Local and Global Error

Theorem [General Error Bound]

Consider an IVP solver for a Lipschitz function with Lipschitz constant L in which the local truncation error satisfies the bound $e_i \leq ch^{k+1}$ for some constants c and k. Then the **global errors** E_i satisfy the bound

$$E_i \leq \frac{ch^k}{L}(e^{Lhi} - 1)$$

The proof is almost identical to the analysis of global error for forward Euler (which is recovered under the choice $k = 1$ and $c = M/2$).

Higher-Order Taylor Methods

Forward Euler is a 1st-order method: local error is $\mathcal{O}(h^2)$ and global error is $\mathcal{O}(h)$.

Taylor methods are one family of techniques to create higher-order accuracy. They rely on higher-order Taylor expansions:

$$y(t) = y(t_0) + (t - t_0)y'(t_0) + \frac{(t - t_0)^2}{2}y''(t_0) + \frac{(t - t_0)^3}{3!}y'''(t_0) + \frac{(t - t_0)^4}{4!}y^{(4)}(t_0) + \dots$$

Taylor Methods

Definition

A kth-order **Taylor method** solves an IVP with the iteration

$$y_{i+1} = y_i + hf(t_i, y_i) + \frac{h^2}{2} f'(t_i, y_i) + \ldots + \frac{h^k}{k!} f^{(k-1)}(t_i, y_i)$$

The “prime” notation $f'(t, y)$ is the **total derivative**, which can be obtained with **partial derivatives** $f_t(t, y)$ and $f_y(t, y)$:

$$f'(t, y) = \frac{d}{dt} f(t, y(t)) = f_t + f_y \frac{\partial y}{\partial t} = f_t + f_y f$$
$$f'(t_i, y_i) = f_t(t_i, y_i) + f_y(t_i, y_i) f(t_i, y_i)$$

(See note for $f''(t, y)$)

Local and Global Error

Theorem

A kth-order Taylor method has **local truncation error** $e_i \leq ch^{k+1}$ for all i, where $c = \frac{M}{(k+1)!}$ and $|y^{(k+1)}(t)| \leq M$.

Theorem

The **global error** E_i for a k-th order Taylor method is bound by

$$E_i \leq \frac{ch^{(k)}}{L}(e^{Lhi} - 1)$$

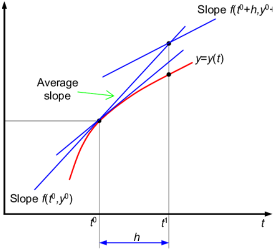
(See notes for proof.)

Explicit Trapezoidal Method

Definition

The **explicit trapezoidal method** solves an IVP with the iteration

$$\tilde{y}_{i+1} = y_i + hf(t_i, y_i)$$
$$y_{i+1} = y_i + \frac{h}{2}\left[f(t_i, y_i) + f(t_{i+1}, \tilde{y}_{i+1})\right]$$



Local and Global Error

Theorem

The **local truncation error** e_i for the explicit trapezoidal method is third order, $e_i = \mathcal{O}(h^3)$. and if $|y'''(t)| \leq M$ for some constant M, then $e_i \leq ch^3$ for all i for some c

Theorem

Considering an IVP with a Lipschitz function with Lipschitz constant L, the **global error** E_i for the explicit trapezoidal method is bound by

$$E_i \leq \frac{ch^2}{L}(e^{Lhi} - 1)$$

(See notes for proof.)