CS 225

Graph Theory

Fall 2020

Graph: A graph G is a nonempty set $V(G)$ of vertices, and a set $E(G)$ of edges, where each edge is associated with a set of either one or two vertices, called the **endpoints** of that edge.

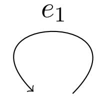
Here is a graph vith $V(G) = \{v_1, v_2\}$ and $E(G) = \{e_1\} = \{\{v_1, v_2\}\}$:

$$v_1 \overset{e_1}{—} v_2$$

The **edge to endpoint** function sends an edge to its set of endpoints. If we define an edge as the set of its endpoints (which we can do for simple graphs), this is pretty straightforward.

An edge with just one endpoint is a loop (note that the loop drawn here is directed due to how my drawing program is working at the moment. Please ignore the arrow on the loop edge):

e_1



v_1

Two or more *distinct* edges with the same set of endpoints are said to be **parallel**:

$$v_1 \overset{e_1}{\underset{e_2}{=}} v_2$$

Two vertices connected by an edge are adjacent. In the picture below, v_1 and v_2 are adjacent vertices, and v_2 is adjacent to itself.

$$v_1 \overset{e_1}{—} v_2 \circlearrowleft e_2$$

An edge is **incident** on each of its endpoints, and two edges incident to the same endpoint are **adjacent**. Below, edges e_1 and e_2 are adjacent since they share the endpoint v_2.

$$v_1 \overset{e_1}{—} v_2 \overset{e_2}{—} v_3$$

An isolated vertex:

$$v_1$$

A **directed graph** has vertices and edges, but the edges are now ordered pairs. We can represent a directed graph as vertices and edges where the edges have arrows. Below is a graph with vertices $V(G) = \{v_1, v_2\}$ and directed edges $D(G) = \{e_1\} = \{(v_1, v_2)\}$.

$$v_1 \xrightarrow{e_1} v_2$$

If G is a graph and v is a vertex of G, then the **degree** of v, $\deg(v)$ is the number of edges that are incident on v, with an edge that is a loop counted twice. Below v_1 has degree 1 and v_2 has degree 3:

$$v_1 \overset{e_1}{—} v_2 \circlearrowleft$$

The **total degree** of a graph is the sum of degrees of all vertices of the graph. In the graph below, the total degree is 4.

$$v_1 \overset{e_1}{—} v_2 \circlearrowleft$$

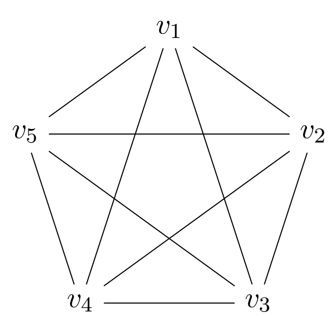
Theorem 4.9.1: The handshake theorem If G is any graph then the sum of degrees of all the vertices of G is equal to twice the number of edges of G.

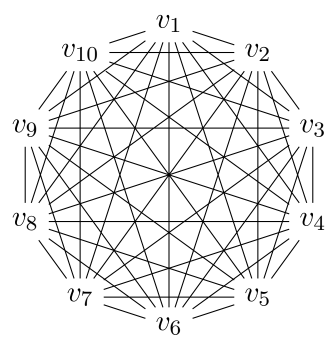
Corollary 4.9.2: The total degree of a graph is even.

Proposition 4.9.3: In any graph there is an even number of vertices of odd degree. (if not then Corollary 4.9.2 would be false)

A **simple** graph is a graph that does not have any loops or parallel edges.

If n is a positive integer, then a **complete graph on n vertices**, denoted K_n, is a simple graph with n vertices and exactly one edge connecting each pair of distinct vertices. Below are K_5 and K_{10}:



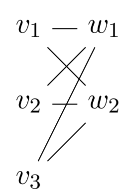


Example 4.9.9 K_n has $\dfrac{n(n-1)}{2}$ edges.

Let m and n be positive integers. A **complete bipartite graph on** (m,n) **vertices**, denoted $K_{m,n}$, is a simple graph whose vertices are divided into two distinct subsets, V with m vertices and W with n vertices, in such a way that

1. Every vertex of $K_{m,n}$ velongs to one of V or W but not both
2. There is exactly one edge from each vertex of V to each vertex of W
3. There is no edge from any one vertex of V to any other vertex of V.
4. There is no edge from any one vertex of W to any other vertex of W.

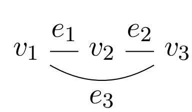
$K_{3,2}$ is shown below:



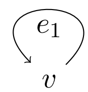
<u>From Chapter 10:</u>

Let G be a graph, and let v and w be vertices of G. A **walk from** v **to** w is a finite alternating sequence of adjacent vertices and edges of G. A walk has the form $v_0e_1v_1e_2\ldots e_nv_n$. The **trivial walk** from v to v consists of the single vertex v.

In the graph below, $v_1e_1v_2e_2v_3$ is a walk from v_1 to v_3. Another walk from v_1 to v_3 is $v_1e_3v_3$.



A **trail** from v to w is a walk from v to w with no repeated edge. In the graph below, ve_1v is a trail from v to v.



A **path** from v to w is a trail that does not contain a repeated vertex. In the graph above, the trail from v to v ve_1v is not a path, but the trivial walk is a path.

A **closed walk** is a walk that starts and ends at the same vertex.

A **circuit** is a closed walk that contains at least one edge and does not contain a repeated edge.

A **simple circuit** is a circuit that does not have any other repeated vertex except the first and last.

A graph H is said to be a **subgraph** of a graph G if and only if every vertex in H is also a vertex in G (i.e. $V(H) \subseteq V(G)$), every edge in H is also an edge in G, and every edge in H has the same endpoints as it has in G (i.e. $E(H) \subseteq E(G)$).

Let G be a graph. Two vertices v and w of G are **connected** if and only if there is a walk from v to w. The graph G is **connected** if and only if all pairs of vertices in G are connected.

Lemma 10.1.1 Let G be a graph. Then

a. If G is connected, then any two distinct vertices of G can be connected by a path.

b. If vertices v and w are part of a circuit in G and one edge is removed from the circuit, then there still exists a trail from v to w in G.

c. If G is connected and G contains a circuit, then an edge of the circuit can be removed without disconnecting G.

A **connected component** of a graph is a connected subgraph of largest possible size.

A graph H is a **connected component** of a graph G if and only if

1. H is a subgraph of G.
2. H is connected.
3. no connected subgraph of G has H as a subgraph and contains vertices or edges that are not in H.

Let G be a graph. An **Euler circuit** for G is a circuit that contains every vertex and every edge of G. That is, an Euler circuit for G is a sequence of adjacent vertices and edges in G that has at least ne edge, starts and ends at the same vertex, uses every vertex of G at least once, and uses every edge of G exactly once.

Theorem 10.1.2: If a graph has an Euler circuit, then every vertex of the graph has positive even degree. Therefore if some vertex of a graph has odd degree, then the graph does not have an Euler circuit.

Theorem 10.1.3: If a graph G is connected and the degree of every vertex of G is positive even integer, then G has an Euler circuit.

Theorem 10.1.4: A graph G has an Euler circuit if and only if G is connected and every vertex of G has positive even degree.

Let G be a graph, and let v and w be two distinct vertices of G. An **Euler trail** from v to w is a sequence of adjacent edges and vertices that starts at v, ends at w, passes through every vertex of G at least once, and traverses every edge of G exactly once.

Corollary 10.1.5: Let G be a graph, and let v and w be two distinct vertices of G. There is an Euler trail from v to w if and only if G is connected, v and w have odd degree, and all other vertices of G have positive even degree.

Given a graph G, a **Hamiltonian circuit** for G is a simple circuit that includes every vertex of G. That is, a Hamiltonian circuit for G is a sequence of adjacent vertices and distinct edges in which every vertex of G appears exactly once, except for the first and last, which are the same.

Proposition 10.1.6: If a graph G has a Hamiltonian circuit, then G has a subgraph H with the following properties:

1. H contains every vertex of G
2. H is connected
3. H has the same number of edges as vertices.
4. Every vertex of H has degree 2.