MAE237 Computational Fluid Dynamics
Homework #4 - Final Project
Nov. 18, 2016
Due Mon. Nov. 28, 2016

1 The Quasi-1D Euler Equations

For the flow in a 1D channel with a given slowly-varying cross-sectional area $S(x)$, we can write the Euler equations as

$$\frac{\partial [\mathbf{W}S(x)]}{\partial t} + \frac{\partial [\mathbf{F}S(x)]}{\partial x} = \mathbf{Q} \tag{1}$$

where

$$\mathbf{W} = \begin{bmatrix} \rho \\ \rho u \\ \rho E \end{bmatrix} \quad \mathbf{F} = \begin{bmatrix} \rho u \\ \rho uu + p \\ \rho uH \end{bmatrix} \quad \mathbf{W} = \begin{bmatrix} 0 \\ pS'(x) \\ 0 \end{bmatrix}$$

p, ρ, u, E and H denote the pressure, density, velocity , total energy and total enthalpy. For a perfect gas

$$E = \frac{p}{(\gamma-1)\rho} + \frac{1}{2}u^2$$

$$H = E + \frac{p}{\rho}$$

where γ is the ratio of specific heats.

2 Solution of the Steady Quasi-1D Flow in a Channel

Solve the above Euler equations using time marching to obtain the steady flow through a quasi-1D channel whose cross-sectional area is given as

$$S(x) = \begin{cases} 1 + 1.5\left(1 - \frac{x}{5}\right)^2 & 0 \le x \le 5 \\ 1 + 0.5\left(1 - \frac{x}{5}\right)^2 & 5 \le x \le 10 \end{cases}$$

The stagnation pressure and stagnation temperature upstream of the channel is $p_0 = 10\ atm$, $T_0 = 288K$. Try the cases with downstream static pressure $p_b = 4\ atm$ and $p_b = 9\ atm$.

Use the following numerical methods:

1. The JST scheme (see Appendix)
2. First-order Steger flux vector splitting
3. First-order Van Leer flux vector splitting
4. First-order Roe Approximate Riemann Solver.

5. 2nd-order scheme with Van Leer flux vector splitting and MUSCL.

6. 2nd-order scheme Roe approximate Riemann solver and MUSCL.

7. 2nd-order scheme with Van Leer flux vector splitting and MUSCL with limiter.

8. 2nd-order scheme Roe approximate Riemann solver and MUSCL with limiter.

You may also solve the problem analytically (see [1]). Compare your computational results with the exact solution. Plot the pressure, density, Mach number along the x axis. Discuss the results.

3 Appendix: The Jameson-Schmidt-Turkel (JST) scheme

3.1 Semi-discretization:

The computational domain is divided into a number of small cells (intervals in 1-dimension). A system of ordinary differential equations can be obtained by applying equation (1) to each cell and approximating the surface integral with a finite volume scheme,

$$\frac{d}{dt}(V_j \mathbf{W}_j) + \mathbf{R}_j = 0 \tag{2}$$

where V_j is the cell volume (Δx_j in 1-dimension), $\mathbf{W}_j$ is the average flow variable over the cell, $\mathbf{R}_j$ is the finite volume approximation for the net flux out of the cell. In 1-dimension, it is

$$\mathbf{R}_j = \mathbf{h}_{j+\frac{1}{2}} - \mathbf{h}_{j-\frac{1}{2}} \tag{3}$$

3.2 Numerical Fluxes:

$\mathbf{h}_{j+\frac{1}{2}}$ is the numerical approximation to the flux $\mathbf{F}$ at the cell interface $j + \frac{1}{2}$, which is usually split into two parts

$$\mathbf{h}_{j+\frac{1}{2}} = \tilde{\mathbf{h}}_{j+\frac{1}{2}} - \mathbf{d}_{j+\frac{1}{2}} \tag{4}$$

The first term on the RHS is a second order approximation to $\mathbf{F}$

$$\tilde{\mathbf{h}}_{j+\frac{1}{2}} = \frac{\mathbf{F}(\mathbf{W}_{j+1}) + \mathbf{F}(\mathbf{W}_j)}{2}; \tag{5}$$

The second term is the the artificial dissipation term

$$\mathbf{d}_{j+\frac{1}{2}} = \epsilon^{(2)}_{j+\frac{1}{2}} R_{j+\frac{1}{2}} \Delta \mathbf{W}_{j+\frac{1}{2}} - \epsilon^{(4)}_{j+\frac{1}{2}} R_{j+\frac{1}{2}} \Delta^3 \mathbf{W}_{j+\frac{1}{2}} \tag{6}$$

where

$$\Delta \mathbf{W}_{j+\frac{1}{2}} = \mathbf{W}_{j+1} - \mathbf{W}_j \tag{7}$$

$$\nu_j = \left| \frac{p_{j+1} - 2p_j + p_{j-1}}{p_{j+1} + 2p_j + p_{j-1}} \right| \tag{8}$$

$$S_{j+\frac{1}{2}} = max(\nu_{j+1}, \nu_j) \tag{9}$$

$$\epsilon^{(2)}_{j+\frac{1}{2}} = min(\alpha_1, \alpha_2 S_{j+\frac{1}{2}}) \tag{10}$$

$$\epsilon^{(4)}_{j+\frac{1}{2}} = max(0, \beta_1 - \beta_2 \epsilon^{(2)}_{j+\frac{1}{2}}) = DIM(\beta_1, \beta_2 \epsilon^{(2)}_{j+\frac{1}{2}}) \tag{11}$$

$R_{j+\frac{1}{2}}$ is the maximum wave speed of the system. For the Euler equations,

$$R_{j+\frac{1}{2}} = |u| + a \tag{12}$$

Usually, $\alpha_1 = \frac{1}{2}$, $\beta_1 = \frac{1}{4}$ to scale the diffusion to the level corresponding to upwinding, while α_2 and β_2 can be adjusted to switch from third order to first order diffusion fast enough near a shock wave. You may experiment on these values.

Eqn. (3) can be written as

$$\frac{d\mathbf{W}_j}{dt} + \mathbf{R}_j(\mathbf{W}) = 0 \quad , \tag{13}$$

where $\mathbf{R}_j$ is the residual

$$\mathbf{R}_j(\mathbf{W}) = \frac{1}{V_j}(\mathbf{Q}_j - \mathbf{D}_j) \quad .$$

$$\mathbf{Q}_j = \tilde{\mathbf{h}}_{j+\frac{1}{2}} - \tilde{\mathbf{h}}_{j-\frac{1}{2}}$$

$$\mathbf{D}_j = \mathbf{d}_{j+\frac{1}{2}} - \mathbf{d}_{j-\frac{1}{2}}$$

$\mathbf{Q}_j$ is called the 2nd order Euler flux and $\mathbf{D}_j$ is called the dissipation flux. It is convenient to write two separate routines, say eflux and dflux, to calculate $\mathbf{Q}_j$ and $\mathbf{D}_j$.

3.3 Allowable Time Step

Regardless whether an explicit or an implicit scheme is used, it is always useful to measure your time step by using

$$\Delta t = CFL \frac{\Delta x}{\text{Maximum Wave Speed}} = CFL \frac{\Delta x}{|u| + a}$$

3.4 Time Marching

Equation (6) can be integrated by a modified Runge-Kutta scheme. Let $\mathbf{W}^n_j$ be the value of $\mathbf{W}_j$ after n time steps. Dropping the subscripts j, we can write a general m stage modified Runge-Kutta scheme to advance a time step Δt as

$$\begin{array}{lclcrl}
\mathbf{W}^{(0)} & = & \mathbf{W}^n & & & \\
\mathbf{W}^{(1)} & = & \mathbf{W}^{(0)} & - & \alpha_1 & \Delta t \mathbf{R}^{(0)} \\
 & \cdots & & & & \\
\mathbf{W}^{(m-1)} & = & \mathbf{W}^{(0)} & - & \alpha_{m-1} & \Delta t \mathbf{R}^{(m-2)} \\
\mathbf{W}^{(m)} & = & \mathbf{W}^{(0)} & - & & \Delta t \mathbf{R}^{(m-1)} \\
\mathbf{W}^{n+1} & = & \mathbf{W}^{(m)} & & &
\end{array}$$

where the residual at each stage is evaluated as a linear combination of the flux and dissipation terms at current and previous stages subject to a consistency requirement. However, in this homework, we will simply use

$$\mathbf{R}^{(m)} = \mathbf{R}(\mathbf{W}^{(m)})$$

that is, the residual evaluated by using the intermediate solution $\mathbf{W}^{(m)}$.

Multistage schemes are chosen because of their extended stability limit and high frequency damping properties which are appropriate for multigrid schemes. Three schemes are often used:

1. 3-stage scheme
$$\alpha_1 = 0.6, \quad \alpha_2 = 0.6$$
2. 4-stage scheme
$$\alpha_1 = 1/4, \quad \alpha_2 = 1/3, \quad \alpha_3 = 1/2$$
3. 5-stage scheme
$$\alpha_1 = 1/4, \quad \alpha_2 = 1/6, \quad \alpha_3 = 3/8, \quad \alpha_4 = 1/2 \quad .$$

3.5 Suggested Reading

1. Anderson, J., ”Modern Compressible Flow with Historical Perspective,” McGraw Hill.
2. Hirsch, Sections 16.4 and 16.6, Chapter 17, and Section 18.3
3. A. Jameson and W. Schmidt and E. Turkel, “Numerical Solution of the Euler Equations by Finite Volume Methods Using Runge-Kutta Time Stepping Schemes”, AIAA Paper 81-1259, July, 1981.
4. A. Jameson, “The Present Status, Challenges, and Future Developments in computational Fluid Dynamics”, AIAA Paper 1999.
5. Other Materials/papers in the course dropbox.