1 The Truncated Power Basis

1.1 Piecewise Polynomial Functions

Let $\xi = \{\xi_1 < \xi_2 < \cdots < \xi_{l+1}\}$ be a set of strictly increasing series of points, and let k be a positive integer. Further, let $P_1, \ldots, P_l$ denote a sequence of l polynomials of order k. Then the correponding piecewise polynomial (pp) function of order k is defined as follows:

$$f(x) = P_i(x) \text{ if } \xi_i < x < \xi_{i+1}$$

for $i = 1, \ldots, l$. $\{\xi\}$ are known as the breakpoints of f. At the interior breakpoints, $\xi_2, \ldots, \xi_l$, the function value is defined by specifying f to be right continuous; that is,

$$f(\xi_i) = f\left(\xi_i^+\right), \quad i = 2, \ldots, l$$

However, in a sense, without this specification, the function has two values at any interior breakpoint: the value it gets from the polynomial piece to the left of the breakpoint, $f\left(\xi_i^-\right) = P_{i-1}(\xi_i)$, in addition to the value it gets from the polynomial piece to the right of the breakpoint, $f\left(\xi_i^+\right) = P_i(\xi_i)$. To properly define the function, one can specify f to be right-continuous:

$$f(\xi_i) \equiv f\left(\xi_i^+\right) \tag{1}$$

Denote the set of pp functions of order k with breakpoints $\xi = \{\xi_1, \ldots, \xi_{l+1}\}$ by

$$\mathcal{P}_{k,\xi}.$$

$\mathcal{P}_{k,\xi}$ is a linear space having dimension kl, as it consists of l polynomials, each having k polynomial coefficients. The j^{th} derivative of a pp f,

$$D^j f$$

is a pp function of order $k - j$ having the same breakpoint sequence and constructed from the same j^{th} derivatives of the polynomial pieces from which f was constructed. This "definition" dodges much of the complicated discussion of the derivatives of a pp function at its breakpoints and thus must be treated with considerable care in context of the fundamental theorem of calculus.

Proposition pp function, f satisfies

$$f(x) - f(a) = \int_a^x (Df)(t)\, dt \quad \text{for all} \quad x$$

if and only if f is a continuous function.

Consider a piecewise constant function f: by the previous definition, its first derivative is identically zero, and is therefore equal to the usual derivative of f if and only if f is constant.

This prerequisite information is merely for the ability to responsibly refer to the set of piecewise polynomial functions and have a shorthand way of doing so. These means enable us to introduce two sets of basis functions: first, the truncated power basis, followed by B-spline basis functions. We will see that both are closely related, with the former having some properties which leave them unattractive for function approximation and thus present the construction of B-splines and how to use them to construct a representation of $\mathcal{P}_k$. In practice, one typically is given some information about an unknown function, g, and the task is to construct a function $f \in \mathcal{P}_{k,\xi}$ which satisfies conditions that g also satisfies, and in addition, has a certain number of continuous derivatives. These conditions define a subspace of $\mathcal{P}_{k,\xi}$, $\mathcal{P}_{k,\xi,\nu}$ for which we will need a corresponding basis.

For illustrative purposes, consider the task of smoothing a histogram using parabolic splines. Suppose we are given points

$$\tau_1 < \tau_2 < \cdots < \tau_{n+1}$$

and non-negative numbers $h_1, h_2, \ldots, h_n$, with h_i denoting the height of the histogram over the interval (τ_i, τ_{i+1}). The histogram is an approximate representation of some underlying density function, g. Letting $\Delta\tau_i = \tau_{i+1} - \tau_i$, one may interpret $h_i \Delta\tau_i$ as (approximately) equal to the integral of g over $[\tau_i, \tau_{i+1}]$. One may impose the following interpolation conditions on our smooth function, f:

$$\int_{\tau_i}^{\tau_{i+1}} f(x)\,dx = h_i \Delta\tau_i$$

for $i = 1, \ldots, n$. Let f be a piecewise polynomial of order 3 having continuous first derivative:

$$f \in \mathcal{P}_{3,\xi} \cap \mathcal{C}^{(1)}$$

Choose the breakpoint sequence ξ to coincide with $\tau = \{\tau_1, \ldots, \tau_{n+1}\}$. If g is smooth and vanishes outside its support, $[\tau_1, \tau_{n+1}]$, then

$$g^{(j)}(\tau_1) = g^{(j)}(\tau_{n+1}) = 0,$$

for $j = 0, 1, \ldots, d$, where d characterizes the extent of the smoothness of g, we may also wish to require f to obey two additional interpolation constraints:

$$f(\tau_1) = f(\tau_{n+1}) = 0,$$

giving a total of $n+2$ interpolation conditions. These, along with the $2(n-1)$ continuity conditions yield a total $3n$ constraints on the $3n$ polynomial coefficients,

$$c_{ji} \equiv D^{j-1} f\left(\xi_i^+\right).$$

These conditions lead to the system of equations:

$$
\begin{array}{llllllllll}
c_{11} & & & & & & & & = & 0 \\
c_{11}+ & c_{21}\dfrac{\Delta\tau_1}{2!}+ & c_{31}\dfrac{(\Delta\tau_1)^2}{3!} & & & & & & = & h_1 \\
c_{11}+ & c_{21}\Delta\tau_1+ & c_{31}\dfrac{2\,(\Delta\tau_1)^2}{3!} & - & c_{12} & & & & = & 0 \\
\vdots & c_{21}+ & c_{31}\Delta\tau_1 & - & & c_{22} & & & = & 0 \\
 & & & & c_{12}+ & c_{22}\dfrac{\Delta\tau_2}{2}+ & c_{32}\dfrac{(\Delta\tau_2)^2}{3!} & & = & h_2 \\
 & & & & c_{12}+ & c_{22}\Delta\tau_2+ & c_{32}\dfrac{(\Delta\tau_2)^2}{2} & \dots & = & 0 \\
 & & & & & c_{22}+ & c_{32}\Delta\tau_2 & \dots & = & 0 \\
 & & & & & & & & & \\
 & & & & & & & \ddots & &
\end{array}
\tag{2}
$$

One may quickly see that this system is two-thirds homogeneous; that is, for every integral interpolation constraint, we have two continuity constraints that lead to zeros on the right hand side of the equality. To solve 2, the homogeneous equations are solved, leaving a reduced set of (non-homogeneous) equations. To do this, one may construct a set of linearly independent functions $\phi_1, \phi_2, \dots$ of the same size as the number of interpolation constraints which satisfy the homogeneous equations. The smoother, f, is then constructed within this subspace of $\mathcal{P}_{3,\xi}$ and has form

$$f = \sum_j \alpha_j \phi_j.$$

The construction of f as a linear combination of the $\{\phi_j\}$ constitutes finding a basis for the subspace of the piecewise polynomials comprised of functions in this space which satisfy the homogeneous equations in 2.