Finite Difference Methods

Solve a BVP by using finite difference approximations to construct a large system in which all the approximations $y_0, y_1, y_2, \dots, y_N$ are solved at the same time.

Example: 3 finite differences to estimate the slope at t_i:

$$
\begin{aligned}
y'(t) &= \frac{y(t+h)-y(t)}{h} - \frac{h}{2}y''(\xi) \quad (\textit{forward difference})\\
y'(t) &= \frac{y(t)-y(t-h)}{h} + \frac{h}{2}y''(\xi) \quad (\textit{backward difference})\\
y'(t) &= \frac{y(t+h)-y(t-h)}{2h} - \frac{h^2}{6}y'''(\xi) \quad (\textit{centered difference})
\end{aligned}
$$

(See notes.)

Higher-Order Finite Differences

Example: Approximating the second derivative

$$y''(t) = \frac{y(t+h) - 2y(t) + y(t-h)}{h^2} - \frac{h^2}{12} y^{(4)}(\xi)$$

(See notes.)

Finite Difference Methods

Consider a 2nd-order boundary value problem

$$\begin{cases} y''(t) & = f(t,y) \\ y(a) & = y_a \\ y(b) & = y_b. \end{cases}$$

The finite difference method is as follows:

- Choose the number N of steps and let $h = (b-a)/N$ denote the step size
- Define a mesh $t_0, t_1, \ldots, t_N$ so that $t_0 = a$ and $t_N = b$.
- Define an approximation in which $y_0 = y(a)$, $y_N = y(b)$, and the remaining y_i values are found by approximating $y''(t)$ and $f(t,y)$ at each t_i and y_i using finite differences.

Example 1: Linear BVP

$$\begin{cases} y''(t) &= 4y \\ y(0) &= 1 \\ y(1) &= 3. \end{cases}$$

Let y_i estimate $y(t_i)$ and set $y_0 = 1$ and $y_N = 3$. We obtain the other estimates through a finite difference equation:

$$y''(t_i) \approx \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} = 4y_i$$

or

$$y_{i+1} + (-2 - 4h^2)y_i + y_{i-1} = 0$$

(See notes and code.)

Newton's Root-finding Method

Definition

Newton's method for root finding uses the iterative process

$$x_{t+1} = x_t - \frac{f(x_t)}{f'(x_t)} \qquad \text{(scalar } x\text{)} \qquad (1)$$

$$\mathbf{x}^{(t+1)} = \mathbf{x}^{(t)} - [J(\mathbf{x}^{(t)})]^{-1} F(\mathbf{x}^{(t)}), \qquad \text{(vector } \mathbf{x}\text{)} \qquad (2)$$

where $J(\mathbf{x})$ is the **Jacobian matrix** for vector function $F(\mathbf{x}) = [f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_n(\mathbf{x})]$ in which $J_{ij}(\mathbf{x}) = \frac{\partial}{\partial x_j} f_i(\mathbf{x})$.

For non-repeated roots, Newton's method converges quadratically: $e_{t+1} = c e_t^2$.

(See notes.)

Example 2: Nonlinear BVP

$$\begin{cases} y''(t) &= y'(t) + cos(y) \\ y(0) &= 0 \\ y(\pi) &= 1. \end{cases}$$

Let y_i estimate $y(t_i)$ and set $y_0 = 1$ and $y_N = 1$. We obtain the other estimates through a finite difference equation that uses the following finite differences

$$y''(t_i) \approx \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}$$

$$y'(t_i) \approx \frac{y_{i+1} - y_{i-1}}{2h}$$

(See notes and code.)