Finite Element Method (FEM) for Heat Transfer Equation

$$\frac{d}{dx}\left(k(x)\frac{du(x)}{dx}\right) = -100x \tag{1}$$

$$u(2) = 0 \tag{2}$$

$$\frac{du(0)}{dx} + u(0) = 20 \tag{3}$$

$$k(x) = \begin{cases} 1 & \text{for } x \in [0,1] \\ 2x & \text{for } x \in (1,2] \end{cases} \tag{4}$$

$$\text{Where } u \text{ is the function you are looking for} \tag{5}$$

$$[0,2] \ni x \mapsto u(x) \in \mathbb{R} \tag{6}$$

Calculations

$$\begin{cases} -(ku')' = f \\ u(2) = 0 \\ u'(0) + u(0) = 20 \end{cases} \tag{7}$$

$$\begin{aligned} &x \in [0,2] \\ &v \in V \\ &V = \{f \in H^1, f(2) = 0\} \end{aligned} \tag{8}$$

$$-(ku')' = f \tag{9}$$

$$-(ku')'v = fv \tag{10}$$

$$-\int_\Omega (ku')'v = \int_\Omega fv \tag{11}$$

$$-ku'v\Big|_\Omega + \int_\Omega ku'v' = \int_\Omega fv \tag{12}$$

$$k(0)u'(0)v(0) - \underbrace{k(2)u'(2)v(2)}_{0} + \int_\Omega ku'v' = \int_\Omega fv \tag{13}$$

$$k(0)u'(0)v(0) + \int_\Omega ku'v' = \int_\Omega fv \tag{14}$$

$$u'(0) = 20 - u(0) \tag{15}$$

$$-u(0)k(0)v(0) + 20k(0)v(0) + \int_\Omega ku'v' = \int_\Omega fv \tag{16}$$

$$\underbrace{\int_\Omega ku'v' - u(0)k(0)v(0)}_{B(u,v)} = \underbrace{\int_\Omega fv - 20k(0)v(0)}_{L(v)} \tag{17}$$

$$u = \tilde{u} + w, \tilde{u} = 0 \text{ and thus } u = w \tag{18}$$

$$\underbrace{\int_\Omega kw'v' - w(0)k(0)v(0)}_{B(w,v)} = \underbrace{\int_\Omega fv - 20k(0)v(0)}_{L(v)} \tag{19}$$

Original problem:
Find $w \in V$ that satisfies
$B(w, v) = L(v), \ \forall v \in V$

Approximate problem:
$\forall v_h \in V_h$, find $w \in V_h$ that satisfies
$B(w_h, v_h) = L(v_h), \ \forall v_h \in V_h$
$w \approx w_h = \sum_{i=1}^{n+1} w_i e_i$
$B\left(\sum_{i=1}^{n+1} w_i e_i, v_j\right) = L(v_j)$ for $v_j = e_j$ for $j = 1, 2, \ldots, n+1$

$$\sum_{i=1}^{n+1} w_i B(e_i, e_j) = L(e_j) \tag{20}$$

$$\text{We impose the Dirichlet condition at two: } u(2) = 0 \tag{21}$$

$$\begin{bmatrix} B(e_1, e_1) & B(e_1, e_2) & \cdots & 0 \\ B(e_2, e_1) & B(e_2, e_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_{n+1} \end{bmatrix} = \begin{bmatrix} L(e_1) \\ L(e_2) \\ \vdots \\ 0 \end{bmatrix} \tag{22}$$

Requirements

Python, `numpy`, `matplotlib`, and `scipy`.