CS 225

Graph Theory

Fall 2020

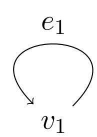
Graph: A graph G is a nonempty set $V(G)$ of vertices, and a set $E(G)$ of edges, where each edge is associated with a set of either one or two vertices, called the **endpoints** of that edge.

Here is a graph vith $V(G) = \{v_1, v_2\}$ and $E(G) = \{e_1\} = \{\{v_1, v_2\}\}$:

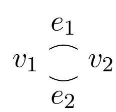
$$v_1 \overset{e_1}{—} v_2$$

The **edge to endpoint** function sends an edge to its set of endpoints. If we define an edge as the set of its endpoints (which we can do for simple graphs), this is pretty straightforward.

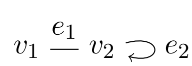
An edge with just one endpoint is a loop (note that the loop drawn here is directed due to how my drawing program is working at the moment. Please ignore the arrow on the loop edge):



Two or more *distinct* edges with the same set of endpoints are said to be **parallel**:



Two vertices connected by an edge are adjacent. In the picture below, v_1 and v_2 are adjacent vertices, and v_2 is adjacent to itself.



An edge is **incident** on each of its endpoints, and two edges incident to the same endpoint are **adjacent**. Below, edges e_1 and e_2 are adjacent since they share the endpoint v_2.

$$v_1 \overset{e_1}{—} v_2 \overset{e_2}{—} v_3$$

An isolated vertex:

$$v_1$$

A **directed graph** has vertices and edges, but the edges are now ordered pairs. We can represent a directed graph as vertices and edges where the edges have arrows. Below is a graph with vertices $V(G) = \{v_1, v_2\}$ and directed edges $D(G) = \{e_1\} = \{(v_1, v_2)\}$.

$$v_1 \xrightarrow{e_1} v_2$$

If G is a graph and v is a vertex of G, then the **degree** of v, $\deg(v)$ is the number of edges that are incident on v, with an edge that is a loop counted twice. Below v_1 has degree 1 and v_2 has degree 3:

$$v_1 \overset{e_1}{—} v_2 \circlearrowleft$$

The **total degree** of a graph is the sum of degrees of all vertices of the graph. In the graph below, the total degree is 4.

$$v_1 \overset{e_1}{—} v_2 \circlearrowleft$$

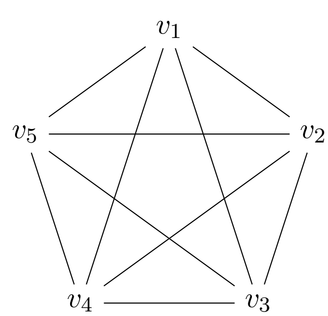
Theorem 4.9.1: The handshake theorem If G is any graph then the sum of degrees of all the vertices of G is equal to twice the number of edges of G.

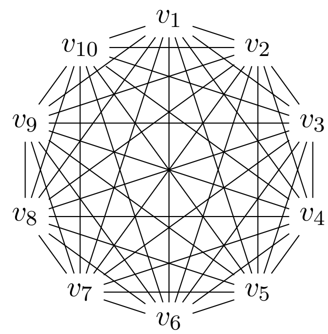
Corollary 4.9.2: The total degree of a graph is even.

Proposition 4.9.3: In any graph there is an even number of vertices of odd degree. (if not then Corollary 4.9.2 would be false)

A **simple** graph is a graph that does not have any loops or parallel edges.

If n is a positive integer, then a **complete graph on n vertices**, denoted K_n, is a simple graph with n vertices and exactly one edge connecting each pair of distinct vertices. Below are K_5 and K_{10}:





Example 4.9.9 K_n has $\dfrac{n(n-1)}{2}$ edges.

Let m and n be positive integers. A **complete bipartite graph on** (m, n) **vertices**, denoted $K_{m,n}$, is a simple graph whose vertices are divided into two distinct subsets, V with m vertices and W with n vertices, in such a way that

1. Every vertex of $K_{m,n}$ velongs to one of V or W but not both
2. There is exactly one edge from each vertex of V to each vertex of W
3. There is no edge from any one vertex of V to any other vertex of V.
4. There is no edge from any one vertex of W to any other vertex of W.

$K_{3,2}$ is shown below: