Nonparametric Covariance Estimation for Longitudinal Data via Penalized Tensor Product Splines

Tayler A. Blake* Yoonkyung Lee†

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1 Appendix

Proof. Using the properties of reproducing kernels, we can rewrite ϕ^* as an inner product of itself with R:

$$
\begin{aligned}
\phi^*\left(l_j, m_j\right) &= \left\langle R\left(\left(l_j, m_j\right), (\cdot,\cdot)\right), \phi^*(\cdot,\cdot)\right\rangle \\
&= \Bigg\langle R_0\left(\left(l_j, m_j\right), (\cdot,\cdot)\right) + R_1\left(\left(l_j, m_j\right), (\cdot,\cdot)\right), d_0 + d_1 k_1(\cdot) \\
&\qquad\qquad + \sum_{i=1}^{N_{\phi^*}} c_i R_1\left(\left(l_i, m_i\right), (\cdot,\cdot)\right) + \rho\left((\cdot,\cdot)\right) \Bigg\rangle \\
&= \left\langle R_0\left(\left(l_j, m_j\right), (\cdot,\cdot)\right), d_0 + d_1 k_1(\cdot)\right\rangle + \left\langle R_0\left(\left(l_j, m_j\right), (\cdot,\cdot)\right), \sum_{i=1}^{N_{\phi^*}} c_i R_1\left(\left(l_i, m_i\right), (\cdot,\cdot)\right)\right\rangle \\
&\quad + \left\langle R_0\left(\left(l_j, m_j\right), (\cdot,\cdot)\right), \rho\left((\cdot,\cdot)\right)\right\rangle + \left\langle R_1\left(\left(l_j, m_j\right), (\cdot,\cdot)\right), d_0 + d_1 k_1(\cdot)\right\rangle \\
&\quad + \left\langle R_1\left(\left(l_j, m_j\right), (\cdot,\cdot)\right), \sum_{i=1}^{N_{\phi^*}} c_i R_1\left(\left(l_i, m_i\right), (\cdot,\cdot)\right)\right\rangle + \left\langle R_1\left(\left(l_j, m_j\right), (\cdot,\cdot)\right), \rho\left((\cdot,\cdot)\right)\right\rangle \\
&= \left\langle R_0\left(\left(l_j, m_j\right), (\cdot,\cdot)\right), d_0 + d_1 k_1(\cdot)\right\rangle + \left\langle R_1\left(\left(l_j, m_j\right), (\cdot,\cdot)\right), \sum_{i=1}^{N_{\phi^*}} c_i R_1\left(\left(l_i, m_i\right), (\cdot,\cdot)\right)\right\rangle \\
&\quad + \underbrace{\left\langle R_0\left(\left(l_j, m_j\right), (\cdot,\cdot)\right), \rho(\cdot,\cdot)\right\rangle}_{0} + \underbrace{\left\langle R_1\left(\left(l_j, m_j\right), (\cdot,\cdot)\right), \rho(\cdot,\cdot)\right\rangle}_{0} \\
&= d_0 + d_1 k_1(\cdot) + \sum_{i=1}^{N_{\phi^*}} c_i R_1\left(\left(l_i, m_i\right), \left(l_j, m_j\right)\right)
\end{aligned}
$$

*The Ohio State University, 1958 Neil Avenue, Columbus, OH 43201
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Rewriting the data fit functional, we have that

$$\begin{aligned} & \sum_{i=1}^{N}\sum_{j=1}^{n_i}\sigma_{ij}^{-2}\left(y\left(t_{ij}\right)-\sum_{k=1}^{j-1}\phi^*\left(t_{ij},t_{ik}\right)y\left(t_{ik}\right)\right)^2 \\ = & \sum_{i=1}^{N}\sum_{j=1}^{n_i}\sigma_{ij}^{-2}\left(y\left(t_{ij}\right)-\sum_{k=1}^{j-1}\left\langle R\left(\left(l_{jk}^i,m_{jk}^i\right),\left(\cdot,\cdot\right)\right),\phi^*\left(\cdot,\cdot\right)\right\rangle y\left(t_{ik}\right)\right)^2 \end{aligned}$$

which is free of ρ. Consider the contribution of any nonzero ρ to $J\left(\phi^*\right)$:

$$\begin{aligned} J\left(\phi^*\right) &= ||P_1\phi^*||^2 \\ &= \left\langle \sum_{i=1}^{N_{\phi^*}} c_i R_1\left(\left(l_i,m_i\right),\left(\cdot,\cdot\right)\right)+\rho\left(\cdot,\cdot\right), \sum_{j=1}^{N_{\phi^*}} c_j R_1\left(\left(l_j,m_j\right),\left(\cdot,\cdot\right)\right)+\rho\left(\cdot,\cdot\right)\right\rangle \\ &= ||\sum_{i=1}^{N_{\phi^*}} c_i R_1\left(\left(l_i,m_i\right),\left(\cdot,\cdot\right)\right)||^2+||\rho||^2 \end{aligned}$$

Thus, including ρ in ϕ^* only increases the penalty without improving (decreasing) the data fit functional, so we indeed have that the minimizer of (**??**) has the form

$$\phi^*\left(l,m\right)=d_0+d_1k_1\left(l\right)+\sum_{i=1}^{N_{\phi^*}} c_i R_1\left(\left(l,m\right),\left(l_i,m_i\right)\right) \tag{1}$$

□