Calculus and Thomas the Tank Engine: Relating theoretical math to a real world problem

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PURPOSE

The purpose of this project is to apply principles of calculus and physics (specifically, those of related rates and the conservation of energy) to a real world problem in order to demonstrate the importance of these two disciplines in solving everyday issues—not just theoretical word-problems. The problem to be examined in this project involves a runaway train on an incline, and the buffer stop that must be strong enough to halt the train's descent. An inclined track was picked rather than a horizontal track, because this achieves a constant acceleration due to gravity.

The scenario in this project is as follows: A train starts rolling from rest at the top of a hill. The hill has a 1% grade—in other words, for every 100 meters that the hill reaches horizontally, it rises 1 meter. The train rolls 15 meters down the slope to a spring buffer stop, which must stop the train in 5 meters. Thus, from rest, the train has a total of 20 meters to travel before it stops. If trains of different masses roll down this same hill under the same parameters, the spring at the bottom of the hill will need to be changed. This is because a spring that is exactly strong (or resistant) enough to stop a train that has a mass of 1 kilogram within 5 meters is not strong enough to stop another train that is twice as heavy in that same distance. The research question that this project was designed to answer is below:

How will the required spring strength change as the mass of the train that must be stopped changes?

SIGNIFICANCE

The title of this paper is derived from the children's television series Thomas and Friends, in which train accidents occur frequently. While the accidents in the show are humorous, train crashes in the real world can be devastating. This research is important because it shows how calculus can be taken beyond the theoretical and applied to stopping a heavy, runaway train from causing such damage. Trains carry a lot of momentum, and in the case of a brake failure, or operator error, a moving train poses a huge risk. Officials say that the recent train crash in Hoboken, New Jersey was due to such an operator error. According to CBS News, the train entered the station traveling two to three times the speed limit, injuring over a hundred people and killing one bystander when it “smashed through a concrete-and-steel bumper and knocked out pillars, causing a section of the station’s outdoor roof to collapse” (“Official: Investigators have speed,” 2016). After the crash, the operator was diagnosed with sleep-apnea, a fatigue inducing disorder (“Sleep Apnea,” 2016). While trains are designed to be able to slow down and stop safely, accidents like this show the serious need for extra safety mechanisms—mechanisms like the buffer stop in this project, designed to stop a runaway train as it descends a hill.

BACKGROUND

Before the methodology of this project is outlined, it is important to explain several pertinent concepts, both from calculus and physics.

The discipline of calculus is used mainly to analyze rates of change. Most often, the rates analyzed are a quantity that changes over a period of time. Speed is an example of a rate with respect to time. Speed measures a change in distance with respect to change in time. In this project, however, changing quantities were analyzed with respect to a range of masses for the train on the hill. These rates are described as rates with respect to mass.

Calculus is also used to relate one rate to another, so that if the value for one rate is known, the other can be calculated. In order for this to work, the two rates must represent two quantities that change with respect to a common variable. As stated above, this variable is the mass of the train, not time.

To find the relation between two rates of change, an equation must first be set up that includes variables that represent the quantities whose rates of change are to be related. In this project, the equation was built using a formula for kinetic energy, a formula for work, and the principle of conservation of energy. Kinetic energy is the energy of a moving object. Work is a type of energy caused by applying a net force over a distance. The principle of the conservation of energy states that in any system the mechanical energy (the sum of kinetic energy, potential energy, and work done by outside forces) remains constant. In other words, as one type of energy increases, the others will decrease in proportion, keeping the total constant. In this project, the only forces taken into account were the forces of gravity and the spring buffer-stop upon a train. Forces of air resistance and friction were ignored, for simplicity's sake. Thus, the kinetic energy of the train rolling down the hill is converted completely to work performed by the spring in stopping the train. The force applied against the spring is this work. This total conversion of kinetic energy to work is the conservation of energy mentioned above. Since there is a complete conversion of energy, the formulas for the two types of energy are equal to each other. This equivalence is show below, in Equation 1. Kinetic energy is on the left side of the equation, and work is on the right side.

Equation 1: $\frac{1}{2}mv^2 = \frac{1}{2}kx^2$

where

- m = mass of the train in kilograms. This is a variable. Since different trains of different masses are considered in this scenario, the "train mass" changes.

- v = pre-calculated velocity in meters per second at the point of collision between the spring and the train. Air resistance and friction are ignored in this scenario, so this velocity stays constant for all trains, regardless of mass. The calculation for this constant is shown in Appendix A.
- k = spring constant. A spring constant is a measurement of the stiffness/resistance of a spring, and is the force that the spring exerts divided by the distance it is compressed measured in newtons per meter. This is a variable because it changes for the different springs needed to stop trains of different masses.
- x = the distance the spring will be compressed, in this case 5 meters. This is a constant for all the trains, regardless of mass.

Physics can also be used to predict the instantaneous velocity of an object when it reaches the bottom of a hill after starting at the top of the hill. Equation 2 uses values for initial velocity (v_0), acceleration (a), and distance traveled (Δx: not to be confused with x in Equation 1) to predict the instantaneous velocity (v) of an object after it has traveled Δx. This equation is very useful because it is not dependent on time—thus the time it takes the object to travel the distance Δx does not need to be measured.

$$\text{Equation 2: } v^2 = {v_0}^2 + 2a\Delta x$$

where

- a = acceleration measured in meters-per-second-per-second
- v = final velocity measured in meters-per-second
- v_0 = initial velocity measured in meters-per-second
- Δx = change in distance between initial and final points measured in meters. These points

correspond to the initial and final velocities.

METHODOLOGY

Equation 1 (shown again below) was used as the starting point to construct a related rates equation.

$$\text{Equation 1: } \frac{1}{2}mv^2 = \frac{1}{2}kx^2$$

For simplicities sake, the ½'s were canceled. The result is Equation 1.5

$$\text{Equation 1.5: } mv^2 = kx^2$$

Both sides of the equation were differentiated (a calculus method for finding rates) with respect to the mass of the train. This yields Equation 3 (shown below) which relates the two rates dm/dm and dk/dm. These are the rates with respect to the common variable of mass (*m*), as discussed above, in the "Background" section.

$$\text{Equation 3: } \frac{dm}{dm}v^2 = \frac{dk}{dm}x^2$$

where

- *dm/dm* = the change in train mass with respect to the same change in mass. This rate is equal to 1.
- *v* = velocity of the train when it hits the spring buffers stop measured in meters-per-second. As defined for Equation 1, this is constant for all trains. The calculation of this velocity is shown in Appendix A.
- *dk/dm* = the change in strength of the spring with respect to the change in train mass measured in newtons-per-meter-per-kilogram. This rate is constant just like *dm/dm* above. This rate, however, is unknown. Equation 3 will be used below to solve for this unknown.

To find the change in spring strength with respect to mass (*dk/dm*), the values for *dm/dm*, *v*, and *x* (listed below) are simply plugged into Equation 3:

Equation 3 (shown again): $\frac{dm}{dm}v^2 = \frac{dk}{dm}x^2$

- *dm/dm* = 1
- *v* = 1.7146 meters-per-second (this value is calculated in Appendix A)
- *x* = 5 meters (as stated in the "Purpose" section above)

Substituting these values into Equation 3:

$$1(1.7146^2) = \frac{dk}{dm}5^2$$

$$\frac{1(1.7146^2)}{25} = \frac{dk}{dm}$$

$\frac{dk}{dm} = 0.1176$ newtons-per-meter-per-kilogram

This rate is important because it can be used to predict the strength of a spring (represented by *k*) necessary to stop any given mass, within the parameters defined in this project, without using Equation 1. All that is needed is the rate *dk/dm* and the mass value. The mass value must be multiplied by the rate *dk/dm*. This yields equation 4:

Equation 4: $m\left(\frac{dk}{dm}\right) = k$

To find the value of the spring constant *k* (the strength of the spring) needed to stop a train of 1 kilogram, the rate *dk/dm* found above is multiplied by this mass value according to Equation 4:

$(1)(0.1176) = k = 0.1176$ newtons-per-meter

An examination of the units of measurement for *dk/dm* and *m* shows why multiplying the given mass value by the rate *dk/dm* yields the corresponding value for the strength of the spring. *dk/dm* is measured in newtons-per-meter (N/m) over kilograms (m). The units for *dk/dm* are expressed as a fraction below:

$$\frac{N}{m^2}$$

When the given mass is multiplied by *dk/dm*, the below result is achieved:

$$\frac{N}{m^2}(m)$$

The units for mass cancel, leaving the units for the strength of the spring (*k*):

$$\frac{N}{m}$$

Thus, the result of the multiplication is the corresponding value for *k.*

This value for *k* can be evaluated for accuracy (and thus the accuracy of the rate *dk/dm*) by using Equation 1.5 to find the value for *k* that the mass of 1 kilogram should yield. The values for velocity *v* and compression distance *x* are the same as defined in Appendix A and the "Purpose" section of this paper, respectively.

Equation 1.5: $mv^2 = kx^2$

Substituting the above values for the variables *m, v,* and *x*:

$$(1)(1.7146)^2 = k5^2$$

$$\frac{(1)(1.7146)^2}{5^2} = 0.1176 \text{ newtons-per-meter}$$

Since both Equation 1 and Equation 4 yield the same value for the strength of the spring, Equation 4 is a valid method of utilizing the determined value for *dk/dm*.

The value of 0.1176 newtons-per-meter for *k* means that the spring buffer stop must be able to apply a force of 0.1176 newtons for every meter that it is compressed by the 1-kilogram train, if the spring is to stop the train in 5 meters.

In order to apply Equation 4 to a real-world situation, a range of mass values that reflects actual freight trains was determined. This range was determined through correspondence with Gary Fairbanks, Staff Director for the Motive Power & Equipment Division of the Federal Railroad Administration. Mr. Fairbanks suggested using 10,000 gross tons as a base value for this range, while some freight trains can weigh up to 18,000 gross tons (G. Fairbanks, personal communication, October 31, 2016). 1 gross ton is equal to 907.19 kilograms, so 10,000 gross tons is a little over 9,000,000 kilograms, and 20,000 gross tons is about 18,000,000 kilograms. This range of 9- to 18-million kilograms was split into 10 different values, and applied to Equation 4. The results of these calculations are shown in Table 1:

Table 1

dk/dm	Mass in Kilograms	Spring Strength *k* measured in Newtons-per-meter
0.1176	9,000,000	1,058,400
0.1176	9,900,000	1,164,240
0.1176	10,800,000	1,270,080
0.1176	11,700,000	1,375,920
0.1176	12,600,000	1,481,760
0.1176	13,500,000	1,587,600
0.1176	14,400,000	1,693,440
0.1176	15,300,000	1,799,280
0.1176	16,200,000	1,905,120
0.1176	17,100,000	2,010,960
0.1176	18,000,000	2,116,800

CONCLUSION

In answer to the research question posed in the “Purpose” section, the strength of a spring must increase as the mass of the train which it must stop increases, according to the values in Table 1. By applying principles of physics to a runaway train on a hill, it is possible to predict the strength of a spring required to stop a train of any mass. After setting parameters for the distance that the train travels and the distance the spring is compressed while stepping the train, calculus can be used to determine a much simpler equation for predicting the necessary strength for the spring, given any mass.

Appendix A

Calculating the Velocity of a Train as it Collides with the Spring at the Bottom of the Hill

As discussed in the “Background” section, Equation 2 (shown again below) is used to determine the velocity of a train as it collides with a spring buffer stop.

Equation 2: $v^2 = {v_0}^2 + 2a\Delta x$

The train, irrespective of mass, starts from rest 15 meters from the spring. Thus, its initial velocity is zero, and the value of Δx is 15 meters. These values were plugged into Equation 2. But before the instantaneous velocity could be determined, the acceleration down the hill had to be found.

In this case, it is assumed that the train’s engine is not running, so the only acceleration is due to gravity. The acceleration of an object falling straight down is 9.8 meters-per-second-per-second (the change in velocity, meters-per-second, with respect to change in time). Since the train is not traveling straight down, however, its acceleration is less that 9.8 meters-per-second-per-second. Because of this, geometry and trigonometry must be used to determine its acceleration due to gravity. The formula for this is shown below:

Formula 1: $Acceleration\ of\ a\ train\ down\ a\ slope = 9.8 \sin\theta$

where θ = angle between slope of hill and horizontal “base” of hill

Also using trigonometry, the angle between the hill and its base can be found. As stated in the “Purpose” section, the hill rises vertically 1 meter for every 100 meters it runs horizontally. For a right triangle like this hill, the trigonometric function tan θ is equal to the ratio of the vertical side length to the horizontal side length. Thus:

Formula 2: $\tan\theta = \frac{1}{100}$

where θ = angle between slope of hill and horizontal “base” of hill

Solving for θ:

$$\theta = arctan\frac{1}{100}$$

$$\theta = 0.573$$

Substituting this value into Formula 1:

$$Acceleration\ of\ a\ train\ down\ a\ slope = 9.8 \sin 0.573$$

$$9.8 \sin 0.573 = 0.0980$$

Thus, the train accelerates down the hill at a rate of 0.0980 meters-per-second-per-second.

This value, along with the values for v_0 (initial velocity, equal to zero) and Δx (distance between the train's staring point and the spring, equal to 15 meters) from above, are substituted into Equation 2:

$$v^2 = {v_0}^2 + 2a\Delta x$$

$$v^2 = 0^2 + 2(.0980)(15)$$

$$v = \sqrt{0^2 + 2(.0980)(15)}$$

$$v = 1.7146$$

Thus, the velocity of each train when it collides with the spring at the bottom of the hill is 1.7146 meters-per-second.

Appendix B

List of Equations & Formulas

Equation 1: $\frac{1}{2}mv^2 = \frac{1}{2}kx^2$

Equation 1.5: $mv^2 = kx^2$

Equation 2: $v^2 = {v_0}^2 + 2a\Delta x$

Equation 3: $\frac{dm}{dm}v^2 = \frac{dk}{dm}x^2$

Equation 4: $m\left(\frac{dk}{dm}\right) = k$

Formula 1: $Acceleration\ of\ a\ train\ down\ a\ slope = 9.8\sin\theta$

Formula 2: $\tan\theta = \frac{1}{100}$

RESOURCES

Professors Amy Harris, Sarada Moturu, and Martha Donnelly; Lone Star College System

Larson, R., & Edwards, B. (2012). *Calculus* (10th ed.). Boston, MA: Brooks/Cole.

Young, H. D., & Freedman, R. A (2012). *Sear's and Zemanksy's University Physics: With Modern Physics* (13th ed.). San Francisco, CA: Addison-Wesley.

WORKS CITED

"Official: Investigators have speed estimate of Hoboken train at time of crash." (2016, Oct. 4). *CBSNews.* Retrieved from http://www.cbsnews.com/news/official-investigators-have-speed-estimate-of-hoboken-train-at-time-of-crash/

"Sleep apnea warning issued by regulators after deadly New Jersey train crash." (2016, Oct. 21). *CBSNews.* Retrieved from http://www.cbsnews.com/news/sleep-apnea-warning-regulators-new-jersey-train-crash/