Runge-Kutta (RK) Methods

Recall that the Trapezoidal method has a smaller error than forward Euler, but note it requires finding the slope at two different times: t_i and t_{i+1} (i.e., 2 "steps").

Definition

Runge-Kutta (RK) methods are explicit "multistage" methods for solving an IVP that take the form

$$y_{i+1} = y_i + h\tilde{f}(t_i, y_i),$$

where $\tilde{f}(t_i, y_i)$ is "best" slope that is obtained as a weighted average of k different slopes (i.e., k "stages").

For the trapezoidal method, $\tilde{f}(t_i, y_i) = \frac{1}{2}\left[f(t_i, y_i) + f(t_{i+1}, \tilde{y}_{i+1})\right]$

Midpoint Methods

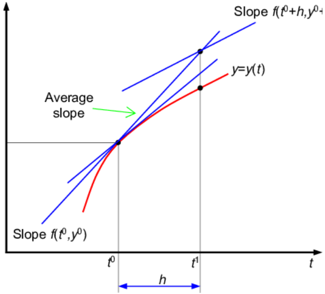
Definition

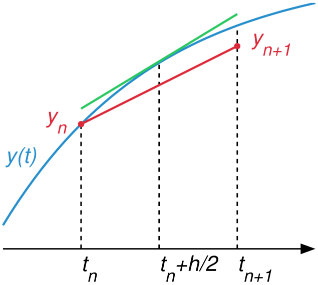
The **midpoint method** is a 2-stage method that solves an IVP with the iteration

$$\begin{aligned}\tilde{y}_{i+1/2} &= y_i + \frac{h}{2} f(t_i, y_i) \\ y_{i+1} &= y_i + hf(t_i + h/2, \tilde{y}_{i+1/2})\end{aligned}$$

2nd Order Runge-Kutta Method (RK2)

The Trapezoidal and Midpoint methods are Runge-Kutta methods with 2 stages





(See note.)

Local and Global Error

Midpoint's error is similar to trapezoidal's:

Theorem

The **local truncation error** e_i for the midpoint method is third order, $e_i = \mathcal{O}(h^3)$. and if $|y'''(t)| \leq M$ for some constant M, then $e_i \leq Ch^3$ for all i for some C

Theorem

Considering an IVP with a Lipschitz function with constant L, the **global error** E_i for the midpoint method is bound by

$$E_i \leq \frac{Ch^2}{L}(e^{Lhi} - 1)$$

(See notes for proof.)

4th Order Runge-Kutta Method (RK4)

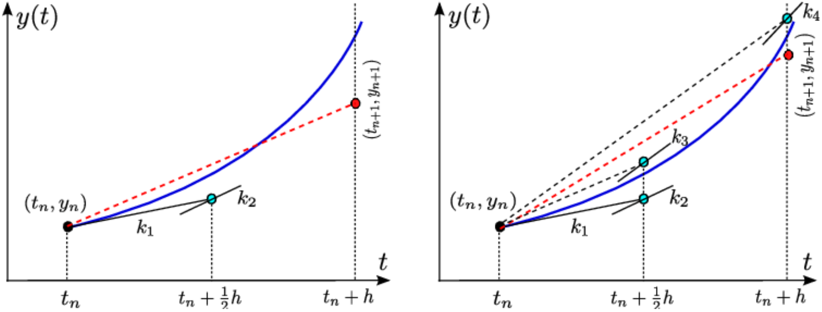
Definition

The **4th-order Runge-Kutta method (RK4)** solves an IVP with $y_0 = y(t_0)$ and the iteration

$$y_{i+1} = y_i + \frac{h}{6}\left[k_1 + 2k_2 + 2k_3 + k_4\right]$$
$$k_1 = f(t_i, y_i)$$
$$k_2 = f\left(t_i + \frac{h}{2}, y_i + \frac{h}{2}k_1\right)$$
$$k_3 = f\left(t_i + \frac{h}{2}, y_i + \frac{h}{2}k_2\right)$$
$$k_4 = f\left(t_i + h, y_i + hk_3\right)$$

4th Order Runge-Kutta Method (RK4)

Comparison of RK2 and RK4



4th Order Runge-Kutta Method (RK4)

Example: Solve

$$y'(t) = (t - y)/2$$

over $t \in [0, 10]$ with initial condition $y(0) = 1$.

(See notes and code.)

Error for 4th Order Runge-Kutta (RK4)

Theorem

The **local truncation error** e_i for RK4 is fifth order, $e_i = \mathcal{O}(h^5)$ so that $e_i \leq ch^5$ for all i for some c

Theorem

Given an IVP with a function $f(t, y)$ with Lipschitz constant L, the **global error** E_i for the midpoint method is bound by

$$E_i \leq \frac{ch^4}{L}(e^{Lhi} - 1)$$

(See notes for context.)

RK Pairs and Adaptive Step Sizes

Thus far, we studied approximation error $E_i = |y(t_i) - y_i|$ using the true function $y(t)$, which is typically unknown.

In practice, we must approximate E_i by comparing one approximation y_i to a more accurate approximation, such as one obtained using smaller time steps or a higher-order method.

Definition

RK Pairs allow one to efficiently and simultaneously develop both a higher-accuracy and a lower-accuracy predict together.

RK Pairs and Adaptive Step Sizes

Definition

The **RK 2/3 Method** simultaneously provides both 2nd-order and 3rd-order approximations to an IVP with $y_0 = y(t_0)$ and using the iteration

$$
\begin{aligned}
y_{i+1} &= y_i + h\frac{(s_1 + s_2)}{2} \qquad\qquad (\textit{trapezoidal}) \\
z_{i+1} &= z_i + \frac{h}{6}\left[s_1 + s_2 + 4s_3\right] \\
s_1 &= f(t_i, y_i) \\
s_2 &= f\left(t_i + h, y_i + hs_1\right) \\
s_3 &= f\left(t_i + 0.5h, y_i + 0.5h\frac{(s_1 + s_2)}{2}\right)
\end{aligned}
$$

RK Pairs and Adaptive Step Sizes

One can estimate the error $E_{i+1} = |y_{i+1} - z_{i+1}|$ and adjust the time step size h to ensure the error is smaller that some tolerance ϵ.

For example, if $E_{i+1} > \epsilon$ is too large, one can decrease h by a factor qh with $q < 1$ to decrease the new error $\hat{E}_i$ below ϵ

If we know the method is order k, i.e., $E_i \leq ch^k$ for some constant c, then $\hat{E}_i$ for step size qh becomes $\hat{E}_i \leq c(qh)^k = q^k E_i < \epsilon$ if

$$q < (\epsilon / E_i)^{1/k}.$$