Nonparametric covariance estimation for longitudinal data via tensor product smoothing

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The data:

$$Y_i = (Y_{i1}, Y_{i2}, \ldots, Y_{im})', \qquad i = 1, \ldots, N$$

associated with measurement times

$$t_1 < t_2 < \cdots < t_m.$$

The flaming hoops:

- Covariance matrices (and their estimates) should be positive definite.
  - Constrained optimization is a headache.
- The $\{t_{ij}\}$ may be suboptimal.
  - Observation times may not fall on a regular grid, may vary across subjects.
- More dimensions, more problems (maybe.)
  - Sample covariance matrix falls apart when m is large.

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Covariance dress-up: the modified Cholesky decomposition

$$Y = (Y_1, \ldots, Y_M)' \sim \mathcal{N}(0, \Sigma).$$

For any positive definite Σ, we can find T which diagonalizes Σ:

$$D = T\Sigma T', \quad T = \begin{bmatrix} 1 & 0 & \ldots & & \\ -\phi_{21} & 1 & & & \\ -\phi_{31} & -\phi_{32} & 1 & & \\ \vdots & & & \ddots & \\ -\phi_{M1} & -\phi_{M2} & \ldots & -\phi_{M,M-1} & 1 \end{bmatrix} \tag{1}$$

The matrix T is the *Cholesy factor* of the precision matrix.

Now, for the cutest part:

Okay, really:

Imagine regressing Y_j on its predecessors:

$$y_j = \begin{cases} e_1 & j = 1, \\ \sum_{k=1}^{j-1} \phi_{jk} y_k + \sigma_j e_j & j = 2, \ldots, M \end{cases} \tag{2}$$

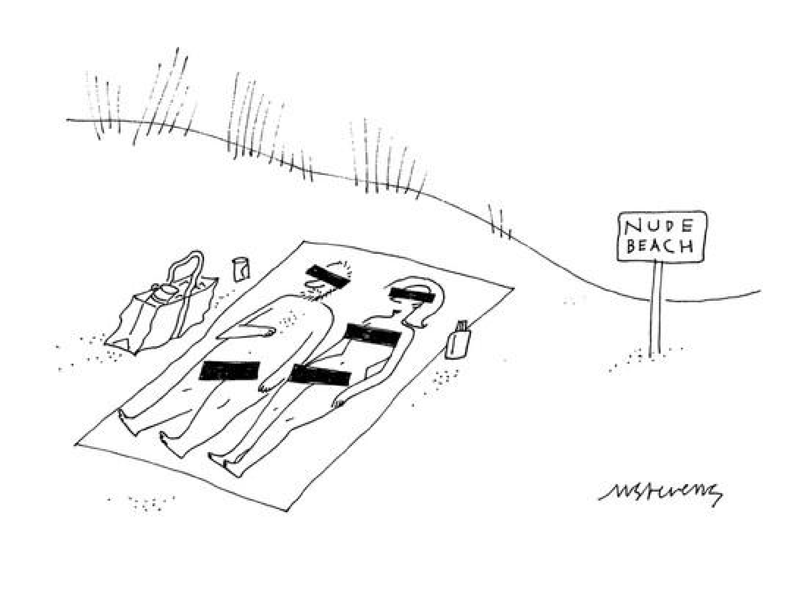
In matrix form:

$$e = TY, \tag{3}$$

and taking covariances on both sides:

$$D = diag\left(\sigma_1^2, \ldots, \sigma_M^2\right) = T\Sigma T'. \tag{4}$$

No constraints on the ϕ_{jk}s!



The regression model tool box: a deep treasure chest of luxury.

Model Y_j, e_j as

$$Y_j = Y\left(t_j\right), \quad e_j = e\left(t_j\right), \\ e\left(s\right) \sim \mathcal{WN}\left(0,1\right),$$

Swap the standard regression model 2 for a varying coefficient model:

$$\phi_{jk} = \phi\left(t_j, t_k\right),$$

$$y\left(t_j\right) = \sum_{k=1}^{j-1} \phi\left(t_j, t_k\right) y\left(t_k\right) + \sigma\left(t_j\right) e\left(t_j\right) \tag{5}$$

The $\{\phi_{jk}\}$ are called *generalized autoregressive parameters.*
The $\left\{\sigma_j^2\right\}$ are called the *innovation variances.*

(Iterated) penalized maximum likelihood estimation

1. Fix $\sigma_{ij}^2 = \sigma_{ij0}^2$, $i = 1, \ldots, N$,$j = 1, \ldots, M$.
2. Find $\phi_0 = arg\, \underset{\phi}{min} - 2L_\phi\left(\phi, y_1, \ldots, y_N\right) + \lambda J\left(\phi\right)$
3. Fix $\phi = \phi_0$.
4. Find $\sigma_0^2 = arg\, \underset{\sigma^2}{min} - 2L_\sigma^2\left(\sigma^2, y_1, \ldots, y_N\right) + \lambda J\left(\sigma^2\right)$

$$-2L_\phi\left(\phi, y_1, \ldots, y_N\right) = \sum_{i=1}^{N} \sum_{j=2}^{m_i} \sigma_{ij0}^{-2} \left(y_{ij} - \sum_{k=1}^{j-1} \phi\left(t_{ij}, t_{ik}\right) y_{ik} \right)^2$$

Regularization of $\phi(s,t)$ is more intuitive if we transform the s-t axis.

Rotate the input axes:

$$
\begin{aligned}
l &= s - t \\
m &= \frac{1}{2}(s+t).
\end{aligned}
$$

Then ϕ becomes

$$
\begin{aligned}
\phi^*(l,m) &= \phi^*\left(s-t, \frac{1}{2}(s+t)\right) \\
&= \phi(s,t).
\end{aligned}
$$

Take $\hat{\phi}^*$ to be the minimizer of

$$
-2L + \lambda J(\phi^*)
$$

Smooth ANOVA models

Decompose

$$\phi^* (l, m) = \mu + \phi_1 (l) + \phi_2 (m) + \phi_{12} (l, m), \tag{7}$$

so Model 5 becomes

$$y (t_j) = \sum_{k=1}^{j-1} \Big[\mu + \phi_1 (l_{jk}) + \phi_2 (m_{jk}) + \phi_{12} (l_{jk}, m_{jk}) \Big] y (t_k) + \sigma (t_j) e (t_j) \tag{8}$$

We can use B-splines to construct the model basis.

Represent the main effects as

$$\begin{aligned} \phi_1(l) &= \sum_{c=1}^{c_l} B_c(l; q_l)\, \theta_{lc}, \\ \phi_2(m) &= \sum_{c'=1}^{c_m} B_{c'}(m; q_m)\, \theta_{mc'}, \end{aligned} \tag{9}$$

and the interaction term by the tensor product of the marginal bases **??** and **??**:

$$\phi_{12}(l, m) = \sum_{c=1}^{c_l} \sum_{c'=1}^{c_m} B_c(l; q_l)\, B_{c'}(m; q_m)\, \theta_{cc'} \tag{10}$$

PS-ANOVA model basis

In matrix notation, Model 8 becomes

$$E\left[Y|W\right] = WB\theta,$$

where W is the matrix of covariates holding the past values of Y, and B is the B-spline regression basis:

$$B = [\ 1_p \mid B_l \mid B_m \mid B_{lm}\ ] \tag{11}$$

where

$$\begin{aligned} B_{lm} &= B_m \,\square\, B_l \\ &\equiv \left(B_m \otimes 1'_{c_l}\right) \odot \left(1'_{c_m} \otimes B_l\right). \end{aligned}$$

Decomposition of ϕ^* *for* $d_l = d_m = 2$

	$\{1\}$	$\{m\}$	$\{B_{j'}(m)\}$
$\{1\}$	$\{1\}$	$\{m\}$	$\{B_{j'}(m)\}$
$\{l\}$	$\{l\}$	$l \times m$	$l \times \{B_{j'}(m)\}$
$\{B_j(l)\}$	$\{B_j(l)\}$	$m \times \{B_j(l)\}$	$\{B_j(l)\, B_{j'}(m)\}$

Nested PS-ANOVA

Re-express ϕ_{12}:

$$\phi_{12}\left(l,m\right) = g_{1}\left(l\right)\left[\sum_{r=1}^{d_m-1} m^{r}\right] + \left[\sum_{r=1}^{d_l-1} l^{r}\right] g_{2}\left(m\right) + h\left(l,m\right),$$

For $d_l = d_m = 2$,

$$\phi_{12}\left(l,m\right) = g_{1}\left(l\right)\ m + l\ g_{2}\left(m\right) + h\left(l,m\right)$$

with basis:

$$B = \Big[\ 1 \mid B_1 \mid B_2 \mid B_3 \mid B_4 \mid B_5\ \Big], \tag{12}$$

where

$$B_3 = m \square B_1 \qquad B_5 = B_2 \square B_1$$

$$B_4 = B_2 \square l$$

Nested PS-ANOVA

$$P = \text{blockdiag}\left(0,\ P_1,\ P_2,\ P_3,\ P_4,\ P_5\right)$$