CFRM 405: Mathematical Methods for Quantitative Finance

L3: Derivative Securities

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CFRM Computational Finance & Risk Management

Outline

Announcements

Derivative Securities

Options

Arbitrage-free pricing

Put-Call parity for European options

Forward Contracts

Futures Contracts

Announcements

- Online Office Hours : Wednesdays 7:00-8:00 pm$< \textit{Adekoya} >$ and
- Tuesdays 5:00-6:00 pm$< \textit{Liu} >\ldots$
- `https://hangouts.google.com/group/hsPNoBpNV6fwfVUA6` $< \textit{Wed} >$
- `https://hangouts.google.com/group/mDwodcS2LtS42nAWA`$< \textit{Tue} >$

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  Index value (e.g., DOW, S&P 500, etc.)

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- Common Derivatives
  - Forward contracts
  - Futures contracts
  - Swaps
  - Options

Outline

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For $t < T$

- $C(t)$ = Price of the call option
- $P(t)$ = Price of the put option

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  Put Option is:
  - In the money (ITM) : $S(t) < K$

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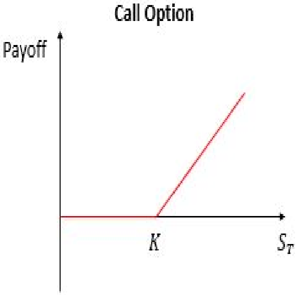
  - At the money (ATM) : $S(t) = K$
  - Out of the money (OTM) : $S(t) > K$
- Definition also applies at the expiration date T

Options

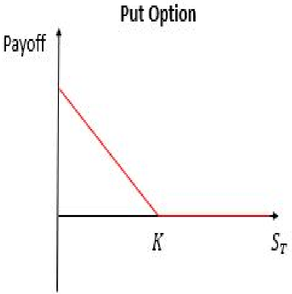
	$S(T) < K$	$S(T) = K$	$S(T) > K$
Call	OTM	ATM	ITM
Put	ITM	ATM	OTM

Payoff at Maturity (not profit, long position)

$$C(T) = \begin{cases} S(T) - K & \text{if } S(T) > K \\ 0 & \text{if } S(T) \leq K \end{cases}$$



$$P(T) = \begin{cases} 0 & \text{if } S(T) \geq K \\ K - S(T) & \text{if } S(T) < K \end{cases}$$



Outline

Arbitrage-free pricing

Arbitrage opportunity - An investment opportunity guaranteed to earn money without any risk involved

- They exist in the market but many have little practical value
- Trading costs and constant moves of the market tend to eliminate arbitrage opportunities

In an arbitrage-free market, one can make conclusions on relationships between the prices of various securities based on **The Law of One Price**

Arbitrage-free pricing

Theorem (The Law of One Price)

If two portfolios are guaranteed to have the same value at a future time $\tau > t$ regardless of the state of the market at time τ, then they must have the same value at time t. If portfolio A is more valuable than portfolio B at a future time τ regardless of the state of the market at time τ, then it is also more valuable at time $t < \tau$.

Lemma

If the value $V(T)$ of a portfolio at time T in the future is independent on the state of the market at T, then

$$V(t) = V(T)e^{-r(T-t)}$$

where $t < T$ and r is the constant risk-free rate (compounded continuously)

Example

How much are European options worth if the value of the underlying asset is 0?

- If $S(t) = 0$, then $S(\tau) \not> 0$ for $\tau > t$. In particular $S(T) = 0$
- Otherwise, buy asset at t for 0 dollars and sell as a soon as asset is worth something. (Arbitrage)

Thus

$$C(T) = 0$$
$$P(T) = K \text{ (the premium)}$$

By the Lemma

$$C(t) = 0$$
$$P(t) = Ke^{-r(T-t)}$$

Outline

Put-Call parity for European options

Let $C(t)$ and $P(t)$ be the price of the European call and put option at time t with maturity T and strike K,

Put-Call parity On a non-dividend paying asset with spot price $S(t)$

$$P(t) + S(t) - C(t) = Ke^{-r(T-t)}$$

And on an asset that pays dividends continuously at rate q with spot price $S(t)$

$$P(t) + S(t)e^{-q(T-t)} - C(t) = Ke^{-r(T-t)}$$

Proof- non dividends

Consider a portfolio with

- long 1 put option
- long 1 unit of the asset
- short 1 call option

Then the value of the portfolio at time t (if liquidating portfolio) is

$$V_{port}(t) = P(t) + S(t) - C(t)$$

At maturity T

$$V_{port}(T) = P(T) + S(T) - C(T)$$

	$P(T)$	$C(T)$	$V_{port}(T)$
$S(T) < K$	$K - S(T)$	0	$K - S(T) + S(T) - 0 = K$
$S(T) \geq K$	0	$S(T) - K$	$0 + S(T) - (S(T) - K) = K$

Proof- non dividends

Thus at maturity T

$$V_{port}(T) = K$$

Now using the Lemma

$$V_{port}(t) = Ke^{-r(T-t)}$$

Therefore the put-call parity formula says that

$$P(t) + S(t) - C(t) = Ke^{-r(T-t)}$$

Example

Consider a portfolio with the following positions:

- long one call option with strike $K_1 = 45$ for \$4
- long one call option with strike $K_2 = 55$ for \$9
- short two call options with strike $K_3 = 50$ for \$6

1) Draw the payoff diagram at maturity of the portfolio, i.e., plot the graph of the value of the portfolio at maturity $V(T)$ as a function of $S(T)$

2) What is the profit and loss at maturity?

2) When will the portfolio be profitable?

All options are on the same asset with the same maturity T and interest rates are zero

Solution

$$V(T) = C_1(T) + C_2(T) - 2C_3(T)$$

	$S(T) \leq 45$	$45 < S(T) \leq 50$	$50 < S(T) \leq 55$	$S(T) > 55$
$C_1(T)$	0	$S(T) - 45$	$S(T) - 45$	$S(T) - 45$
$C_2(T)$	0	0	0	$S(T) - 55$
$C_3(T)$	0	0	$S(T) - 50$	$S(T) - 50$
$V(T)$	0	$S(T) - 45$	$55 - S(T)$	0

Cost to set up portfolio is

$\text{Cost}(T) = \$4 + \$9 - \$12 = \$1$ (zero interest rate)

The profit and loss is given by

$$\text{P\&L} = V(T) - \text{Cost}(T) = V(T) - 1$$

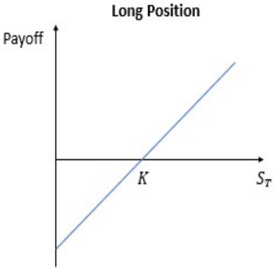
	$S(T) \leq 45$	$45 < S(T) \leq 50$	$50 < S(T) \leq 55$	$S(T) > 55$
P&L(T)	-1	$S(T) - 46$	$54 - S(T)$	-1

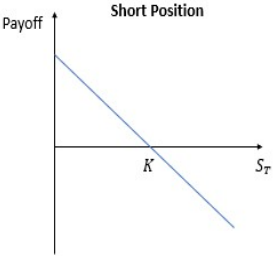
Therefore, portfolio is profitable when $46 < S(T) \leq 54$

Outline

Forward Contracts

- A **Forward Contract** is an agreement between two parties to buy or sell the underlying asset at a specified future time T for a specified price K (delivery price).
- **Note**: K is **not** the price of the contract
- Take the long position $\rightarrow$ to buy the asset
- Take the short position $\rightarrow$ to sell the asset





Forward price

The forward price F is the market price that would be agreed today for delivery of the asset at a specified maturity date

It equals K at the start, but changes as time goes on

The forward price F of a forward contract with maturity T and struck at time 0

- Non-dividend-paying underlying asset with spot price $S(0)$

$$F = S(0)e^{rT}$$

- Asset pays dividend continuously at the rate q with spot price $S(0)$

$$F = S(0)e^{(r-q)T}$$

Value of Forward contract

The value of a forward contract with maturity T and struck at time $t > 0$ with delivery price K

- Non-dividend-paying underlying asset

$$F(t) = S(t) - Ke^{-r(T-t)}$$

- Asset pays dividend continuously at the rate q

$$F(t) = S(t)e^{-q(T-t)} - Ke^{-r(T-t)}$$

Example

A long forward contract on a non-dividend paying stock was entered into some time ago. It currently has six months to maturity. The risk- free interest with continuous compounding is 10% per annum, the stock price is $25, and the delivery price is $24. What is the six-month forward price, what is the value of the forward contract, and what is the payoff of the forward contract at maturity if the stock price is $30?

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$$Payoff = S(T) - K = 30 - 24 = \$6$$

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Futures Contracts

- Has a similar structure as forward contract but needs delivery of asset for the futures price
- i.e., forward contracts can be settled in cash without the delivery of a physical asset
- Forward price and Futures price are the same when the risk-free interest rate is constant and the same for all maturities.
- In reality, interest rates vary unpredictably, so forward and futures prices will vary

Futures	Forwards
Traded on exchange	Over-the-counter
Range of delivery dates	Specified delivery date
Settled in a margin account daily	Settled in cash at maturity

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