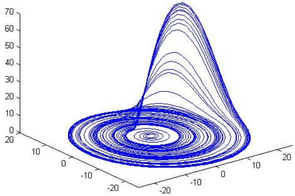
Solving a System of ODEs

How do we solve a system, such as a Rossler system with 3 variables $x(t)$, $y(t)$ and $z(t)$ and 3 coupled differential equations?



$$\begin{cases} \frac{dx}{dt} = f_1(x,y,z) = -y - z \\ \frac{dy}{dt} = f_2(x,y,z) = x + ay \\ \frac{dz}{dt} = f_3(x,y,z) = b + z(x-c). \end{cases}$$

Solving a System of ODEs

Definition

A **system of ODEs** is a set of ODEs describing coupled functions $y_1(t), \ldots, y_p(t)$. Letting $\mathbf{y}(t) = [y_1(t), \ldots, y_p(t)]$, each equation takes the form

$$\frac{dy_j}{dt} = f_j(t, \mathbf{y}(t)), \quad j = 1, 2, \ldots, p$$

and the entire system can be written as

$$\frac{d\mathbf{y}(t)}{dt} = F(t, \mathbf{y}(t)),$$

where $F(t, \mathbf{y}(t)) = [f_1(t, \mathbf{y}(t)), \ldots, f_p(t, \mathbf{y}(t))]$.

Forward Euler for a System of ODEs

Definition

An **initial value problem (IVP) for a system of ODEs** aims to solve for the functions $y_j(t)$ over a domain $t \in [t_0, T]$, and it requires a set of initial conditions $y_1(t_0), \dots, y_n(t_0)$.

Definition

The **forward Euler method for a system of ODEs** approximates the solutions $y_j(t)$ at times $t_i = t_0 + hi$ with $h = (T - t_0)/N$ by setting $y_{0,j} = y_j(t_0)$ and iterating

$$y_{i+1,j} = y_{i,j} + hf_j(t_i, \mathbf{y}_i), \quad i = 0, 1, 2, \dots, N, \quad j = 1, 2, \dots, p$$

where $\mathbf{y}_i = [y_{i,1}, \dots, y_{i,p}]$.

Forward Euler for a System of ODEs

Example: Solve the susceptible-infected-susceptible (SIS) epidemic model. Let $S(t)$ be the fraction of susceptible persons and $I(t)$ be the fraction of infected persons. Let $\mathbf{y}(t) = [S(t), I(t)]$ and define

$$\frac{dS(t)}{dt} = f_1(t, [S(t), I(t)]) = -\beta I(t)S(t) + \gamma I(t)$$
$$\frac{dI(t)}{dt} = f_2(t, [S(t), I(t)]) = \beta I(t)S(t) - \gamma I(t)$$

where $\beta > 0$ is the transmission rate and $\gamma > 0$ is the healing rate.

(See notes and code.)

Existence and Uniqueness of a Solution

Lipschitz continuity also applies to vector-valued functions

Definition

A vector-valued function $F(t, \mathbf{y})$ is **Lipschitz continuous** over a region $\mathcal{S} = [t_0, T] \times \mathcal{D}$ if there exists a **Lipschitz constant** $L > 0$ such that

$$||F(t, \mathbf{y}) - F(t, \mathbf{z})|| \leq L||\mathbf{y} - \mathbf{z}||$$

for any $t \in [t_0, T]$ and $\mathbf{y}, \mathbf{z} \in \mathcal{D}$.

Theorem

Assume $F(t, \mathbf{y})$ is Lipschitz continuous over a region $\mathcal{S} = [t_0, T] \times \mathbb{R}^p$. Then the IVP with initial condition $\mathbf{y}(t_0) = \mathbf{y}^{(0)}$ has exactly one solution $\mathbf{y}(t)$ over $t \in [t_0, T]$.

Forward Euler for Higher-order ODEs

We have so far focused on solving 1st order differential equations:

$$y'(t) = \frac{dy}{dt}(t) = f(t, y(t))$$

We can also solve a k-th order ODE

$$y^{(k)}(t) = \frac{d^k y}{dt^k}(t) = f(t, y(t), y'(t), y''(t), \ldots, y^{(k)}(t))$$

by transforming it into a system of ODEs

$$\begin{aligned}
y_1(t) &= y(t) \\
y_2(t) &= y'(t) = \frac{dy}{dt}(t) \\
y_3(t) &= y''(t) = \frac{d^2 y}{dt^2}(t) \\
&\vdots \\
y^{(k)}(t) &= \frac{d^k y}{dt^k}(t)
\end{aligned}$$

Forward Euler for Higher-order ODEs

Example: Solve

$$y'''(t) = \frac{d^3y}{dt^3}(t) = y(t) + 2y'(t) + y''(t)$$

with initial conditions $y(0) = 1$, $y'(0) = 1$ and $y''(0) = 0$

***Note this is a linear, third-order differential equation.

(See notes and code.)