Collocation Method

Given a 2nd-order BVP

$$\begin{cases} y''(t) & = f\big(t, y(t), y'(t)\big) \\ y(a) & = y_a \\ y(b) & = y_b, \end{cases}$$

we approximate the function as

$$y(t) = \sum_{j=0}^{N} c_j \phi_j(t) = \sum_{j=0}^{N} c_j t^j,$$

In principle, one can choose other basis functions besides polynomials $\phi_j(t) = t^j$.

Collocation Method

First we define a mesh of time points, $t_0 = a, t_1, \dots, t_{N-1}, t_N = b$, where $t_i = a + ih$ and $h = (b-a)/N$.

Then, we obtain the unknown coefficients c_j by satisfying the two boundary conditions

$$y(a) = \sum_{j=0}^{N} c_j a^j, \qquad y(b) = \sum_{j=0}^{N} c_j b^j$$

and then by solving the differential equation at the interior time points

$$y''(t_i) = f\big(t_i, y(t_i), y'(t_i)\big), \qquad i = 1, 2, \dots, N-1$$

Collocation Method

These equations use that

$$y(t) = \sum_{j=0}^{N} c_j \phi_j(t) = \sum_{j=0}^{N} c_j t^j$$

$$y'(t) = \sum_{j=0}^{N} c_j \phi_j'(t) = \sum_{j=0}^{N} c_j \; j t^{j-1}$$

$$y''(t) = \sum_{j=0}^{N} c_j \phi_j''(t) = \sum_{j=0}^{N} c_j \; j(j-1) t^{j-2}$$

Collocation Example

$$\begin{cases} y''(t) &= 4y(t) \\ y(0) &= 1 \\ y(1) &= 3. \end{cases}$$

Observe for each t_i that the differential equation implies

$$\sum_{j=0}^{N} c_j \, j(j-1)t_i^{j-2} = 4\sum_{j=0}^{N} c_j t_i^j$$

(See notes and code.)

Finite Elements Method

Similar to collocation, we numerically solve a 2nd-order BVP $y''(t) = f\big(t, y(t), y'(t)\big)$ by approximating the function as

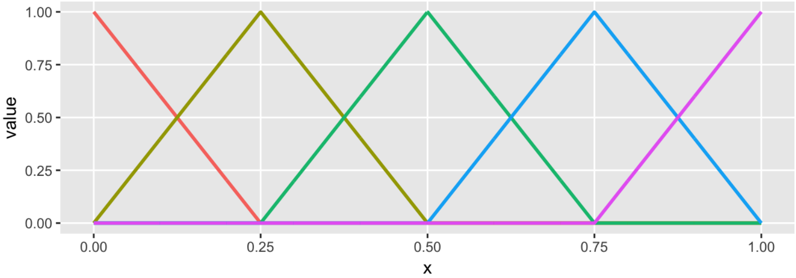
$$y(t) = \sum_{j=0}^{N} c_j \phi_j(t),$$

and we again must solve for the coefficients c_j.

- For collocation we assumed $\phi_j(t) = t^j$, whereas for **finite elements** we assume $\phi_j(t)$ are B-splines.
- Moreover, for collocation we obtained the c_j values by evaluating the system at times t_i; however, for the **Galerkin finite element** method the c_j values minimize the squared error of the differential equation *along the solution*.

Piecewise Linear B-Splines

Each $\phi_j(t)$ is a "tent" function that is zero everywhere, except for the range $t \in [t_{j-1}, t_{j+1}]$, over which it forms a linear tent.



Observe that $\phi_j(t_i) = 1$ when $i = j$ and $\phi_j(t_i) = 0$ when $i \neq j$. Therefore if $y(t) = \sum_{j=0}^{N} c_j \phi_j(t)$, then $y(t_i) = c_i$.

Piecewise Linear B-Splines

$$\phi_0(t) = \begin{cases} \frac{t_1 - t}{t_1 - t_0} & \text{for } t_0 \leq t \leq t_1 \\ 0 & \text{otherwise} \end{cases}$$

$$\phi_N(t) = \begin{cases} \frac{t - t_{N-1}}{t_N - t_{N-1}} & \text{for } t_{N-1} \leq t \leq t_N \\ 0 & \text{otherwise} \end{cases}$$

$$\phi_i(t) = \begin{cases} \frac{t - t_{i-1}}{t_i - t_{i-1}} & \text{for } t_{i-1} \leq t \leq t_i \\ \frac{t_{i+1} - t}{t_{i+1} - t_i} & \text{for } t_i \leq t \leq t_{i+1} \\ 0 & \text{otherwise} \end{cases}$$

(See code.)

Galerkin Finite Element Method

Consider solving $y''(t) = f\left(t, y, y'\right)$ with the assumed form $y(t) = \sum_{j=0}^{N} c_j \phi_j(t)$. To minimize the squared error, the residual $y''(t) - f\left(t, y, y'\right)$ must be orthogonal to the basis functions so that

$$\int_a^b \Big(y''(t) - f(t, y, y')\Big)\phi_i(t)dt,$$

which is called the **weak form** of the BVP.

(See notes.)

Galerkin Finite Element Method

This can also be simplified using integration by parts"

$$\begin{aligned}\int_a^b f(t,y,y')\phi_i(t)dt &= \int_a^b y''(t)\phi_i(t)dt \\ &= y'(b)\phi_i(b) - y'(a)\phi_i(a) - \int_a^b y'(t)\phi_i'(t)dt \\ &= 0 - \int_a^b y'(t)\phi_i'(t)dt,\end{aligned}$$

which assumes i is neither 0 or N and uses that $\phi_i(a) = \phi_i(t_0) = 0$ and $\phi_i(b) = \phi_i(t_N) = 0$.

Galerkin Finite Element Method

The coefficients c_j in $y(t) = \sum_{j=0}^{N} c_j \phi_j(t)$ can thus be solved as:

$$j = 0: \qquad y(a) = \sum_{j=0}^{N} c_j \phi_j(a) = c_0$$

$$j = N: \qquad y(b) = \sum_{j=0}^{N} c_j \phi_j(b) = c_N$$

$$\text{other } j: \qquad \int_a^b f(t, y, y')\phi_i(t)dt + \int_a^b y'(t)\phi_i'(t)dt = 0$$

Galerkin Finite Element Method

Importantly, these integrals can be simplified after using the following properties of piecewise-linear B-splines

$$\langle \phi_i, \phi_j \rangle = \int_a^b \phi_i(t)\phi_j(t)dt = \begin{cases} 2h/3 & \textit{if } j = i \\ h/6 & \textit{if } |j-i| = 1 \\ 0 & \textit{otherwise} \end{cases}$$

$$\langle \phi_i', \phi_j' \rangle = \int_a^b \phi_i'(t)\phi_j'(t)dt = \begin{cases} 2/h & \textit{if } j = i \\ -1/h & \textit{if } |j-i| = 1 \\ 0 & \textit{otherwise} \end{cases}$$

Collocation Methods

Example:

$$\begin{cases} y''(t) &= 4y(t) \\ y(0) &= 1 \\ y(1) &= 3. \end{cases}$$

Note that we consider the equations

$$\begin{aligned} 0 &= \int_0^1 \phi_i(t) f(t, y, y') dt + \int_0^1 \phi_i'(t) y'(t) dt \\ &= \int_0^1 \left[\phi_i(t) 4 \sum_{j=0}^{N} c_j \phi_j(t) + \phi_i'(t) \sum_{j=0}^{N} c_j \phi_j'(t) \right] dt \end{aligned}$$

(See notes and code.)