

Contemporary Pre-Calculus Through Applications

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Contents

1	Data	1
1.1	1
2	Transformations of Functions	3
2.1	3
3	Combining Functions	5
3.1	5
4	Exponential Functions	7
4.1	Recursive Functions	7
4.2	Using Recursion to Understand Loans and Investments	12
4.3	Geometric Growth and Decay	17
4.4	Summing Geometric Growth	24
4.5	Compound Interest (or the Magic of Compounding)	30
4.6	Exponential Functions	35
5	Logarithmic Functions	45
5.1	Introduction to Logarithms	45
5.2	Scientific Scales	50
5.3	Graphing Logarithmic Functions	53
5.4	Solving Exponential and Logarithmic Equations	62
6	Parametric Functions	67
6.1	67
7	Intro to Trigonometric Functions	69
7.1	69
8	More Trigonometric Functions	71
8.1	71
9	Combinations of Functions	73
9.1	73
10	Matrices	75
10.1	75

Chapter 1

Data

Introduction to this chapter

1.1

Chapter 2

Transformations of Functions

Introduction to this chapter

2.1

Chapter 3

Combining Functions

Introduction to this chapter

3.1

Chapter 4

Exponential Functions

4.1 Recursive Functions

In a previous chapter we learned that a function is a special sets of ordered pairs. In most of the examples in the preceeding chapters, functions were described by an algebraic expression that could be evaluated for a particular input value resulting in a unique output value. Such algebraic expressions are called closed form or explicit expressions. For these functions, the relationship $y = f(x)$ is used to show how the y -value is related to the given x -value. For example, the function $f(x) = x^2 + 6x$ is an explicit function. This notation tells us that any particular numerical value for x is paired with the y -value equal to $x^2 + 6x$. So 1 is paired with 7, since $f(1) = (1)^2 + 6(1) = 7$, and -3 is paired with -9 , since $f(-3) = (-3)^2 + 6(-3) = -9$.

In this section we will investigate functions that are defined recursively. The domain values for these functions are positive whole numbers, and each range value is defined in terms of the preceding range value, rather than in terms of an x -value.

Example 4.1.1 (Ibuprofen in the blood stream). Joan has a headache and decides to take a 200mg ibuprofen tablet for pain relief. The drug is absorbed into her system and stays in her system until the drug is metabolized and filtered out by the liver and kidneys. Ibuprofen is rapidly metabolized. Every four hours, Joan's body removes 67% of the ibuprofen that was in her body at the beginning of that four-hour time period. How much of the ibuprofen will remain in her system 24 hours after taking the 200mg tablet?

Solution. One way to generate values for the amount of ibuprofen in Joan's system is to use an iterative process. In any iterative process the current value of a variable is used to determine the next value. In this example, we generate a new amount of ibuprofen by subtracting the amount of ibuprofen filtered out of Joan's system from the amount that was previously there. Since Joan begins with 200mg of ibuprofen, we write

$$D_0 = 200$$

where D_0 is used to represent the amount of ibuprofen present at the start of the process

We will use D_1 to represent the amount of ibuprofen left after four hours. The subscripts are used to count the steps, or iterations, in the iterative process. In this problem the subscript represents the number of four-hour time periods since Joan took the tablet. In four hours, her kidneys have filtered out 67% of the drug from her bloodstream, so we write

$$D_1 = D_0 - .67D_0 = 200 - .67 \cdot 200 = 66$$

The amount of drug in her body after a second four-hour time period is represented by D_2 .

$$D_2 = D_1 - 0.67D_1 = 66 - .67 \cdot 66 = 21.78$$

Similarly, we know that successive values of the amount of drug in her body can be generated by

$$D_3 = D_2 - 0.67D_2 = 7.187$$

$$D_4 = D_3 - 0.67D_3 = 2.372$$

and, in general,

$$D_n = D_{n-1} - 0.67D_{n-1}, n = 1, 2, 3, \dots$$

Using a spreadsheet or calculator, we can generate successive values of D_n as shown in Figure 2. Note that values in the table are rounded to three decimal places but that exact values were used in all computations. The amount of drug in Joan's body drops to less than 1 mg between the fourth and fifth time periods. If she takes a single 200 mg dose, Joan will have only about 0.258 mg remaining in her body 24 hours later.

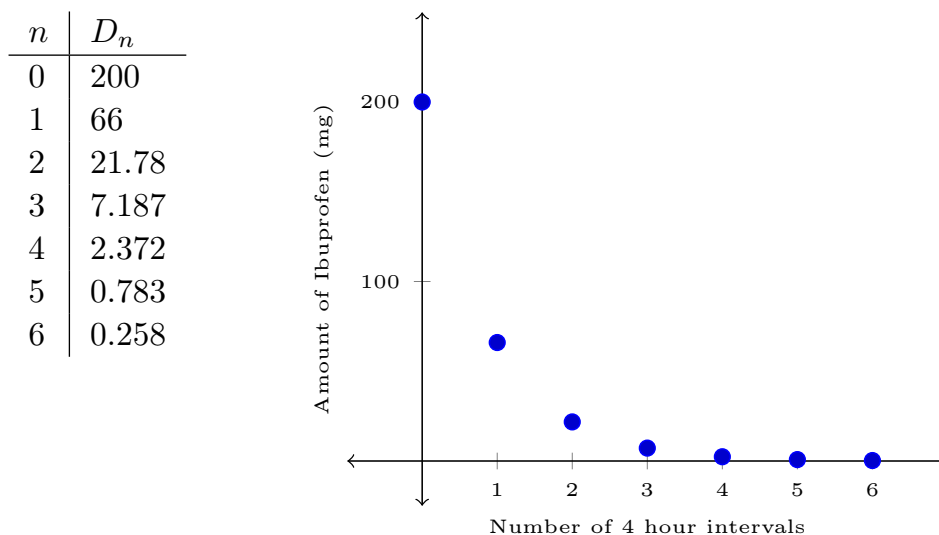


Figure 4.1.2: Amount of drug in Joan's body (Single 200 mg dose)

The graph in Figure 2 shows the ordered pairs (n, D_n) generated by the recursive system

$$\begin{aligned} D_0 &= 200 \\ D_n &= D_{n-1} - 0.67D_{n-1}, n = 1, 2, 3, \dots \end{aligned}$$

Each point on the graph shows the amount of ibuprofen in Joan's body at the end of a four-hour time period. Notice that there is obvious curvature in this graph. The amount of drug in Joan's body does not decrease by the same number of milligrams during each time interval.

4.1.1 Class Practice

1. Modify the recursive system used in Example 1 as appropriate to answer the following questions:

- Suppose Joan takes tablets that contain 250 milligrams of ibuprofen. How much ibuprofen would be in her body after 4, 8, 12, 16, 20, and 24 hours?
- Suppose Joan's kidneys filter only 50% of the drug in a four hour time period. If Joan takes a 200 mg tablet every 4 hours, how much ibuprofen would she have in her system after 4, 8, 12, 16, 20, and 24 hours?

2. In each recursive system, the domain of D_n is $n = 1, 2, 3, \dots$ and D_n represents the amount of ibuprofen in Joan's system after n four-hour time periods. For each system, find how much drug remains after 24 hours, and identify the rate at which Joan's system filters out the drug.

- $D_0 = 300, D_n = D_{n-1} - 0.8D_{n-1}$

- (b) $D_0 = 150, D_n = D_{n-1} - 0.2D_{n-1}$
- (c) $D_0 = 500, D_n = 0.2D_{n-1}$
- (d) $D_0 = 500, D_n = 0.5D_{n-1}$

Example 4.1.3 (Repeated dose of ibuprofen). Joan strained her knee playing tennis and her doctor has prescribed ibuprofen to reduce the inflammation and control pain. Joan is instructed to take two 200-milligram ibuprofen tablets every 4 hours for three days. Joan doesn't like taking medicine, so she decides to take only one tablet every four hours for six days. After the six days, Joan's knee has not responded to the medication. Naturally, she knew that the knee would take longer to respond to the reduced treatment, but she did not expect no response at all. What could have happened?

Solution. In this situation, Joan did not take just a single 200 mg tablet. Every four hours she took another 200 mg tablet, and we can modify our recursive system to model this behavior. At the end of the n th four-hour period, Joan's body has filtered 67% of the drug that was in her body after $n-1$ st four-hour period. In addition, 200 mg from the new tablet have been added into her body. The recursive system representing the amount of ibuprofen in Joan's body if she takes one tablet every four hours is

$$D_0 = 200$$

$$D_n = D_{n-1} - 0.67D_{n-1} + 200, n = 1, 2, 3, \dots$$

The subscript n represents the number of four-hour time periods that have elapsed since Joan took the first dose. By iterating the recursive system we can generate values of D_n that represent the amounts of drug in Joan's body at the end of each four-hour period, assuming 67% of the drug is filtered in a four-hour period. These values, rounded to two decimal places, are shown together with a graph in [Figure 4](#)

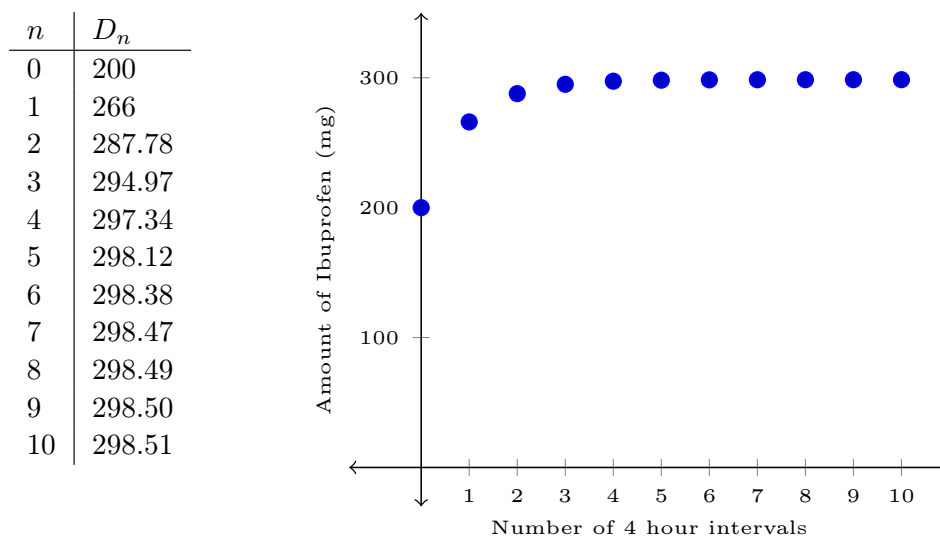


Figure 4.1.4: Amount of drug in Joan's body (Reoccurring 200 mg doses)

The points shown in [Figure 4](#) represent the amount of the drug in Joan's body immediately after she takes a tablet. Between consecutive doses, we know that the level of the drug declines. We assume that the level "jumps up" at the moment she takes another tablet, and the recursive system enables us to compute these values. If we record the drug levels only after she takes a tablet, then we see that these values reach an equilibrium of approximately 298.51 mg. To see why this equilibrium has been reached, consider how much of the 298.51 mg of the drug will be filtered in four hours. Joan's kidneys will filter out 67% of the 298.51 mg in her body, or approximately 200 mg, which will be replaced when she takes the next

tablet. Equilibrium occurs because the amount of drug taken into the body is the same as the amount filtered prior to taking the next tablet.

Suppose the drug Joan is taking has a therapeutic level of 450 mg. This means that there must be at least 450 milligrams of the drug in her body for Joan to receive the benefits of the drug. No wonder she thought the drug was not working. It wasn't!

4.1.2 Exercises

1. Modify the recursive system used in [Example 1](#) as appropriate to answer the following questions:

- Suppose Joan takes tablets that contain 250 milligrams of ibuprofen. How much ibuprofen would be in her body after 4, 8, 12, 16, 20, and 24 hours?
- Suppose Joan's kidneys filter only 50% of the drug in a four hour time period. If Joan takes a 200 mg tablet every 4 hours, how much ibuprofen would she have in her system after 4, 8, 12, 16, 20 and 24 hours?

2. If Joan takes her medication every 4 hours, determine the amount of drug in Joan's body after 2 days (twelve 4-hour time periods) and the equilibrium level resulting from each of the following recursive systems. Plot the ordered pairs you generate on a graph. Note that in some cases the initial dosage and subsequent doses are not the same size. In each exercise, the domain of D_n is $n = 1, 2, 3, \dots$

- $D_0 = 200, D_n = D_{n-1} - 0.4D_{n-1} + 200$
- $D_0 = 800, D_n = 0.6D_{n-1} + 200$
- $D_0 = 600, D_n = 0.4D_{n-1} + 200$
- $D_0 = 600, D_n = D_{n-1} - 0.4D_{n-1} + 200$
- $D_0 = 600, D_n = 0.6D_{n-1} + 300$
- $D_0 = 600, D_n = 0.4D_{n-1} + 300$

3. Each of the systems in exercise 2 can be written in the form

$$D_0 = a$$

$$D_n = (1 - r)D_{n-1} + b, n = 1, 2, 3, \dots$$

- What does r represent in the context of Joan's medication? Why is r between 0 and 1?
- By looking back at the results of exercise 2 and by trying other variations, determine the effect of changing a , r , and b on the amount of drug in Joan's body after 5 days.
- Determine the equilibrium level in terms of a , r , and b . You can recognize that equilibrium has been reached if $D_n = D_{n-1}$.
- Use the result of part c to determine the equilibrium level of a drug if you take 200 mg every 4 hours and your kidneys filter out 50% of the drug in your body every 4 hours.

4. A company has \$10,000 worth of equipment and for tax purposes they want to figure the depreciation of the equipment over a 10-year time period. One method is to reduce the value each year by the same dollar amount. A second method is to decrease the value of the equipment each year by the same percent of the current value each year.

- Using the first method, generate a table and graph for the value of the equipment if it is decreased each year by \$2000.
- Using the second method, generate a table and graph for the value of the equipment if it is decreased by 20% each year.

5. One of the primary responsibilities of the manager of a swimming pool is to maintain the proper concentration of chlorine in the swimming pool. The concentration should be between 1 and 2 parts per million (ppm). If the concentration gets as high as 3 ppm swimmers experience burning eyes. If the concentration gets below 1 ppm, the water will become cloudy, which is unappealing. If it drops below 0.5 ppm, algae begin to grow. During a period of one day, 15% of the chlorine present in the pool dissipates (mainly due to the sun).

- (a) If the chlorine content starts at 2.5 ppm and no additional chlorine is added, how long will it be before the water becomes cloudy?
- (b) If the chlorine content starts at 2.5 ppm and the equivalent of 0.5 ppm of chlorine is added daily, what will happen to the level of chlorine in the pool in the long run?
- (c) If the chlorine content starts at 2.5 ppm and the equivalent of 0.1 ppm of chlorine is added daily, what will happen to the level of chlorine in the pool in the long run?
- (d) How much chlorine must be added daily for the chlorine level to stabilize at 1.8 ppm?

6. The Fish and Wildlife Division monitors the trout population in a stream that is under their jurisdiction. Their research indicates that natural predators, together with pollution and fishing, are causing the trout population to decrease at a rate of 20% per month. They propose to introduce additional trout into the stream each month. Assume the current population is 300. Use tables and graphs to investigate the following:

- (a) What will happen to the trout population over the next ten months with no replenishment program?
- (b) What is the long-term result of introducing 100 trout into the stream each month?
- (c) Investigate the result of changing the number of trout introduced each month. What is the impact on the long-term population of the number of trout added each month?
- (d) Investigate the impact on the long-term behavior of the population of changing the initial population. What is the effect of the initial population?
- (e) What is the impact of the rate of decrease in the population during the replenishment program?
- (f) There are three parameters in this problem: the initial number of trout, the rate of decrease, and the number of trout added each month. Which parameter seems to have the most influence on the long-term behavior? Explain briefly.

7. Drugs generally have a therapeutic range rather than a single therapeutic level. In other words, a drug is effective if the level in the body is between two values. At concentrations below this range, too little of the drug is present to have a measurable effect, and concentrations above this range may be toxic. The level of drug in the body peaks just after the drug is taken, while the drug level is at a minimum just before the dosage is taken. Suppose Joan takes an anti-inflammatory drug at the prescribed dosage of 440 mg every 12 hours and her kidneys filter 60% of the drug from her body every 12 hours. Use tables and graphs to investigate the following:

- (a) Generate a sequence of values for the level of drug in Joan's body just before each dose.
- (b) In the long-run, the level of drug in Joan's body will range between what two values?
- (c) Suppose the therapeutic range of the anti-inflammatory drug is between 300 mg and 800 mg. What adjustment, if any, needs to be made in Joan's dosage to stay within this range in the long run?

8. Suppose Morgan wants to buy a television that costs \$549. He has a part-time job, and he is able to save \$85.00 each week. Suppose he accumulates the money at home.

- (a) Write a recursive system that can be used to determine how much Morgan has saved over time.
- (b) Use the recursive system to generate values for the amount of money Morgan has saved in thirteen weeks.

- (c) Make a graph to show the amount Morgan has saved versus the number of weeks he has been saving.
 - (d) The points graphed in part (c) should appear linear. Explain why the situation implies that these ordered pairs are linear.
 - (e) Write an explicit function of the form $A = f(t)$ that could be used to generate the same ordered pairs you graphed in part (c).
 - (f) State the domain and the range of the explicit function within the context of this problem. Compare them to the domain and range of the recursively defined function.
9. Now suppose that Morgan deposits his savings in a bank that will pay 0.02% interest each week.
- (a) Write a recursive system that can be used to determine how much Morgan has saved over time.
 - (b) Use the recursive system to generate values for the amount of money Morgan has in the bank after 13 weeks of saving.
 - (c) Make a graph to show the amount Morgan has saved versus the number of weeks he has been saving.
 - (d) Are the ordered pairs graphed in part (c) linear? Explain why or why not.
 - (e) State the domain and range of the recursively defined function.

4.2 Using Recursion to Understand Loans and Investments

Recursive systems are useful for finding solutions to various types of problems.

Suppose you are interested in purchasing a car and need a \$10,000 loan. The lending agency is going to charge you interest each month and you are going to make a payment each month. You plan to pay \$230 each month until the loan is paid off. Suppose the interest rate is 0.45% per month (approximately 5.4% per year). How long will it take you to repay the loan? What is the total amount you will have to repay? You can answer these questions using recursion.

When making monthly payments to repay a loan, an interest payment is charged on all of the money that is owed at the end of each month. In this example, at the end of the first month you will owe $(0.0045)(10,000) = \$45.00$ in interest. After making the first payment you will owe $\$10,000 + \$45 - \$230 = \9815 . The amount you still owe on the loan at the end of a month is equal to the amount you owed previously plus the interest minus the amount of your payment. This is expressed in a recursive system as:

$$L_0 = 10,000, L_n = L_{n-1} + (0.0045) \cdot L_{n-1} - 230, \text{ where } n = 1, 2, 3, \dots$$

where L_n is the amount you owe on the loan after n months. This amount is also known as the *principal*, or the *outstanding balance*. If we iterate this system, the values we generate represent the outstanding balance at the end of each month.

$$L_1 = L_0 + (0.0045) \cdot L_0 - 230 = 10,000 + (0.0045) \cdot 10,000 - 230 = 9,815$$

$$L_2 = L_1 + (0.0045) \cdot L_1 - 230 = 9,815 + (0.0045) \cdot 9,815 - 230 = 9,629.17$$

$$L_3 = L_2 + (0.0045) \cdot L_2 - 230 = 9,629.17 + (0.0045) \cdot 9,629.17 - 230 = 9,442.50$$

Note that values have been rounded to the nearest cent. All the decimal places on the calculator were retained in the computations.

After three months, you will owe \$9,442.50 on the loan. Notice that you have paid a total of $\$230 \cdot 3 = \690 , but only $\$10,000 - \$9442.50 = \$557.50$ was applied towards decreasing the principal of the loan. The remaining \$132.50 was payment of interest.

We can continue generating values of L_n using a more compact form of the iterative equation,

$$L_n = (1.0045) \cdot L_{n-1} - 230$$

which yields

$$L_4 = (1.0045) \cdot L_3 - 230 = \$9,254.99$$

$$L_5 = (1.0045) \cdot L_4 - 230 = \$9,066.64$$

$$L_6 = (1.0045) \cdot L_5 - 230 = \$8,877.44$$

After six months, you owe \$8,877.44 of the original \$10,000 principal. You have paid \$1,380 in interest, and this payment has reduced the outstanding balance by \$1,122.56. Over \$250 was interest on the loan.

If we continue the payments of \$230 for four years (48 months), the final payment will probably not bring the outstanding balance to exactly zero dollars. Although it is possible to calculate a loan payment that will exactly pay off the loan in 48 equal payments, it is common practice for the lender to make all of the payments, except for the last one, a whole dollar amount, or even round these payments to the nearest five or ten-dollar amounts. Doing this will almost certainly make the final payment different from the rest. This final payment is known as the balloon payment.

The graph and partial table in Figure 4.2.1 show that it will take 47 payments to get the balance down to \$341.20. After one additional month the balance will be $\$341.20 \cdot 1.0045 = \342.74 if we round in the usual manner. Thus, the balloon payment would be \$342.74.

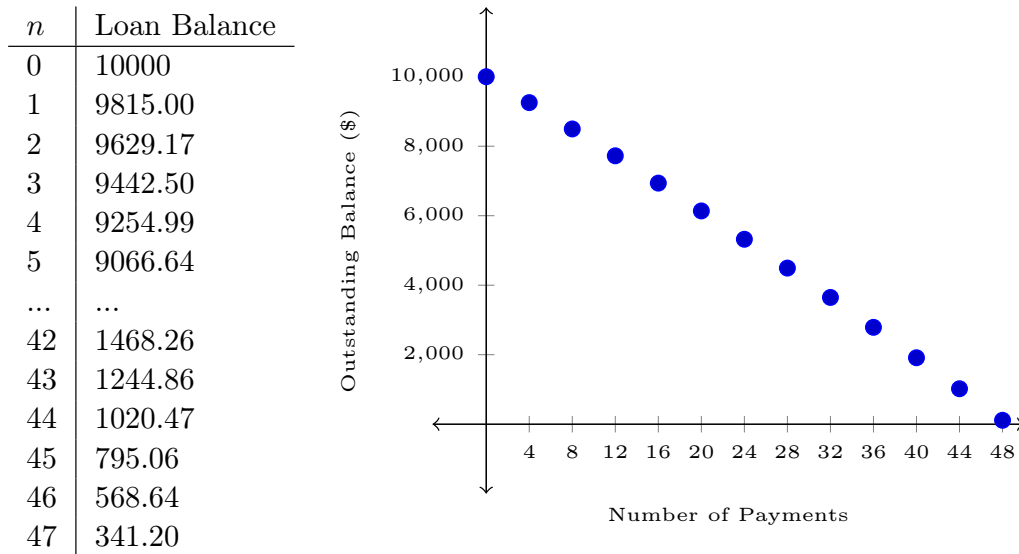


Figure 4.2.1: Loan Ammortization

Since the first 47 payments were each \$230, the total amount paid is $47 \cdot \$230 + \$342.74 = \$11,152.74$. We see that it costs \$1,152.74 to borrow \$10,000 for 48 months.

The process of paying off a loan is known as *amortizing the loan*, or *loan amortization*. When we study loan amortization, we are interested in the amount borrowed, the interest rate, the payment, the length of time it will take to repay the loan and the total amount the borrower will have to repay.

We can generalize the recursive system used to determine the total amount repaid as follows. If we borrow an amount A and let r represent the interest rate per time period and P the amount of the payment during each time period, we can describe the amount owed after n time periods with the system

$$L_0 = A, L_n = L_{n-1} + r \cdot L_{n-1} - P, n = 1, 2, 3, \dots$$

or

$$L_0 = A, L_n = (1 + r)L_{n-1} - P, n = 1, 2, 3, \dots \quad (4.2.1)$$

You should complete practice problem 1 at the end of the section at this time

When borrowers take out a loan, they typically know the amount they want to borrow, the interest rate they will have to pay, and the length of time they have to pay off the loan. We are interested in determining the payment that will allow the borrower to pay back the loan in the required time. For most loans, the borrower makes payments every month. Thus, a car loan that needs to be repaid in 5 years requires 60 monthly payments and a mortgage loan that is repaid in 15 years requires 180 monthly payments. For simplicity's sake, our first few examples will assume that payments are made annually (that is, once per year) rather than monthly.

Example 4.2.2 (Finding the Yearly Payment Needed to Pay Off a Loan). Suppose you buy a car and take out a loan of \$22,000 at 6.5% annual interest to be paid back over four years. What is the yearly payment you must make to pay off the loan in four equal payments?

Solution. Referring to [equation \(4.2.1\)](#) we have

$$L_0 = 22,000, L_n = (1 + 0.065)L_{n-1} - P$$

and our goal is to find the value of P so that $L_4 = 0$

If there were no interest charged, you would have to make a payment of \$5,500 each year to repay the \$22,000. Since you must pay interest on the loan, \$5,500 per year is obviously too small a payment and we conclude that \$5,500. We can confirm that \$5,500 is too small a payment with the recursive system:

$$L_0 = 22,000, L_n = (1 + 0.065)L_{n-1} - 5,500$$

n	L_n
0	\$22,000
1	$\$22,000(1 + 0.065) - \$5,500 = \$17,930$
2	$\$17,930(1 + 0.065) - \$5,500 = \$13,595.45$
3	$\$13,595.45(1 + 0.065) - \$5,500 = \$8,979.15$
4	$\$8,979.15(1 + 0.065) - \$5,500 = \$4,062.80$

Table 4.2.3

This confirms that if we make an annual payment of \$5,500 for each of 4 years, the outstanding balance on a loan of \$22,000 is \$4,062.80 at the end of the 4 years. So \$5,500 is too small an annual payment to pay off the loan.

With a 6.5% annual interest rate, we know that the interest for the first year is \$1,430 (6.5% of \$22,000), so a reasonable guess for the payment might be $\$5500 + \$1430 = \$6930$. However, in the second year we will need to pay less than \$1430 in interest because the outstanding balance has decreased. Therefore, a payment of \$6930 per year will result in an overpayment for the last three years. The calculations below show that \$6930 is indeed too large an annual payment; that is, $P < 6,930$.

$$L_0 = 22,000, L_n = (1 + 0.065)L_{n-1} - 6,930$$

n	L_n
0	\$22,000
1	$\$22,000(1 + 0.065) - \$6,930 = \$16,500$
2	$\$16,500(1 + 0.065) - \$6,930 = \$10,642.50$
3	$\$10,642.50(1 + 0.065) - \$6,930 = \$4,404.26$
4	$\$4,404.26(1 + 0.065) - \$6,930 = -\$2,239.46$

Table 4.2.4

We have established that \$5500 is too small a payment and \$6930 is too large a payment; that is, $5,500 < P < 6,930$. Our next logical guess for a payment might be mid-way between these two payments, or \$6215. We can do recursive calculations to see if this payment is too small, which would result in a positive outstanding balance after 4 years, or if this payment is too large, resulting in a negative outstanding balance after 4 years.

The outstanding principal is modeled by the recursive system

$$L_0 = 22,000, L_n = (1 + 0.065)L_{n-1} - 6,215$$

The successive principals are shown in the table.

n	L_n
0	\$22,000
1	$\$22,000(1 + 0.065) - \$6,215 = \$17,215$
2	$\$17,215(1 + 0.065) - \$6,215 = \$12,118.98$
3	$\$12,118.98(1 + 0.065) - \$6,215 = \$6,691.71$
4	$\$6,691.71(1 + 0.065) - \$6,215 = \$911.67$

Table 4.2.5

An annual payment of \$6,215 is not sufficient to pay off the loan in four years, as a balance of \$911.67 remains after the four payments. We conclude that $6,215 < P < 6,930$ and we need to pay more than \$6,215. If we again select the average of two payments, one that is too large and one that is too small, we will have arrived at a better guess. We know \$6,930 is too large and \$6,215 is too small, so our next guess will be $\frac{6930+6215}{2} = \$6,572.50$. The successive balances would now be as shown.

n	L_n
0	\$22,000
1	$\$22,000(1 + 0.065) - \$6,572.50 = \$16,857.50$
2	$\$16,857.50(1 + 0.065) - \$6,572.50 = \$11,380.74$
3	$\$11,380.74(1 + 0.065) - \$6,572.50 = \$5,547.99$
4	$\$5,547.99(1 + 0.065) - \$6,572.50 = -\$663.90$

Table 4.2.6

We see that a payment of \$6,572.50 is too large since after 4 payments you have paid \$663.90 more than necessary. We now know that the appropriate payment P is in the interval $6,215 < P < 6,572.50$.

We can continue the process of averaging two payments, one that is too large and one that is too small. Eventually we find that three payments of \$6,393.75 and a fourth balloon payment of $\$6,393.75 + \123.89 will bring the outstanding balance to zero dollars.

The process used to find an appropriate payment is called *binary search*. We began the process by finding an interval that contained the appropriate payment. This interval is bounded by a payment that is too low and a payment that is too high. We found the midpoint of the interval and determined if that payment was too large or too small. If the "midpoint payment" is too high, that is, if we have a negative balance after four payments, then consider a new interval that is the lower half of the previous interval; otherwise, take the new interval to be the upper half of the previous interval. We continue bisecting, which results in a smaller and smaller interval that we know contains the appropriate payment. We stop bisecting the interval as soon as we have found a payment that results in a balance of zero, or as close to zero as we need.

4.2.1 Exercises

1. How long will it take to amortize a loan and how much will the loan cost under each of the following conditions? In each case L_0 , r and P represent the initial amount borrowed,

the monthly interest rate, and the monthly payment, respectively. Determine the balloon payment for each scenario as well.

- (a) $L_0 = \$5000$, $r = 1\%$ and $P = \$200$
- (b) $L_0 = \$5000$, $r = 1.5\%$ and $P = \$200$
- (c) $L_0 = \$5000$, $r = 0.5\%$ and $P = \$200$
- (d) $L_0 = \$5000$, $r = 1\%$ and $P = \$250$

2. Maisha opens a retirement account on her 35th birthday with a deposit of \$2,400. Each year on her birthday, she plans to deposit an additional \$2,400. The account earns interest at a rate of 10% annually.

- (a) How much will Maisha have saved by the time she retires at age 65?
- (b) Suppose Maisha wants to have \$1 million in the account by age 65. To the nearest hundred dollars, how much should she deposit each year?
- (c) Suppose Maisha starts saving ten years earlier, at age 25. To the nearest hundred dollars, how much should she deposit each year to have \$1 million at retirement?

3. You and your parents need to borrow money to pay for your college tuition. You are looking for an education loan for \$50,000. The Village Bank offers a 15-year loan at 7% annual interest. The Hometown Bank offers a 20-year loan at 6% annual interest. Which loan is better? Explain what criteria you used to decide.

4. Isaac wants to buy a car and is shopping for a four year (48 month) loan. If he needs to borrow \$24,000 and the loan charges 4.6% annual interest, what must be his annual payment to pay off the loan in the required 4 years?

5. Kyle and Taylor have taken out a loan for \$175,000 to buy their first house. They have a 15-year mortgage and an annual interest rate of 3.9%.

- (a) The lending agency tells them that their monthly payment is \$1,290, and so they pay \$1,290 every month for 179 months. What must be their final payment (the 180th) so that the loan is paid off? In total, how much did it cost them to borrow \$175,000 for 15 years?
- (b) Kyle and Taylor recognize that they may save money in the long run if they make payments larger than what the lending agency requires. They decide to pay \$2,000 for the first 5 years (60 months) and then pay the required \$1,290 per month until the loan is paid off. How long will it take them to bring their outstanding balance to zero. In total, how much did it cost them to borrow \$175,000 for 15 years?

6. Terry has his heart set on owning a Tesla electric car. He will sell his current car to raise some of the money, but still needs to take out a \$55,000 loan to be able to afford a Tesla. His credit union will charge him 4.2% interest per year, a special rate for an energy efficient car. He will make equal monthly payments in order to pay off the loan in 72 months. Your task is to determine what his monthly payment needs to be. You will make several educated guesses about how much he needs to pay each month in order to finish paying off the loan and bring his outstanding balance down to zero with 72 payments.

- (a) In a table like the one shown, record the size of the monthly payment and the balance Terry still owes at the end of 72 months. The first row of the table has been completed. It indicates that if Terry makes a payment of \$950 each month, he will have overpaid by about \$6,900 after the 72nd payment.

Monthly Payment	Outstanding Balance after 72 Monthly Payments
\$950	−\$6905.08

Table 4.2.7

- (b) Make a scatterplot that shows monthly payment on the horizontal axis and outstanding balance on the vertical.
- (c) Describe how you can take advantage of the shape of the scatterplot to find the monthly payment that will yield a zero outstanding balance after 72 months.
- (d) Confirm that a monthly payment of \$865.51 will pay off the loan (the balance will round to zero). What does this number have to do with the scatterplot you made in (b)?

4.3 Geometric Growth and Decay

Some recursively defined functions have important applications in life. One of the simplest, yet most important, represents *geometric decay*. In geometric decay, the value of the function at time n is directly proportional to the value at time. This relationship suggests a recursive definition for geometric decay. The first example from Section 1, Example 1, involved the amount of ibuprofen in Joan's system at time n if she takes a single 200 mg tablet. Since her body filters out 67% of the ibuprofen present, the amount remaining after n time intervals is always 33% of the amount remaining after $n - 1$ time intervals.

In that case, we calculated the amount of drug in Joan's body with the recursive system

$$D_0 = 200, D_n = 0.33 \cdot D_{n-1}, n = 1, 2, 3, \dots$$

Example 4.3.1 (Fading Blue Jeans). Blue jeans fade when they are washed. Suppose a pair of jeans loses 2% of its color with each washing. How much of the original color is left after 50 washes?

Solution. In this problem, we use the system

$$C_0 = 1, C_n = 0.98 \cdot C_{n-1}, n = 1, 2, 3, \dots$$

where C_n is the amount of color remaining in the jeans after n washings

We use 1 as the initial value to represent all or 100% of the color. Since we want to measure the amount of color remaining, the coefficient in the recursive equation is 0.98. With 50 iterations of the equation for C_n , we find that the jeans have about 36% of their original color left after 50 washings.

Example 4.3.2 (Electrical Power Demand). The amount of electrical power used by a community is increasing by 5% per year. This year, they used 500 thousand kilowatt-hours of electrical power. How many years will it take until the electrical power consumption for this community has doubled?

Solution. This is an example of *geometric growth*, since the amount of power is increasing each year. We use the recursive system

$$P_0 = 500, P_n = 1.05 \cdot P_{n-1}, n = 1, 2, 3, \dots$$

Since the amount of power used is increasing by 5% each year, next year the citizens of the community will use 105% of what they used this year. By iterating the equation for P_n , we find that after 14 years the community will use about 990 thousand kwh per year and after 15 years they will use about 1039 thousand kwh per year. The amount of power required by the community will have doubled in a little less than 15 years. Note that this conclusion assumes that the demand for electrical power continues to increase by 5% per year.

Example 4.3.3 (Radioactive Decay). Potassium-42 is a radioactive element that is often used in biological experiments as a tracer element. Potassium-42, like all radioactive elements, decays into a non-radioactive form at a rate proportional to the amount present. Potassium-42 loses 5.545% of its mass every hour. If 1 milligram of potassium-42 is initially present in an animal, at what time will only 0.1 milligram be present?

Solution. Note that the effect of losing 5.545% is equivalent to retaining 94.455%. We use the recursive system

$$P_0 = 1, P_n = 0.94455 \cdot P_{n-1}, n = 1, 2, 3, \dots$$

We are interested in finding the amount of time until P_n is less than or equal to 0.1. Iterating the equation for P_n , we find that $P_{40} = 0.1021$ and $P_{41} = 0.0964$. So after 40 hours, 0.1021 mg of potassium-42 remain, and after 41 hours, 0.0964 mg remain. Sometime between 40 and 41 hours we expect to have only 0.1 mg of potassium-42 remaining.

The recursive system we used to generate amounts of potassium-42 does not allow us to determine the amount present between the 40th hour and the 41st hour. The amount of potassium-42 changes incrementally between the 40th and the 41st hours, but the recursive equation we have used to represent this phenomenon cannot give us information about potassium levels between P_{40} and P_{41} .

Each of the three previous examples uses a recursive system that can be written as

$$Y_0 = a, Y_n = (1 + k) \cdot Y_{n-1}, n = 1, 2, 3, \dots$$

If $k > 0$, this system represents geometric growth (with growth rate k); as n increases, the value of Y_n increases. If $k < 0$, the system represents geometric decay (and k is the decay rate); as n increases, the value of Y_n decreases. In either case, the next value of Y_n depends entirely on the value of k and the old value of Y_n .

Geometric growth that can be described with a recursive system can also be described by an explicit function. To demonstrate, we will iterate the system used in [Example 3](#) to describe electricity consumption

$$\begin{aligned} P_0 &= 500 \\ P_1 &= (1 + 0.05)P_0 \\ P_2 &= (1 + 0.05)P_1 \\ P_3 &= (1 + 0.05)P_2 \\ P_4 &= (1 + 0.05)P_3 \end{aligned}$$

We can rewrite each of these equations in terms of P_0 , which yields

$$\begin{aligned} P_1 &= (1.05)P_0 \\ P_2 &= (1.05)P_1 = (1.05)(1.05)P_0 = (1.05)^2P_0 \\ P_3 &= (1.05)P_2 = (1.05)(1.05)^2P_0 = (1.05)^3P_0 \\ P_4 &= (1.05)P_3 = (1.05)(1.05)^3P_0 = (1.05)^4P_0 \end{aligned}$$

If we continue this process, we see that the n^{th} term is given by

$$P_n = (1.05)^n P_0$$

In general, we can convert the recursive system for geometric growth or decay, namely

$$Y_0 = a, Y_n = (1 + k) \cdot Y_{n-1}, n = 1, 2, 3, \dots$$

to an explicit function in terms of a, k , and n , as follows:

$$\begin{aligned} Y_0 &= a \\ Y_1 &= (1 + k)Y_0 = (1 + k)a \end{aligned}$$

$$Y_2 = (1+k)Y_1 = (1+k)(1+k)a = a(1+k)^2$$

$$Y_3 = (1+k)Y_2 = (1+k)a(1+k)^2 = a(1+k)^3$$

If we continue this process, we see that the n^{th} term is given by

$$Y_n = a(a+k)^n$$

Note the distinction between the recursive equation

$$Y_n = Y_{n-1}(1+k)$$

and the explicit equation

$$Y_n = a(1+k)^n$$

The recursive equation shows that each value of Y_n is obtained from the preceding value by multiplying by $(1+k)$. The explicit equation uses an exponent to represent this repeated multiplication.

If we wanted the value of Y_{100} , the recursive equation would require that values of Y_1, Y_2, Y_3 , and so forth up to Y_{99} , all be calculated. In contrast, the explicit equation $Y_n = a(1+k)^n$ allows us to calculate Y_{100} without requiring any intermediate values.

The explicit equation can be rewritten using the more traditional functional notation:

Definition 4.3.4 (Explicit Function for Geometric Growth).

$$Y(n) = a(1+k)^n$$

In equation 4 the independent variable n is in the exponent, so this function is an exponential function. The exponential function $Y(n) = a(1+k)^n$ is the closed form representation of the recursive system $Y_0 = a, Y_n = (1+k)Y_{n-1}$

Example 4.3.5. Suppose you plan to make a one-time deposit into a bank account that will earn 0.45% monthly interest. How large must this deposit be so that you will have a college fund of \$75,000 available after 18 years or 216 months?

Solution. We can solve this problem using the recursive model $S_0 = a, S_n = (1.0045)S_{n-1}$ but we will have to guess and check to find the appropriate value of S_0 that gives of $S_{216} = 75,000$. Using the closed form $S(n) = a(1.0045)^n$, we need to find the value of a such that $75,000 = a(1.0045)^{216}$. Solving for a gives the equation $a = \frac{75,000}{(1.0045)^{216}}$, or $a = \$28,436.36$. We see that the closed form is useful when we do not need all the intermediate values that the recursive form generates.

Example 4.3.6 (Continuous Versus Discrete). In Example 3 we considered the amount of Potassium-42 that remains present during the decay process. We used the recursive system

$$P_0 = 1, P_n = 0.94455P_{n-1}, n = 1, 2, 3, \dots$$

to determine the amount of Potassium-42 remaining after n 1-hour time intervals.

We could also use the explicit function

$$P(t) = 1 \cdot (0.944355)^t$$

to represent the amount of Potassium-42 remaining t hours after the decay process begins. Use each of these representations to determine how much Potassium remains after 12 hours.

Solution. This is what our work looks like if we use the recursive system:

$$\begin{aligned} P_0 &= 1 \\ P_1 &= 0.94455 \cdot P_0 = 0.94455 \\ P_2 &= 0.94455 \cdot P_1 = 0.94455 \cdot 0.94455 = 0.892175 \\ P_3 &= 0.94455 \cdot P_2 = 0.94455 \cdot 0.892175 = 0.847036 \end{aligned}$$

$$P_4 = 0.94455 \cdot P_3 = 0.94455 \cdot 0.847036 = 0.7959757$$

...

$$P_{12} = 0.94455 \cdot P_{11} = 0.94455 \cdot 0.53392 = 0.5043121$$

If we use the explicit function:

$$P(12) = 1(0.94455)^{12} = 0.5043121$$

Both representations tell us that in 12 hours the amount of Potassium-42 will have decreased from 1 milligram to about 0.5 milligram.

We need to be aware of the advantages and disadvantages of each representation that we used in [Example 5](#). The recursive system is inherently discrete. Values of P_n can be calculated only for positive integer values of n , where n counts the number of 1-hour time intervals that have elapsed since we measured 1 milligram of potassium-42. The recursive representation is not able to tell us anything about the amount of potassium present between P_2 and P_3 , in fact, $P_{2.7}$ is not even defined.

The explicit function $P(t) = 1 \cdot (0.944355)^t$ uses t as a variable whose domain is all positive real numbers and $P(2.7)$ is well-defined and meaningful. The explicit function can tell us the amount of Potassium-42 present at any time on the continuum between zero and infinity.

Graphs of both the recursive and the explicit representations are shown below

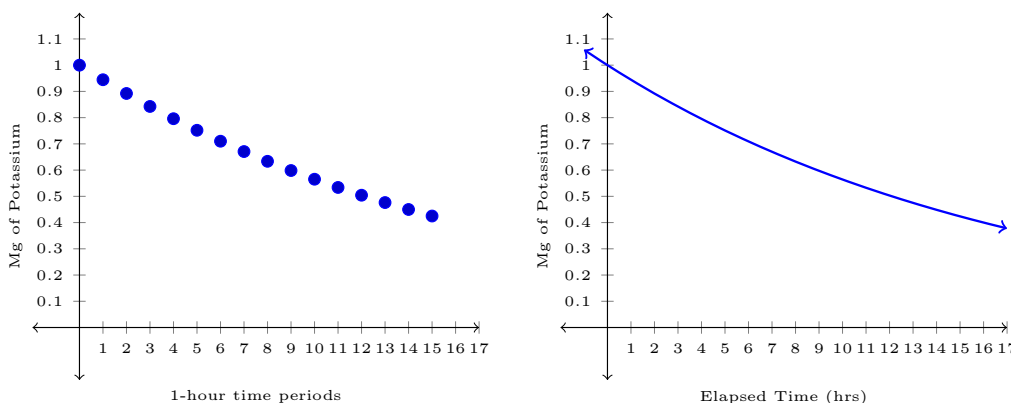


Figure 4.3.7: Mg of Potassium Over Time: Discrete and Continuous Models

Note that in the graph on the left, the horizontal scale shows the number of 1-hour time intervals that have elapsed since P_0 . On the right, the horizontal scale shows elapsed time. This is consistent with the fact that the recursive system has domain $n = 1, 2, 3, \dots$ and the explicit function has domain all positive real numbers.

[Example 6](#) shows that we must pay attention to issues of domain when we are choosing between using recursive and explicit representations of a particular phenomenon. Some phenomena by their very nature have a discrete (and thus discontinuous) domain, and these are often best represented recursively. Other phenomena are by nature continuous and may be better represented by an explicit function. Of course, with appropriate care about interpretations, we can choose to use a continuous function to represent a discrete phenomenon. We can also use a discrete representation for a continuous phenomenon. If we choose to do this, we need to pay particular attention to the way that we interpret our calculations.

4.3.1 Class Practice

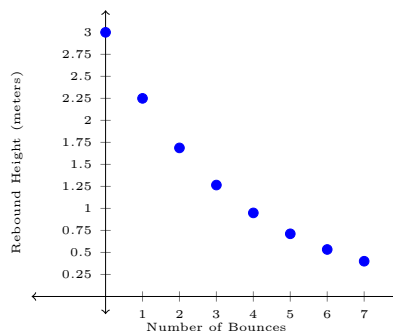
1.

When a basketball is released and drops to a hard surface, its path looks something like what is shown in the photo. Each time that it bounces, it rebounds to 75% of the height from which it was released (assuming that the ball is correctly inflated.) If a ball is dropped from 3 meters, we can write the recursive system:

$$H_0 = 3, H_n = 0.75H_{n-1}$$



The graph shows the rebound height on the vertical axis and bounce number on the horizontal. Sketch a graph of the associated explicit function $h(t) = 3(0.75)^t$ where $h(t)$ is the height at time t . Explain how the graph you sketch is related to the discrete graph that is shown.



2. In [Example 1](#) we used the recursive system $C_0 = 1, C_n = 0.98C_{n-1}$ to represent the amount of color left in blue jeans after n washings. The graph shows remaining color on the vertical axis and number of washings on the horizontal axis. Sketch a graph of the associated explicit function $C(t) = 1(0.98)^t$ where t represents elapsed time. Write a few sentences to explain how and why your graph differs from the discrete graph shown.

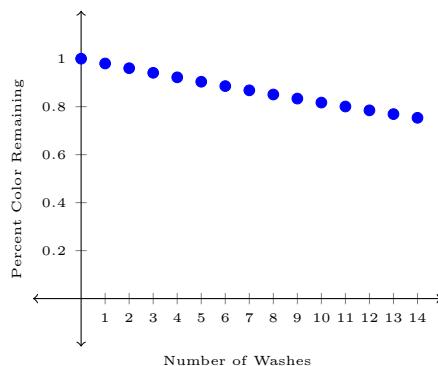


Figure 4.3.8

Example 4.3.9. Rebecca starts working for a company at a salary of \$40,000 per year. Based on the company's history, she can expect raises of 3.5% each year on the anniversary of her employment. When will she first make \$50,000?

Solution. We can use the recursive system $A_0 = 40,000, A_n = 1.035A_{n-1}, n = 1, 2, 3, \dots$ to determine when her salary will equal or exceed \$50,000. Rebecca's first pay raise will result in a salary of \$41,400. Her sixth raise will bring her salary up to \$49,170.21 and her seventh raise will put her pay over \$50,000 at \$50,891.17.

We could also use a closed form function to solve for the time at which her salary will reach \$50,000. This function is $S(t) = 40,000(1.035)^t$. By evaluating this function, we find that $S(6) = 49,170.21$ and $S(7) = 50,891.17$. Both the recursive model and the closed form model inform us that Rebecca will make more than \$50,000 with her 7th raise.

If we wanted to know what Rebecca would make if she stayed with this company for 30 years, it would be easier to use the closed form and substitute $t = 30$. In many cases either the recursive system or the closed form could be used to arrive at the same answer.

In cases where we need to predict far into the future, it is more efficient to use the closed form. In cases where we want to see all of the intermediate values, as would be the case for the balance due on a loan after each payment, it would be to our advantage to use the recursive system. In Rebecca's case, if we choose to use an explicit representation we must limit the domain to integer values since her pay raises occur only one time per year.

Example 4.3.10 (Doubling Time). Suppose the population of a certain type of bacteria is known to grow geometrically and increases by about 26% every hour. How much time will it take for a population of 150 million cells to grow to 300 million? How long will it take the population to double again to 600 million? When will the population reach 1200 million (another doubling)?

Solution. Since the population is growing by 26% per hour, we can use the recursive system

$$P_0 = 150, P_n = (1.26)P_{n-1}$$

We also have the option of using the explicit function $P(t) = 150 \cdot (1.26)^t$. For integer values of t , these two representations give roughly identical values for the number of bacteria cells.

Time (Hours)	Cells (Millions)	Time (Hours)	Cells (Millions)
0	150	5	476.37
1	189	6	600.22
2	238.14	7	756.28
3	300.06	8	952.92
4	378.07	9	1200.67

The population took about 3 hours to double from 150 to 300 million. In another 3 hours, it had doubled again, and in another 3 hours there was yet another population doubling. This population is said to have a *doubling time* of 3 hours. Note that the first doubling corresponds to an increase of 150 cells, the next doubling is an increase of 300 cells, and the third doubling is an increase of 600 cells. For each of these doublings, the elapsed time is the same (3 hours) but the increase, measured in cells per hour, is not the same.

Example 10 shows that a population that experiences geometric growth has a doubling time. It is also true that populations that experience geometric decay have a *half-life*. A half-life is the amount of time it takes for a population size to be halved. Populations that experience other types of growth, such as linear, quadratic or logistic, do not have a doubling time.

The exponential function you studied in Chapter 2, $f(t) = 2^t$, has a doubling time of 1 time unit. This is because

$$\begin{aligned} f(1) &= 2^1 = 2 \\ f(2) &= 2^2 = 4 \\ f(3) &= 2^3 = 8 \\ f(4) &= 2^4 = 16 \end{aligned}$$

A doubling of function-values takes place with each increase of 1 unit in t

The function $g(t) = 2^{\frac{1}{3}t}$ is a horizontal stretch of the function $f(t) = 2^t$, and this transformation makes $g(t)$ have a doubling time of 3 time units:

$$\begin{aligned} g(3) &= 2^{\frac{1}{3} \cdot 3} = 2^1 = 2 \\ g(6) &= 2^{\frac{1}{3} \cdot 6} = 2^2 = 4 \\ g(9) &= 2^{\frac{1}{3} \cdot 9} = 2^3 = 8 \end{aligned}$$

Since $g(t) = 2^{\frac{1}{3}t}$ has a doubling time of 3 time units, in Example 10 we could have used the function $y = 150 \cdot 2^{\frac{1}{3}t}$ to model the number of bacteria cells present at time t . You can confirm that the two functions $y = 150 \cdot 2^{\frac{1}{3}t}$ and $P(t) = 150 \cdot (1.26)^t$ produce roughly the same ordered pairs.

4.3.2 Exercises

In exercises 1 through 4, identify whether the growth (or decay) that is described is discrete or continuous. Write either a recursive system or an explicit function to represent the phenomenon. Use the most appropriate form to answer each question.

1. Research City is growing by 14% each year. If the population of the city is approximately one million people and the rate of growth continues at 14% annually, what will Research City's population be 15 years from now?
2. The population of Coastal City grows by 3% each year due only to births and deaths among current residents. The population is currently one million. Each year 15,000 more people move into the city than move out, resulting in a net gain of 0.015 million people. How long will it take Coastal City's population to reach 1.8 million people?
3. Each year hunting and natural predators combine to cause the population of rabbits in the meadow to decrease by 5%. If the year begins with 230 rabbits and the population continues to decrease by 5% each year, how many rabbits will there be in this meadow in 50 years?
4. Each year the population of rabbits in the meadow decreases by 5%. Farmer Dan decides to help the rabbit population by releasing 5 new rabbits into the meadow each year. If the year begins with 230 rabbits, describe what will happen to the population over the next ten years.
5. An annual inflation rate of $k\%$ means that items will cost $k\%$ more next year than they cost this year. Based on a yearly inflation rate of 3%, estimate the cost of the following items in 10, 20, 30, and 40 years.

Item	Cost Today
Jeans	\$45.00
Hamburger	\$3.90
Car	\$29,000
Textbook	\$75.00
Movie Ticket	\$9.00

Table 4.3.11: Cost of Items

6. How much money would you need to invest now in an account that receives 0.5% monthly interest so that in 20 years you will have \$50,000?
7. The number of cells in a certain bacteria colony triples every hour. Write an explicit function that models this growth. By what factor does the population grow in half an hour?
8. Thorium-234 is a radioactive material whose half-life is 25 days. Write an explicit function for the amount of thorium-234 left after t days. What percent of an original amount is left after 300 days?
9. The population of The Peoples Republic of China in 2015 was a little over 1.39 billion and growing at a rate of about 0.5% annually.
 - (a) Write a recursive system to model the population
 - (b) Find an explicit function to model the population
 - (c) To the nearest year, how long will it take the population to double? Assume that the growth rate remains 0.5% per year.
 - (d) Use the doubling time you found in part (c) to write an transformation of the function $y = 2^x$ to represent China's population. (Use x to represent the number of years elapsed since 2015.).

- (e) Write a few sentences to compare and contrast the models you found in parts a, b, and d.
- 10.** When you are 40 years old your rich Uncle Harry leaves you \$10,000 in his will when he dies. His death makes you realize it is time to start saving for your own retirement. Your goal is to deposit enough in a retirement account when you are between the ages of 40 and 65 that you can "pay yourself" a comfortable amount each year when you are over 65.
- (a) On your 40th birthday you invest the \$10,000 in an account that pays 3.5% annual interest. You also decide to make a yearly deposit in the account of \$1,000. What will your balance be when you turn 65? Give the amount in your account when you turn 65 and write the equations you use to arrive at your answer.
- (b) Will you have enough money in your account when you are 65 to pay yourself \$5,000 per year from age 65 to age 80. While you are withdrawing money, the balance in the account continues to earn 3.5% annual interest. Give a yes or no answer and write the equations you use to support your answer.
- 11.** Thomas has two plans for saving to buy a car. In Plan A, he will make an initial deposit of \$50 and then he will deposit \$33 each week in an account that earns 0.1% interest per week. In Plan B, he will make an initial deposit of \$50 and then each week he will put \$30 in an account that earns 0.45% each week. Write recursive equations to show the amounts Thomas would have under each of the plans. Write down the account balances under each plan during weeks 1 – 3 and during weeks 50 – 52
- 12.** In this section you have seen that the recursive system $P_0 = a, P_n = (1 + k) \cdot P_{n-1}, n = 1, 2, 3, \dots$ can be written as a closed form exponential function $P(n) = a \cdot (1 + k)^n$. What recursive system can be written as a closed form linear function of the form $y(n) = a + kn$?

4.4 Summing Geometric Growth

In 2012 the world wide rate of crude oil consumption was about 89,837 thousand barrels per day and was increasing at an annual rate of about 1.42% ¹. As a consequence, the world consumed about 32.8 billion barrels of crude oil in 2012 and about 33.3 billion barrels in 2013. We can use the explicit function $A(n) = 32.8(1.0142)^n$, where n represents the number of years since 2012, to find that the amount of oil consumed in 2042 will be a little over 50 billion barrels, since $A(30) = 32.8(1.0142)^{30} = 50.07$. That is certainly a large amount, but knowing the amount in any one year does not really tell the whole story. A more important question: What is the total amount of crude oil that will be consumed between 2012 and 2042? This total represents the quantity of oil that will be depleted from world oil reserves over this 31-year time period.

To determine the total amount of oil consumed from 2012 to 2042, we want to find the sum:

$$T = A(0) + A(1) + A(2) + \dots + A(30)$$

We can rewrite the equation for T as:

$$T = 32.8 + 32.8(1.0142) + 32.8(1.0142)^2 + \dots + 32.8(1.0142)^{30}$$

Notice that each term in the sum is 1.0142 times the previous term. This sum is an example of a geometric series, which is a sum in which each successive addend is found by multiplying the previous term by some fixed value. Or, put another way, a geometric series is the sum of a geometric sequence.

A more general way to represent a geometric series is:

$$S = a + ar + ar^2 + ar^3 + \dots + ar^n \quad (4.4.1)$$

¹SOURCE: www.eia.gov

This series has first term a and each subsequent term is obtained from the preceding term by multiplying by r . The ratio of two consecutive terms in a geometric series is always r , so r is known as the common ratio.

We can find a formula for the sum of a geometric series without actually adding all of the terms, instead we use algebra in a clever way. First, we multiply both sides of equation (1) by r , which yields:

$$Sr = ar + ar^2 + ar^3 + ar^4 + \dots + ar^{n+1} \quad (4.4.2)$$

Most of the terms on the right sides of equations (4.4.1) and (4.4.2) are the same. If we now subtract equation (4.4.2) from equation (4.4.1), meaning we subtract the left side of (4.4.2) from the left side of (4.4.1) and the same with the right sides, we obtain the new equation:

$$S - Sr = a - ar^{n+1}$$

Solving this equation to isolate S , we have:

$$S = \frac{a - ar^{n+1}}{1 - r} \quad (4.4.3)$$

Equation (4.4.3) represents the sum of the terms a through ar^n of a geometric series with first term a and common ratio r , assuming $r \neq 1$.

Note that if $r = 1$ the series

$$S = a + ar + ar^2 + ar^3 + \dots + ar^n$$

is equivalent to

$$S = a + a + a + a + \dots + a$$

and the sum of this series is simply $S = (n + 1)a$

Returning to the question regarding world crude oil consumption from 2012 through 2042, we need to find the sum of terms 0 through 30 of a geometric series with initial term 32.8 and common ratio 1.0142. This sum is given by

$$T = \frac{32.8 - 32.8(1.0142)^{31}}{1 - (1.0142)} \approx 1,266 \text{ billion barrels of oil}$$

In the world's top 17 oil producing countries, estimates of the total proven oil reserves that are recoverable with 2012 technology are in the range of 1200 to 1300 billion barrels. Therefore, if oil consumption continues to increase at the rate observed in 2012, proven oil reserves will be almost exhausted by 2042.

You will encounter a wide variety of situations in mathematics classes and in other disciplines that require finding the sum of a geometric series. This happens frequently enough that you should be sure to include in the collection of mathematical tools you have available to you either the formula for the sum of a geometric series, or the method used here to arrive at that sum.

Below are two additional examples from the financial world where this concept is used.

Example 4.4.1 (Value of an Annuity). Suppose your parents want to set aside money for college tuition for a younger sibling. They begin saving when she is twelve by opening an account with an initial deposit of \$100. At the beginning of each month for six years thereafter, they deposit an additional \$100. An account into which regular payments are made (or from which regular withdrawals are made) is called *an annuity*. The account into which they place the money earns 0.5% monthly interest, which is added to the account at the end of each month. How much money will be in the account at the end of six years?

Solution. The initial deposit earns interest for 72 months, which means that in 72 months the first \$100 deposited has grown to a value of $\$100(1.005)^{72}$. The money deposited at the beginning of the second month earns interest for 71 months and grows to $\$100(1.005)^{71}$.

Each successive deposit earns interest for one month less than the previous deposit. We will assume that your parents close the account on the day that they make the final payment of \$100; this means that the final \$100 deposit earns no interest. The timeline in Figure 2 shows each deposit together with the amount of interest earned by each deposit.

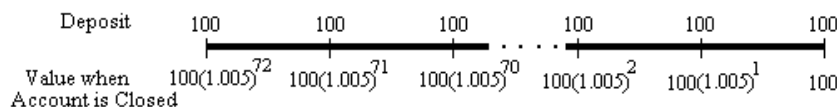


Figure 4.4.2: Timeline for interest and deposit in an annuity (REDO GRAPHIC)

The balance of the annuity after six years is the sum of the values of all 73 deposits, which is

$$B = 100 + 100(1.005) + 100(1.005)^2 + 100(1.005)^3 + \dots + 100(1.005)^{72}$$

This is a geometric series with initial term 100 and common ratio 1.005, so we can use equation (4.4.3) to write the sum of this series as

$$B = \frac{100 - 100(1.005)^{73}}{1 - 1.005} \quad (4.4.4)$$

which equals \$8,784.09, the balance of the annuity after six years. Note that only \$7,300 was deposited, so over 6 years the deposits have earned over \$1,400 in interest.

Example 4.4.3 (Paying Off a Loan). Suppose you borrow \$20,000 to buy a car, and you agree to pay back the loan over 48 months at 0.4% interest per month. You want to determine the monthly payment P that you must make in order to pay off the loan. You know how to find the payment using a "guess and check" method. Now, since you are able to find the sum of a geometric series, you can write and solve an equation to find the payment.

Solution. Your outstanding balance can be represented with the recursive system

$$L_0 = 20,000, L_n = 1.004L_{n-1} - P, n = 1, 2, 3, \dots$$

Where L_n is the amount still owed on the loan after n months and P is the monthly payment in dollars. We know that after one month,

$$L_1 = 1.004(20,000) - P$$

After two months, the amount still owed is given by

$$L_2 = 1.004L_1 - P$$

which, by substitution is equivalent to

$$L_2 = 1.004(1.004 \cdot 20,000 - P) - P$$

or

$$L_2 = 1.004^2(20,000) - 1.004P - P$$

Similarly, since $L_3 = 1.004L_2 - P$, we have:

$$L_3 = 1.004(1.004^2(20,000) - 1.004P - P) - P$$

$$L_3 = 1.004^3(20,000) - 1.004^2P - 1.004P - P$$

We can continue to iterate to find the outstanding balance after any number of months. For instance,

$$L_5 = 1.004^5(20,000) - 1.004^4P - 1.004^3P - 1.004^2P - 1.004P - P$$

In general, we see that after n months the outstanding balance is given by:

$$L_n = 1.004^n(20,000) - P(1.004^{n-1} + 1.004^{n-2} + \dots + 1.004^2 + 1.004 + 1) \quad (4.4.5)$$

Embedded in the right side of equation (4.4.5) is the geometric series

$$1 + 1.004 + 1.004^2 + 1.004^3 + \dots + 1.004^{n-2} + 1.004^{n-1}$$

which has first term 1 and common ratio 1.004. Using equation (4.4.3), we know that

$$1 + 1.004 + 1.004^2 + 1.004^3 + \dots + 1.004^{n-2} + 1.004^{n-1} = \frac{1 - 1.004^n}{1 - 1.004}$$

Substituting into equation (4.4.5) we obtain the following representation for L_n :

$$L_n = 1.004^n(20,000) - P\left(\frac{1 - 1.004^n}{1 - 1.004}\right) \quad (4.4.6)$$

We want 48 payments to reduce the outstanding balance to zero, that is, our goal is $L_{48} = 0$. This means that we can solve the following equation to find the value of P .

$$0 = 1.004^{48}(20,000) - P\left(\frac{1 - 1.004^{48}}{1 - 1.004}\right)$$

Solving for P , we find

$$P = 1.004^{48}(20,000)\left(\frac{1 - 1.004^{48}}{1 - 1.004}\right)$$

or, equivalently

$$P = \frac{1.004^{48}(20,000)(0.004)}{1.004^{48} - 1}$$

which means that $P \approx 458.78$. The loan will be paid off in 48 months with a monthly payment of \$458.78. You should verify using the recursive system

$$L_0 = 20,000, L_n = 1.004L_{n-1}, n = 1, 2, 3, \dots$$

that the balance after 48 payments, L_{48} is zero. Note that due to rounding, your answer may not be exactly zero. The final balloon payment may be a few cents more or less than \$458.78

It is worthwhile to take a moment to study equation (4.4.6), which effectively gives an explicit function for $L(n)$.

$$L(n) = 1.004^n(20,000) - P\left(\frac{1 - 1.004^n}{1 - 1.004}\right)$$

Recall that the monthly interest rate was 0.4%, the amount borrowed was \$20,000, $L(n)$ is the amount still owed after n payments, and P is the monthly payment. You should conclude that if the monthly interest rate is $k\%$ and the amount borrowed is A , then $L(n)$, the amount still owed after n payments, will be given by

$$L(n) = (1 + k)^n A - P\left(\frac{1 - (1 + k)^n}{1 - (1 + k)}\right) \quad (4.4.7)$$

If you want the outstanding balance to be zero after n^* months, you can solve to find the value of P for which $L(n^*) = 0$.

4.4.1 Exercises

1. Consider the geometric series $100 + 100(1.05) + 100(1.05)^2 + 100(1.05)^3 + \dots$

(a) Find the sum of the first 10 terms of the series

- (b) Find the sum of the first 100 terms of the series
 - (c) Find the sum of the first 500 terms of the series
- 2.** Consider the geometric series $200 + 200(0.8) + 200(0.8)^2 + 200(0.8)^3 + \dots$
- (a) Find the sum of the first 50 terms of the series
 - (b) Find the sum of the first 500 terms of the series
 - (c) Find the sum of the first 1000 terms of the series
 - (d) Make a prediction about the sum of the first million terms. Compare your prediction with the actual sum.
- 3.** Consider the geometric series $3 + 1.5 + 0.75 + 0.375 + \dots$
- (a) Find the sum of the first 10 terms of the series
 - (b) Find the sum of the first 50 terms of the series
 - (c) How many terms of the series must be added to bring the sum to at least 5.99999?
- 4.** Consider the geometric series with first term $\frac{9}{10}$ and common ratio $\frac{1}{10}$
- (a) Find the sum of the first 10 terms of the series
 - (b) Find the sum of the first 100 terms of the series
 - (c) Explain if it is meaningful to talk about adding infinitely many of the terms in this series
- 5.** Let $T(n)$ represent the sum of the first n terms of the series $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \dots$
- (a) Find the sum of $T(10)$ without actually adding terms.
 - (b) Find an explicit expression for $T(n)$ in terms of n .
 - (c) Describe how the values of $T(n)$ behave as n increases without bound.
- 6.** Let $T(n)$ represent the sum of the first n terms of the series $0.3 + 0.03 + 0.003 + 0.0003 + \dots$
- (a) Find an explicit expression for $T(n)$ in terms of n .
 - (b) Describe how the values of $T(n)$ behave as n increases without bound.
- 7.** Let $S(n)$ represent the sum of the first n terms of the series with common ratio r . For what values of r is it meaningful to talk about adding infinitely many terms of the series?
- 8.** Find the sum of each geometric series.
- (a) $3 + 6 + 12 + \dots + 192$
 - (b) $1 + 0.2 + (0.2)^2 + (0.2)^3 + \dots + (0.2)^{14}$
 - (c) $4 + 4^2 + 4^3 + \dots + 4^45$
 - (d) $3 + \frac{3}{5} + \frac{3}{25} + \frac{3}{125} + \dots + \frac{3}{5^{20}}$
 - (e) $3 - \frac{3}{5} + \frac{3}{25} - \frac{3}{125} + \dots + \frac{3}{5^{20}}$
- 9.** In the oil consumption problem at the beginning of this section, the consumption of oil is increasing at a rate of 1.42% per year. By how many billions of barrels will the total consumption of oil from 2012 to 2042 be reduced if the rate of increase is reduced to 1%?
- 10.** The number of new cars sold at a car dealer each year grows at a rate of 1.5% a year. This year, the dealer sold 200 cars. Assume that the 1.5% annual growth rate continues.
- (a) How many cars will the dealer sell 15 years from now
 - (b) What is the total number of cars that will have been sold between this year and 15 years in the future?

11. The landfill in Jefferson County can hold 22.6 million cubic yards of compacted garbage. In 2015 the city of Jefferson and the surrounding communities contributed 550,000 cubic yards of garbage to the landfill. The following year the total amount was about 560,000 cubic yards, which represents a 1.8% increase. This rate of Increase is actually smaller than the population growth rate of the county, so residents are reducing their per capita contributions to the land fill. ²

- (a) Find the total amount of garbage, in cubic yards, that the country residents will contribute to the landfill over the next 10 years, from 2015 thru 2024
- (b) In what year will the landfill reach its capacity, if the annual growth rate of garbage volume remains at 1.8%?
- (c) Suppose that new recycling program can decrease the number of cubic yards of garbage that the country produces. In 2015 residents contributed 550,000 cubic yards of garbage to the landfill and annual rate of increase is brought down to 1%. How many years will the landfill last under these conditions?
- (d) How long would the landfill last if the annual rate of increase of garbage contribution was brought down to zero?

12. You have an annuity to which you initially depot \$1,000 and you add \$150 each month. The account earns interest at 0.35% monthly interest rate; interest earned is added to the account balance at the end of each month.

- (a) Write a recursive model for the situation.
- (b) How much money will you have after 3 years?
- (c) How much money will you have after n months? Your answer should be written in the form of an explicit expression.

13. Compare these two investment strategies to save for retirement by determining the account balance at age 65:

Strategy I: For the 8 years from age 44 to age 52, invest \$300 per month and earn interest at a rate of 0.75% per month. Do not save any more money after age 52, and make no withdrawals from the account.

Strategy II: For the 16 years from age 49 to age 65, invest \$300 per month and earn at a rate of 0.75% per month. Make no withdrawals from the account.

14. Start by writing an explicit function that expresses Y_n in terms of n if the values of Y_n are generated by the recursive system $Y_0 = A, Y_n = aY_{n-1} + b, n = 1, 2, 3, \dots$ Use the method for summing a geometric series. Then:

- (a) Use your explicit function to determine the balance in an annuity after 25 years if an initial deposit of \$50,000 and regular annual deposits of \$1,000 have all earned interest at a 3.5% annual rate.
- (b) Use your explicit function to determine the outstanding balance after 60 months on a loan of \$55,000 if payments of \$600 are made each month and the interest rate is 2.4% per month.

15. Earl and Larry are twins. They both went to work at the age of 24 at the same job. On their birthday each year (starting with their 25th birthday), they received identical bonuses of \$2,000. However, they decided to deal with their finances differently.³

Earl's Strategy: Earl decided to invest his money early in life, and each year, he took his bonus and put it in a savings account that earned 6.6% annual interest. At the age of 34, Earl decided to live life a little and began spending his bonuses (beginning with the one he received at age 35). He did not, however, use any of the money he already had in his savings

²Adapted from Mathematics Modeling our Worlds COMAP course 2

³Adapted from: Smith, Keith, Tim and Tom's Financial Adventure, HImap Pull-Out Section, COMAP Consortium, Fall, 1991

account. He let it continue growing until He turned 60 years old, without depositing any new money into his savings account.

Larry's Strategy: Larry decided to have some fun when he was young and spent his bonuses up until he turned 34. After he turned 34 and received his bonus, he realized that he should start thinking about the future and put some money away for retirement. He began to invest his bonuses (starting with the one he received at age 35) in a savings account that earned 6.6% annual interest. He did this until the age of 60, including the bonus he received on his 60th birthday.

Your task is to compare Earl's and Larry's investment strategies. Your analysis should include the amount each twin has in his bank account on their 60th birthday as well as how much bonus money each twin was able to spend between the ages of 25 and 60. Which investment strategy would you recommend? Your comparison should include both *recursive equations and explicit functions* that represent the amount of money Earl and Larry have in their respective accounts at any time.

4.5 Compound Interest (or the Magic of Compounding)

In the previous section, we encountered a situation in which money was deposited into an interest-bearing account, and the interest was added to the account at regular intervals. For example, suppose Maria deposits \$1,000 in an account that earns interest at a rate of 5% annually. After one year, the balance of the account will increase to \$1,050. After two years the balance will be \$1,102.50 and so on according to the recursive equation

$$A_n = A_{n-1} + 0.05A_{n-1}$$

where A_n is the account balance after n years. A_n is known as the *future value* of the initial deposit of \$1,000 after n years. The interest in Maria's account is *compounded annually*, which means that the interest that has been earned during a year is credited to the account at the end of that year.

Rather than adding earned interest into an account at the end of each year, financial institutions often use quarterly, monthly, or daily compounding. For example, a bank that compounds monthly would add one month's interest into the account at the end of every month. As a consequence, during the second month, the initial deposit and the first month's interest will both earn interest. During the third month, the initial deposit and two months of interest payments will earn interest.

What effect does the frequency of compounding have on the future value of Maria's \$1,000 deposit? We will rely on the formula $I = prt$, where I is the amount of interest earned in one compounding time period, p the principal, r the annual interest rate, and t the time (in years) between successive compoundings. We see that at the end of the first month, the amount of interest earned is $I = 1000 \cdot 0.05 \cdot 1/12 \approx 4.17$. The account balance after one month is

$$\$1,000 + \$1,000(0.05)\frac{1}{12} = \$1,000\left(1 + \frac{0.05}{12}\right) = \$1,004.17$$

At the end of the second month, the account has a balance of

$$\$1,004.17\left(1 + \frac{0.05}{12}\right) = \$1,008.35$$

At the end of the third month, the account has a balance of

$$\$1,008.35\left(1 + \frac{0.05}{12}\right) = \$1,012.55$$

At the end of the first year, which contains twelve compounding periods, the balance will be

$$\$1,000\left(1 + \frac{0.05}{12}\right)^{12} = \$1,051.16$$

Notice that this amount is higher than the balance of \$1,050.00 that we expect at the end of one year using annual compounding. This increased balance is the result of Maria earning interest on interest payments that have been deposited into the account during the year.

Example 4.5.1 (Investment Accounts). Financial institutions often offer a variety of investment accounts from which their customers can choose.

1. What will be Felix's account balance after 1 year if he deposits \$100 in an account that pays 5% annual interest that is compounded quarterly? What annual interest rate, compounded only one time per year, would result in the same account balance?
2. What will be Felix's account balance after 1 year if he deposits \$100 in an account that pays 4.9% annual interest that is compounded daily? What annual interest rate, compounded only one time per year, would result in the same account balance?
3. Would Felix be better off investing money in an account that earns 5.1% annual interest that is compounded annually, or an account that earns 5% that is compounded every hour?

Solution.

1. If 5% annual interest is compounded quarterly (four times per year) then the account balance is multiplied by $(1 + \frac{0.05}{4})$ each quarter. The account balance after n quarters is given by $A(n) = 100(1 + \frac{0.05}{4})^n$. After 1 year, the balance will be $A(4) = 100(1 + \frac{0.05}{4})^4 \approx \105.09 . Felix would have ended up with the same account balance if he earned an interest rate of 5.09% that was compounded once per year.
2. When 4.9% annual interest is compounded daily (365 times per year) the balance after n days is given by $A(n) = 100(1 + \frac{0.049}{365})^n$. After 1 year, the balance will be $A(365) = 100(1 + \frac{0.049}{365})^{365} \approx \105.02 . An annual interest rate of 5.02% compounded once per year would have yielded the same balance as 4.9% compounded daily.
3. Felix can compare these accounts even if he does not know how much money he has to invest or how much time he will leave his money in the account. Use A to represent his initial deposit. If 5.1% interest is compounded once per year, his balance is multiplied by (1.051) every year, and after n years his balance will be $A(1.051)^n$ dollars. Now consider 5% annual interest being compounded every hour. Since each compounding period is $\frac{1}{8760}$ year, his balance is multiplied by $(1 + \frac{0.05}{8760})$ every hour. Since there are 8760 hours in a year, the account balance is multiplied by $(1 + \frac{0.05}{8760})^{8760}$ every year. Note that $(1 + \frac{0.05}{8760})^{8760} \approx 1.0513$, so after n years Felix's account balance will be $A(1.0513)^n$ dollars. Felix is better off investing in the account that pays 5% interest compounded hourly because $A(1.0513)^n > A(1.051)^n$ for all positive values of A and of n .

The different interest rates and compounding frequencies in [Example 1](#) highlight the need for a way to compare these accounts. In part (a) we saw that 5% compounded quarterly has the same “effect” on an account balance as 5.09% compounded once per year. So we call 5.09% the *effective annual yield*, or *effective annual interest rate* of 5% compounded quarterly. It is the interest rate that if compounded only once per year would yield the same balance as another rate with more frequent compounding. In part (b) we saw that 4.8% annual interest compounded daily has an effective annual yield of 5.02%. In (c), we saw that 5% annual interest compounded once per hour has an effective annual yield of 5.13%.

4.5.1 Class Practice

1.
 - (a) Write a recursive system to represent the future value of an initial deposit A_0 that earns $r\%$ annual interest that is compounded quarterly. Let A_n represent the value after n years

- (b) Write an explicit function to represent the future value of an initial deposit A_0 that earns $r\%$ annual interest that is compounded quarterly. Let $A(n)$ represent the value after n years
2. Write representations as described in [Exercise 1](#), but use monthly compounding
3. Write representations as described in [Exercise 1](#), but use k compounding periods per year

[Table 2](#) shows the future value of \$1,000 earning 5% annual interest for various compounding frequencies. (All table entries are rounded to the nearest dollar.) You should be able to confirm the entries in the table. For instance,

$$\$1,647 \approx \$1,000 \left(1 + \frac{0.05}{12}\right)^{12 \cdot 10} = \$1,000 \left(1 + \frac{0.05}{12}\right)^{120}$$

Years (N)	Yearly $k = 1$	Quarterly $k = 4$	Monthly $k = 12$	Weekly $k = 52$	Daily $k = 365$	Hourly $k = 8,760$	By the Minute $k = 525,600$
1	\$1,050	\$1,051	\$1,051	\$1,051	\$1,051	\$1,051	\$1,051
10	\$1,629	\$1,634	\$1,647	\$1,648	\$1,649	\$1,649	\$1,649
25	\$3,386	\$3,463	\$3,481	\$3,488	\$3,490	\$3,490	\$3,490
50	\$11,467	\$11,495	\$12,119	\$12,168	\$12,180	\$12,182	\$12,183

Table 4.5.2: Impact of the Frequency of Compounding

Reading down any column of [Table 2](#), we notice that the future value increases as the number of years increases. This is to be expected. But reading across any row reveals that the future value seems to level off as the frequency of compounding increases. For any number of years, there appears to be an upper limit to the future value of the account. We see that increasing the frequency of compounding is associated with an increase in the future value, but we don't expect the future values associated with compounding "by the second" to be appreciably larger than the values displayed in the right-most column of the table for "by the minute" compounding.

Each entry in [Table 2](#) can be calculated with the explicit function

$$A(N) = 1,000 \left(1 + \frac{0.05}{k}\right)^{kN} \text{ or } A(N) = 1000 \left(1 + \frac{0.05}{k}\right)^{kN}$$

where $A(N)$ is the future value of N years.

The value of N is fixed in each row of [Table 2](#), in the first row $N = 1$, in the second $N = 10$, and so on. This means that increases in the future values in a particular row result solely from the value of the expression $\left(1 + \frac{0.05}{k}\right)^k$. Since the future values associated with each value of N appear to have a limit, we suspect that the quantity $\left(1 + \frac{0.05}{k}\right)^k$ has some limiting value as k becomes very large. [Table 3](#) provides values of this quantity for k -values associated with increasing frequency of compounding. Correct to six decimal places, the limiting value appears to be 1.051271

	Yearly $k = 1$	Quarterly $k = 4$	Monthly $k = 12$	Weekly $k = 52$	Daily $k = 365$	Hourly $k = 8,760$	By the Minute $k = 525,600$
$\left(1 + \frac{0.05}{k}\right)^k$	1.050000	1.050945	1.051162	1.061246	1.051267	1.051271	1.051271

Table 4.5.3: Values of $\left(1 + \frac{0.05}{k}\right)^k$ rounded to six decimal places

We can gain some understanding of this limiting value if we consider the case where the interest is 100%.

	$k = 10$	$k = 100$	$k = 1,000$	$k = 10,000$	$k = 100,000$	$k = 1,000,000$
$(1 + \frac{1.00}{k})^k$	2.593742	2.704814	2.716924	2.718146	2.719268	2.718280

Table 4.5.4: Values of $(1 + \frac{1.00}{k})^k$ rounded to six decimal places

Mathematicians began exploring the behavior of $(1 + \frac{1}{k})^k$ early in the eighteenth century. In essence they found that the value of $(1 + \frac{1}{k})^k$ approaches a limiting value as k gets larger and larger. Mathematicians define this limiting value, which is 2.71828..., as the number e in honor of the Swiss mathematician Leonhard Euler. In mathematical notation, we can write

$$\lim_{k \rightarrow \infty} \left(1 + \frac{1}{k}\right)^k = e$$

which is read, “the limit of $(1 + \frac{1}{k})^k$ as k approaches infinity is the number e .”

Based on the numerical values in Table 3, we see that $(1 + \frac{0.05}{k})^k$ approaches a limiting value of about 1.051271 as k approaches infinity. You can use a calculator to confirm that this value is equal to $e^{0.05}$. We write

$$\lim_{k \rightarrow \infty} \left(1 + \frac{0.05}{k}\right)^k = e^{0.05}$$

We can also use a calculator to compare the values of $e^{0.02}$ and $(1 + \frac{0.02}{k})^k$ for large values of k , as well as several more values of r . What we find supports the statement:

$$\lim_{k \rightarrow \infty} \left(1 + \frac{r}{k}\right)^k = e^r$$

Based on the relationship between limiting values and e , we can use the exponential function with base e to arrive at useful and accurate approximations for future values when interest is compounded frequently. For example, if 3.2% annual interest is compounded 5,000 times per year, the future value in N years of an initial deposit of \$1,000 is $\$1,000(1 + \frac{0.032}{5000})^{5000N}$ which can be approximated by $\$1,000e^{0.032N}$. When this approximation is used, the future values are the result of *continuous compounding*. It is useful to think of continuous compounding as compounding that occurs at every instant. It may be physically impossible to add interest into an account in a way that is truly continuous, but continuous compounding refers to the limiting case of compounding that is more and more frequent.

The equation

$$\text{Future Value} = A_0(e^r)^N = A_0e^{rN}$$

gives the balance after N years in an account with continuous compounding, initial deposit, and annual interest rate r . Note that since the compounding is continuous, the future value changes with any change in time, no matter how small, so it is reasonable to substitute any positive real number for N .

The continuous function $F(N) = A_0e^{rN}$ describing the future value of an account with continuous compounding is an example of an important class of functions known as exponential functions. All exponential functions are transformations of the exponential function $y = e^x$. This function is closely related to the tool-kit function $y = 2^x$ and can be used to describe many "real-world" situations, from the cooling of coffee sitting on a table to the increase of carbon-dioxide in the atmosphere. Over the next several sections of this chapter, you will become more aquatinted with this important function and the situations in which it is a useful model.

4.5.2 Exercises

1. Compute the balance that results when \$2,000 is deposited for one year in an account paying 4% annual interest compounded quarterly. How does this balance compare to an approximation using continuous compounding of 4% annual interest? Make the comparable comparison if the annual interest rate is 12%.

2. If Jack invests \$250 at an annual interest rate of 7.5%, what is the future value after two years if the interest is compounded quarterly? monthly? weekly? daily? continuously?
- 3.
- Which has the greater future value after 5 years, \$1,000 invested at 8% with yearly compounding or \$1,000 invested at 7.75% with quarterly compounding? Use graphs of appropriate functions to determine if the number of years affects which deposit has a greater future value.
 - Which has the greater future value after 5 years, (1,000 invested at 8% compounded yearly or \$800 invested at 9% compounded yearly? Does the number of years affect which deposit has the greater future value?
4. It has been said that the island of Manhattan was purchased for \$24 in 1626. Suppose the \$24 had been invested at 6% annual interest compounded quarterly. What would it be worth today?
5. If we denote the effective annual rate by R , then R satisfies the equation

$$A(1 + R) = A\left(1 + \frac{r}{k}\right)^k$$

where A is the initial deposit, k is the number of compounding periods per year, and r is the annual interest rate. Solve the equation for $R = \left(1 + \frac{r}{k}\right)^k - 1$.

6. Use the concept of effective annual yield to compare a 7.25% certificate of deposit with quarterly compounding to a 7% certificate with monthly compounding. Which option will provide a better return on your investment?
7. In this exercise you will look at the relationship between interest rate and doubling time.
- Complete the table. Assume that the interest is compounded monthly. The doubling time should be accurate to the nearest hundredth of a year.

Annual Interest (%)	Doubling Time (Years)
1.5	
3.2	
4.0	
5.3	
7.2	
8.0	
9.4	
12.0	
20.5	

- Make a scatterplot of the set of ordered pairs (rate, time to double). For example, if you found that a 5.8% interest rate compounded monthly takes 12.9 years to double, you would have the ordered pair (5.8, 12.9).
- Based on the shape of your scatterplot, identify a toolkit function that would be a good model for your scatterplot. *Hint:* An exponential function would NOT be a good model
- Find a model for the ordered pairs (rate, time to double). To do this, you should decide by what factor this toolkit function must be vertically stretched so that it will fit your scatterplot.
- According to your model, what is the doubling time for a 2% interest rate that is compounded monthly? What is the doubling time for 10%? For 14%? What interest rate will cause an investment to double in 70 years?

8. Suppose that when you were born your parents estimated they would need \$50,000 for college expenses. The best interest rate they could find was offered on a certificate of deposit paying 6% annual interest compounded monthly.

- What is the effective annual interest rate for this account?
- How much money should your parents have invested to have a balance of \$50,000 on your eighteenth birthday?

9. The amount of money that would have to be invested today to yield some specified amount in the future is called the *present value* of that future amount. (In exercise 8 you found the present value of \$50,000 to be paid in eighteen years.) Suppose a professional athlete signs a one-year contract for \$2,000,000 and agrees to be paid over a period of five years. At the beginning of each of the next five years he or she will be paid \$400,000. What is the present value of this contract assuming a 6% interest rate compounded annually? Another way to ask this question is: how much money should the team management deposit in an account earning 6% annual interest when the contract is signed to guarantee that they can pay this five-year deal? Since the athlete is paid at the beginning of each year, assume that the present value of the first payment of \$400,000 is \$400,000.

10. Suppose the interest rate is 5% in the athletic contract discussed in Exercise 9. What is the present value of this contract? Does this make sense when compared with your answer to Exercise 9? What would you expect to be true for an interest rate of 3%?

4.6 Exponential Functions

In Chapter 2 we graphed the toolkit function $f(x) = 2^x$ and in the previous section we introduced the function $f(x) = e^x$. Both of these functions are examples of exponential functions of the form $f(x) = b^x$, where $b > 0$ and $b \neq 1$. The positive real number b is called the *base* of the exponential function.

We observe that the graph of $f(x) = 2^x$ contains the points $(0, 1)$ and $(1, 2)$ and that the graph of $f(x) = e^x$ contains the points $(0, 1)$ and $(1, e)$. We know that for any positive base b , $b^0 = 1$, so graph of $f(x) = b^x$ always contains the point $(0, 1)$. Likewise, since $b^1 = b$ and $b(-1) = \frac{1}{b}$, we know that these graphs will also contain the points $(1, b)$ and $(-1, \frac{1}{b})$. In addition, all of the graphs will have the x -axis as an asymptote. When the base is greater than 1 this asymptote is the negative part of the x -axis. Figure 4.6.1 shows a graph of $f(x) = e^x$, along with the toolkit function $g(x) = 2^x$ on the axes on the left. The graph on the right illustrates $y = 3^x$, $y = 4^x$, and $y = 10^x$.

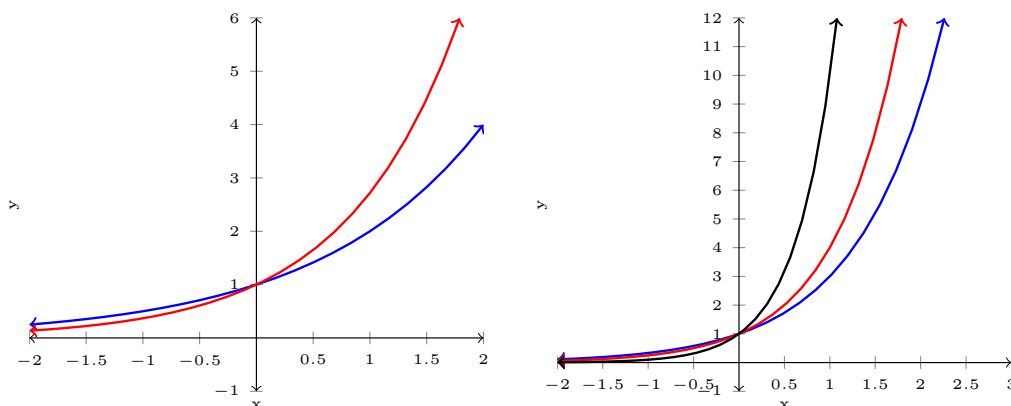


Figure 4.6.1: Graph of the functions $f(x) = e^x$ and $g(x) = 2^x$, and of $y = 3^x$, $y = 4^x$, and $y = 10^x$.

You should recall the laws of exponents

$$b^{m+n} = b^m \cdot b^n$$

and

$$b^{mn} = (b^m)^n = (b^n)^m$$

These laws can be used to explain why exponential functions whose equations appear quite different may in fact be the same function and have the same graph. The following examples show two such cases.

Example 4.6.2. Graph the function $f(x) = 3^{-x}$ and the function $g(x) = (\frac{1}{3})^x$. Explain why the graphs have the appearance they do.

Solution.

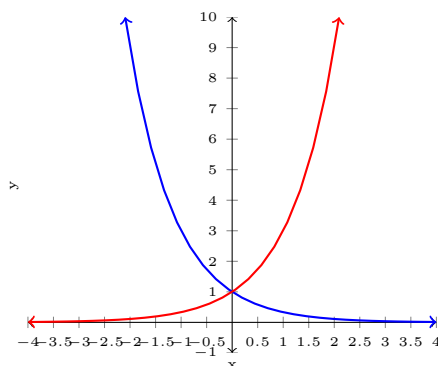


Figure 4.6.3: $h(x) = 3^x$ and $h(-x) = 3^{-x}$

Recall from Chapter 2 that the graph of $h(-x)$ is the reflection about the y -axis of the graph of $h(x)$. Thus the graph of the function $f(x) = 3^{-x}$ is the reflection of the graph of $y = 3^x$ about the y -axis. The graph of $f(x) = 3^{-x}$ is shown in Figure 3; it is decreasing, contains the points (0, 1) and $(-1, 3)$ and approaches the asymptote $y = 0$ as x approaches infinity.

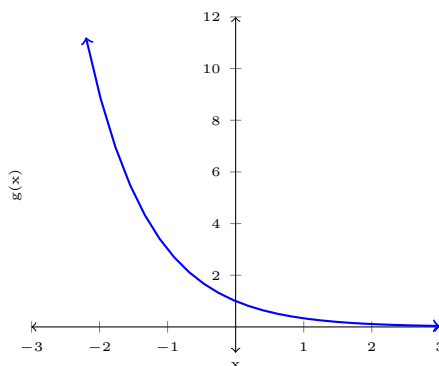


Figure 4.6.4: Graph of $g(x) = (\frac{1}{3})^x$

Figure 4 shows the graph of $g(x) = (\frac{1}{3})^x$. This is an exponential function with base $\frac{1}{3}$. The graph contains the points (0, 1) and $(1, \frac{1}{3})$. Because the base is less than 1, the graph is decreasing. To understand why $f(x) = g(x)$ for all x , note that the rules of exponents tell us that $(\frac{1}{3})^x = (3^{-1})^x = 3^{-1 \cdot x} = 3^{-x}$. Exponential functions like $g(x) = (\frac{1}{3})^x$, with a base between zero and one, and functions like $f(x) = 3^{-x}$, with a base greater than one and a negative coefficient in the exponent, are both used to model *exponential decay*.

Example 4.6.5. Use ideas you know about transformations to help you graph the functions $g(x) = 4 \cdot 2^x$ and $k(x) = 2^{x+2}$. Why do these graphs look the way they do?

Solution. The graph of $g(x) = 4 \cdot 2^x$ is a vertical stretch by a factor of 4 of the function $y = 2^x$. The graph is shown on the left in Figure XX. The point (0, 1) stretches up to (0, 4)

and the point $(1, 2)$ stretches up to $(1, 8)$. The graph of $k(x) = 2^{x+2}$ is a horizontal shift 2 units to the left of the function $y = 2^x$. The point $(0, 1)$ moves left to $(-2, 1)$ and the point $(1, 2)$ moves left to $(-1, 2)$. You should notice that the graphs of $g(x)$ and $k(x)$ look identical; this is because $4 \cdot 2^x = 2^2 \cdot 2^x = 2^{x+2}$.

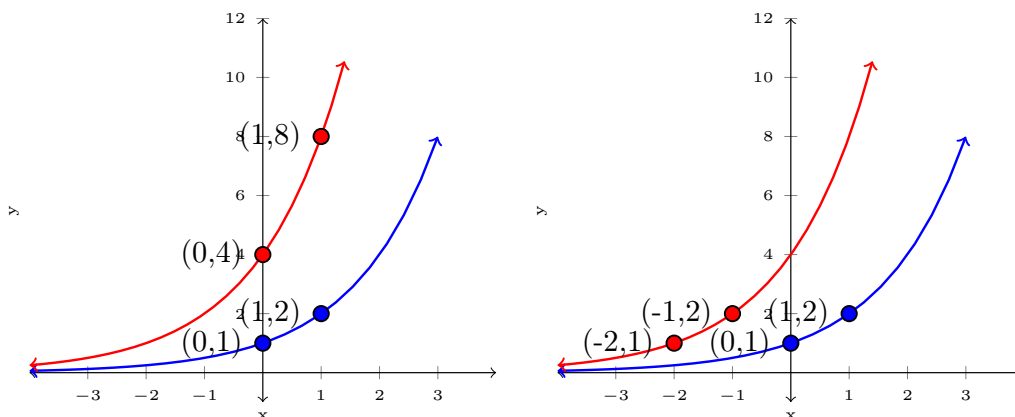


Figure 4.6.6: $g(x) = 4 \cdot 2^x$ and $k(x) = 2^{x+2}$

Although the function definitions look different, $g(x) = 4 \cdot 2^x$ and $k(x) = 2^{x+2}$ are actually two different ways of writing the same function.

Other transformations of exponential functions, such as horizontal compressions and vertical shifts, result in curves that are important in modeling various types of growth and decay.

Example 4.6.7 (Growth of a Tumor). In a laboratory experiment, the growth of a tumor in a mouse was monitored over time. Using data analysis techniques, the function $s = 0.32e^{0.11d}$ was found to be a good model for the growth. In this equation, d represents the number of days since monitoring began, and s represents the size of the tumor in cubic centimeters. Graph the function over an appropriate domain. What is the initial size of the tumor? Determine the size after ten days and after twenty days.

Solution. The graph of $s = e^d$ must be compressed vertically by a factor of 0.32 and stretched horizontally by a factor of $\frac{1}{0.11} \approx 9$ to obtain the graph shown in Figure 8. The graph contains the point $(0, 0.32)$ since $(0, 1)$ has been stretched horizontally by a factor of 9 and stretched vertically by a factor of 0.32. We can estimate that another point has approximate coordinates $(9, 0.32e) \approx (9, 0.87)$ since $(1, e)$ has been stretched horizontally by a factor of 9 and stretched vertically by a factor of 0.32. Based on the context, the appropriate domain is non-negative values of d . Although we would expect there to be physical limitations on the size of the tumor, we do not know what this upper limit is, so we have not attempted to show this in the graph.

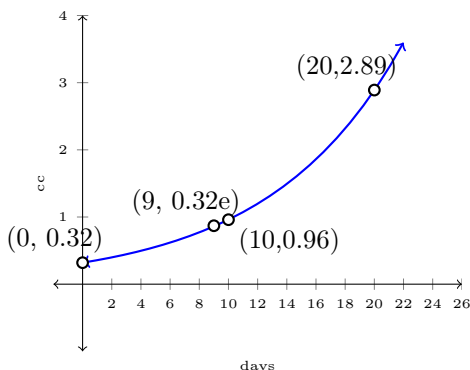


Figure 4.6.8: Graph of $s = 0.32e^{0.11d}$

The initial size of the tumor is 0.32 cc since this is the s -value associated with $d = 0$. Substituting $d = 10$ and $d = 20$ into the equation $s = 0.32e^{0.11d}$ reveals that the size of the tumor is 0.96 cc after ten days and 2.89 cc after twenty days.

Example 4.6.9 (Cooling Coffee). When a cup of hot coffee is placed in a cool room, the coffee begins to cool and will continue to do so until it reaches room temperature. If the initial temperature of the coffee is 85°C and the room temperature is 24°C , find an exponential function that models the temperature at any time t .

Solution. The cooling process can be reasonably well-modeled using exponential decay. We can use the function $y = e^{-x}$ to model exponential decay, but we must transform the curve so that it has a horizontal asymptote at 24, not 0, and has a y -intercept of 85, not 1. The graph of $y = e^{-x} + 24$ will be shifted up 24 units and will have an asymptote at $y = 24$ instead of $y = 0$. The graph of $y = e^{-x} + 24$ contains the point $(0, 25)$; the y -intercept is 1 unit above the horizontal asymptote. The function that models the coffee's temperature must have a y -intercept that is $85 - 24 = 61$ units above the asymptote. This means that we must vertically stretch the graph by a factor of 61 before shifting the entire graph up 24. This can be done by multiplying the e^{-x} by 61. The graph of the function $y = 61e^{-x} + 24$ is shown in Figure 10

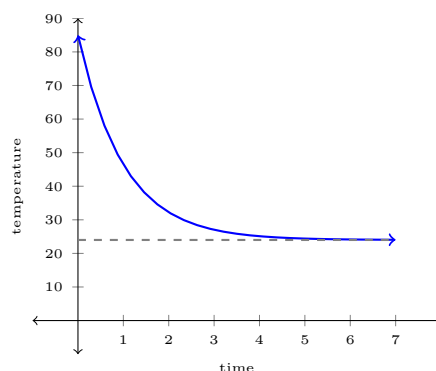


Figure 4.6.10: Graph of $y = 61e^{-x} + 24$

Objects cool at different rates, depending on their composition and shape. In graphing the function in Figure XX we did not take into account how quickly the coffee cooled to room temperature. We can see from the graph that it seemed to cool very quickly, nearly reaching room temperature in 4 minutes, which is unrealistically fast. In real life, coffee takes much longer to cool. To model this slower cooling process, we need to apply a horizontal stretch to the functional model. This would result in a model of the form $y = 61e^{-kx} + 24$, where $0 < k < 1$. To find the particular value of k in a given situation requires us to know more than just the initial and room temperatures.

Example 4.6.11. When you start to learn a new skill, such as typing, your proficiency begins rather low and grows toward some maximum level. At first your proficiency grows quickly, but as you near the maximum level, your gains grow more slowly. Exponential curves that display this behavior are called *learning curves*. Such functions can be modeled with transformed exponential functions. Consider the function $f(x) = 45 - 35e^{-0.3x}$, when x represents the number of weeks of practice and $f(x)$ represents typing speed in words per minute. Sketch a graph of this function and use the graph to describe the specific characteristics of this learning curve.

Solution. The domain of this function based on the situation described above is $x \geq 0$. To sketch this function, we begin with the graph of $g(x) = 35e^{-0.3x}$, which we know is a decreasing function with a y -intercept of 35.

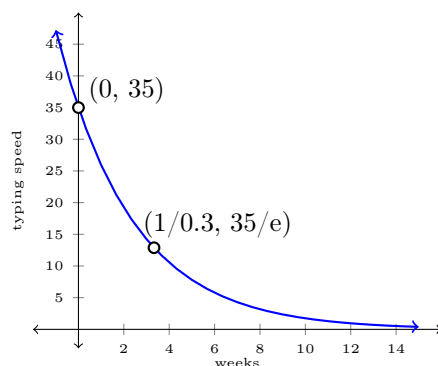


Figure 4.6.12: The graph of $g(x) = 35e^{-0.3x}$

Next, we flip function g about the x -axis to get the graph of $-g(x) = -35e^{-0.3x}$.

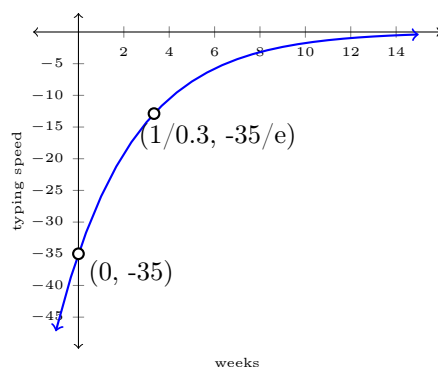


Figure 4.6.13: The graph of $y = -g(x) = -35e^{-0.3x}$

Finally, we have a vertical translation of $y = -g(x)$ up 45 units. The resulting graph is the desired function $f(x) = 45 - 35e^{-0.3x}$.

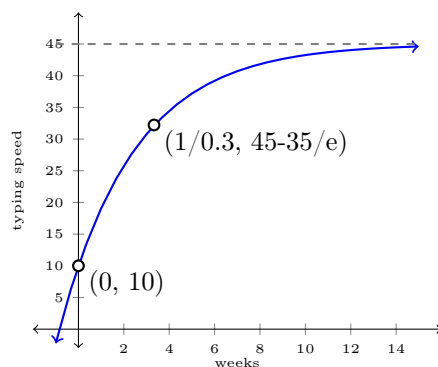


Figure 4.6.14: Graph of $f(x) = 45 - 35e^{-0.3x}$

When $x = 0$, the value of the function is 10, so you start typing at 10 words per minute. In the first five weeks of practice you improve all the way up to 37 words per minute, but in the next five weeks you improve only 6 more words per minute. At this point you type a little over 43 words per minute but gain very little with additional practice. The horizontal asymptote at 45 indicates that there is an upper limit of 45 words per minute on your typing speed.

Example 4.6.15. Use composition to sketch the graph of $f(x) = e^{\frac{1}{x-1}}$.

Solution. Function f is a composition of

$$g(x) = \frac{1}{x-1} \text{ and } h(x) = e^x$$

Since $f(x) = h(g(x))$ and the outer function $h(x)$ has no domain restrictions, the domain of the composite function $f(x)$ is the same as the domain of the inner function $g(x)$. Therefore, x cannot equal 1. Also, since $h(x) < 0$ for all values of x , we know that $h(g(x)) > 0$ for all x in the domain of g . The graphs of $g(x) = \frac{1}{x-1}$ and $h(x) = e^x$ are shown in Figure 16

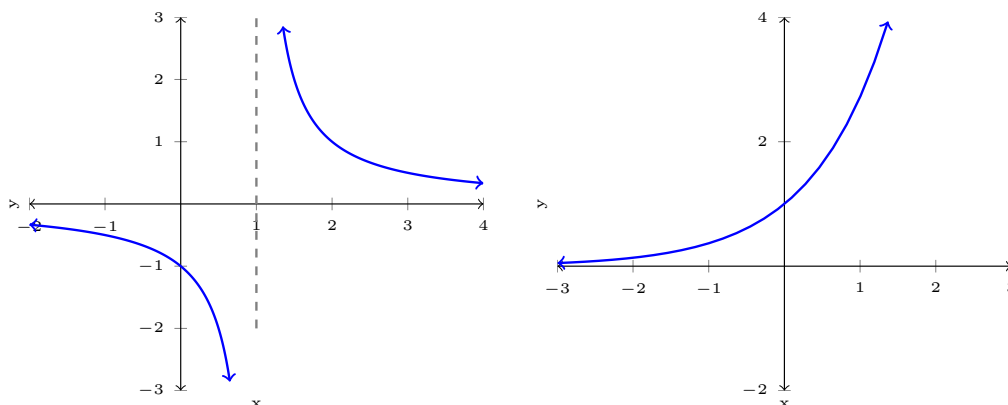


Figure 4.6.16: Graph of the functions $f(x) = e^x$ and $g(x) = 2^x$, and of $y = 3^x$, $y = 4^x$, and $y = 10^x$.

Since $g(x)$ never equals zero, the input to $h(x)$ is never zero, so the y -value of $h(g(x))$ is never one. It is also important to note that $g(x)$ has $y = 0$ as a horizontal asymptote. As x -values increase or decrease without bound, $g(x) = \frac{1}{x-1}$ approaches zero; so $f(x) = e^{\frac{1}{x-1}}$ approaches e^0 , or 1. This produces a horizontal asymptote in the composition at $y = 1$.

Now consider the behavior of $f(x)$ close to $x = 1$. As x approaches 1 from the left, $\frac{1}{x-1}$ decreases without bound, and the value of $f(x) = e^{\frac{1}{x-1}}$ approaches 0 from above. The open circle on the graph of the composition is used to signify this. As x approaches 1 from the right, $\frac{1}{x-1}$ increases without bound, and the value of $f(x) = e^{\frac{1}{x-1}}$ also increases without bound. The graph of the composition $f(x) = e^{\frac{1}{x-1}}$ is shown in Figure 17

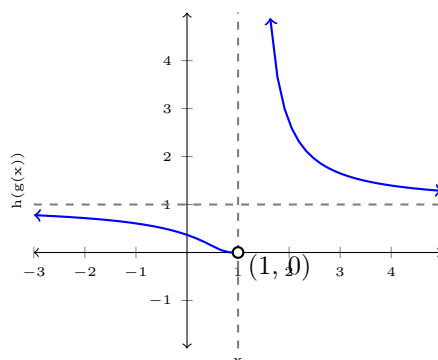


Figure 4.6.17: Graph of $f(x) = 45 - 35e^{-0.3x}$

4.6.1 Exercises

1. Using the limit notation introduced in the previous section, we can write

$$\lim_{x \rightarrow \infty} b^x = 0$$

provided $b > 1$. This is read "the limit of b raised to the x power, as x approached negative infinity, is 0". This means that any base b that is greater than 1, when raised to a power that is more and more negative, produces a value closer and closer to zero. What characteristic of the graph of $y = b^x$ is explained by this limit statement?

2. Use technology to graph the function $y = b^x$ for $b = 2, 5, 7$ and 10 on the same axes. You may be able to accomplish this using a slider in some dynamic algebra/geometry applications. You should see that the graph of each of the functions contains the point $(0, 1)$, has the negative x -axis as an asymptote and is increasing in quadrant I. The following questions will help you decide if the graphs of $y = 2^x, y = 5^x, y = 7^x$, and $y = 10^x$, are so similar in shape that they are all actually transformations of each other.

- There is no vertical stretch or compression that can transform the graph of $y = 2^x$ into the graph of $y = 5^x$. Explain why.
- There is no vertical shift that can transform the graph of $y = 2^x$ into the graph of $y = 7^x$. Explain why.
- There is no horizontal shift that can transform the graph of $y = 5^x$ into the graph of $y = 10^x$. Explain why.
- Explain why there *could* be a horizontal stretch or compression that can transform the graph of $y = 2^x$ into the graph of $y = 5^x$.
- The graph of $y = 5^x$ can be horizontally stretched by a factor of k so that the stretched graph is identical to the graph of $y = 2^x$. Use the graphs you made to estimate the size of the number k , i.e. the size of the stretch factor.

4.6.2 Exercises

1. As we illustrated in Examples 1 and 2, we can often apply laws of exponents to simplify graphing exponential functions. Describe two different ways to graph the following equations as transformations of an exponential function. Which way seems easier to you? Rewrite each equation to make your choice of transformation obvious, and sketch a graph. Identify the horizontal asymptote and the points that correspond to $(0, 1)$ and $(1, b)$.

- $y = e^{2x}$
- $y = 4 \cdot 2^{x-1}$
- $y = (-4) \cdot 2^x$
- $y = 4^{-x-2}$

2. Consider the four expressions $3^{3x-2}, 3^{3(x-2/3)}, 27^{x-2/3}$, and $\frac{1}{9} \cdot 27^x$

- Explain why these four expressions are all equal.
- Graph $y = 3^{3x-2}$ using one of the expressions that is equal to $y = 3^{3x-2}$. Label the coordinates of the points that correspond to $(0, 1)$ and $(1, b)$.
- Which expression did you choose to graph? Why?

3. Graph each function that is a transformation of $y = b^x$. Label asymptote(s) and points that correspond $(0, 1)$ and $(1, b)$.

- $y = 5^{x-7}$
- $y = e^{1.61x}$
- $y = 5 + 3^{2x}$
- $y = 1 + 2e^{-x}$
- $y = -8 + 0.5^x$
- $y = 3 - 6^{-x}$
- $y = 8^{2x-2}$
- $y = 7(3^x)$

- (i) $y = -3(2^{2x}) + 1$
- (j) $y = |-5 + 3^{x+2}|$

4. A typical worker at a supermarket bakery can decorate $f(t)$ cakes per hour after t days on the job, where:

$$f(t) = 10 \cdot (1 - e^{-\frac{1}{4}})$$

- (a) Sketch a graph of $f(t)$. Restrict your domain to meaningful values of t .
- (b) How many cakes can a newly-employed worker decorate in an hour?
- (c) After eight days, how many cakes can a worker decorate in an hour?
- (d) Based on this graph, after a worker has decorated cakes for a very long time, how many cakes can he or she decorate in an hour?

5. Rheumatoid arthritis patients are treated with large doses of aspirin. Research has shown that the concentration of aspirin in the bloodstream increases for a short period of time after the drug is administered and then decreases exponentially. For a typical patient, this relationship is given by $a = 14.91e^{-0.18t}$, where t represents the number of hours since peak concentration and a represents the concentration of aspirin measured in milligrams per cubic centimeter of blood.

- (a) Graph the function over an appropriate domain. Label the coordinates of the points that correspond to $t = 0$ and $t = 1$.
- (b) Determine the peak concentration of aspirin.
- (c) Determine the amount of aspirin remaining four hours after peak concentration.
- (d) Use a computer- or calculator-generated graph to determine the time at which the concentration is 5 mg per cc of blood

6. In a classroom experiment students made fudge the old-fashioned way. They cooked sugar, milk, and chocolate until it reached 234° Fahrenheit. Then they removed the mixture from the heat, added butter and vanilla, and let the mixture cool to 110° . As they monitored the temperature, they kept records and used data analysis techniques to determine a mathematical model for the temperature of the fudge as a function of time. Their model is $F = 154e^{-0.00063t} + 77$, where F is the temperature of the fudge in degrees Fahrenheit and t represents the number of seconds elapsed since stirring in the butter.

- (a) Graph this function over an appropriate domain. Interpret the y -intercept and the horizontal asymptote.
- (b) Determine the temperature of the mixture after thirty minutes.
- (c) Use your graph to determine how long it takes for the temperature to reach 110°

7. McKenzie invests \$1,000 in a local bank at an annual rate of 5.14% compounded quarterly.

- (a) Use a graph to determine when their investment will be worth \$1,500
- (b) Suppose instead that the 5.14% is compounded continuously. When will they have \$1,500?

8. Newton's law of cooling states that the change in temperature of an object as it cools is proportional to the difference between the current temperature and the ambient temperature. Suppose the initial temperature of a cup of hot coffee was 92°C and the ambient temperature was 3°C . The temperature of this coffee could be modeled with the following iterative system:

$$T_0 = 92, T_n = T_{n-1} - k(T_{n-1} - 3)$$

In this situation, n represents the number of minutes that have elapsed

- (a) Using the initial temperature of 92°C and a value of $k = 0.02$, use your calculator to generate values of the temperature over time and to produce a graph of these values. Explain the shape of the graph in terms of the iterative equation.
- (b) There are several steps involved in constructing a closed form equation to model this situation. Start with the decreasing exponential function $f(x) = e^{-x}$. Sketch a graph of this function. What is the range of this function? Where are its y -intercept and asymptote?
- (c) Transform f so that it has a horizontal asymptote of $y = 3$ and a y -intercept of 92
- (d) Now transform f so that it decreases less quickly. This transformed function should produce a graph similar to the one you generated in part (a).

9. Suppose, because of limitations in food and living space, a population of field mice experiences constrained growth. This means that the population cannot increase beyond a certain maximum number M . The change in population is directly proportional to the product of the current population and the difference between the maximum population M and the current population. This can be expressed by the following recursive equation:

$$P_n = P_{n-1} + kP_{n-1}(M - P_{n-1})$$

where P_n represents the population after n months, and k the constant of proportionality. Populations that grow in this manner are said to exhibit *logistic growth*.

- (a) Assume that the initial population is 100, the maximum population is 1,000, and $k = 0.0006$. Use your calculator to generate values of the population over time and to produce a graph of these values. Explain, in terms of the recursive equation, why the graph is shaped the way it is.
- (b) We can use the graph from part (a) to help determine an explicit function to model this logistic growth. Begin with the function $f(x) = 1 + e^{-x}$. Sketch a graph of this function. What is the range of this function? What is the y -intercept?
- (c) Now graph $g(x) = \frac{1}{1+e^{-x}}$. What is the range of this function? What is the y -intercept?
- (d) Apply transformations to f so that it has a y -intercept of 100 and grows less quickly. The horizontal asymptote should be $y = 1000$. This transformed function should produce a graph similar to the one you generated in part (a). This means that you have written an explicit function that models the logistic growth described at the beginning of the problem

10. Use the ideas of composition to sketch a graph of each function without using technology

- (a) $f(x) = \frac{1}{e^x}$
- (b) $f(x) = 3^{|x|}$
- (c) $f(x) = 2^{\frac{1}{x}}$
- (d) $f(x) = 2^{(x^2)}$
- (e) $f(x) = e^{x^2 - 2x}$
- (f) $f(x) = 3^{\sin(x)}$
- (g) $f(x) = e^{-x^2}$
- (h) $f(x) = \frac{1}{1+2e^{-x}}$

11. The cane toad, a native of South America, was introduced to Australia in 1935 as a way to control the destructive sugar cane beetle. It has no natural predators in Australia and it reproduces very rapidly. In June 1935, 102 cane toads were brought to Australia. Twenty-one months later, the population of cane toads had grown to about 65,000. Let t represent the number of years since June 1935, m represent the number of months since June 1935, and C represent the number of cane toads present.

- (a) Write an explicit function, $C_1(m)$, that expresses the toad population in terms of m . To do this, you will need to find the monthly percentage rate at which the toad population grew.
- (b) Modify the function you wrote in (a) to obtain a function, $C_2(t)$, that expresses the toad population in terms of t . Note that the population when $m = 12$ is the same as the population when $t = 1$. Use properties of exponentials to write your function in the form $C_2(t) = a \cdot b^t$, for some constants a, b . Interpret the information given by the base, b .
- (c) Use either $C_1(m)$ or $C_2(t)$ to determine how long it takes the cane toad population to double. Using this information, write a function $C_3(t)$ that uses 2 as a base to express population as a function of t .
- (d) Use each of your three models to predict how many cane toads there would have been 5 years after the landing of those initial 102 toads, if the growth had continued unchecked. Of course, you should get approximately the same answer from all the models. If they are not exactly the same, explain why this is the case.
- (e) Pretend you are an Australian reporter who got interested in the cane toad situation just a year after they landed, and, being a mathematician too, you investigated the situation by developing the models above. Write a short "newspaper" article in which you use some of that information to tell your readers about this situation. Use your 5-year prediction as the clincher, by relating that number to some geographical or other fact about Australia, to make it "real". For example, you could compare that number of cane toads there will be to the population of Australia that year, or find some other measure of "cane toad density". Be creative! Cite the reference from which you get your Australian information.

Chapter 5

Logarithmic Functions

5.1 Introduction to Logarithms

In 2002, about 2.8 million ninth graders were enrolled in a math class. By the time these students reached their first year of college, only 250,000 of them were enrolled in a math class. In 2010, this group of students earned their bachelor's degrees and only 10,000 of them were enrolled in a math class. In 2012, earning M.S. degrees, 3,000 students were enrolled in a math class. In 2016, working on Ph.D. degrees, only 500 of them were enrolled in a math class. The data are given below in [Table 1](#)

Year	Math Class Enrollment
2002	2,800,000
2006	250,000
2010	10,000
2012	3,000
2016	500

Table 5.1.1: Enrollment in mathematics classes over time

A scatterplot of the data is shown in [Figure 2](#). Notice that the vertical scale required to show all of the data makes it difficult to distinguish the smaller y -values from each other.

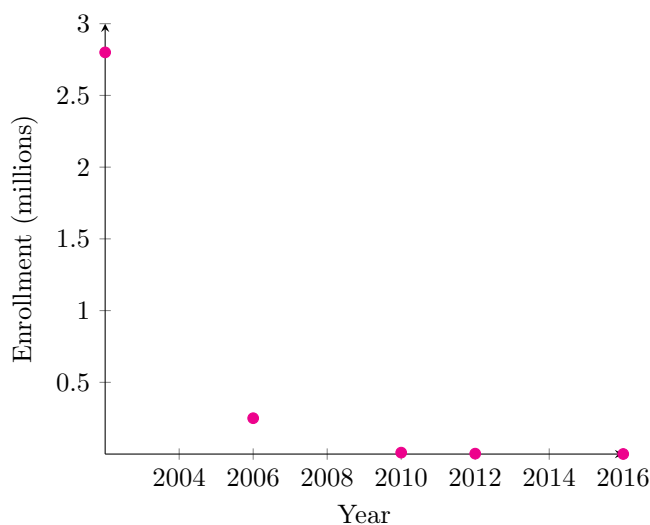


Figure 5.1.2: Scatterplot of mathematics class enrollment

The mathematics class enrollment for this group of students appears to be decreasing

exponentially. Figure 3 shows a graph in which the vertical scale has been altered so that it has equal spacing between $10^1, 10^2, 10^3$, etc. rather than having equal spacing between 1, 2, 3, etc. The consequence of this alteration is that the graph appears linear. Notice it is much easier to distinguish between the smaller y -values using this new scale.

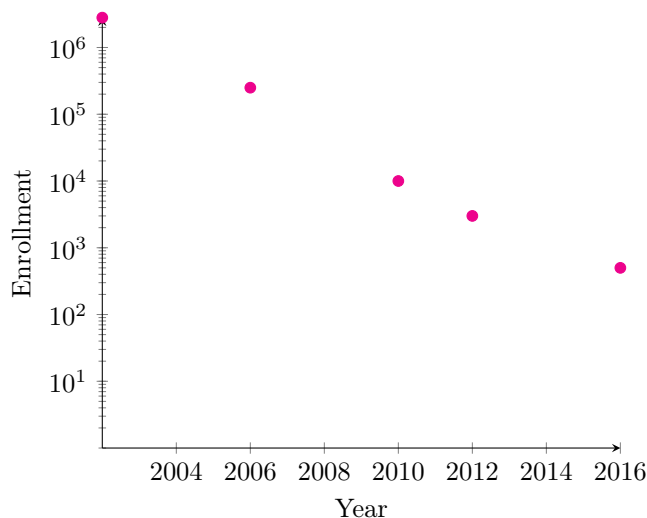


Figure 5.1.3: Enrollment with vertical scale using powers of 10

The alteration of the scale on the vertical axis was a standard technique for graphing exponential data sets before the age of calculator and computer technology. Data were plotted on special graph paper which had the vertical scale re-written as powers of 10. Modern technology allows us to adjust the data instead of the coordinate system so that the y -value of the data represents the exponent on the base used in the scale on the y -axis. This eliminates the need for special graph paper. The process of adjusting the data is called *re-expression*.

To re-express the data, we first need to define *logarithms*. Logarithms allow us to rewrite any exponential equation so that the exponent is isolated.

If we write $c = \log_{10}(a)$, then we mean that c is the exponent that must be put on 10 to yield a . Thus, $\log_{10}(a)$ is, by definition, the number c so that $10^c = a$.

In general, $b^c = a$ is equivalent to $\log_b(a) = c$. That is, the logarithm with base b of a is defined to be the exponent needed such that $b^c = a$.

So $\log_{10}(1000) = 3$, (since $10^3 = 1000$), $\log_{10}(10) = 1$ (since $10^1 = 10$) and $\log_{10}(0.1) = -1$ (since $10^{-1} = 0.1$).

For the mathematics enrollment data, instead of plotting points (year, enrollment), we will plot points (year, $\log(\text{enrollment})$). In particular, instead of (2,010, 10,000), we will plot the point (2,010, 4). We use the y -coordinate of 4 since $\log_{10}(10,000) = 4$. Values for other y -coordinates can be found using a calculator to evaluate logarithms; the values of these coordinates cannot be determined using familiar algebraic techniques and require the use of technology. For example, $\log_{10}(250,000) = 5.39794$ because $10^{5.39794} = 250,000$. Re-expressing the entire set of data results in the following table and graph.

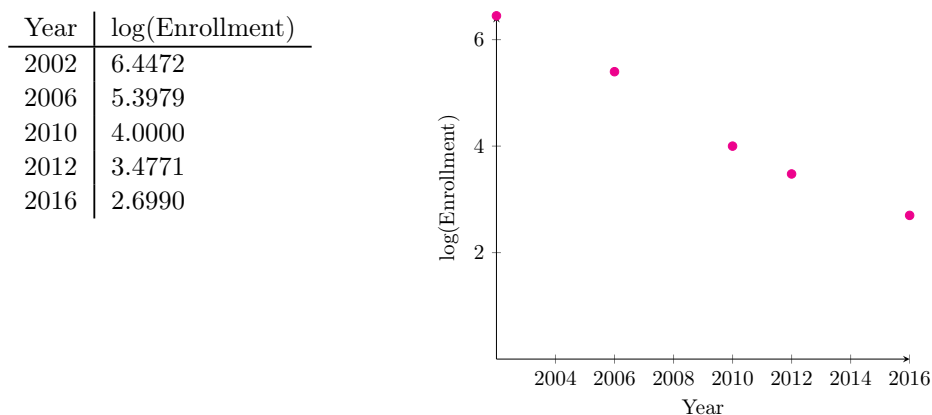


Figure 5.1.4: Enrollment in mathematics classes over time, log-scale

This new spacing creates what is called a logarithmic scale. Note that every increase of 1 year on the horizontal axis corresponds to a decrease in enrollment by a factor of 10 on the vertical axis. Every increase of 2 years corresponds to a decrease in enrollment by a factor of 100 and every increase of k years is associated with a decrease in enrollment by a factor of 10^k .

The logarithm used in the math enrollment example is called a base 10 logarithm; the logarithmic equation $\log_{10} x = y$ and the exponential equation $10^y = x$ express the same information about 10 and x and y . In both equations the number 10 is called the base. We can write logarithm and exponential equations with bases other than 10: we write $\log_2 8 = 3$ (read "the log base 2 of 8 is 3") because $2^3 = 8$ and we write $\log_9 3 = \frac{1}{2}$ (read "the log base 9 of 3 is one-half") because $9^{1/2} = 3$. Recall that a logarithm is an exponent; $\log_b a$ represents the exponent on b that produces a .

Logarithms with base 10 are referred to as common logarithms; the conventional notation omits the base, e.g. we write $\log(100) = 2$ instead of $\log_{10}(100) = 2$. Logarithms with base e are referred to as natural logarithms, and written with the special notation \ln (read "el en"). (e.g. $\log_e(2) = \ln(2) = 0.6931$). Most of our work with logarithms will be in one of these two bases.

Example 5.1.5. Convert each logarithmic equation into an equivalent exponential equation, and each exponential equation into an equivalent logarithmic equation.

1. $10^{-3} = 0.001$
2. $x^y = z$
3. $\log_2 \frac{1}{16} = -4$
4. $\log_a b = c$

Solution. We use the definition of logarithm to rewrite each of these equations

1. $10^{-3} = 0.001$ is equivalent to $\log_{10}(0.001) = -3$
2. $x^y = z$ is equivalent to $\log_x z = y$
3. $\log_2 \frac{1}{16} = -4$ is equivalent to $2^{-4} = \frac{1}{16}$
4. $\log_a b = c$ is equivalent to $a^c = b$

Notice that when the equation is expressed in exponential form, the value of the logarithm is the exponent.

Since the values of logarithms are themselves exponents, it is not surprising that there are laws of logarithms which are closely related to the laws of exponents. The laws of exponents and the corresponding laws of logarithms are stated here and discussed below:

Product	$b^r \cdot b^s = b^{r+s}, b > 0$	$\log_b(r \cdot s) = \log_b(r) + \log_b(s), r, s, b > 0, b \neq 1$
Quotient	$\frac{b^r}{b^s} = b^{r-s}, b > 0$	$\log_b\left(\frac{r}{s}\right) = \log_b(r) - \log_b(s), r, s, b > 0, b \neq 1$
Power	$(b^r)^s = b^{r \cdot s}, b > 0$	$\log_b(r^s) = s \cdot \log_b(r), r, s, b > 0, b \neq 1$
Equality	$b^r = b^s$, if and only if $r = s$	$\log_b(r) = \log_b(s)$, if and only if $r = s$

There are two other important formulas relating exponents and logarithms.

$$b^{\log_b r} = r \text{ and } \log_b b^r = r$$

Practice problems 1 and 2 at the end of the section will ask you to use the definition of a logarithm to explain why these formulas are true.

Example 5.1.6. Show that $\log_b(r \cdot s) = \log_b(r) + \log_b(s)$ is true for all positive values of r, s , and b (provided that $b \neq 1$)

Solution. We will show that $b^{\log_b(r \cdot s)}$ is equal to $b^{\log_b(r) + \log_b(s)}$; that is b raised to the left hand side is equal to b raised to the right hand side. This will allow us to conclude that the exponents on b are equal.

First consider the expression $b^{\log_b(r \cdot s)}$. One of our logarithm properties tell us that:

$$b^{\log_b(r \cdot s)} = r \cdot s$$

What about $b^{\log_b(r) + \log_b(s)}$? We know how to re-write an expression that involves a sum in the exponent:

$$b^{\log_b(r) + \log_b(s)} = b^{\log_b(r)} \cdot b^{\log_b(s)}$$

We know that:

$$\begin{aligned} b^{\log_b r} &= r \text{ and } b^{\log_b s} = s \\ \text{so} \\ b^{\log_b(r) + \log_b(s)} &= b^{\log_b(r)} \cdot b^{\log_b(s)} = r \cdot s \end{aligned}$$

We have shown that $b^{\log_b(r \cdot s)} = r \cdot s$ and $b^{\log_b(r) + \log_b(s)} = b^{\log_b(r)} \cdot b^{\log_b(s)} = r \cdot s$. Two expressions that are both equal to $r \cdot s$ are equal to each other, so $b^{\log_b(r \cdot s)} = b^{\log_b(r) + \log_b(s)}$. From this, we can conclude that the exponents on b are equal. Therefore,

$$\log_b(r \cdot s) = \log_b(r) + \log_b(s)$$

Example 5.1.7. Re-write the exponential equation $A = e^{kt}$ into an equivalent logarithmic equation. Then, solve for t . Assume that A and k are both constants.

Solution. Recall that by definition, $b^c = a$ is equivalent to $\log_b(a) = c$. So in this exponential equation, $A = e^{kt}$ is equivalent to

$$\ln(A) = kt$$

To solve for t , divide both sides of the equation by the constant k ,

$$t = \frac{1}{k} \cdot \ln(A)$$

Example 5.1.8. Rewrite the expression $\log_4(x) + \log_4\left(\frac{8}{x^2}\right)$ as a single logarithm

Solution. $\log_4 x + \log_4\left(\frac{8}{x^2}\right) = \log_4\left(x \cdot \frac{8}{x^2}\right) = \log_4\left(\frac{8}{x}\right)$

It is worth noting here that these properties of logarithms make difficult products, quotients, and powers much easier to compute. Products are turned into sums, quotients into differences, and powers into products. Scientists, starting with the 17th century astronomers, used logarithms to do many of the difficult and lengthy calculations needed for many of the important discoveries they made. Imagine the time it would take to divide a number like

8,497.231 by 0.00097388 without a calculator. With tables of logarithms, this division problem could be turned into a subtraction problem. Some of the most widely published books for nearly three centuries were tables of logarithms, and one of the first applications of the modern computer was to generate more accurate and precise tables of logarithms. The "handheld calculator" for generations of students was the slide rule, which was based on logarithms. Of course, with the spread of computers and calculators, very few people still use logarithms for computational purposes. However, logarithms are still extremely useful for solving equations that involve exponents.

5.1.1 Practice Problems

1. Use the definition of logarithm to explain why $b^{\log_b r} = r$ is true.
2. Use the definition of logarithm to explain why $\log_b b^r = r$ is true.

5.1.2 Exercises

1. Convert each equation to exponential form
 - (a) $\log_4 64 = 3$
 - (b) $\log_5 1 = 0$
 - (c) $\log 10,000 = 4$
 - (d) $\log_{1/2} 8 = -3$
2. Convert each equation to logarithmic form.
 - (a) $e^1 = e$
 - (b) $99^0 = 1$
 - (c) $\sqrt{16} = 4$
 - (d) $\sqrt[3]{64} = 4$
3. Evaluate the following without a calculator.
 - (a) $\log_3 81$
 - (b) $\ln \frac{1}{e^3}$
 - (c) $\log_2 \sqrt[5]{2^3}$
 - (d) $\log(0.01)$
4. Determine the two integers between which each logarithm falls. For example, we can conclude that $2 < \log_2 7 < 3$
 - (a) $\log 12$
 - (b) $\log 1.2$
 - (c) $\log_4 5$
 - (d) $\log_7 21$
 - (e) $\log_{0.5} \frac{1}{7}$
 - (f) $\log_5 1000$
 - (g) $\ln 27$
 - (h) $\ln \frac{1}{3}$
5. Prove the following laws of logarithms
 - (a) $\log_b \left(\frac{r}{s}\right) = \log_b(r) - \log_b(s), r, s, b > 0, b \neq 1$
 - (b) $\log_b \left(\frac{1}{a}\right) = -\log_b(a), a > 0, b \neq 1$
6. Evaluate each of the following. State any restrictions on values of the variables

- (a) $\log_2 2^5$
- (b) $7^{\log_7(a+3)}$
- (c) $\log_{64} 8$
- (d) $(-3)^{\log_{-3} 5}$

7. Solve the following equations for x . Use the definition of logarithm to rewrite each equation in a different form. Give exact values (rather than decimal approximations) whenever possible

- (a) $\log_x 27 = -3$
- (b) $\log_{64} x = \frac{2}{3}$
- (c) $x = 5^{2 \cdot \log_5 6}$
- (d) $\log_6 x^2 = -2$
- (e) $\log_2 (x^2 + 5x + 10) = 4$
- (f) $\log_x 9 = 2$

8. Use the properties of logarithms and exponents to help you solve each of the following equations.

- (a) $\log_2 8 + \log_2 32 = \log_2 x$
- (b) $\log_2 8 - \log_2 32 = \log_2 x$
- (c) $\log_{10} 7 + \log_{10} 32 = \log_2 x$
- (d) $\log_6 \frac{1}{216} + \log_6 36 = \log_6 x$
- (e) $\log_8 \frac{1}{2} + \log_8 x = \log_8 32$
- (f) $\log_9 3 - \log_9 x = \log_9 \frac{1}{27}$
- (g) $\log_2 32 = 5 \cdot \log_2 x$
- (h) $\log_3 81^{1/4} = x \cdot \log_3 81$
- (i) $\log_4 x = 3 \cdot \log_4 2$
- (j) $\log_5 \frac{1}{125} = -1 \cdot \log_5 x$
- (k) $\log_6 36 - \log_6 x = \log_6 6$
- (l) $\log_5 25 + \log_5 \frac{1}{125} = \log_5 x$
- (m) $\log_2 \frac{1}{8} + \log_2 x = \log_2 4$
- (n) $\log_2 8^{1/3} = x \cdot \log_2 8$

5.2 Scientific Scales

Scientists commonly use logarithms to create a simple scale for data; this is beneficial when some values in the data set are many times larger than other values in the data set. For example, the pH of a solution is determined by the concentration of hydrogen ions in the solution, measured in moles per liter. Solutions with a high concentration of hydrogen ions are acidic, while solutions with a low concentration are basic. The concentration of hydrogen ions in household ammonia is approximately 0.0000000000005 moles/liter, while the concentration of hydrogen ions in lemon juice is ten billion times larger at approximately 0.005 moles/liter.

To describe these vastly different values on the same scale, logarithms are used. The pH of a solution is defined to be

$$\text{pH} = -\log(H^+), \text{ where } H^+ \text{ is the concentration of hydrogen ions.}$$

So we say that household ammonia has a pH of $-\log(0.0000000000005) = 12.3$, while lemon juice has a pH of $-\log(0.005) = 2.3$. The pH of pure water is about 7. Solutions with a pH less than 7 are acidic, while solutions with a pH greater than 7 are basic. By using logarithms, chemists take values that vary by huge amounts and place them on a simple scale that ranges from about 0 to 14, which is much easier to understand.

5.2.1 The Decibel Scale for Sound

The intensity of a sound is dependent on the actual energy carried in the sound wave and is measured in units of power per unit area, or $\frac{\text{watts}}{\text{meter}^2}$. The greater the intensity, the louder the perceived sound. The intensities of some common sounds are given in [Table 1](#). Physicists define a loudness level (measured in decibels) that is related to the actual intensity, I (measured in Watts / meter²), of the sound by the following equation:

$$\text{Loudness Level (dB)} = 10 \cdot \log \left(\frac{I \text{ W/m}^2}{10^{-12} \text{ W/m}^2} \right)$$

The threshold of human hearing is a sound with intensity 10^{-12} W/m^2 . It is defined to be of loudness level 0 dB and serves as a baseline by which other sounds are measured. You can use the equation above to verify that the loudness is 0 dB when the intensity is 10^{-12} W/m^2 .

Sound	Intensity (W/m^2)	Loudness Level (dB)
Threshold of hearing	10^{-12}	0
Normal breathing	10^{-11}	10
Rustling leaves	10^{-10}	20
Whisper	10^{-9}	30
Quiet Library	10^{-8}	40
Quiet radio	10^{-7}	50
Ordinary conversation	10^{-6}	60
Busy street traffic	10^{-5}	70
Factory	10^{-4}	80
Niagara Falls	10^{-3}	90
Siren (at 30 m)	10^{-2}	100
Loud thunder	10^{-1}	110
Rock concert (at 2 m)	1	120
Jet plane takeoff (at 30 m)	10	130
Rupture of eardrum	10^2	160

Table 5.2.1: Intensity and Loudness Level of some sounds

A sound of 10 decibels has ten times the intensity of a 0 decibel sound; a sound of 20 decibels has one hundred times the intensity of a 0 dB sound, or ten times the intensity of a 10 dB sound. Similarly, a sound of 30 dB has one thousand times the intensity of a 0 dB sound or ten times the intensity of a 20 dB sound. When the loudness level increases by an increment of 10 decibels, the actual sound intensity is multiplied by a factor of ten. Therefore, the intensity of a 60 dB sound (ordinary conversation) is not double the intensity of a 30 dB sound (a whisper). Rather, the sound intensity of conversation is three factors of ten, or 1000, times the intensity of a whisper, and the intensity of a whisper is 1000 times that of the least audible sound. Equal differences in loudness level, measured in decibels, correspond to equal ratios of sound intensity.

Example 5.2.2. Compare the decibel level of two sirens at a distance of 30 meters to the decibel level of a single siren at the same distance.

Solution. The total intensity of two simultaneous sounds is the sum of the individual intensities of the two sounds.

Note that the total decibel level is not the sum of the individual decibel levels. When the intensity level doubles, the decibel level increases by $10 \cdot \log 2 \approx 3\text{dB}$

5.2.2 Exercises

1. At a party with 25 people, everyone is talking at once. Use Table 1 to determine the sound level in decibels.
2. The sound intensity level in large cities has been increasing by about one decibel annually. To what percent increase in intensity does one decibel correspond? If this annual increase continues, in how many years would the sound intensity double?
3. The Richter scale measures the magnitude of an earthquake in terms of the total energy released by the earthquake. One form of Richter's equation is

$$M = \frac{2}{3} \cdot \log(E) - 2.9$$

where M is the magnitude and E is the energy in joules of the earthquake.

- (a) If an earthquake releases 10^{13} joules of energy, what is the magnitude on the Richter scale?
- (b) A very powerful earthquake occurred in Colombia on January 31, 1906, and measured 8.6 on the Richter scale. Approximately how many joules of energy were released?
4. According to seismologists, an earthquake which registers 2 on the Richter scale is hardly perceptible while an earthquake which measures 5 on this scale is capable of shattering windows and dishes and is generally classified as "minor." The San Francisco earthquake in 1989 caused great damage and registered 7.1 on the Richter scale.
 - (a) Compare the energy release of the San Francisco earthquake to a level 5 minor earthquake. That is, calculate the ratio of their released energies.
 - (b) If an earthquake releases ten times as much energy as the San Francisco earthquake, what would it measure on the Richter scale?
 - (c) If an earthquake releases twice as much energy as the San Francisco earthquake, what would it measure on the Richter scale?
5. In the first edition of *Contemporary Precalculus Through Applications*, the authors erroneously stated that "each number on the Richter scale represents an earthquake ten times as strong as one of the next lower number". Megan Bisk, a student at Wachusett Regional High School in Holden, Massachusetts, corrected us and proved that a 0.67 increase on the Richter scale represents an earthquake 10 times as strong.
 - (a) Prove that Megan was correct.
 - (b) Show that the correct statement is: "each number on the Richter scale represents an earthquake approximately 32 times as strong as one of the next lower number".
6. An empty auditorium has a sound level of 40 dB (due to heating and air-conditioning and outside noise). On Saturday, 100 students are taking the SAT. While they are working on the test, the only sounds are labored breathing and pencils rapidly moving across the paper. The noise level then rises to 60 dB (not counting the groans). If each student contributes equally to the total noise, what would be the noise level if only 25 students were taking the test?
7. Because of the dissolved carbon dioxide in the atmosphere, the pH of rain and snow is lower than that of distilled water. The pH of distilled water is 7 while that of rain and snow is around 5.6.
 - (a) Determine the concentration of hydrogen ions in distilled water and in rain and snow.
 - (b) How many times greater is one than the other?

8. The basic astronomical unit of brightness is *magnitude*. The perceived, or apparent, magnitudes of two stars (m_1 and m_2) are related to their actual intensities (I_1 and I_2) by the equation

$$m_1 - m_2 = 2.5 \cdot \log \left(\frac{I_2}{I_1} \right)$$

If a bright star has intensity I_1 and a dim star has intensity I_2 , then $I_2 < I_1$ and $0 < \frac{I_2}{I_1} < 1$. Since the logarithm of a number between zero and one is negative, the quantity $(m_2 - m_1)$ will be negative. Thus $m_1 < m_2$, and this implies that the magnitude of the first, brighter star is smaller. We see, then, that small magnitudes are associated with brighter, more intense light sources. In 1856, Norman Pogson defined the magnitude scale such that the first magnitude, $m_1 = 1$, corresponds to the brightest-appearing stars and the sixth magnitude, $m_2 = 6$, corresponds to the faintest stars visible to the naked eye.

So the intensity of a star of magnitude one is 100 times the intensity of a star with magnitude six. The magnitude system has been extended to negative magnitudes for bright objects like the sun and the moon and to positive magnitudes beyond 6 for stars visible only with a telescope. Table 3 illustrates the magnitude scale.

Object	Apparent Magnitude
Sun	-26
Full moon	-13
Venus at brightest	-4.6
Jupiter at brightest	-2.9
Mars at brightest	-2.6
Sirius, the brightest star	-1.5
Polaris, the north star	2
Faintest star visible with 7 x 35 binoculars	8
Faintest star visible with 8-inch telescope	14
Stars barely visible in largest telescopes	28

Table 5.2.3: Levels of Brightness

- Using Table 3 determine how much brighter is a full moon than Venus?
- To observe stars beyond the sixth magnitude, a telescope is required. Telescopes, however, also have limitations. The *limiting magnitude* of a telescope is the magnitude of the faintest star that can be seen with the telescope. A telescope with lens diameter D meters has a limiting magnitude L , given by the formula:

$$L = 17.1 + 5.1 \cdot \log(D)$$

Find the lens diameter of a telescope with limiting magnitude of 11.1

5.3 Graphing Logarithmic Functions

5.3.1 Defining the Logarithm Function

Since we can find the base 10 logarithm of any positive real number, we can use logarithms to define a function. We define a logarithmic function as follows:

$$f(x) = \log(x)$$

this function pairs every positive x -value with the base 10 logarithm of that x .

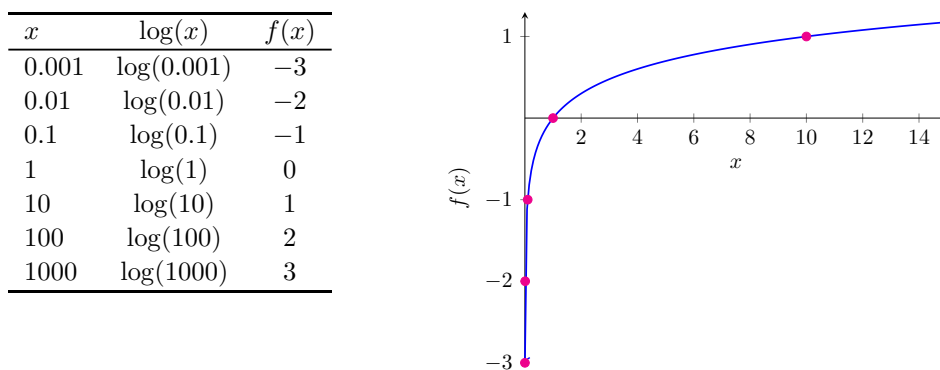


Figure 5.3.1: Table of values and graph of $f(x) = \log(x)$

The ordered pairs highlighted on the graph are all of the form $(a, \log(a))$. Consider the pairs with the input and output interchanged: $(-3, 0.001)$, $(-2, 0.01)$, $(-1, 0.1)$, $(0, 1)$, $(1, 10)$, $(2, 100)$, $(3, 1000)$. These new pairs are associated with the base 10 exponential function $g(x) = 10^x$.

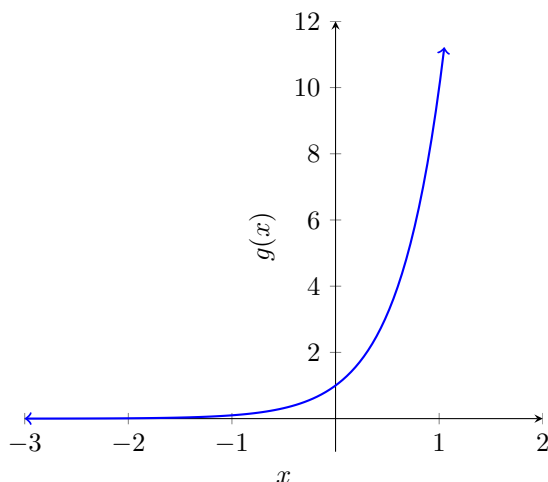


Figure 5.3.2: Graph of $g(x) = 10^x$

Suppose a pair (a, b) is on the graph of $f(x) = \log(x)$: it must be true that $b = \log(a)$. But if $b = \log(a)$, then it is true that $a = 10^b$. When $a = 10^b$, then $g(b) = 10^b = a$, which implies that the pair (a, b) is on the graph of $g(x) = 10^x$.

These observations let us conclude that $f(x) = \log(x)$ and $g(x) = 10^x$ are inverse functions. For every point (a, b) on the graph of $f(x) = \log(x)$, the pair (b, a) is on the graph of $g(x) = 10^x$. As you should expect, the composition of the functions $f(x)$ and $g(x)$ yields the identity function.

$$\begin{aligned}
 f(g(x)) &= \log(10^x) = x \\
 &\text{and} \\
 g(f(x)) &= 10^{\log(x)} = x
 \end{aligned}$$

In [Figure 3](#) we show the graphs of the exponential function $f(x) = e^x$ and its inverse, $g(x) = \log_e(x) = \ln(x)$, which is obtained by reflecting the graph of $y = e^x$ across the line $y = x$.

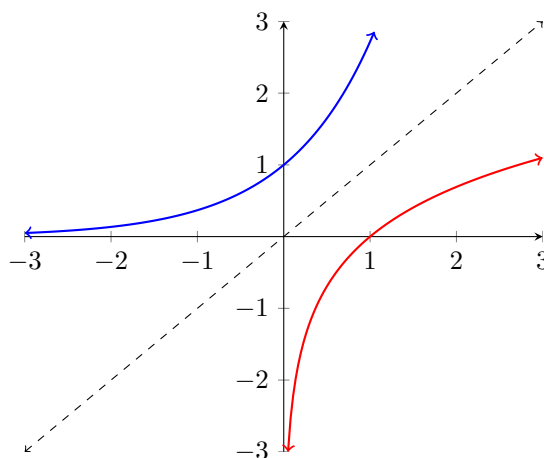


Figure 5.3.3: Exponential and Logarithmic Functions as Inverses

The domain of our new logarithmic function is the same as the range of the exponential function, and the range of the logarithmic function is the domain of the exponential function. Thus, the logarithmic function has the positive real numbers as its domain and all real numbers as its range.

Example 5.3.4. Find the inverse of $f(x) = 10^x + 3$. Specify the domain and range of $f(x)$ and $f^{-1}(x)$.

Solution. The function $f(x)$ is one-to-one with domain all real numbers and range $y > 3$. To find $f^{-1}(x)$, we first must solve for x in the equation $y = 10^x + 3$ as follows:

$$\begin{aligned} y &= 10^x + 3 \\ y - 3 &= 10^x \\ \log_{10}(y - 3) &= x \end{aligned}$$

Note that we had to subtract 3 from the right-hand side of the equation before using the definition of logarithm to isolate x . Now we can write $f^{-1}(x) = \log_{10}(x - 3) = \log(x - 3)$. The domain of $f^{-1}(x)$ is the range of $f(x)$, or $x > 3$; the range of $f^{-1}(x)$ is the domain of $f(x)$, or all real numbers. A graph of both functions is shown in Figure 5. The graph of $y = \log(x - 3)$ is the reflection of the graph of $y = 10^x + 3$ across the line $y = x$.

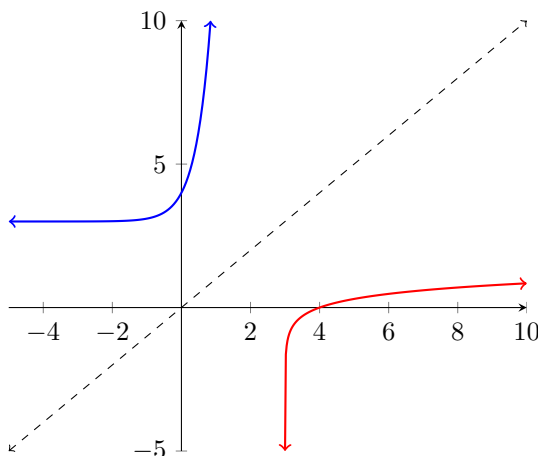


Figure 5.3.5: Exponential and Logarithmic Functions as Inverses

5.3.2 Compositions of Exponential and Logarithmic Functions

Since the exponential and logarithmic functions are inverses and both are one-to-one functions, we can simplify the compositions of these two functions. If $f(x) = 10^x$, then $f^{-1}(x) = \log x$, and we expect that

$$f(f^{-1}(x)) = x.$$

We know, however, that some restrictions on x might apply. For what x -values are these equations valid? First consider $f(f^{-1}(x)) = 10^{\log x}$.

This is a function composition with domain restrictions on the inner function $f^{-1}(x) = \log(x)$. For $f^{-1}(x) = \log(x)$ to be defined, we must have $x > 0$. So we can conclude that

$$f(f^{-1}(x)) = 10^{\log x} = x \text{ if } x > 0$$

You might want to illustrate this statement using some specific examples. For example, according to this statement $10^{\log 100} = 100$ but $10^{\log(-2)} \neq -2$. In the first example, we know that $\log 100 = 2$ using the definition of logarithms and the fact that $10^2 = 100$. So $10^{\log 100} = 10^2 = 100$ as expected. In the second example, we need to find the value a for which $\log(-2) = a$. Again using the definition of logarithm, we want $10^a = -2$. However, no power of 10 is negative. Therefore $\log(-2)$ does not exist and consequently $10^{\log(-2)} \neq -2$.

Now consider

$$f^{-1}(f(x)) = \log(10^x)$$

There are no domain restrictions on the inner function $f(x) = 10^x$ of this composition because $f(x) = 10^x$ is defined for all real numbers. Furthermore, since values of 10^x are always positive, we can take the common logarithm of any number of the form 10^x . So we can conclude that

$$f^{-1}(f(x)) = \log(10^x) = x \text{ if } x \text{ is any real number.}$$

According to this statement, $\log(10^3) = 3$ and $\log(10^{-2}) = -2$. We can verify these easily since $\log(10^3) = \log(1000)$ and $\log(10^{-2}) = \log(\frac{1}{100}) = -2$. Use your calculator to verify that $\ln(e^5) = 5$ and to graph the function $y = \log(10^x)$ and $y = \ln(e^x)$.

Now use your calculator to evaluate $\log(e^3)$. Are you surprised that the result is not 3? The bases of the logarithmic function and the exponential function must be the same for their composition to be the identity function. In general:

$$\begin{aligned} b^{\log_b x} &= x, \text{ if } x > 0 \\ \text{and} \\ \log_b(b^x) &= x, \text{ if } x \text{ is any real number.} \end{aligned}$$

5.3.3 Graphing Logarithm Functions

When we consider exponential functions with bases greater than one, we know that the exponential functions with larger bases grow more rapidly than those with smaller bases. Since the graph of a logarithmic function is the reflection of an exponential function about the line $y = x$, we would expect log functions with larger bases to grow more slowly than those with smaller bases. Figure 6 shows graphs of the functions $y = 3^x$ with $y = \log_3 x$ on the left and $y = 10^x$ along with $y = \log_{10} x$ on the right. For the logarithmic functions you can see that the larger base produces a curve that increases more slowly.

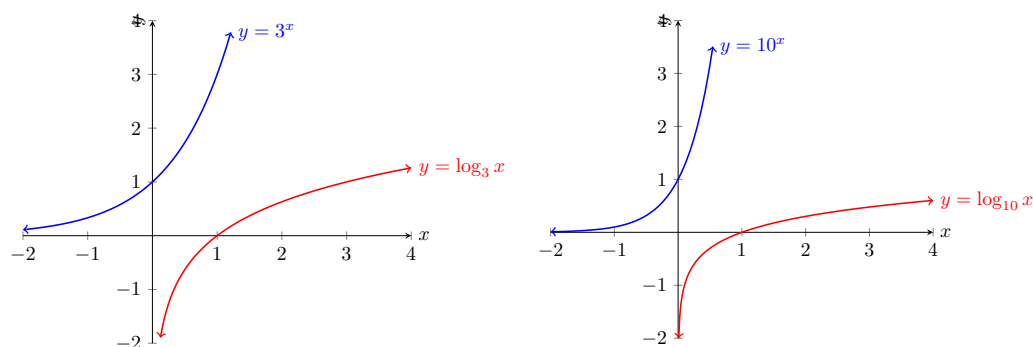


Figure 5.3.6: Exponential and Logarithmic Functions with Base 3 and 10

Notice that the graph of $f(x) = \log_3 x$ appears to be a vertical stretch of the graph of $f(x) = \log_{10} x$ and the graph of $y = 3^x$ appears to be a horizontal stretch of $y = 10^x$. You will see more on this relationship in the exercises at the end of this section.

We know that the base for a logarithm must be positive, but so far we have looked only at graphs of logarithmic functions with bases greater than 1. Now we will examine the graph of $y = \log_{\frac{1}{3}} x$. We know that the function $g(x) = \left(\frac{1}{3}\right)^x$ is a decreasing exponential function. The graph of this function contains the points $(0, 1)$ and $(1, \frac{1}{3})$ and the positive x -axis is a horizontal asymptote for the graph. The inverse of this function is $g^{-1}(x) = \log_{\frac{1}{3}}(x)$ which therefore contains the points $g(x) = \left(\frac{1}{3}\right)^x$ is a decreasing exponential function. The graph of this function contains the points $(1, 0)$ and $(\frac{1}{3}, 1)$ and has the positive y -axis as a vertical asymptote. The graphs of these functions are shown in Figure 7.

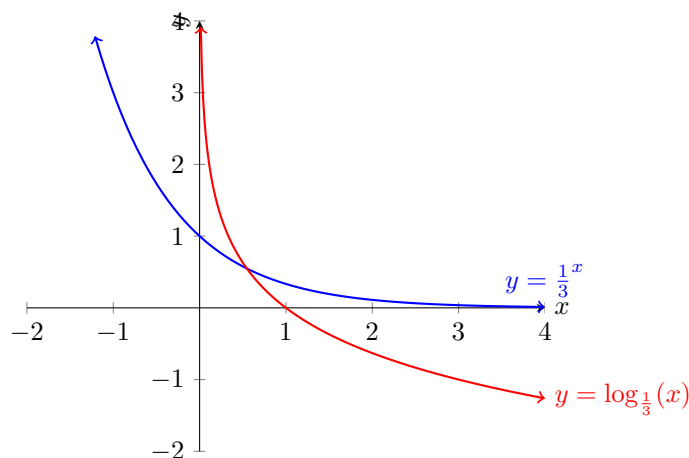


Figure 5.3.7: Graphs of $g(x) = \left(\frac{1}{3}\right)^x$ and $g^{-1}(x) = \log_{\frac{1}{3}}(x)$

The logarithmic function $f(x) = \log_b x$, $b > 0, b \neq 1$ should now be added to our toolkit of functions.

To graph an equation that is a transformation of a logarithmic function, we will apply the same techniques that we used most recently for exponential functions. The most useful characteristics for graphing transformations of a logarithmic function are the points $(1, 0)$ and $(b, 1)$ and the vertical asymptote $x = 0$. The special characteristics of logarithm graphs are listed in List 8.

Characteristics of Logarithmic Functions of the form $y = \log_b x$, $b > 0, b \neq 1$

1. The point $(1, 0)$ is on every graph.
2. The point $(b, 1)$ is on every graph.
3. The y -axis ($x = 0$) is an asymptote for each curve.
4. The domain of each function is the set of real numbers greater than 0.
5. The range of each function is the set of real numbers.
6. Each function is one-to-one

List 5.3.8: Characteristics of Logarithmic Functions

Example 5.3.9. Graph $f(x) = \ln(x) + 1$.

Solution. The graph is provided in [Figure 10](#). It has exactly the same shape as the graph of $y = \ln x$, but the graph has been shifted up one unit. The vertical asymptote remains the same. However, the special point $(1, 0)$ moves to $(1, 1)$ and the point $(e, 1)$ is shifted up to $(e, 2)$.

Notice that the expression $\ln(x) + 1$ can be rewritten as $\ln(x) + \ln(e)$ and this can be further simplified to $\ln(e \cdot x)$ using the laws of logarithms. The graph of $f(x) = \ln(x) + 1$ is the same as the graph of $f(x) = \ln(e \cdot x)$. More generally $\ln(x) + k$ is equivalent to $\ln(x) + \ln(e^k)$ which is equal to $\ln(e^k \cdot x)$. So a vertical shift of k units transforms the graph in exactly the same way as a horizontal compression (or stretch) using e^k .

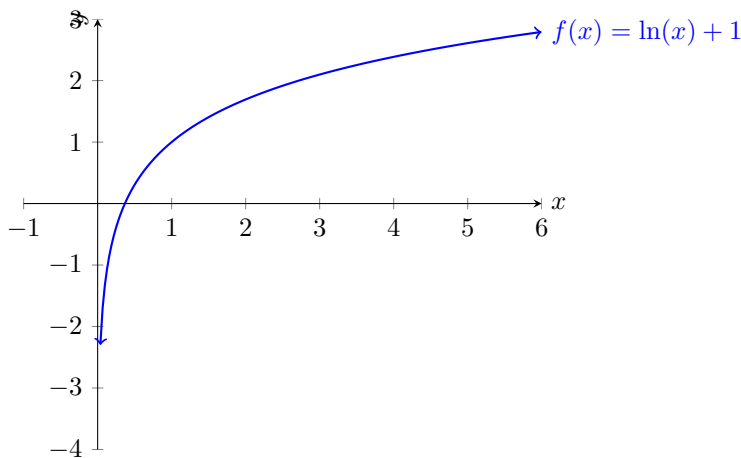


Figure 5.3.10: Graph of $f(x) = \ln(x) + 1 = \ln(e \cdot x)$

Example 5.3.11. Graph $f(x) = \log_2(x^2)$. This expression can be simplified using the laws of logarithms to give the expression $g(x) = 2 \log_2(x)$. Are the two functions f and g truly identical?

Solution. Notice first that x^2 is greater than zero for all $x \neq 0$. This means that the domain of $f(x) = \log_2(x^2)$ will be all real numbers except zero. Notice also that $f(-x) = f(x)$, so the graph of this function is symmetric about the y -axis. We might be tempted to simplify $\log_2(x^2)$ and rewrite it as $2 \log_2(x)$. However, when we do this, we change the domain and now can use only positive values of x . For positive values of x the graphs of $f(x) = \log_2(x^2)$ and $g(x) = 2 \log_2(x)$ are identical, as shown in [Figure 12](#). Both graphs contain the points $(1, 1)$ and $(2, 2)$. The graph of $f(x) = \log_2(x^2)$ also contains the points $(-1, 0)$ and $(-2, 2)$. The functions f and g are not identical because they have different domains.

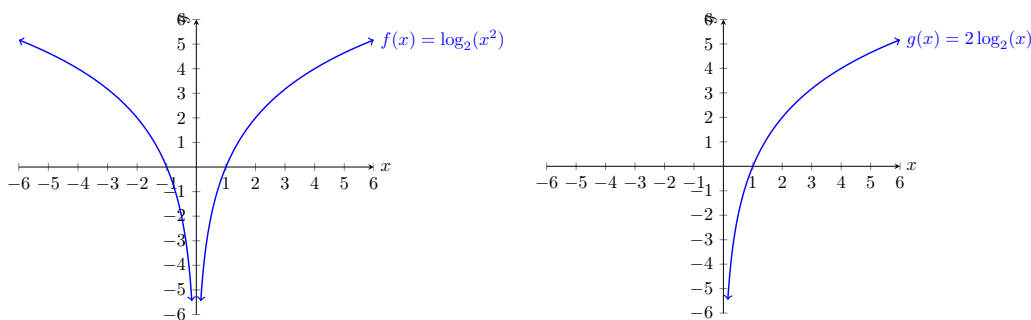


Figure 5.3.12: Graphs of $f(x) = \log_2(x^2)$ and $g(x) = 2 \log_2(x)$

Example 5.3.13. Sketch the graph of $f(x) = \ln(x^2 - 3x - 4)$.

Solution. We can approach this graph as the composition $f(x) = g(h(x))$ where $h(x) = x^2 - 3x - 4$ and $g(x) = \ln(x)$. Domain plays an important role in this composition. The graph of $h(x)$ is shown in Figure 14 and can be used to analyze the domain of $f(x)$ and the flow of the composition. Since $\ln(x)$ is defined only for $x > 0$, the domain of the composition is restricted to those values of x for which $h(x) = x^2 - 3x - 4$ is positive. That is, the domain is restricted to those values of x for which the graph of $h(x) = x^2 - 3x - 4$ is above the x -axis. This happens when $x > 4$ or $x < -1$. In the log function $y = \ln x$, as x gets close to zero, the y -values tend toward negative infinity, making the y -axis a vertical asymptote. In this graph, the values of the inner function $h(x)$ approach zero as x gets close to 4 and to -1 . Therefore, the graph of f will have vertical asymptotes at $x = 4$ and at $x = -1$.

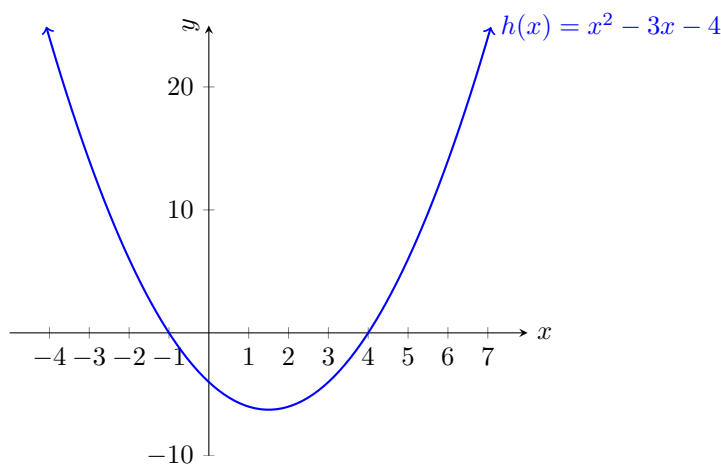


Figure 5.3.14: Graph of $h(x) = x^2 - 3x - 4$

Since $\ln(1) = 0$, the values of x for which $x^2 - 3x - 4 = 1$ are zeros of f . These are $x = \frac{3 \pm \sqrt{29}}{2}$, or $x \approx -1.2$ and $x \approx 4.2$. Notice that the y -values of f are negative for those x -values where $0 < x^2 - 3x - 4 < 1$. When $a \geq 1$, $\ln(a) < a$. Therefore, for x -values where $x^2 - 3x - 4 > 1$, the graph of $f(x) = \ln(x^2 - 3x - 4)$ is below the graph of $h(x) = x^2 - 3x - 4$. Notice that the graph of $h(x) = x^2 - 3x - 4$ is symmetric about $x = \frac{3}{2}$. This symmetry is also present in the graph of f . The graphs of f and h are shown in Figure 15.

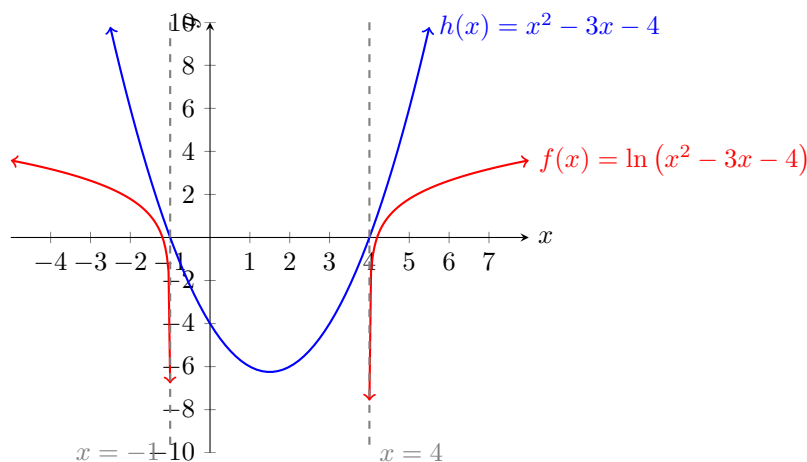


Figure 5.3.15: Graph of $f(x) = \ln(x^2 - 3x - 4)$ and $h(x) = x^2 - 3x - 4$

5.3.4 Exercises

1. Find the inverse of the following functions and specify the domain of each.

- (a) $f(x) = e^{-5x}$
- (b) $f(x) = 3^{x-1}$
- (c) $f(x) = \ln(2x)$
- (d) $f(x) = \log_2(x - 3)$

2.

- (a) What attributes of the logarithm function allows us to say that if $\ln r = \ln s$ then $r = s$ as long as $r, s > 0$?
- (b) What attributes of the logarithm function allows us to say that if $r = s$, then $\ln r = \ln s$ as long as $r, s > 0$?
- (c) For the toolkit functions, is it true that if $f(x) = f(s)$ then $r = s$?

3. Computer scientists compare the speeds of some algorithms by finding the time required to complete a task as a function of the number of input items. A typical algorithm may include sorting n items into ascending order, for example. Suppose there are three algorithms, A, B, and C for sorting items in a list. The amount of time each of the algorithms A, B, and C takes is a function T of the number of input items n . Three functions for T_A , T_B , and T_C are shown below.

$$T_A(n) = 10,000 + 2\ln(n)$$

$$T_B(n) = 100 + 3n^2$$

$$T_C(n) = 0.001e^n$$

- (a) Rank the speed of the algorithms when sorting $n = 10$ items.
 - (b) Rank the speed of the algorithms when sorting $n = 10$ items.
 - (c) When these algorithms are used in industry they often have thousands of input items. Which algorithm would be best for industrial use? Write a few sentences to justify your answer.
4. Use a computer or calculator to graph $y = \log(x)$ and $y = \ln(x)$ on the same axes.
- (a) Write a paragraph in which you compare these graphs. What special features do they share? In what ways are they different?
 - (b) Use graphs as needed to complete the following:

- i. For what values of x are values of $\log(x)$ negative? zero? positive?
- ii. For what values of x are values of $\ln(x)$ negative? zero? positive?
- iii. For what values of x are values of $\log(x)$ between zero and one? equal to one? greater than one?
- iv. For what values of x are values of $\ln(x)$ between zero and one? equal to one? greater than one?

5. Use the entries in the table below to compare the rate of graph of $f(x) = \sqrt{x}$ and $g(x) = \ln(x)$.

x	$f(x) = \sqrt{x}$	$g(x) = \ln(x)$
0.1	0.316	-2.303
0.7	0.837	-0.357
1	1.000	0
1.5	1.225	0.405
5	2.236	1.609
10	3.162	2.303
25	5.000	3.219
100	10	4.605
1,000	31.623	6.908
1,000,000	1000.000	13.816

Figure 5.3.16: Comparison of $f(x) = \sqrt{x}$ and $g(x) = \ln(x)$

The functions $f(x) = \sqrt{x}$ and $g(x) = \ln(x)$ are the inverses of the functions $h(x) = x^2$ and $k(x) = e^x$, respectively. Graph both $h(x)$ and $k(x)$ on the same axes and compare the rate at which they are increasing when $x > 0$. How can these graphs be used to support the conclusions you made when comparing the rate of growth of $f(x) = \sqrt{x}$ and $g(x) = \ln(x)$

6. For what values of b is $f(x) = \log_b(x)$ an increasing function? A decreasing function?
7. Graph each function. Label the points that correspond to $(1,0)$ and $(b,1)$ and the asymptote that corresponds to $x = 0$. Identify the domain and range of each.
- (a) $y = 3 + \log(x)$
 - (b) $y = \log_{1/2}(x)$
 - (c) $y = 4 \log_3(x)$
 - (d) $y = \ln(x - 5)$
 - (e) $y = \ln(5x)$
 - (f) $y = \log(-x)$
 - (g) $y = |\log_4(x)|$
 - (h) $y = \ln(4x - 3)$
 - (i) $y = -3 \log(x) + 5$
 - (j) $y = -(3 \log(x) + 5)$

8. Consider the function $f(x) = \log_2 \sqrt{\frac{x}{8}}$.

- (a) Graph f by rewriting it as $f(x) = \frac{1}{2} \log_2\left(\frac{x}{8}\right)$.
- (b) We can use another law of logarithms to write this function in the form:

$$f(x) = \frac{1}{2} (\log_2(x) - \log_2(8)) = \frac{1}{2} \log_2(x) - \frac{1}{2} (3) = \frac{1}{2} \log_2(x) - \frac{3}{2}$$

Use this form of the equation to graph the function.

- (c) Which form seems easier for graphing by hand?

9. Use properties of logarithms or composition of functions to sketch a graph of each of the following. Identify the property you use or the two functions that compose to form f .

(a) $f(x) = \log\left(\frac{1}{x}\right)$

(b) $f(x) = \ln(\sqrt[3]{x})$

(c) $f(x) = \frac{1}{\ln(x)}$

(d) $f(x) = \log|x|$

(e) $f(x) = \ln(x^2 - 5)$

(f) $f(x) = \ln(\sin x)$

(g) $f(x) = \log(10^{-x^2+2})$

(h) $f(x) = 10^{\log(-x^2+2)}$

10. Graph $f(x) = \ln(x)$ and $g(x) = \log_{10}(x)$ on the same axis and relate the two using transformations.

5.4 Solving Exponential and Logarithmic Equations

In the preceding chapter, a problem about aspirin dosage provided the function $a = 14.91e^{-0.18t}$ and asked you to use a graph to determine the time at which the aspirin concentration is 5 mg per cc of blood. What happens when we attempt to solve the equation $5 = 14.91e^{-0.18t}$ algebraically? Dividing both sides by 14.91 gives approximately $0.335 = e^{-0.18t}$. How do we isolate the variable t ? We can say that t is the number, which when multiplied by -0.18 and used as the exponent of e produces 0.335. Our goal is to write this statement using mathematical symbols.

To rewrite $0.335 = e^{-0.18t}$ so that t is isolated, we can use logarithms. Due to their inverse relationship, logarithms allow us to rewrite an exponential equation so that the exponent is isolated, as defined earlier in this chapter. In the aspirin dosage problem we can rewrite $0.335 = e^{-0.18t}$ as $\log_e(0.335) = -0.18t$.

We can use the calculator to evaluate $\log_e(0.335) = \ln(0.335) \approx -1.09$ and solve the resulting equation.

$$\begin{aligned} -1.09 &\approx -0.18t \\ t &\approx 6.1 \end{aligned}$$

to determine that the aspirin level is 5 mg per cc of blood after approximately 6.1 hours.

Example 5.4.1. Solve each equation for x :

1. $\log_3(3) = 5$

2. $\log_x(20) = 3$

3. $x = 7^{3\log_7(2)}$

Solution.

1. $\log_3(x) = 5$ can be rewritten as $3^5 = x$, so $x = 243$.

2. $\log_x(20) = 3$ can be rewritten as $x^3 = 20$, so $x = \sqrt[3]{20} \approx 2.7144$

3. $x = 7^{3\log_7(2)}$ can be rewritten as $x = 7^{\log_7(2^3)} = 7^{\log_7(8)}$, so $x = 8$

Solving equations that involve exponents or logarithms often requires techniques that involve composition as well as laws of exponents and logarithms.

Example 5.4.2. The population of the world, as of November 1, 2016, was estimated by the U. S. Census Bureau to be approximately 7.348 billion and was growing at an annual rate of approximately 1.13%. If this rate were to remain constant, how long would it take for the population of the world to reach 10 billion?

Solution. If we assume that the rate of growth is a constant percent (of the current population) then the function that models this growth is $P(t) = 7.348(1.0113)^t$, where t represents the number of years since November 1, 2016, and $P(t)$ represents population in billions. Since we need to determine the value of t for which $P(t) = 10$, we need to solve the equation $10 = 7.348(1.0113)^t$. Applying the log of both sides of this equation yields the following:

$$\begin{aligned}\log(10) &= \log(7.348(1.0113)^t) \\ \log(10) &= \log(7.348) + t \log(1.0113)\end{aligned}$$

Solving for t gives us $t = \frac{\log(10) - \log(7.348)}{\log(1.0113)} \approx 27.4243$. Thus we can estimate that the population of the world will reach 10 billion in just a little less than 27 years and 6 months, sometime in April of 2044. Note that we could have first divided by 7.348, then taken the log of both sides of the equation $\frac{10}{7.348} = (1.0113)^t$, and solved for t to obtain the same solution.

In the solution above we applied the common logarithm. We can obtain the same answer by taking the natural logarithm of both sides as follows:

$$\begin{aligned}10 &= 7.348(1.0113)^t \\ \frac{10}{7.348} &= (1.0113)^t \\ \ln\left(\frac{10}{7.348}\right) &= \ln(1.0113^t) \\ \ln\left(\frac{10}{7.348}\right) &= t \ln(1.0113)\end{aligned}$$

Solving for t gives us:

$$t = \frac{\ln\left(\frac{10}{7.348}\right)}{\ln(1.0113)} \approx 27.4243$$

Example 5.4.3. If a group of people are all asked to memorize a list of nonsense words, the percent of those who remember all of the words after t hours can be approximated by the logarithmic function $f(t) = 1 - k \ln(t + 1)$, where k is a constant that depends on several factors, such as the length of the list of words. If after 5 hours only 20% of the members of a group can remember all of the words, determine the value of k . Using this value of k , determine what percent remember all of the words after 1 hour. At what time will no one in the group be able to remember all of the words?

Solution. Note first that $f(0) = 1$, which tells us that initially 100% of the people in the group know all the words. To determine the value of k we need to solve the equation $0.20 = 1 - k \ln(t + 1)$ for k when $t = 5$.

$$\begin{aligned}0.20 &= 1 - k \ln(5 + 1) \\ -0.80 &= -k \ln(6) \\ k &= \frac{0.80}{\ln(6)} \approx 0.446\end{aligned}$$

Using this value of k we now need to determine the value of $f(1)$.

$$\begin{aligned}f(1) &= 1 - 0.446 \ln(1 + 1) \\ &= 1 - 0.446 \ln(2) \\ &\approx 0.691\end{aligned}$$

Since $f(1) \approx 0.691$, after one hour about 69% of the group still remembers all of the words. Finally, to determine how long it will take until no one in the group can remember all the words, we need to solve the following equation for t :

$$\begin{aligned} 0 &= 1 - 0.446 \ln(t + 1) \\ -1 &= -0.446 \ln(t + 1) \\ \frac{1}{0.446} &= \ln(t + 1) \end{aligned}$$

At this point in the solution, we can exponentiate both sides of the equation, so that:

$$\begin{aligned} e^{\frac{1}{0.446}} &= e^{\ln(t+1)} \\ e^{\frac{1}{0.446}} &= t + 1 \\ t &= e^{\frac{1}{0.446}} - 1 \end{aligned}$$

Evaluating t , we find that $t \approx 8.4$. Thus, after about 8.4 hours, no one in the group can remember all of the words. For values of t larger than 8.4, the function f would produce negative values for the percent, which would not make sense in this context.

5.4.1 Change of Base

Mara was writing a report on the life of Marie Curie. She found that in 1921, President Warren G. Harding presented Marie Curie with 1 gram of radium on behalf of the women of the United States. Although not essential to her report, two of Mara's references gave a function to describe the decay of radium. One text gave the function describing the amount of radium left after t years to be $R_1(t) = 2^{-0.0006039t}$ and the other as $R_2(t) = 1 \cdot e^{-0.0004186t}$. Mara was curious about the two different functions given in her references. After thinking about the two functions for a while, Mara realized that they were the same function, only written in two different bases.

We know that $3^4 = 9^2$ and that $2^{-2} = \left(\frac{1}{16}\right)^{frac{12}{16}}$. Using a calculator, we can see that $5^2 \approx e^3.22$ and that $2^{-0.0006039} \approx e^{-0.0004186}$. How do we change from one exponential base to another? When is it useful to do so? These questions are very important when we write exponential functions as models of real-world phenomena. Equivalent functions may look different because they are written with a different base.

Let's investigate how we can express $f(x) = 2^x$ as an exponential function with base e . We need to express 2 as a power of e , say e^k . So we must solve the equation $2 = e^k$ to find k . Both sides of this equation are positive, so we can use the technique of taking logarithms of both sides of the equation. Since the exponential component has a base of e , choosing a natural log will simplify the resulting work that follows:

$$\begin{aligned} 2 &= e^k \\ \ln(2) &= \ln(e^k) \\ \ln(2) &= k \\ 0.693 &\approx k \end{aligned}$$

So,

$$2 = e^k = e^{\ln(2)} \approx e^{0.693}$$

and we can write:

$$f(x) = 2^x \approx (e^{0.693})^x = e^{0.693x}$$

This explains the difference in the two functions Mara found, Since

$$R_1(t) = 2^{-0.0006039t} = \left(e^{\ln(2)}\right)^{(-0.0006039)t} = e^{(-0.0006039)(\ln(2))t} = e^{-0.0004186t}$$

Since exponential and logarithmic functions are inverses, you might suspect that we can also change the base of logarithmic functions. Let's try to rewrite $g(x) = \log_2(x)$ with a natural logarithm. Let

$$g(x) = y = \log_2(x)$$

and then rewrite this equation as

$$2^y = x$$

We now take the natural logarithm of both sides (which are positive):

$$\begin{aligned}\ln(2^y) &= \ln(x) \\ y \ln(2) &= \ln(x) \\ y &= \frac{\ln(x)}{\ln(2)} \approx 1.44 \ln(x)\end{aligned}$$

Therefore,

$$g(x) = \log_2(x) \approx 1.44 \ln(x)$$

Example 5.4.4. Evaluate $\log_3(7)$. Round your approximation to the thousandths place.

Solution. Certain calculators and computers can compute logarithms of any base. However, many applications or computers are not be able to directly evaluate $\log_3(7)$. We can use the techniques above to change the base and then evaluate using a base, such as e or 10, that the computer can calculate.

Let $y = \log_3(7)$, and rewrite this equation in exponential form, $3^y = 7$.

Since we now have an exponential equation, we can apply either the common log or the natural log of both sides of the equation as shown below.

5.4.2 Exercises

- 1.
- 2.
- 3.
- 4.
- 5.
- 6.
- 7.
- 8.
- 9.
- 10.

Chapter 6

Parametric Functions

Introduction to this chapter

6.1

Chapter 7

Intro to Trigonometric Functions

Introduction to this chapter

7.1

Chapter 8

More Trigonometric Functions

Introduction to this chapter

8.1

Chapter 9

Combinations of Functions

Introduction to this chapter

9.1

Chapter 10

Matrices

Introduction to this chapter

10.1