PHYS 123: Analytical Mechanics
Lecture 1: Lagrange Multipliers
July 19, 2015

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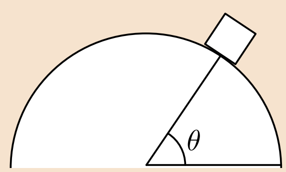
1.1 Block on a half-sphere

Let's consider a basic kinematics asdf problem which we've solve previously using classical Newtonian methods.

Example: Block on a half-sphere

Consider a half-sphere with a block sitting on its top. Using Lagrange multiplier methods, we can determine when the block will leave the surface of the half-sphere and at what angle from the vertical.



First, consider the general lagrangian

$$\mathcal{L} = T - U$$

where

$$\mathcal{L} = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - mgz$$

so,

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{y}}\right) &= 0 \\ 0 - \frac{\mathrm{d}}{\mathrm{d}t}\left(m\dot{y}\right) &= 0 \\ m\ddot{y} &= 0 \end{aligned}$$

Now, note that $y_0 = 0$ and $\dot{y}(0) = 0$, so we can eliminate $y(t)$

$$\mathcal{L} = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) - mgz$$

if we then switch to spherical coordinates

$$\mathcal{L} = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - mgr\cos(\theta)$$

utilizing the constraint that $r - a = 0$ such that

$$\mathcal{L}^* = \mathcal{L} + \lambda(r - a)$$

where λ is a force constraint only satisfied when the block is in contact with the surface of the half-sphere. If we use the Euler-Lagrange equation in terms of $\mathcal{L}^*$

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial r} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}^*}{\partial \dot{r}}\right) &= 0 \\ mr\dot{\theta}^2 - mgr\cos(\theta) + \lambda - m\ddot{r} &= 0 \\ m\ddot{r} - mr\dot{\theta}^2 &= \lambda - mg\cos(\theta) \end{aligned} \tag{1.1}$$

where we can see that λ in Eq. (1.1) is in fact just the normal force. Now, we consider

$$\begin{aligned} \frac{\partial \mathcal{L}^*}{\partial \theta} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}^*}{\partial \dot{\theta}}\right) &= 0 \\ mgr\sin(\theta) - \frac{\mathrm{d}}{\mathrm{d}t}\left(mr^2\dot{\theta}\right) &= 0 \\ mgr\sin(\theta) - m\left(2r\dot{r}\dot{\theta} + r^2\ddot{\theta}\right) &= 0 \end{aligned} \tag{1.2}$$

and, finally,

$$\begin{aligned} \frac{\partial \mathcal{L}^*}{\partial \lambda} &= r - a = 0 \\ r &= a \\ \dot{r}(t) &= 0 \longrightarrow \ddot{r}(t) = 0 \end{aligned} \tag{1.3}$$

so, we can solve for the equation of motion with these three equations (i.e. Equation (1.1), (1.2) and (1.3)) and applying the condition that $\dot{r} = 0$ and $\ddot{r} = 0$, we have

$$\begin{aligned} -mr\dot{\theta}^2 &= \lambda - mg\cos(\theta) \\ mgr\sin(\theta) - mr^2\ddot{\theta} &= 0 \\ r &= a \end{aligned}$$

and then substituting in the condition $r = a$ and integrating over time

$$\int_0^t mga\sin(\theta)\dot{\theta}\,\mathrm{d}\,t = \int_0^t ma^2\ddot{\theta}\dot{\theta}\,\mathrm{d}\,t$$

and performing a change of base (i.e. $t \to \theta$)

$$\begin{aligned} \int_0^\theta mga\sin(\theta)\, d\theta &= \int_0^\theta ma^2\dot{\theta}\, d\dot{\theta} \\ -mga\cos(\theta)\Big|_0^\theta &= \frac{1}{2}ma^2\dot{\theta}^2 \\ mga(1-\cos(\theta)) &= \frac{1}{2}ma^2\dot{\theta}^2 \end{aligned}$$

1.2 Generalized Momentum

Consider the Lagrangian

$$\mathcal{L} = \frac{1}{2}m\left(\dot{x}^2 + \dot{z}^2\right) - mgz \tag{1.4}$$

and

$$\frac{\partial \mathcal{L}}{\partial \dot{x}} = m\dot{x} = \mathbf{p}_x$$

with the Euler-Lagrange equation

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) &= 0 \\ 0 - \frac{\mathrm{d}}{\mathrm{d}t}\left(\mathbf{p}_x\right) &= 0 \end{aligned}$$

Considering translational invariance in the x-direction (i.e. conservation of momentum $\mathbf{p}_x$), we let

$$\mathcal{L} = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - mgr\cos(\theta)$$

In this case, $\mathbf{p}_r$ is not conserved since the momentum is dependent on the radius,

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial \dot{r}} &= m\dot{r} = \mathbf{p}_r \\ \frac{\partial \mathcal{L}}{\partial \dot{\theta}} &= mr^2\dot{\theta} = \mathbf{p}_\theta \end{aligned}$$

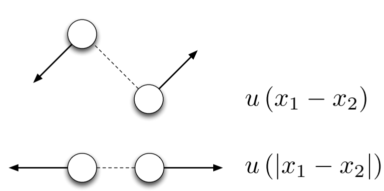


Figure 1.1: Our standard convention for momentum