Math 215 Linear Algebra
Project 1

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Exercise 1

We have the following four equations, one for each satellite:

$$2x+4y+0z-2(0.047)^2(19.9)t = 1^2+2^2+0^2-0.047^2(19.9)^2+x^2+y^2+z^2-0.047^2t^2 \tag{E1}$$

$$4x+0y+4z-2(0.047)^2(2.40)t = 2^2+0^2+2^2-0.047^2(2.40)^2+x^2+y^2+z^2-0.047^2t^2 \tag{E2}$$

$$2x+2y+2z-2(0.047)^2(32.6)t = 1^2+1^2+1^2-0.047^2(32.6)^2+x^2+y^2+z^2-0.047^2t^2 \tag{E3}$$

$$4x+2y+0z-2(0.047)^2(19.9)t = 2^2+1^2+0^2-0.047^2(19.9)^2+x^2+y^2+z^2-0.047^2t^2 \tag{E4}$$

Exercise 2

We can see there is a quadratic term that appears on the right hand side of all the equations. We can subtract Equation (E1) from each equation to eliminate that quadratic term.

(L1) = (E2) - (E1), (L2) = (E3) - (E1), and (L3) = (E4) - (E1)

$$2x - 4y + 4z + (0.077315)t = 3.86206225 \tag{L1}$$

$$0x - 2y + 2z - (0.056109)t = -3.47285075 \tag{L2}$$

$$2x - 2y + 0z + 0t = 0 \tag{L3}$$

Exercise 3

Now we can represent equations L1, L2, and L3 as an augmented matrix.

$$\left[\begin{array}{cccc|c} 2 & -4 & 4 & 0.077315 & 3.86206225 \\ 0 & -2 & 2 & -0.056109 & -3.47285075 \\ 2 & -2 & 0 & 0 & 0 \end{array}\right]$$

Then we can use Octave to solve.

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octave:1>A = [2, -4, 4, 0.077315, 3.86206225; 0, -2, 2, -0.056109, -3.47285075;
              2, -2, 0, 0, 0]
octave:2>rref(A)
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We get the resulting matrix in reduced echelon form.

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0.09477 & 5.40388 \\ 0 & 1 & 0 & 0.09477 & 5.40388 \\ 0 & 0 & 1 & 0.06671 & 3.66746 \end{array}\right]$$

We can see that x, y, and z are basic variables, while t is a free variable.

We get the following solution:

$$\begin{cases} x = 5.40388 - 0.09477t \\ y = 5.40388 - 0.09477t \\ z = 3.66746 - 0.06671t \\ t = t \end{cases}$$

Exercise 4

The next step is to plug our variables x, y, z into the equation (E1) to solve for t. This should give us a quadratic in terms of t which we can use to find the location of our GPS receiver.

$$\begin{aligned} 2(5.40388 - 0.09477t) + 4(5.40388 - 0.09477t) \\ + 0(3.66746 - 0.06671t) - 0.087918t = \\ 1^2 + 2^2 + 0^2 - 0.047^2(19.9)^2 + (5.40388 - 0.09477t)^2 + \\ (5.40388 - 0.09477t)^2 + (3.66746 - 0.06671t)^2 - 0.047^2t^2 \end{aligned}$$

We can begin simplifying this equation:

$$\begin{aligned}10.8078 - 0.18954t + 21.6155 - 0.37908t - 0.087918t = \\ 5 - 0.87478609 + 0.00898135t^2 - 1.02425t + 29.2019 \\ +0.00898135t^2 - 1.02425t + 29.2019 \\ +0.00445022t^2 - 0.489313t + 13.4503 - 0.002209t^2\end{aligned}$$

And we can simplify further:

$$32.4233 - 0.656538t = 0.0202039t^2 - 2.53781t + 75.9793$$

This gives us the following quadratic equation:

$$0.0202039t^2 - 1.88127t + 43.556 = 0$$

We solve this to get the roots:

$$(t = 43.1304), (t = 49.9838)$$

To see which time value is correct, we can plug each one into the x, y, z definitions and see if the x,y,z values work in the following equation that we know is true for any point on earth at sea level:

$$x^2 + y^2 + z^2 = 1$$

Let's start with t = 43.1304.

$$\begin{cases} x = 5.40388 - 0.09477(43.1304) = 1.316468854 \\ y = 5.40388 - 0.09477(43.1304) = 1.316468854 \\ z = 3.66746 - 0.06671(43.1304) = 0.790231016 \\ t = 43.1304 \end{cases}$$

$$1.316468854^2 + 1.316468854^2 + 0.790231016^2 = 4.0906 \neq 1$$

We can see that t = 43.1304 is not the correct time value.

Now let's try t = 49.9838

$$\begin{cases} x = 5.40388 - 0.09477(49.9838) = 0.666915274 \\ y = 5.40388 - 0.09477(49.9838) = 0.666915274 \\ z = 3.66746 - 0.06671(49.9838) = 0.333040702 \\ t = 49.9838 \end{cases}$$

$$0.666915274^2 + 0.666915274^2 + 0.333040702^2 = 1.000$$

It looks like this time value is correct! Hooray!

So the location of our GPS receiver in x,y,z coordinates is:

$$(x, y, z) = (0.666915274, 0.666915274, 0.333040702)$$

Or in fractional form:

$$(x, y, z) = (\frac{2}{3}, \frac{2}{3}, \frac{1}{3})$$

Exercise 5

For this section we want to convert our x,y,z coordinates to meters (using the fact that the radius of earth is 6,377,563.396 meters). Then we want to convert into latitude and longitude coordinates.

Our x,y,z values in meters:

$$x_{meters} = x * 6,377,563.396 = 0.666915274 * 6,377,563.396 = 4,253,294.4396$$

$$y_{meters} = y * 6,377,563.396 = 0.666915274 * 6,377,563.396 = 4,253,294.4396$$

$$z_{meters} = y * 6,377,563.396 = 0.333040702 * 6,377,563.396 = 2,123,988.1904$$

Now let's use the longitude equation to calculate longitude.

$$\lambda = \arctan(\frac{x}{y}) = \arctan(\frac{4,253,294.4396}{4,253,294.4396}) = \frac{\pi}{4} = 45^\circ$$

Next we need to calculate latitude, which will be more complicated. For our calculations we will use the Airy 1830 ellipsoidal to represent the Earth, this gives a = 6,377,563.396m and b = 6,356,256.910m. We also know the following:

$$p = \sqrt{x^2 + y^2} = 6,015,066.6812$$

$$e = \frac{a^2 - b^2}{a^2} = 0.0066705$$

We begin by calculating initial values.

$$\phi_0 = \arctan(\frac{z}{p(1-e)}) = \arctan(\frac{2,123,988.1904}{5,974,943.1789}) = 19.57^\circ$$

$$v_0 = \frac{a}{\sqrt{1 - e\sin^2(\phi_0)}} = \frac{6,377,563.396}{\sqrt{1 - (0.0066705)\sin^2(19.57^\circ)}}$$

$$= \frac{6,377,563.396}{0.9985473)} = 6,386,841.5607$$

Now we can use Octave to solve the following iterative equations for our latitude coordinate:

$$\phi_i = \arctan(\frac{z + ev_{i-1}\sin(\phi_{i-1})}{p}) \quad i = 1, 2, 3, ...$$

$$v_i = \frac{a}{\sqrt{1 - e\sin^2(\phi_i)}} \quad i = 1, 2, 3, ...$$

First Iteration:

octave:1> *atan*(((2123988.1904) - (0.0066705)*(6386841.5607)*sin(19.57)) / (6015066.6812))

$$\phi_1 = \arctan(\frac{z + ev_0\sin(\phi_0)}{p}) = \arctan(\frac{z + e(6,386,841.5607)\sin(19.57°)}{p}) = 0.33528$$

$octave : 2 >\ (6,377,563.396)/(sqrt(1 - (0.0066705) * (sin(0.33528)^2)))$

$$v_1 = \frac{a}{\sqrt{1 - e\sin^2(\phi_1)}} = \frac{a}{\sqrt{1 - e\sin^2(0.33528)}} = 6,379,867.48$$

Second Iteration:

octave:3> *atan*(((2123988.1904) - (0.0066705)*(6379867.48)*sin(0.33528)) / (6015066.6812))

$$\phi_2 = \arctan(\frac{z + ev_1\sin(\phi_1)}{p}) = \arctan(\frac{z + e(6379867.48)\sin(0.33528)}{p}) = 0.33737$$

$octave : 4 >\ (6377563.396)/(sqrt(1 - (0.0066705) * (sin(0.33737)^2)))$

$$v_2 = \frac{a}{\sqrt{1 - e\sin^2(\phi_2)}} = \frac{a}{\sqrt{1 - e\sin^2(0.33737)}} = 6,379,895.21$$

Third Iteration:

octave:5> *atan*(((2123988.1904) - (0.0066705)*(6379895.21)*sin(0.33737)) / (6015066.6812))

$$\phi_3 = \arctan(\frac{z + ev_2\sin(\phi_2)}{p}) = \arctan(\frac{z + e(6,379,895.21)\sin(0.33737)}{p}) = 0.33736$$

$octave : 6> (6377563.396)/(sqrt(1 - (0.0066705) * (sin(0.33736)^2)))$

$$v_3 = \frac{a}{\sqrt{1 - e\sin^2(\phi_3)}} = \frac{a}{\sqrt{1 - e\sin^2(0.33736)}} = 6,379,895.08$$

We convert ϕ_3 to degrees:

$$\phi_3 = 0.33736 \text{ rad } = 19.3293^\circ$$

Now we have our final answer, the location of our GPS receiver in latitude and longitude coordinates:

$$\boxed{\begin{array}{l} \phi = 19.3293^\circ \ latitude \\ \lambda = 45^\circ \ longitude \end{array}}$$