CS 225

Some Definitions and Theorems from Chapter 4

By Nathan Taylor

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From 4.1:

Assumptions:

- Basic laws of algebra from appendix A (I assume we do not have to cite these–I listed some on the last page of this document)
- Properties of equality: $x = x$, $x = y \implies y = x$, $x = y$ and $y = z \implies x = z$
- Principle of substitution: if $x = y$, then we can substitute in y wherever x appears.
- The integers are closed under addition, subtraction and multiplication. (The book also mentions that there is no integer between 0 and 1–does that need to be an assumption??)

Definitions:

- An integer n is **even** if and only if it can be written as $n = 2k$ for some integer k.
- An integer n is **odd** if and only if it can be written as $n = 2k + 1$ for some integer k.
- An integer n is **prime** if and only if $n > 1$ and for all positive integers r and s, if $n = rs$, then either r or s equals n.
- An integer n is **composite** if and only if $n > 1$ and $n = rs$ for some integers r and s with $1 < r < n$ and $1 < s < n$.

Theorem 4.1.1

The sum of any two even integers is even.

From 4.2:

Theorem 4.2.1:

The difference of any odd integer and any even integer is odd.

From 4.3:

Definition:

A real number r is **rational** if and only if there are integers a and b with $b \neq 0$ so that $r = \dfrac{a}{b}$

Property:

The zero product property is: if neither of two real numbers is zero, then their product is also not zero. (this is T11 in appendix A)

Theorem 4.3.1:

Every integer is a rational number.

Theorem 4.3.2:

The sum of any two rational numbers is a rational number.

Corollary 4.2.3:

The double of a rational number is a rational number.

Result of Exercise 12:

The square of any rational number is a rational number.

Result of Exercise 13:

The negative of any rational number is a rational number.

Result of Exercise 14:

The cube of any rational number is a rational number.

Result of Exercise 15:

The product of any two rational numbers is a rational number.

Result of Exercise 17:

The difference of any two rational numbers is a rational number.

From 4.4:

Definition:

If n and d are integers, then n is **divisible by** d if and only if $d \neq 0$ and $n = kd$ for some integer k.

Some synonyms for "d divides n":

- n is a multiple of d
- d is a factor of n
- d is a divisor of n
- d divides n
- $d \mid n$

The notation $d \nmid n$ means "d does not divide n".

Theorem 4.4.1:

For all integers a and b, if a and b are positive and a divides b then $a \leq b$.

Theorem 4.4.2:

The only divisors of 1 are 1 and -1.

Theorem 4.4.3:

For all integers a, b and c, if a divides b and b divides c, then a divides c.

Theorem 4.4.4:

Any integer $n > 1$ is divisible by a prime number.

Theorem 4.4.5:

Given any integer $n > 1$ there exists a positive integer k, distinct prime numbers $p_1, p_2 \ldots, p_k$, and positive integers $e_1, e_2, \ldots, e_k$ such that

$$n = p_1^{e_1} p_2^{e_2} \ldots p_k^{e_k} = \prod_{i=1}^{k} p_i^{e_i}$$

and any other expression for n as a product of prime numbers is identical to this except perhaps for the order in which the factors are written.

Definition:

Given any integer $n > 1$, the **standard factored form** of n is an expression of the form

$$n = p_1^{e_1} p_2^{e_2} \dots p_k^{e_k} = \prod_{i=1}^{k} p_i^{e_i}$$

where k is a positive integer, $p_1, \dots, p_k$ are prime numbers, $e_1, \dots, e_k$ are positive integers, and $p_1 < p_2 < \dots < p_k$.

From 4.7:

Theorem 4.7.1:

There is no greatest integer.

Theorem 4.7.2:

There is no integer that is both even and odd.

Theorem 4.7.3:

The sum of any rational number and any irrational number is irrational.

Proposition 4.7.4:

For every integer n, if n^2 is even then n is even.

From 4.8:

Theorem 4.8.1:

$\sqrt{2}$ is irrational.

Proposition 4.8.2:

$1 + 3\sqrt{2}$ is irrational.

Proposition 4.8.3:

For any integer a and any prime number p, if $p \mid a$ then $p \nmid (a + 1)$.

Theorem 4.8.4:

The set of prime numbers is infinite (i.e. there are infinitely many prime numbers).

From Appendix A:

- Field Axioms for real numbers:
  - F1 Commutative laws for addition and multiplication
  - F2 Associative laws for addition and multiplication
  - F3 Distributive laws
  - F4 Existence of an identity for addition and multiplication
  - F5 Existence of additive inverses (negative numbers)
  - F6 Existence of multiplicative inverses (reciprocals)
- Other theorems:
  - T1 Cancellation law for addition: $a + b = a + c \implies b = c$
  - T2 Possibility of subtraction: There is a unique solution to $a + x = b$ (namely $x = b - a$)
  - T3 $b - a = b + (-a)$
  - T4 $-(-a) = a$
  - T5 $a(b - c) = ab - ac$
  - T6 $0 \cdot a = a \cdot 0 = 0$
  - T7 Cancellation law for multiplication: $ab = ac$ and $a \neq 0 \implies b = c$
  - T8 Possibility of division: If $a \neq 0$, there is a unique solution to $ax = b$. (namely $x = b/a$)
  - T9 If $a \neq 0$, $\dfrac{b}{a} = b \cdot a^{-1}$
  - T10 If $a \neq 0$, $(a^{-1})^{-1} = a$
  - T11 Zero product property: If $ab = 0$, then $a = 0$ or $b = 0$
  - T12 $(-a)b = a(-b) = -(ab)$, $(-a)(-b) = ab$,
    $$-\frac{a}{b} = \frac{-a}{b} = \frac{a}{-b}$$
  - T13 Equivalent fractions property: $\dfrac{a}{b} = \dfrac{ac}{bc}$ if $b \neq 0$ and $c \neq 0$
  - T14 Fraction addition: $\dfrac{a}{b} + \dfrac{c}{d} = \dfrac{ad + bc}{bd}$ if $b \neq 0$ and $d \neq 0$
  - T15 Multiplying fractions: $\dfrac{a}{b} \cdot \dfrac{c}{d} = \dfrac{ac}{bd}$
  - T16 Dividing fractions: $\dfrac{a/b}{c/d} = \dfrac{a}{b} \cdot \dfrac{d}{c}$
- Order Axioms (see the appendix on page A-2)
- Order Theorems (see the appendix A-3)
- Least upper bound axiom (see the appendix A-3)