}x^3)+(-\tfrac{1}{12}\sqrt{3}+\tfrac{1}{18}\sqrt{3}x-\tfrac{1}{144}\sqrt{3}\,x^2)y$$
$$+(-\tfrac{1}{18}+\tfrac{1}{144}x)y^2+\tfrac{7}{1296}\sqrt{3}\,y^3,$$
$$r_8(x,y)=(\tfrac{5}{9}-\tfrac{1}{12}x^2)+(-\tfrac{1}{6}\sqrt{3}+\tfrac{1}{72}\sqrt{3}\,x^2)y+\tfrac{1}{36}y^2+\tfrac{1}{648}\sqrt{3}\,y^3,$$
$$r_9(x,y)=r_7(-x,y), \qquad r_{10}(x,y)=r_7(-x,-y),$$
$$r_{11}(x,y)=r_8(x,-y), \qquad r_{12}(x,y)=r_7(x,-y),$$
$$r_{13}(x,y)=(\tfrac{8}{9}-\tfrac{2}{3}x+\frac{1}{6}x^2-\tfrac{1}{72}x^3)+(-\tfrac{1}{18}+\tfrac{1}{72}x)y^2$$
$$+\tfrac{1}{324}\sqrt{3}\,y^3,$$
$$r_{14}(x,y)=(\tfrac{8}{9}-\tfrac{1}{3}x+\tfrac{1}{144}x^3)+(-\tfrac{1}{3}\sqrt{3}+\tfrac{1}{9}\sqrt{3}\,x-\tfrac{1}{144}\sqrt{3}\,x^2)y$$
$$+(\tfrac{1}{9}-\tfrac{1}{48}x)y^2-\tfrac{5}{1296}\sqrt{3}\,y^3,$$
$$r_{15}(x,y)=(\tfrac{8}{9}-\tfrac{1}{3}x)+(-\tfrac{1}{3}\sqrt{3}+\tfrac{1}{9}\sqrt{3}\,x)y+(\tfrac{1}{9}-\tfrac{1}{36}x)y^2$$
$$-\tfrac{1}{324}\sqrt{3}\,y^3,$$
$$r_{16}(x,y)=r_{15}(-x,y), \qquad r_{17}(x,y)=r_{14}(-x,y),$$
$$r_{18}(x,y)=r_{13}(-x,y), \qquad r_{19}(x,y)=r_{13}(-x,-y),$$
$$r_{20}(x,y)=r_{14}(-x,-y), \qquad r_{21}(x,y)=r_{15}(-x,-y),$$
$$r_{22}(x,y)=r_{15}(x,-y), \qquad r_{23}(x,y)=r_{14}(x,-y),$$
$$r_{24}(x,y)=r_{13}(x,-y).$$

In [4], it is observed that the transformation

$$\begin{aligned} a_1(x-x_0)+b_1(y-y_0) &= -\tfrac{1}{2}\eta_1 x' - \tfrac{1}{6}\sqrt{3}\,\eta_1 y', \\ a_2(x-x_0)+b_2(y-y_0) &= -\tfrac{1}{2}\eta_2 x' + \tfrac{1}{6}\sqrt{3}\,\eta_2 y' \end{aligned} \tag{4.1}$$

maps the grid partition

$$\sqrt{3}x' + y' - 2\sqrt{3}j = 0,\ \sqrt{3}x' - y' - 2\sqrt{3}j = 0,\ y' + \sqrt{3}j = 0, \tag{4.2}$$

$-\infty < j < \infty$, onto the grid partition

$$\begin{aligned} a_1(x-x_0)+b_1(y-y_0)+j\eta_1 &= 0 \\ a_2(x-x_0)+b_2(y-y_0)+j\eta_2 &= 0 \\ a_3(x-x_0)+b_3(y-y_0)+j\eta_3 &= 0, \end{aligned} \tag{4.3}$$

$-\infty < j < \infty$, for all pairwise linearly independent ordered pairs (a_1, b_1), (a_2, b_2), (a_3, b_3) and all η_1, η_2, η_3 satisfying

$$(a_2 b_3 - a_3 b_2)\,\eta_1 = (a_1 b_3 - a_3 b_1)\,\eta_2 = (a_1 b_2 - a_2 b_1)\,\eta_3.$$

Hence, the bivariate B-spline $D(x, y)$ can be linearly transformed to any crosscut triangulation of the first kind. Two important ones are triangulated rectangles with only one diagonal for each rectangular cell. The supports of such transformations of $D(x, y)$ are given in Figs. 4 and 5.

In addition, B-spline series can be obtained from the B-splines $B(x, y)$, $C(x, y)$, and $D(x, y)$. Let $B_{ij}(x, y) = B(x-i,\ y-j)$, $C_{ij}(x, y) = C(x-2i,\ y-2j)$, and $D_{ij}(x, y) = D(x-2i, y-\sqrt{3}j)$. Then we have

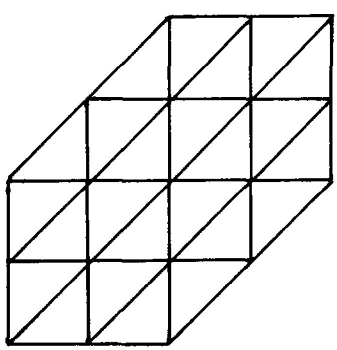

FIGURE 4

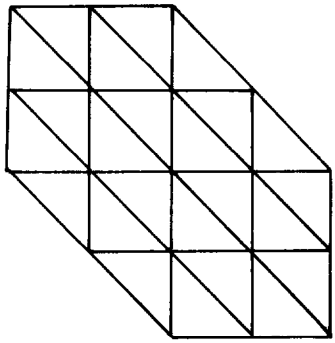

FIGURE 5

THEOREM 4.1. *For all* $(x, y) \in \mathbb{R}^2$,

$$\sum_{ij} B_{ij}(x, y) = \sum_{ij} C_{ij}(x, y) = \sum_{ij} D_{ij}(x, y) = 1.$$

From Theorem 3.2, we already have $\sum B_{ij} \equiv 1$. That the $\{D_{ij}(x, y)\}$ is also a partition of unity is contained in [4]. In [4], we also prove that the variation-diminishing bivariate spline operator corresponding to $\{D_{ij}\}$ preserves all linear polynomials. This is not true, however, for $\{B_{ij}\}$ and $\{C_{ij}\}$. To prove that $\sum C_{ij}(x, y) = 1$ for all (x, y), we first note that $C(0, 0) = \frac{5}{12}$, $C(2, 0) = \frac{5}{48}$, $C(2, 2) = \frac{1}{24}$, $C(1, 1) = C(1, -1) = \frac{1}{4}$, $C(\frac{2}{3}, 0) = \frac{119}{324}$, $C(\frac{4}{3}, 0) = \frac{317}{1296}$, $C(\frac{8}{3}, 0) = \frac{17}{1296}$, $C(2, \frac{2}{3}) = \frac{131}{1296}$, $C(2, \frac{4}{3}) = \frac{13}{162}$, and $C(\frac{8}{3}, 2) = \frac{1}{162}$. Hence, the values of $\sum C_{ij}(x, y)$ at the grid-points are $4(\frac{1}{24}) + 4(\frac{5}{48}) + \frac{5}{12} = 1$ and $4(\frac{1}{4}) = 1$, and its value at the center of each triangular cell is $2(\frac{13}{162}) + 2(\frac{131}{1296}) + \frac{317}{1296} + 2(\frac{1}{162}) + \frac{119}{324} + \frac{17}{1296} = 1$. Also, the values of the partial derivatives of $\sum C_{ij}(x, y)$ with respect to both x and y at the grid-points can be verified to be zero. That is, we have $\sum C_{ij}(x, y) = 1$ for all (x, y).

5. APPLICATION TO INTEGRATION QUADRATURE

Let $R = \{(x, y): 0 \leqslant x,\ y \leqslant 1\}$, $x_i = i/n$ and $y_j = j/n$, i, $j = 0, \ldots, n$. It is natural to use the variation-diminishing bivariate spline operators associated with the B-splines in Section 4 to obtain integration quadratures of the form:

$$\int_R \int f(x, y)\, dx\, dy \doteq \sum_{i=0}^{n} \sum_{j=0}^{n} a_{ij} f(x_i, y_j).$$

Since the only B-spline whose corresponding variation diminishing bivariate spline operator preserves all functions in $\mathbb{P}_1$ is $D(x,y)$, we expect that $D(x,y)$ gives the "best" integration quadrature. Indeed, the B-spline function $B(x,y)$ only gives the product trapezoidal formula, while we shall see that the integration quadrature derived from $D(x,y)$ is "better" than both the product trapezoidal and product Simpson formulas (cf. [12]) for functions with oscillations in the directions of $x+y=0$ or $x-y=0$.

The linear transformation that takes the grid partition in Fig. 3 onto the grid partition given in Fig. 5 with vertices at $(2/n, 0)$ $(2/n, 2/n)$, $(0, 2/n)$, $(-2/n, 0)$, $(-2/n)$, $-2/n)$, and $(0, -2/n)$ is

$$x=\frac{1}{2n}x'+\frac{\sqrt{3}}{6n}y', \qquad y=\frac{\sqrt{3}}{3n}y'$$

With this transformation and the bivariate cubic B-spline $D(x,y)$, we obtain the integration quadrature

$$\int_R\int f(x,y)\,dx\,dy \doteq I_n(f), \tag{5.1}$$

$$\begin{aligned}
I_n(f)=\frac{1}{360n^2}\Big\{&66[f(0,1)+f(1,0)]+170[f(0,0)+f(1,1)]\\
&+177\left[f\left(0,\frac{n-1}{n}\right)+f\left(\frac{1}{n},1\right)+f\left(\frac{n-1}{n},0\right)+f\left(1,\frac{1}{n}\right)\right]\\
&+202\left[f\left(0,\frac{1}{n}\right)+f\left(\frac{1}{n},0\right)+f\left(\frac{n-1}{n},1\right)+f\left(1,\frac{n-1}{n}\right)\right]\\
&+310\left[f\left(\frac{1}{n},\frac{n-1}{n}\right)+f\left(\frac{n-1}{n},\frac{1}{n}\right)\right]\\
&+316\left[f\left(\frac{1}{n},\frac{1}{n}\right)+f\left(\frac{n-1}{n},\frac{n-1}{n}\right)\right]\\
&+335\sum_{j=2}^{n-2}\left[f\left(\frac{1}{n},\frac{j}{n}\right)+f\left(\frac{n-1}{n},\frac{j}{n}\right)+f\left(\frac{j}{n},\frac{1}{n}\right)\right.\\
&\left.+f\left(\frac{j}{n},\frac{n-1}{n}\right)\right]\\
&+360\sum_{j=2}^{n-2}\sum_{i=2}^{n-2}f\left(\frac{i}{n},\frac{j}{n}\right)\Big\}.
\end{aligned}$$

We must remark, however, that when the variation-diminishing bivariate spline operator was applied, the values of $f(x,y)$ at $((i-1)/n, -1/n)$, $(i/n,$

$((n+1)/n)$, $(j/n, -1/n)$, and $(j/n, (n+1)/n)$, $i = 0, \ldots, n+1$ and $j = 0, \ldots, n$ were used. Since $f(x, y)$ is only defined on R, we replace these grid-points by the corresponding closest grid-points in R to obtain the formula $I(f)$. In doing so, it is easy to show that we still have

$$\int_R\int f(x, y)\, dx\, dy = I_n(f)$$

for all $f \in \mathbb{P}_1$. By using the results in [4], it is also easy to show that $I_n(f)$ converges to $\iint f$ for all $f \in C(R)$, and the rates are $o(1/n)$ for $f \in C^1(R)$ and $O(1/n^2)$ for $f \in C^2(R)$. More important is that the integration quadrature is "better" than the product integration formulas for functions f with oscillations in the direction of $x - y = 0$, even if f are not in $C^1(R)$. We give the following example: Let

$$\begin{aligned} f(x, y) &= N(y - 1 + (1/N)) - Nx, &&\text{if} \quad y - x \geqslant (1 - (1/N)), \\ &= 0, &&\text{if} \quad y - x < (1 - (1/N)) \end{aligned}$$

for some positive integer N. Then we have

$$\int_R\int f - I_n(f) = \frac{N}{60}\frac{1}{n^3}$$

for all sufficiently large n divisible by N. For the same values of n, it is easy to verify that the product trapezoidal formula gives an error $\frac{1}{12}n^2$ and the product Simpson's formula gives an error $\frac{1}{9}n^2$ or $\frac{1}{18}n^2$ depending on whether n/N is even or odd.

6. Final Remarks

The bivarite B-splines presented in this paper were constructed by using the conformality conditions of bivariate spline functions at all grid-points as discussed in Section 2. There are other techniques available that can be used for different purposes. For example, in finite element methods, there are procedures used by Fredrickson [8], Powell [10], and Zwart [14]; and in approximation theory, there are methods due to Dahmen [7] and Micchelli [9]. In fact, convolutions of bivariate B-splines also give other bivariate B-splines with larger supports. No matter what methods are being used, however, there is no guarantee that an actual basis, or even a span set, consisting of bivariate B-spline functions can be obtained. There are two difficulties: first one has to determine the dimensions of the bivariate spline spaces $S_k^\mu(\Delta)$, and second there is the problem of linear independence.

Moreover, since the dimensions are usually quite large, there is the problem of finding "enough" B-splines. There are already some results on the dimensions of spaces of multivariate spline functions in [3, 6, 11]. A more general result can be obtained using the method in [6]. For the bivariate quadratic spline space $S_2^1(\Delta)$, where Δ is (a refinement of) the grid partition given in Fig. 2, we have determined a B-spline basis in [5], where bivariate spline identities and approximation properties of corresponding B-spline series are discussed.

REFERENCES

1. C. DE BOOR, Splines as linear combination of B-spline, *in* "Approximation Theory II (G. G. Lorentz, C. K. Chui, and L. L. Schumaker, Eds.), pp. 1–47, Academic Press, New York, 1976.
2. C. K. CHUI AND R. H. WANG, Bases of bivariate spline with crosscut grid partitions, *J. Math. Res. Exposition* **1** (1982), 1–4.
3. C. K. CHUI AND R. H. WANG, On smooth multivariate spline functions, *Math. Comp.*, to appear.
4. C. K. CHUI AND R. H. WANG, Bivariate cubic B-splines relative to crosscut triangulations, submitted for publication.
5. C. K. CHUI AND R. H. WANG, On a bivariate B-spline basis, submitted for publication.
6. C. K. CHUI AND R. H. WANG, Multivariate spline spaces, *J. Math. Anal. Appl.*, in press.
7. W. DAHMAN, On multivariate B-spline, *SIAM J. Numer. Anal.* **17** (1980), 179–190.
8. P. O. FREDERICKSON, Quasi-interpolation, extrapolation, and approximation on the plane, *in* "Proceedings, Conf. Numerical Math." pp. 159–167, Winnipeg, Canada, 1971.
9. C. A. MICCHELLI, A constructive approach to Kergin interpolation in R^k: Multivariate B-splines and Lagrange interpolation, *Rocky Mountain J. Math.* **10** (1980), 485–497.
10. M. J. D. POWELL, Piecewise quadratic surface fitting for contour plotting, *in* "Software for Numerical Analysis" (D. J. Evans, Ed.), pp. 253–271, Academic Press, New York, 1974.
11. L. L. SCHUMAKER, On the dimension of space of piecewise polynomials in two variables, *in* "Multivariate Approximation Theory" (W. Schempp and K. Zeller, Eds.), pp. 390–412, Birkhauser Verlag, Basel, 1979.
12. A. H. STROUD, "Approximate Calculation of Multiple Integrals," Prentice-Hall, Englewood Cliffs, N. J., 1971.
13. R. H. WANG, The structural characterization and interpolation for multivariate splines, *Acta Math. Sinica* **18** (1975), 91–106.
14. P. ZWART, Multi-variate splines with non-degenerate partitions, *SIAM J. Numer. Anal.* **10** (1973), 665–673.