General Multistep Methods

Definition

A numerical method to solve an IVP is called **multistep** if multiple past y_i estimates are used to predict the next y_{i+1} estimate.

Definition

The **Adams-Bashforth** methods are explicit multistep methods that take the form

$$y_{i+1} = \sum_{k=1}^{K} a_k y_{i+1-k} + h \sum_{k=1}^{K} b_k f(t_{i+1-k}, y_{i+1-k})$$

General Multistep Methods

Implicit multistep methods are very similar and take the form:

Definition

The **Adams-Moulton** methods are implicit multistep methods that take the form

$$y_{i+1} = \sum_{k=1}^{K} a_k y_{i+1-k} + h \sum_{k=0}^{K} b_k f(t_{i+1-k}, y_{i+1-k})$$

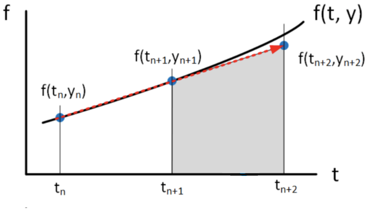
so that there is an implicit term $f(t_{i+1}, y_{i+1})$.

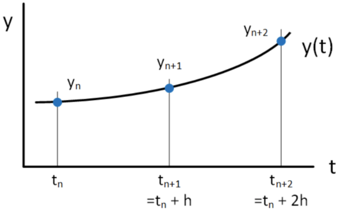
Their derivations are nearly identical to that for the Adams-Bashford methods.

General Multistep Methods

Example: 2nd order Adams-Bashforth method

$$
\begin{aligned}
y_{i+1} &= a_1 y_i + a_2 y_{i-2} + b_1 hf(t_i, y_i) + b_2 hf(t_{i-1}, y_{i-1}) \\
&= y_i + 0 + \frac{3h}{2} f(t_i, y_i) - \frac{h}{2} f(t_{i-1}, y_{i-1}) \\
&= y_i + \frac{h}{2}[3f(t_i, y_i) - f(t_{i-1}, y_{i-1})]
\end{aligned}
$$





General Multistep Methods

3rd order Adams-Bashforth method

$$y_{i+1} = y_i + \frac{h}{12}[23f(t_i, y_i) - 16f(t_{i-1}, y_{i-1}) + 5f(t_{i-2}, y_{i-2})$$

4th order Adams-Bashforth method

$$y_{i+1} = y_i + \frac{h}{24}[55f(t_i, y_i) - 59f(t_{i-1}, y_{i-1}) + 37f(t_{i-2}, y_{i-2}) - 9f(t_{i-3}, y_{i-3})$$

In fact there are many 2nd order Adams-Bashforth methods

$$y_{i+1} = \frac{1}{2}[y_i + y_{i-1}] + \frac{h}{4}[7f(t_i, y_i) - f(t_{i-1}, y_{i-1})]$$

(See notes and code.)

Stability of Multistep Methods

Definition

A multistep method is called **stable** if the roots of polynomial

$$Q(x) = x^K - a_1 x^{K-1} - a_2 x^{K-2} - \dots - a_{K-1} x - a_K$$

have magnitude ≤ 1 (and are simple if $= 1$).

Definition

If there is only root with magnitude 1 and it is 1, then the method is **strongly stable**.

Definition

If there are more roots with magnitude 1, then it's **weakly stable**.

(See notes and code.)

General Multistep Methods

Example: 2-step Adams-Moulton method

$$
\begin{aligned}
y_{i+1} &= a_1 y_i + a_2 y_{i-1} + b_0 hf(t_{i+1}, y_{i+1}) + b_1 hf(t_i, y_i) + b_2 hf(t_{i-1}, y_{i-1}) \\
&= y_i + 0 + \frac{h}{12}[5f(t_{i+1}, y_{i+1}) + 8f(t_i, y_i) - f(t_{i-1}, y_{i-1})] \\
&= y_i + \frac{h}{12}[5f(t_{i+1}, y_{i+1}) + 8f(t_i, y_i) - f(t_{i-1}, y_{i-1})]
\end{aligned}
$$

Claim: It's 3rd order and has bounded local error

$$
e_i \leq \frac{Mh^4}{24}, \quad M = \max_t |y^{(4)}(t)|.
$$

(See notes.)

Stability of Multistep Methods

Let's characterize how error vanishes as h decreases.

Definition

An IVP solver is **consistent** if:

$$\lim_{h \to 0} \max_i \left| \frac{e_i}{h} \right| = 0$$

Definition

An IVP solver is **convergent** if:

$$\lim_{h \to 0} \max_i |E_i| = 0$$

See notes.

Stability of Multistep Methods

Theorem [Dahlquist]

Assuming the initial values are correct, a multistep method is convergent iff it is consistent and stable.

Nearly all IVP solvers are consistent (i.e., at least as good as forward Euler). Thus, understanding a methods stability determines if it is convergent.

Stability of Multistep Methods

In fact there are several types important types of "stability."

Definition

Absolute stability (or linear stability): The system is convergent under the choice $f(t, y) = \lambda y$.

This can checked by examining roots of a polynomial $Q(x, h\lambda)$ associated with the IVP solver under the ansatz $y_i = cx^i$.

Theorem

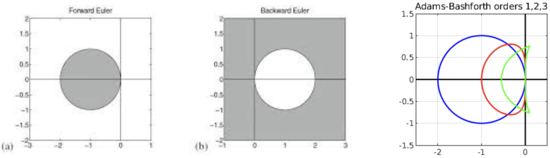
An IVP solver is **absolute stable** iff its characteristic polynomial $Q(x, h\lambda)$ is stable.

(See notes for examples.)

Stability of Multistep Methods

Definition

The **region of absolute stability** for an IVP solver is the domain of the complex parameter space $(\mathbf{Re}(h\lambda), \mathbf{Im}(h\lambda))$ in which the method is absolutely stable.



Above are images of "regions of stability" for three IVP solvers, i.e., the values of $h\lambda$ for which the method is stable

(See notes.)

Stability of Multistep Methods

Definition

A method is **A stable** if its region of absolute stability contains the entire left-half plane. (These are implicit methods that can solve stiff equations with large time steps.)