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OXFORD, OH • EST. 1809

Central Limit Theorem and Different Estimators

College of Engineering and Computing
ECE 345: Engineering Statistics Final Project
Professor: Dr. Zhou
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Ethan Barnes barnese6@miamioh.edu
Deniz Misirlioglu misirld@miamioh.edu
Jared Rozell rozellj@miamioh.edu
Riley Taylor taylo550@miamioh.edu

A) Approximate Theoretical PDF

First we solved for the random variable X in the equation $Y = X^2$. This allows us to find the pdf of x with respect to y. Solving the equation gives us:

$$Y = X^2$$
$$X = \pm\sqrt{Y}$$

Then we solved for the expectation of Y:

$$E(A_n) = E(Y) = \int_0^\infty y f_Y(y) dy$$
$$E(Y) = \int_0^\infty y \frac{1}{\sigma\sqrt{2\pi y}} e^{-\frac{1}{2}(\frac{y}{\sigma^2})} dy = \sigma_x^2$$
$$V(Y) = \int_0^\infty (y-\sigma^2)^2 \frac{1}{\sigma\sqrt{2\pi y}} e^{-\frac{1}{2}(\frac{y}{\sigma^2})} dy = 2\sigma_x^4$$

The pdf of x with respect to y will result in the following:

$$f_x(y) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2}, \mu = 0$$
$$f_x(y) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}}, x^2 = y$$

The complete derived equation:

$$f_x(y) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{y}{2\sigma^2}}, y > 0$$

According to the central limit theorem the approximate theoretical pdf of An is shown as follows:

$$f(A_n) = \frac{1}{\sqrt{2\pi}\sigma_{A_n}} e^{-\frac{(A_n - \mu_{A_n})^2}{2\sigma_{A_n}^2}}, x^2 = y$$

Using the following values of n = 2, 3, 16, 144:

$$f(A_2) = N(\sigma_X^2, \sigma_X^4), y > 0$$

$$f(A_3) = N(\sigma_X^2, \frac{2\sigma_X^4}{3}), y > 0$$

$$f(A_{16}) = N(\sigma_X^2, \frac{\sigma_X^4}{8}), y > 0$$

$$f(A_{144}) = N(\sigma_X^2, \frac{\sigma_X^4}{72}), y > 0$$

B) True PDF of A_n for $n = 2$ without Approximation

$$f(A_n) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{((\frac{1}{2}(x_1+x_2)) - \sigma^2)^2}{2\sigma^4}}, y > 0$$

Calculating the bounds so we can solve for the pdf:

$$y > 0$$

$$z - y > 0$$

$y < z$ and $y > 0$, resulting in $0 > y > z$

$A_2 = \frac{1}{2}(y_1 + y_2)$, transforming into $Z = X + Y$

$$f_z(z) = \int \int_{x+y=z} f_{xy}(x, y) dx dy$$

Since x and y are independent:

$$f_z(z) = \int \int_{x+y=z} f_x(x) f_y(y) dx dy$$

Resulting in a convolution integral of:

$$f_z(z) = \int f_y(y) f_y(z - y) dy$$

Implementing the bounds:

$$\int_0^z f_y(y) f_y(z - y) dy$$

$$f_y(y) = \frac{1}{\sqrt{2\pi y}\sigma} e^{-\frac{y}{2\sigma^2}}, y > 0$$

$$f(A_2) = f(z) = \int_0^z \frac{1}{\sqrt{2\pi y}\sigma} e^{-\frac{y}{2\sigma^2}} \frac{1}{\sqrt{2\pi(z-y)}\sigma} e^{-\frac{(z-y)}{2\sigma^2}} dy$$

$$= \int_0^z \frac{1}{\sqrt{2\pi y}\sigma} \frac{1}{\sqrt{2\pi(z-y)}\sigma} e^{-\frac{z}{2\sigma^2}} dy$$

$$= e^{-\frac{z}{2\sigma^2}} \int_0^z \frac{1}{\sqrt{2\pi y}\sigma} \frac{1}{\sqrt{2\pi(z-y)}\sigma} dy$$

$$\boxed{= \frac{e^{-\frac{z}{2\sigma^2}}}{2\sigma^2}}$$

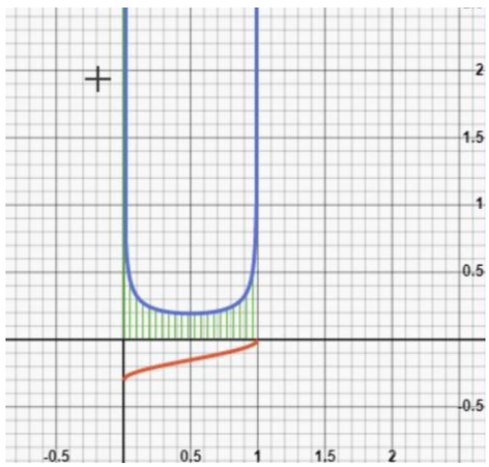

Figure 1: Graph of $f(A_2) = f(z)$

Appendix

Matlab part C using Y

```
%%% Matlab part C using Y. %%%
clear all
x=-3:0.03:3;

% theoretical pdf of standard norm.
y1=normpdf(x, 0, 1);
% generate n pts - normal distribution
bins=100000;
rx=randn(1, bins);
[y2]=hist(rx, x)/bins/(x(2)-x(1));
%{
plot(x, y1, 'o', x, y2, '');
grid; legend('theory', 'simulation')
xlabel('x'),
ylabel('pdf of stand. normal r.v.')
%}
%{
n = 2;
y = x.^2;
ry = rx.^2;
for i = 1:n
    index = randi([1, bins]);
    ysamp(i) = rx(index);
end


%fy = 1/(sqrt(2piy))exp(-y/2);
%An = sum(ysamp)/n;
z = sum(ysamp)/n;
fA2 = exp(-z/2)/2;
plot(y,fA2)
%}
y = [0.01:(.03/2):3.01];
n = 144;
hold on
fA = normpdf(y, 1, 2/n);
plot(y,fA);


fy = 1./(sqrt(2piy)).exp(-y./2);
bins=1000000;

samp=100000;
for samples = 1:samp
    for index = 1:n

        fy = fy/sum(fy); % Make sure probabilities add up to 1.
        cp = [0, cumsum(fy)];
        r = rand;
        ind = find(r>cp, 1, 'last');
        randy = y(ind);                      % https://www.mathworks.com/matlabcentral/
                % answers/506654-select-random-number-from-an-array-with-probabilities
        ysamp(index)=randy;
    end
    An(samples)=1/nsum(ysamp);

end
[y2]=hist(An,y)/(samp/(n*10));
bar(y,y2)
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

Matlab part C using X^2

```
%%% Matlab part C using X^2. %%%
clear all
sz = 0.01;
x=-3:sz:3;
y = [0:(.01/2):3];
n = 16;

% theoretical pdf of standard norm.
y1=normpdf(x, 0, 1);

% theoretical pdf of Y
fA = normpdf(y, 1, 2/n);
plot(y,fA);
hold on

%fy = 1./(sqrt(2piy)).exp(-y./2);

samp=100000;
for samples = 1:samp
    for index = 1:n

        y1 = y1/sum(y1); % Make sure probabilites add up to 1.
        cp = [0, cumsum(y1)];
        r = rand;
        ind = find(r>cp, 1, 'last');
        randx = x(ind);                    % ...
            https://www.mathworks.com/matlabcentral/answers/506654
                % -select-random-number-from-an-array-with-probabilities
        xsamp(index)=randx;
    end

    An(samples)=(1/n)sum(xsamp.^2);

end

% simulated pdf of Y
[y3]=hist(An,y)/(samp*(sz/2));
bar(y,y3)
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```