COSC/MATH 4340: Numerical Methods for Differential Equations

Ordinary Differential Equations (ODEs): Boundary Value Problems (BVPs)

Class: COSC/MATH 4340
Semester: Spring 2024

Instructor: Dane Taylor
Office: Eng 4087
Email: dane.taylor@uwyo.edu

Introduction

Thus far we have considered initial value problems

$$\begin{cases} \frac{dy}{dt} &= f(t, y) \\ y(t_0) &= y_0 \end{cases}$$

for a solution $y(t)$ over some range $t \in [t_0, T]$.

We now turn our focus to boundary value problems (BVPs)

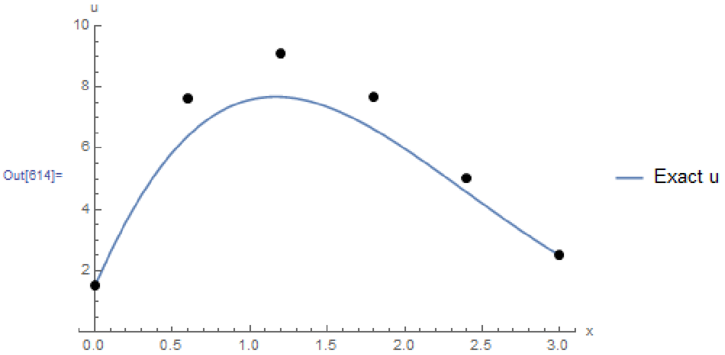
Definition

A **2nd-order boundary value problem** takes the form

$$\begin{cases} y''(t) &= f(t, y) \\ y(a) &= y_a \\ y(b) &= y_b. \end{cases}$$

Introduction

Constraints must be matched on the left and right:
$y(a) = y_a$ and $y(b) = y_b$.



Existence of Solutions

Understanding the existence and uniqueness of solutions to BVPs is much more complicated than for IVPs (which just required a Lipschitz constant).

BVPs often have either 1 solution, no solutions, or infinite solutions.

(See examples)

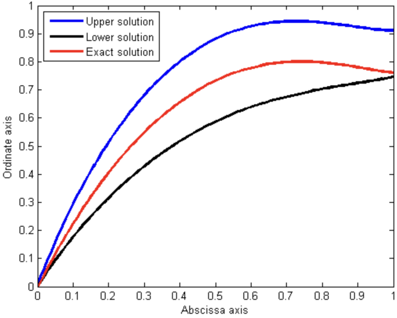
Overview

We will study 3 methods for solving BVPs:

- **Shooting methods**: use IVP numerical methods (e.g., FE, RK, trapezoidal) with $y_0 = y_a$ and iteratively find a solution $\{y_i\}$ so that y_N converges toward y_b.
- **Finite difference methods**: use approximations $y'(t_i) \approx (y_{i+1} - y_i)/h$ and construct a large sparse linear system that must be solved with a linear solver.
- **Finite element methods**: use a functional basis defined over $t \in [a, b]$ (e.g., the set of polynomials,) and then solve for the basis elements (i.e., coefficients)

The Shooting Method

Use an IVP solver such as forward Euler or Runge-Kutta to integrate from $t = a$ to $t = b$ and iteratively adjust the initial condition until the right-hand side is close to y_b.



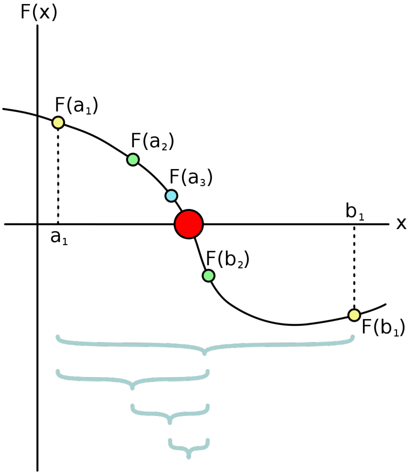
The Shooting Method

Outline of method

- Rewrite the 2nd-order BVP as a 2D system of equations
- Set the initial conditions $y(a) = y_a$ and $y'(a) = s$ for some initial choice s (which will be refined).
- Use a numerical method to solve the IVP over the range $[a, b]$ and define $G(s) = y_N - y_b$ as the error on the right-hand side.
- Use a root-finding algorithm (e.g., bisection method) to approximate the value s^* in which $G(s^*) = 0$.

(See example.)

Recall the Bisection Root-finding Method



Recall the Bisection Root-finding Method

Definition

Starting with any two values a_0 and b_0 such that $G(a_0)G(b_0) < 0$, let $c_0 = (a_0 + b_0)/2$ be the midpoint and compute $G(c_0)$. If $G(a_0)G(c_0) > 0$, define $a_1 = c_0$ and $b_1 = b_0$ (otherwise define $a_1 = a_0$ and $b_1 = c_0$). Then c_i converges to a root and intervals $[a_t, b_t]$ contain the root and become smaller and smaller with t. Iterations stop after a tolerance is reached or after T iterations.

Bisection method is guaranteed to converge when f is continuous, but it is a bit slow.

The error decays as 2^{-t} since the range $[a_t, b_t]$ is halved each iteration.

Recall Root-finding using Fixed Point Iteration

Theorem

Consider $x = g(x)$, and let $g'(x^*)$ denote the derivative of $g(x)$ at $x = x^*$. Then the sequence $x_{t+1} = g(x_t)$ converges to a solution x^* if an only if $|g'(x^*)| < 1$.

To find a root s^* of $G(s)$, we can define $g(s) = s + G(s)$. It follows that $g(s^*) = s^*$ iff $G(s) = 0$.

The convergence rate is $|e_t| \leq M^t |e_0|$, where $M = \max_{\xi \in D} |g'(x)|$ bounds the derivative's magnitude over some region D that contains x^*. This is faster than bisection method when $M < 1/2$.

BVP Problems involving Systems of Equations

Example: Using the shooting method to solve a BVP problem that involves a 2D system of equations:

$$\begin{cases} y_1'(t) &= f_1(t, \mathbf{y}) = y_1(t) - 3y_1(t)y_2(t) \\ y_2'(t) &= f_2(t, \mathbf{y}) = -6(ty_2(t) + \ln(y_1(t))) \\ y_1(0) &= 1 \\ y_2(1) &= -2/3. \end{cases}$$

(See notes and code)