UNIVERSITY OF BERGEN

POROUS MEDIA GROUP

An introduction to spectral methods for solving ODEs

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Numerical methods for ODEs, Karlstad University, 6 Nov 2019.

Introduction

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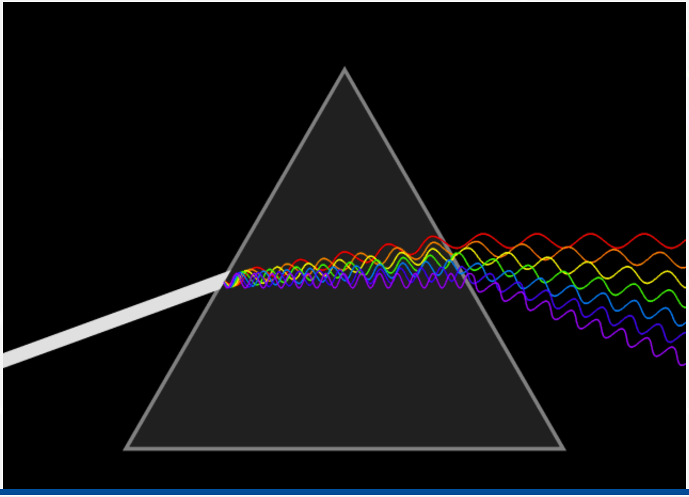

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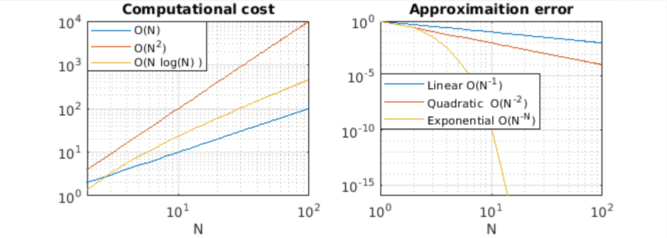
- Exponential convergence rate (Spectral accuracy): $O(h^N)$,$h \approx \frac{1}{N}$
- Computational cost: $O(Nlog(N))$



Outline

Introduction

Orthogonal functions

- Chebyshev polynomials
- Fourier series

Spectral methods

Conclusion

- Recommended references

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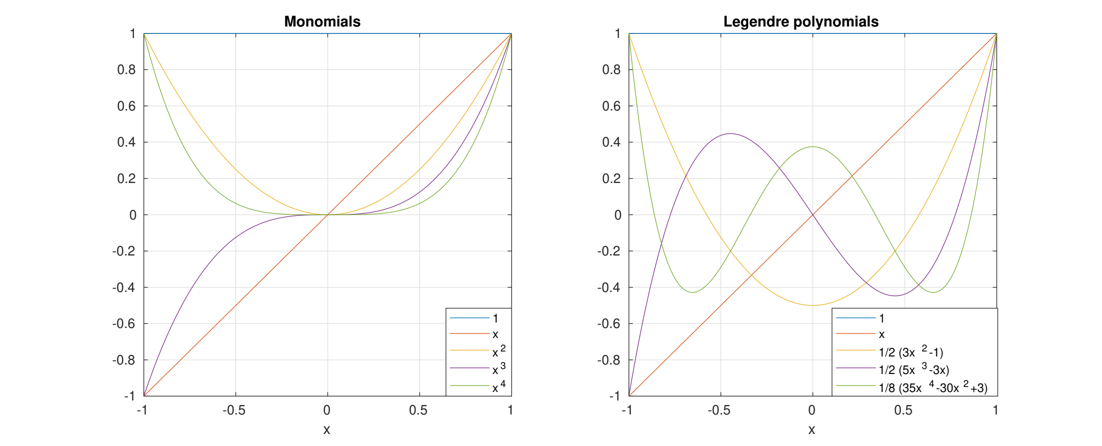
By applying GramSchmidt process to the basis $\{1, x, x^2, ..\}$ the Legendre polynomials are obtained.



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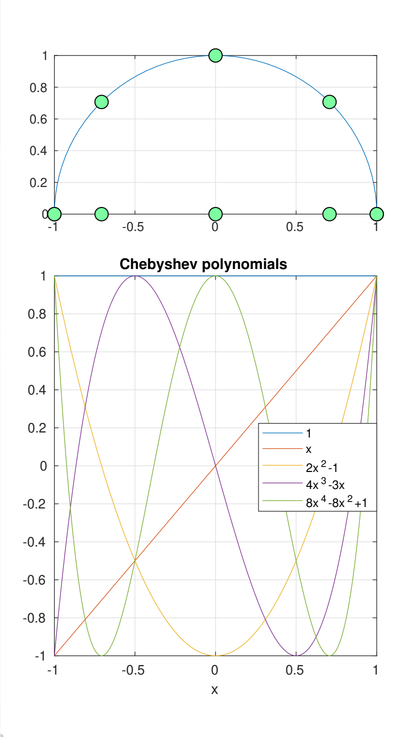
$$T_n(x) = \cos(n\theta(x)),\ \theta(x) := \arccos(x)$$



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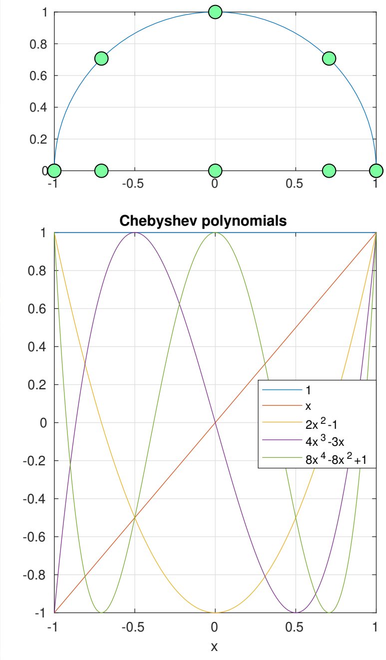

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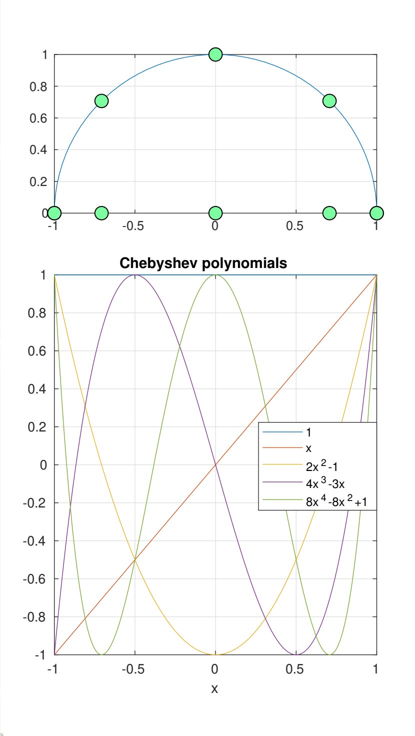
$$\int_{-1}^{1} \frac{T_n(x)T_m(x)}{\sqrt{1-x^2}}\, dx = \begin{cases} 0 & n \neq m \\ \pi & n = m = 0 \\ \frac{\pi}{2} & n = m \neq 0 \end{cases}$$



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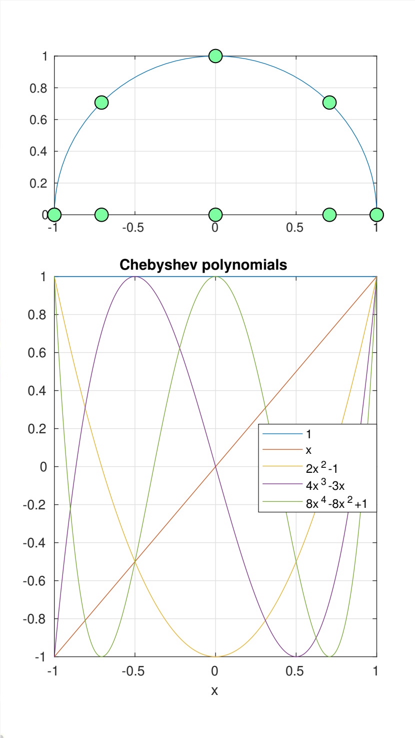
- Derivatives

$$\frac{d^p}{dx^p} T_n(x) = \sum_{k=0}^{N-p} D_{n,k,p}\, T_k(x)$$



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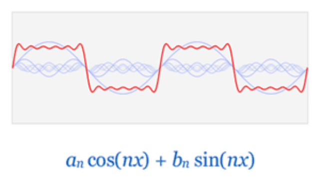

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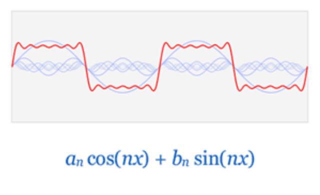

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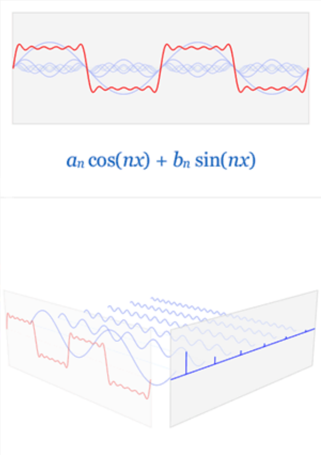

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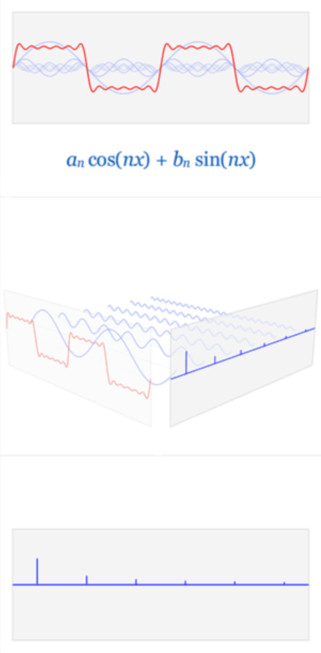

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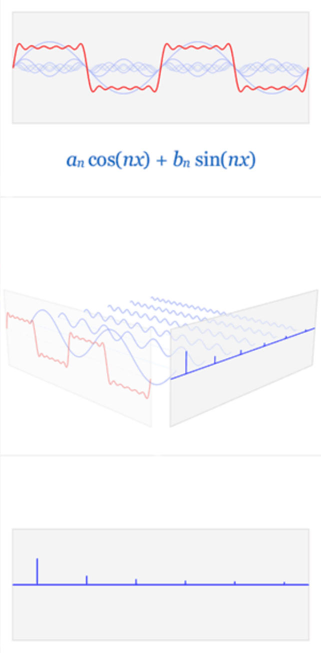

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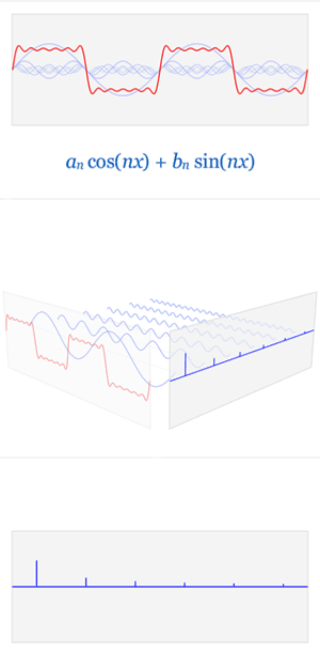

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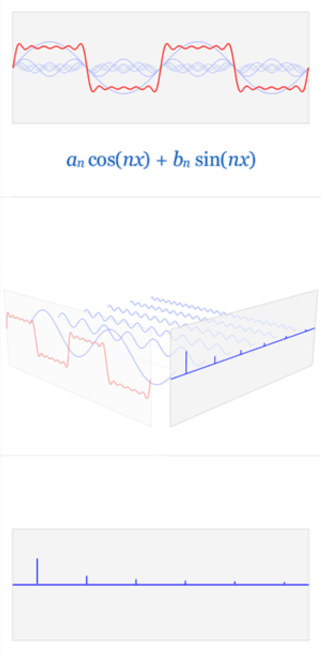

Spectral methods

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Weighted residuals methods: $\int_a^b R(x)\varphi_l(x)w(x)dx = 0,\ \ l = 0 : N$

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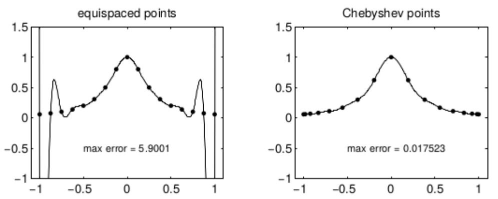
Create an array of points $\{x_l\}_{l=0}^{N}$ such that $a = x_0 < x_1 < ...x_{N-1} < x_N = b$.

$$u^N(a) = 0,\ \ u^N(b) = 0,$$
$$R(x_l) = 0,\ l = 1 : N - 1.$$



Output 9: Degree N interpolation of $u(x) = 1/(1+16x^2)$ in $N+1$ equispaced and Chebyshev points for $N = 16$. With increasing N, the errors increase exponentially in the equispaced case—the Runge phenomenon—whereas in the Chebyshev case they decrease exponentially.

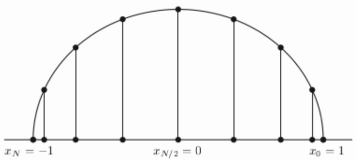


Fig. 5.1. Chebyshev points are the projections onto the x-axis of equally spaced points on the unit circle. Note that they are numbered from right to left.

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Method

$$\sum_{k=0}^{N} c_k \underline{D\phi_k(x_l)} = f(x_l),\ l = 1, \ldots, N-1$$

$$\sum_{k=0}^{N} \underline{(\mathbf{A})_{l,k}}\ c_k = (\mathbf{b})_l,\ l = 1, \ldots, N-1$$

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Example

$$D(\cdot) := (\cdot)'' + (\cdot)' + (\cdot), => \quad u'' + u' + u = f(x),$$

$$\sum_{k=0}^{N} ((\mathbf{D}_2)_{l,k} + (\mathbf{D})_{l,k} + d_l\delta_{lk})\ c_k = b_l,\ l = 1, ..., N-$$

$$(\mathbf{D}_2 + \mathbf{D} + \mathbf{I})\ c = b$$

Differentiation Matrices D

Chebyshev

$$(\mathbf{D})_{ij} = \begin{cases} \frac{2N^2+1}{6}, & i=j=0 \\ -\frac{2N^2+1}{6}, & i=j=N \\ -\frac{x_j}{2(1-x_j^2)}, & i=j; 0 \leq j \leq N \\ (-1)^{i+j}\frac{c_j}{c_j(x_i-x_j)}, & i \neq j \end{cases}$$

$$c_j = \begin{cases} 2 & j=0,N \\ 1 & 1 \leq j \leq N-1 \end{cases}$$

$$(\mathbf{D}_k) = (\mathbf{D})^k$$

the kth differentiation matrix will be the same as applying the first differentiation matrix k times

Fourier

$$(\mathbf{D})_{ij} = \begin{cases} 0 & i=j \\ \frac{1}{2}(-1)^{i-j}cot(\frac{x_i-x_j}{2}), & i \neq \end{cases}$$

$$(\mathbf{D}_2)_{ij} = \begin{cases} -\frac{2N^2+1}{6} & i=j \\ \frac{1}{2}(-1)^{i-j+1}\csc^2(\frac{x_i-x_j}{2}), & i \neq j \end{cases}$$

[0] D. Gottlieb, M. Y. Hussaini, and S. A. Orszag. Introduction: Theory and Applications of Spectral Methods. *Spectral Methods for Partial Differential Equations. SIAM, 1984.*

Collocation method (Pseudospectral method)

Fourier + Fast Fourier Transform

Example

$$D(\cdot) := (\cdot)'' \rightarrow \quad u'' = f(x),$$

$$\sum_{k=0}^{N} c_k {\phi_k}''(x_l) = f(x_l),\ l = 1, ..., N-1$$

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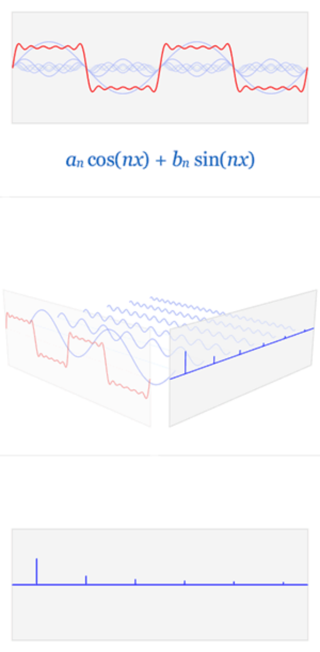

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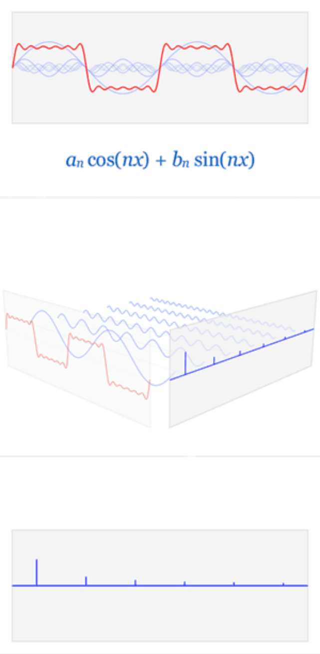

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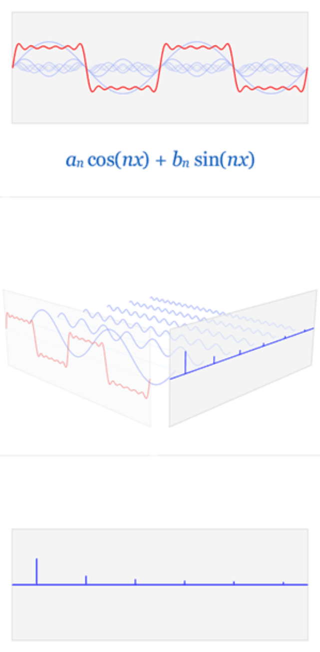

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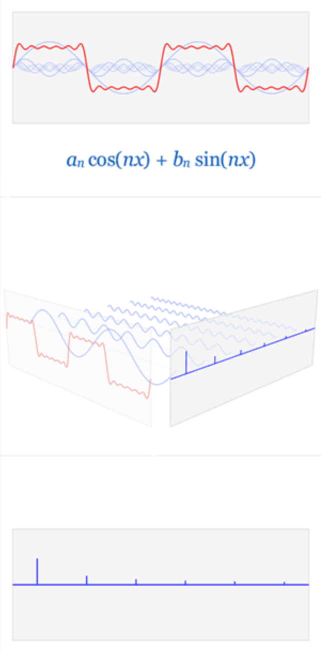
$$\mathcal{F}\left\{\sum_{k=0}^{N} c_k (ik)^2 e^{ikx}\right\} = \mathcal{F}\left\{f(x_l)\right\},\ l = 1, ..., N-1$$

$$c_k (ik)^2 = \hat{f}_l \ \rightarrow \ c_k = \frac{\hat{f}_l}{(ik)^2}$$



Galerkin method

$$\int_a^b R(x)\varphi_l(x)w(x)dx = 0,\ l = 0, .., N$$

Boundary conditions $\phi_k(a) = \phi_k(b) = 0,\ k = 0, ..., N$

$$\sum_{k=0}^{N} \left(\underline{\int_a^b D(\phi_k(x))\ \phi_l(x)\ w(x)\ dx} \right) c_k = \int_a^b f(x)\phi_l(x)\ w(x)\ dx,\ l = 0...N$$

$$\sum_{k=0}^{N} \underline{(\mathbf{A})_{l,k}}\ c_k = (\mathbf{b})_l,\ l = 0, ..., N$$

$$\mathbf{A}c = \mathbf{b}$$

Example

$$D(\cdot) := (\cdot)'' + (\cdot)' + (\cdot), => \quad u'' + u' + u = f(x),$$

$$\sum_{k=0}^{N} ((\mathbf{D}_2)_{l,k} + (\mathbf{D})_{l,k} + d_l\delta_{lk})c_k = \mathbf{b}_l,\ l = 0, ..., N$$

$$(\mathbf{D}_2 + \mathbf{D} + \mathbf{I})\ c = \mathbf{b}$$

Collocation method vs Galerkin method

Collocation method

$$u^N(a) = 0,\ u^N(b) = 0,$$
$$R(x_l) = 0,\ l = 1 : N-1$$

Galerkin method

$$\int_a^b R(x)\varphi_l(x)w(x)dx = 0,\ l = 1 : N-1$$

$$\varphi_l \in \text{span}\{\{\phi_k\}_{k=0}^N\}$$

$$\underline{(\mathbf{A})_{l,k}} = \int_a^b D(\phi_k(x))\ \varphi_l(x)\ w(x)\ dx$$

$$(\mathbf{b})_l = \int_a^b f(x)\varphi_l(x)\ w(x)\ dx,\ l = 0 \ldots N$$

$$\sum_{k=0}^N \underline{(\mathbf{A})_{l,k}}\ c_k = (\mathbf{b})_l,\ l = 0, \ldots, N$$

$$\mathbf{A}c = \mathbf{b}$$

1) Banded matrix A
2) Requires integration for the right hand side
3) Slightly higher acuraccy
4) BC are imposed in the trial functions

Methods

$$\sum_{k=0}^N c_k \underline{D\phi_k(x_l)} = f(x_l),\ l = 1, \ldots, N-1$$

$$\sum_{k=0}^N \underline{(\mathbf{A})_{l,k}}\ c_k = (\mathbf{b})_l,\ l = 1, \ldots, N-1$$

$$\mathbf{A}c = \mathbf{b}$$

1) Dense matrix A
2) No integration need it
3) Easy to implement
4) BC are imposed in the algebraic system

Spectral methods for solvings ODEs

$$\int_a^b R(x)\varphi_l(x)w(x)dx = 0,\ l = 0,..,N$$

$$\int_a^b \left(D \sum_{k=0}^{N} c_k\ \underline{\phi_k}(x) - f(x) \right)\ \underline{\varphi_l}(x)\underline{w}(x)dx = 0$$

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$$\int_a^b R(x)\varphi_l(x)w(x)dx = 0,\ l = 0, .., N$$

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Trial and test function $\{\underline{\phi_k}\}_{k=0}^N, \{\varphi_l\}_{l=0}^N$

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Trial and test function $\{\phi_k\}_{k=0}^N, \{\varphi_l\}_{l=0}^N$

- Collocation

Orthogonality $w(x)$

- Legendre polynomials $P_l^m(x)$
- Laguerre polynomials $L_k(x)$
- Hermite polynomials $H_k(x)$
- Chebyshev polynomials $T_k(x)$
- Trigonometric functions e^{ikx} (Fourier)

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Trial and test function $\{\phi_k\}_{k=0}^N, \{\varphi_l\}_{l=0}^N$

- Collocation
- Galerkin $\{\phi_k\}_{k=0}^N = \{\varphi_l\}_{l=0}^N$
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Spectral methods for solvings ODEs

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- Tau-Lanczos $\{\phi_k\}_{k=0}^N = \{\varphi_l\}_{l=0}^N$
- Petrov-Galerkin $\{\phi_k\}_{k=0}^N \neq \{\varphi_l\}_{l=0}^N$
- Least square $\{\phi_k\}_{k=0}^N, \{\varphi_l\}_{l=0}^M$ with $N \neq M$

Orthogonality $w(x)$

- Legendre polynomials $P_l^m(x)$
- Laguerre polynomials $L_k(x)$
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- Chebyshev polynomials $T_k(x)$
- Trigonometric functions e^{ikx} (Fourier)

Conclusion

Recommended references

- D. Dutykh, A brief introduction to pseudo-spectral methods: application to diffusion problems, *Lecture notes*, 38 pp, 2016. CNRSLAMA, Universit Savoie Mont Blanc, France arXiv:1606.05432v2 [math.NA] 14 Feb 2019
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- M.M. Khader, S. Mziou **Chebyshev spectral method** for studying the viscoelastic slip flow due to a permeable stretching surface embedded in a **porous medium** with viscous dissipation and non-uniform heat generation. *Boundary Value Problems* , (37) 2017.
- V. Vaibhav A **Fast Chebyshev Spectral Method** for Nonlinear Fourier Transform *arXiv:1909.03710 [physics.comp-ph] Submitted on 9 Sep 2019)*

Conlusion

- Spectral methods are a family of methods where the solution of a ODE u is **expanded** into a series of **orthogonal** functions
- The numerical solution is related to its spectrum

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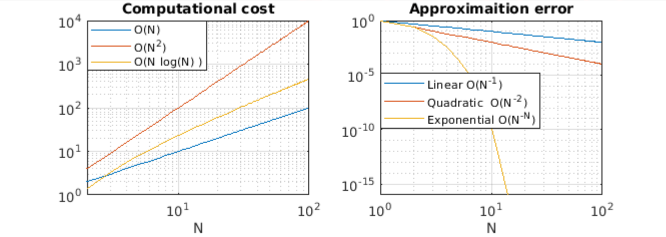
- Spectral accuracy: $O(h^N)$,$h \approx \frac{1}{N}$
- Computational cost: $O(N \log N)$



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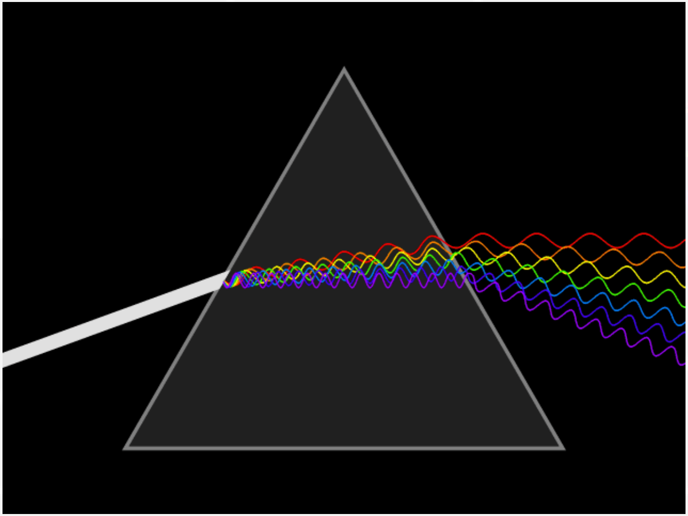

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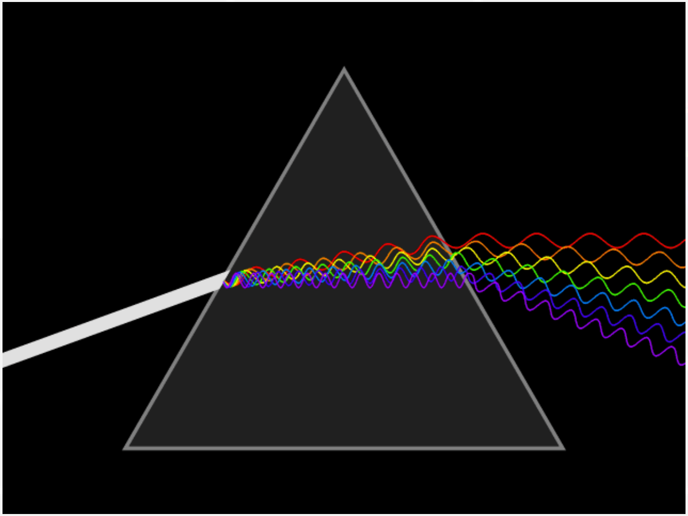

The examples and the slides are available in GitHub https://github.com/tayloris/spectral.git

1.4 Comparisons with Finite Difference Method: Why Spectral Methods are Accurate and Memory-Minimizing

Finite difference methods approximate the unknown $u(x)$ by a sequence of overlapping polynomials which interpolate $u(x)$ at a set of grid points. The derivative of the local interpolant is used to approximate the derivative of $u(x)$. The result takes the form of a weighted sum of the values of $u(x)$ at the interpolation points.

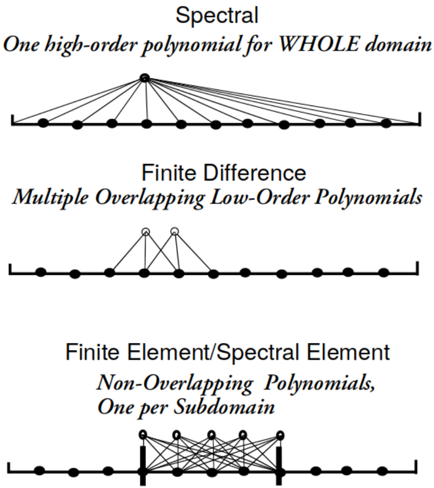

Figure 1.3: Three types of numerical algorithms. The thin, slanting lines illustrate all the grid points (black circles) that *directly* affect the estimates of derivatives at the points shown above the lines by open circles. The thick black vertical lines in the bottom grid are the subdomain walls.

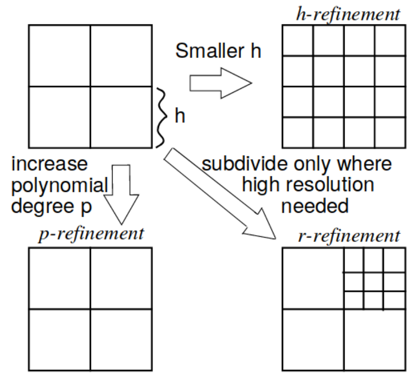

Figure 1.2: Schematic of three types of finite elements [1]

[1] J. P. Boyd. Chebyshev and Fourier spectral methods. *Dover Pubns*, 2001.

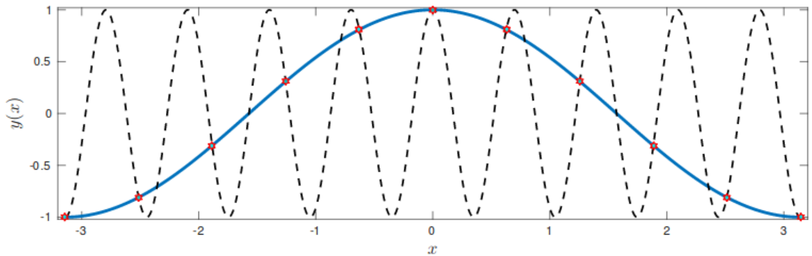

Figure 4. *Illustration of the aliasing phenomenon: two* FOURIER *modes are indistinguishable on the discrete grid. The modes represented here are* $\cos(x)$ *and* $\cos(9x)$ *and the discrete grid is composed of* $N = 11$ *equispaced points on the segment* $[-\pi, \pi]$.

2

[2] D. Dutykh, A brief introduction to pseudo-spectral methods: application to diffusion problems, *Lecture notes*, 38 pp, 2016. CNRSLAMA, Universit Savoie Mont Blanc, France arXiv:1606.05432v2 [math.NA] 14 Feb 2019