CS 225

Set Theory and Logic

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Comparison of Set Theory and Logic

I find the similarities between set theory and logic to be strikingly accurate: both have similar identity laws, distributive laws, DeMorgan's laws, etc. To me the set-theory versions are more intuitive since you can draw venn-diagram pictures to reason about the identities, but since we have covered logic first in CS225, perhaps the logical rules are more fresh in everyone's mind. In any case, I think it is useful to compare the two.

When reading through set theory notation, you may want to draw a venn diagram with sets A and B contained in some universal set $\mathbb{U}$ to help understand the notation.

Here is a table comparing logical symbols/operations to set theoretic ones:

In the table I will refer to two sets A and B which are both contained in some universal set $\mathbb{U}$. You can think of $\mathbb{U}$ as the domain of x.

Note 1: I do not claim that each of these is exactly equivalent! Just a strong relation.

Note 2: In the identities on the next page "True" corresponds to "$\mathbb{U}$" for the most part. In other contexts, it can be useful to think of the set A as "True" if it is not empty. So there may not be exactly one set-theory concept that corresponds exactly to "True".

Symbols in Set Theory vs. Logic

Word	Logic	Set Theory	Set Theory Connection
Or	$\vee$	$\cup$	$\forall\ x \in \mathbb{U}$, $x \in A \cup B$ if and only if $x \in A$ **or** $x \in B$.
And	$\wedge$	$\cap$	$\forall\ x \in \mathbb{U}$, $x \in A \cap B$ if and only if $x \in A$ **and** $x \in B$.
Not	$\sim$	^{c}	$\forall\ x \in \mathbb{U}$, $x \in A^c$ if and only if x is **not** in A.
If / then	$\rightarrow$	$\subseteq$	"$A \subseteq B$" if and only if "$\forall\ x \in \mathbb{U}$, **if** $x \in A$ **then** $x \in B$."
True	$\mathbb{T}$ or T	$\mathbb{U}$ or "not empty"	"$\forall\ x \in \mathbb{U}$, $x \in \mathbb{U}$ is always true" OR "A is nonempty"
False	$\mathbb{F}$ or F	$\varnothing$	$\forall\ x \in \mathbb{U}$, x is not in $\varnothing$.
Equivalent	$\equiv$	$=$ or "if and only if"	

Comparison of Set Theory and Logic Identities

Here is a comparison of set-theoretic versus logical identites (on the next page). Some identities (for example the biconditional) don't seem to translate well, so I have omitted the set-theory version. In the list, p, q, r are propositions, and A, B, C are sets contained in some universal set $\mathbb{U}$.

Identities in Set Theory vs. Logic

Logic	Set Theory	Identity
$\sim(\sim p) \equiv p$	$(A^c)^c = A$	Double Negation
$p \wedge \mathbb{T} \equiv p \quad p \vee \mathbb{F} \equiv p$	$A \cap \mathbb{U} = A \quad A \cup \varnothing = A$	Identity
$p \vee \mathbb{T} \equiv \mathbb{T} \quad p \wedge \mathbb{F} \equiv \mathbb{F}$	$A \cup \mathbb{U} = \mathbb{U} \quad A \cap \varnothing = \varnothing$	Domination
$p \wedge p \equiv p \quad p \vee p \equiv p$	$A \cap A = A \quad A \cup A = A$	Idempotent
$p \vee \sim p \equiv \mathbb{T} \quad p \wedge \sim p \equiv \mathbb{F}$	$A \cup A^c = \mathbb{U} \quad A \cap A^c = \varnothing$	Negation / Complement
$p \vee q \equiv q \vee p \quad p \wedge q \equiv q \wedge p$	$A \cup B = B \cup A \quad A \cap B = B \cap A$	Commutative
$(p \vee q) \vee r \equiv p \vee (q \vee r)$	$(A \cup B) \cup C = A \cup (B \cup C)$	Associative
$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	$(A \cap B) \cap C = A \cap (B \cap C)$	Associative
$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	Distributive
$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	Distributive
$\sim(p \wedge q) \equiv \sim p \vee \sim q$	$(A \cap B)^c = A^c \cup B^c$	DeMorgan's
$\sim(p \vee q) \equiv \sim p \wedge \sim q$	$(A \cup B)^c = A^c \cap B^c$	DeMorgan's
$p \vee (p \wedge q) \equiv p$	$A \cup (A \cap B) = A$	Absorption
$p \wedge (p \vee q) \equiv p$	$A \cap (A \cup B) = A$	Absorption

In the following table, the set-theoretic analogs were added by me. Please do not take my word for these–verify them yourself! I do not guarantee that the set-theoretic analogs are correct (although I believe they are).

Other Identities in Set Theory vs. Logic

Logic	Set Theory	Identity
$p \to q \equiv \sim q \to \sim p$	$A \subseteq B$ if and only if $B^c \subseteq A^c$	Contrapositive
$p \to q \equiv \sim p \vee q$	$A \subseteq B$ if and only if $A^c \cup B = \mathbb{U}$	Implication
$(p \to q) \wedge (p \to \sim q) \equiv \sim p$	If $A \subseteq B$ and $A \subseteq B^c$ then $A = \varnothing$	Absurdity